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Parametrix for a Characteristic Cauchy Problem.

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0. – Introduction, statement of the problem and main results.

In this paper we consider the following second order differential operator with smooth coefficients defined in $R^{n+1} = R_t \times R_x^n$:

$$(0.1)\quad P = t\partial_t^2 - \sum_{i,j=1}^{n} a_{ij}(t,x)\,\partial_{x_i}\partial_{x_j} + \big(\nu(t,x)+1\big)\,\partial_t + \sum_{j=1}^{n} b_j(t,x)\,\partial_{x_j} + b_0(t,x)\,.$$

We assume that the functions a_{ij} are real, $a_{ij} = a_{ji}$, $i, j = 1, \ldots, n$ and that for some $\delta > 0$ we have

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\,\xi_i\xi_j \geqq \delta|\xi|^2$$

for every $(t,x) \in R^{n+1}$, $\xi \in R^n$. For sake of simplicity we shall suppose that all coefficients in (0.1) are constant outside of a compact set.

We are concerned with the Cauchy problem:

$$(0.2)\quad \begin{cases} Pu(t,x) = 0\,, & t > 0 \\ u|_{t=0} = g \in \mathcal{E}'(R^n)\,. \end{cases}$$

One can prove that the Cauchy problem (0.2) is C^∞-well posed iff $\nu(0,x) + 1 \notin \{0, -1, -2, \ldots\}$, which we assume from now on.

We propose to construct a parametrix for pb. (0.2), i.e. an operator $E\colon \mathcal{E}'(R_x^n) \to C^\infty\big([0,T];\mathcal{D}'(R_x^n)\big)$ (for a suitable $T > 0$) such that

$$(0.3)\quad \begin{cases} PE\colon \mathcal{E}'(R_x^n) \to C^\infty\big([0,T]\times R_x^n\big) \\ \gamma E - I\colon \mathcal{E}'(R_x^n) \to C^\infty(R_x^n)\,, \end{cases}$$

where γ denotes the restriction to the hyperplane $t = 0$.

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Actually, under some technical additional conditions, we shall construct a parametrix E with the following properties:

$$(0.4)\qquad WF(\gamma\partial_t^k Eg)\subset WF(g)\,,\qquad g\in \mathcal{E}'(R^n)\,,\quad k=1,2,\ldots$$

(0.5) For every $g\in\mathcal{E}'(R^n)$ and for every $s\in[0,T[$:

$$WF(Eg|_{t=s}) = (\Lambda^+_{2\sqrt{s}}\cup\Lambda^-_{2\sqrt{s}})\ \mathrm{o}\ WF(g)\,,$$

where $\Lambda_t^\pm\subset(T^*R^n\setminus 0)\times(T^*R^n\setminus 0)$ are the two canonical relations defined in the following way: for every $(y,\eta)\in T^*R^n\setminus 0$ let $(x^\pm(t;y,\eta),\xi^\pm(t;y,\eta))$ be the integral curve of the Hamiltonian vector field $H_{\pm a(t,x,\xi)}\big(a(t,x,\xi) = (\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j)^{\frac12}\big)$ issued from (y,η), then

$$(0.6)\qquad \Lambda_t^\pm = \{((x^\pm(t;y,\eta),\xi^\pm(t;y,\eta),(y,\eta))|(y,\eta)\in T^*R^n\setminus 0\}\,.$$

We point out that $\Lambda_t^\pm$ are the usual canonical relations appearing in the Cauchy problem for the wave operator $\partial_t^2-\sum_1^n a_{ij}(t,x)\partial_{x_i}\partial_{x_j}$ (see e.g. J. J. Duistermaat [3]).

Then, modulo uniqueness for pb. (0.2), we obtain from (0.5) a precise description of the singularities of the solutions $u\in C^\infty(\overline{R_t^+};\mathcal{D}'(R_x^n))$ of pb. (0.2), while (0.4) implies that singularities do not scatter along the boundary.

The construction of E is quite long and technical since the usual methods of geometrical optics cannot be applied.

To motivate such a construction consider the following particular case of (0.1)

$$(0.1)'\qquad P_0 = t\partial_t^2 - \Delta_x + (\nu_0+1)\partial_t\,,\qquad \nu_0\in C\,.$$

To solve (0.2) for P_0 we take the Fourier transform $\hat{u}(t,\xi)=\int\exp[-ix\cdot\xi]\cdot u(t,x)\,dx$ of u and obtain

$$(0.7)\qquad \begin{cases} t\partial_t^2\hat{u}(t,\xi)+(\nu_0+1)\partial_t\hat{u}(t,\xi)+|\xi|^2\hat{u}(t,\xi)=0\,, & t>0\\ \hat{u}(0,\xi)=\hat{g}(\xi)\,. \end{cases}$$

Putting $z=2\sqrt{t}\,|\xi|$ and writing $\hat{u}(t,\xi)=t^{-\nu_0/2}v(z,\xi)$, it can be easily seen that v satisfies the Bessel equation

$$(0.8)\qquad z^2\partial_z^2 v(z,\xi)+z\partial_z v(z,\xi)+(z^2-\nu_0^2)v(z,\xi)=0\,.$$

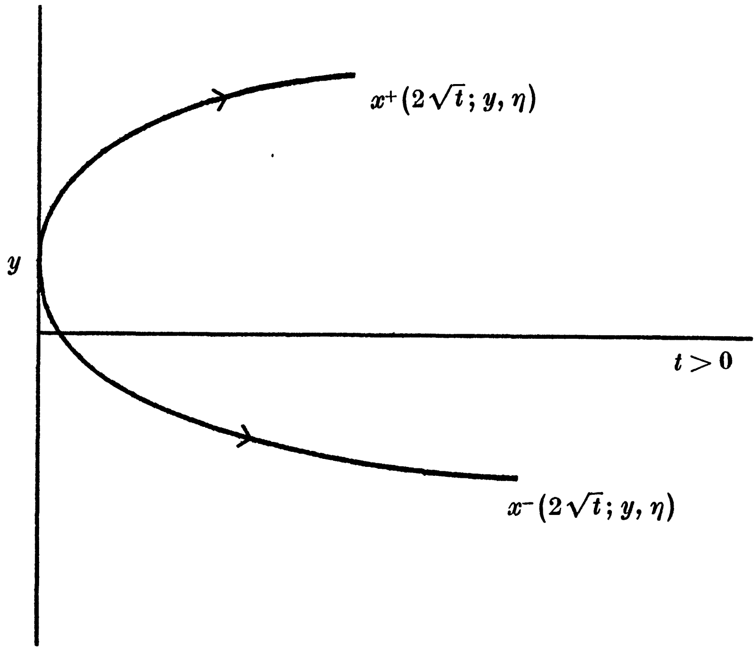

Fig. 1 – The characteristics $x^{\pm}(2\sqrt{t}; y, \eta)$ are tangent to $t = 0$.

Since we are looking for solutions which are smooth in the t variable up to $t = 0$, taking into account the initial condition, we get

$$(0.9)\qquad u(t, x) = E_0 g(t, x) = \int \exp[ix\cdot\xi]\,\tilde{J}_{\nu_0}(2\sqrt{t}\,|\xi|)\hat{g}(\xi)\,\check{d}\xi\,, \qquad (\check{d}\xi = (2\pi)^{-n}d\xi)\,,$$

where

$$(0.10)\qquad \tilde{J}_{\nu_0}(z) = \Gamma(\nu_0 + 1)(z/2)^{-\nu_0}J_{\nu_0}(z)\,, \qquad \nu_0 \notin \{-1, -2, \ldots\}\,,$$

$J_{\nu_0}(z)$ being the usual Bessel function of the first kind.

It can be easily recognized that E_0 extends as a continuous operator from $\mathcal{E}'(R^n)$ into $C^\infty(\overline{R_t^+}; \mathcal{D}'(R_x^n))$ and that $P_0E_0g = 0$, $\gamma E_0 g = g$.

Relation (0.4) is trivially verified. To prove (0.5) we split $J_{\nu_0}(z)$ into a sum of the two Hankel functions $J_{\nu_0}(z) = \frac{1}{2}\big(H_{\nu_0}^{(1)}(z) + H_{\nu_0}^{(2)}(z)\big)$ (see G. Watson [7]).

The functions $H_{\nu_0}^{(1)}(z)$, $H_{\nu_0}^{(2)}(z)$ have the following asymptotic expansion for $z \to +\infty$:

$$H_{\nu_0}^{(1),(2)}(z) \sim (\pi z/2)^{-\frac{1}{2}} \exp\left[\mp i\left(z - \frac{\pi\nu_0}{2} - \frac{\pi}{4}\right)\right] \sum_{j\geqq 0} c_j(\pm 2iz)^{-j}, \qquad c_0 = 1, \tag{0.11}$$

(see G. Watson [7, Sec. 7.2 (5), (6)]).

By (0.11), for every $t > 0$ the operator E_0 splits into a sum of two independent elliptic Fourier integral operators of the form

$$\int \exp\left[i(x\xi \pm 2\sqrt{t}\,|\xi|)\right] b^{\pm}(2\sqrt{t}\,|\xi|)\,\hat{g}(\xi)\,\bar{d}\xi, \tag{0.12}$$

for some $b^{\pm}(z) \in S_{1,0}^{-\mathrm{Re}\,\nu_0 - \frac{1}{2}}(R_z^+)$.

Relation (0.5) is now a straightforward consequence of (0.12) and the calculus of the wave front set (see L. Hörmander [5]).

We remark that the amplitude $\tilde{J}_{\nu_0}(2\sqrt{t}\,|\xi|)$ in (0.9) exhibits a rather different behaviour in the two regions $\sqrt{t}\,|\xi| \lessgtr$ const. More precisely, in the region $\sqrt{t}\,|\xi| <$ const. the parametrix E_0 behaves like a pseudo differential operator (with non-classical symbol), while for $\sqrt{t}\,|\xi| \to +\infty$ E_0 is essentially the sum of two elliptic Fourier integral operators whose phases are hidden in the amplitude $\tilde{J}_{\nu_0}$. This remark suggests that, in the general case (0.2), one should perform two different constructions in the regions $\sqrt{t}\,|\xi| <$ const., $\sqrt{t}\,|\xi| >$ const. respectively.

According to this strategy we collect in Ch. 1 all the formal ingredients we need to construct the parametrix: in particular, in Sect.s 1.1-1.3 and Sect.s 1.4-1.6 we construct a formal parametrix for (0.2) in the region $\sqrt{t}\,|\xi| \lessgtr$ const. respectively, by using suitable integral representations for Bessel's functions. We point out that such a technique has been already used in the literature (see e.g. S. Alinhac [1]).

In Ch. 2 the two formal parametrices are glued together and a precise operator calculus is developed.

Chapter 1

FORMAL THEORY

1.1. – Formal parametrix in the region $\sqrt{t}\,|\xi| <$ const.

The amplitude for the parametrix E_0 in (0.9) has the homogeneity property $\tilde{J}_{\nu_0}(2\sqrt{t/\lambda}\,|\sqrt{\lambda}\xi|) = \tilde{J}_{\nu_0}(2\sqrt{t}\,|\xi|)$, $\lambda > 0$. This suggests that the right homogeneity involved in the problem is of the following type:

$$f(t/\lambda, x, \sqrt{\lambda}\xi) = \lambda^m f(t, x, \xi)\,, \qquad \lambda > 0\,.$$

We are thus led to consider operators of the form

$$\int \exp\,[ix\cdot\xi]\,q(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\,,$$

where the amplitude q is given by an « asymptotic sum » of functions homogeneous in the above sense.

The following definition will be convenient.

DEF. 1.1.1. Let m be a real number.

i) *By* $\mathcal{O}^m$ *we denote the class of the functions*

$$g(x, \xi) \in C^\infty(R^n_x \times \dot{R}^n_\xi) \qquad \textit{such that}$$
$$g(x, \lambda\xi) = \lambda^m g(x, \xi)\,, \qquad \lambda > 0\,.$$

ii) *By* Ψ^m *we denote the class of the functions*

$$f(t, x, \xi) \in C^\infty(\overline{R^+_t} \times R^n_x \times \dot{R}^n_\xi) \qquad \textit{such that}$$
$$f(t/\lambda, x, \sqrt{\lambda}\xi) = \lambda^m f(t, x, \xi)\,, \qquad \lambda > 0\,.$$

It is easy to check that the operator $t^h\partial_t^k\partial_x^\alpha\partial_\xi^\beta$ maps Ψ^m into $\Psi^{m-h+k-|\alpha|/2}$ and that $\mathcal{O}^k \times \Psi^m \ni (g, f) \to gf \in \Psi^{m+k/2}$.

We consider, formally, the following operator:

$$Eg(t, x) = \int \exp\,[ix\cdot\xi]\,q(t, x, \xi)\hat{g}(\xi)\check{d}\xi\,, \qquad g \in C_0^\infty(R^n_x)\,, \tag{1.1.1}$$

where

$$\begin{cases} q(t, x, \xi) \sim \sum_{j\geq 0} q_{-j/2}(t, x, \xi) \\ q_{-j/2} \in \Psi^{-j/2}\,, \qquad j = 0, 1, 2, \ldots \end{cases} \tag{1.1.2}$$

Imposing that Eg satisfies (0.2), we obtain

$$\begin{cases} PEg(t, x) = \int \exp\,[ix\cdot\xi]\,\tilde{q}(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi \\ \tilde{q}(t, x, \xi) = \exp\,[-\,ix\cdot\xi]\,P\big(\exp\,[ix\cdot\xi]\,q(t, x, \xi)\big) \sim 0 \\ \tilde{q}(0, x, \xi) \sim 1\,. \end{cases} \tag{1.1.3}$$

To implement (1.1.3) we write $\tilde{q} \sim \sum_{j\geqq 0} \tilde{q}_{1-j/2}$, with $\tilde{q}_{1-j/2} \in \Psi^{1-j/2}$, $j \geqq 0$. To compute the formal series $\sum_{j\geqq 0} \tilde{q}_{1-j/2}$ in terms of the q's we replace the coefficients of P by their formal Taylor expansions and collect in $\exp[-ix\cdot\xi]\cdot P(\exp[ix\cdot\xi]q)$ all the terms with the same homogeneity degree in the sense of Def. 1.1.1. It is convenient to introduce the following notation

$$(1.1.4)\qquad \begin{cases} A_k(x, \partial_x) = \sum_{i,j=1}^{n} \left(\frac{1}{k!}\,\partial_t^k a_{ji}|_{t=0}\right)\partial_{x_i}\partial_{x_j} \\ B_k(x, \partial_x) = \sum_{j=1}^{n} \left(\frac{1}{k!}\,\partial_t^k b_j|_{t=0}\right)\partial_{x_j}, \\ b_{0,k}(x) = \frac{1}{k!}\,\partial_t^k b_0|_{t=0} \\ \nu_k(x) = \frac{1}{k!}\,\partial_t^k \nu|_{t=0} \\ a(x,\xi) = \sqrt{A_0(x,\xi)} \\ M_k(x,\xi,\partial_x) = \frac{2}{i}\sum_{i,j=1}^{n}\left(\frac{1}{k!}\,\partial_t^k a_{ij}|_{t=0}\right)\xi_i\partial_{x_j}. \end{cases} \qquad k \geqq 0$$

Using (1.1.4) we define the differential operators:

$$(1.1.5)\qquad \begin{cases} L_1 = t\partial_t^2 + (\nu_0(x)+1)\partial_t + A_0(x,\xi) \\ L_{\frac{1}{2}-k} = t^k[M_k(x,\xi,\partial_x) + iB_k(x,\xi)], \qquad k \geqq 0 \\ L_{-k} = t^{k+1}A_{k+1}(x,\xi) + t^k[-A_k(x,\partial_x) + B_k(x,\partial_x) \\ \qquad\qquad + b_{0,k}(x)] + t^{k+1}\nu_{k+1}(x)\partial_t, \qquad k \geqq 0. \end{cases}$$

We note that $L_1\colon \Psi^m \to \Psi^{m+1}$, $L_{\frac{1}{2}-k}\colon \Psi^m \to \Psi^{m+\frac{1}{2}-k}$, $L_{-k}\colon \Psi^m \to \Psi^{m-k}$, $k \geqq 0$.

A straightforward computation yields:

$$(1.1.6)\qquad \tilde{q} \sim L_1 q_0 + (L_1 q_{-\frac{1}{2}} + L_{\frac{1}{2}} q_0) + (L_1 q_{-1} + L_{\frac{1}{2}} q_{-\frac{1}{2}} + L_0 q_0) + \ldots = \sum_{j\geqq 0} \tilde{q}_{1-j/2}$$

with:

$$(1.1.7)\qquad \tilde{q}_{1-j/2} = \sum_{h=0}^{j} L_{1-h/2}\, q_{-j/2+h/2}, \qquad j \geqq 0.$$

Conditions (1.1.3) can thus be rewritten as the following sequence of transport equations

$$(1.1.8)\qquad \begin{cases} L_1 q_0 = 0\,, & t>0 \\ q_0(0,x,\xi)=1 \end{cases}$$

$$(1.1.9)_j\qquad \left\{\begin{array}{l} L_1 q_{-j/2} = -\sum\limits_{h=1}^{j} L_{1-h/2}\, q_{-j/2+h/2}\,, \qquad t>0\,, \\ \\ q_{-j/2}(0,x,\xi)=0\,. \end{array}\right. \qquad j\geqq 1$$

1.2. – The first transport equation: $L_1 q_0 = 0$.

To solve the Cauchy problem (1.1.8) we reduce the equation $L_1 q_0 = 0$ to a Bessel equation. For this purpose we change the variables as follows:

$$(1.2.1)\qquad z = 2\sqrt{t}\, a(x,\xi)$$

$$(1.2.2)\qquad q_0(t,x,\xi) = t^{-\nu_0(x)/2} w(z;x,\xi)\,.$$

Using the relations

$$(1.2.3)\qquad \begin{cases} \partial_t = \dfrac{2A_0}{z}\,\partial_z \\ \partial_t^2 = 4A_0^2\left(\dfrac{1}{z^2}\,\partial_z^2 - \dfrac{1}{z^3}\,\partial\right) \\ t\partial_t = \dfrac{1}{2} z\partial_z \end{cases}$$

we obtain the following Bessel equation for $w(z)$:

$$(1.2.4)\qquad L_1 q_0 = \frac{t^{-\nu_0/2-1}}{4}\,[z^2\partial_z^2 w(z) + z\partial_z w(z) + (z^2 - \nu_0^2)\,w(z)] = 0\,.$$

The Bessel function

$$(1.2.5)\qquad J_{\nu_0(x)}(z) = \left(\frac{z}{2}\right)^{\nu_0(x)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu_0(x)+k+1)} \left(\frac{z}{2}\right)^{2k}, \quad z>0\,,$$

is a solution of (1.2.4). Taking into account (1.2.1) and (1.2.2) we are led to define

$$(1.2.6)\qquad q_0(t,x,\xi)=\Gamma(\nu_0(x)+1)\left(\frac{z}{2}\right)^{-\nu_0(x)}J_{\nu_0(x)}(z)\big|_{z=2\sqrt{t}a(x,\xi)},$$

which is well defined as an element of Ψ^0 satisfying (1.1.8) provided

$$(1.2.7)\qquad \nu_0(x)+1\notin\{0,-1,-2,\dots\},\qquad x\in R^n.$$

From now on we assume that (1.2.7) is satisfied.

1.3. – The other transport equations.

To solve the Cauchy problems $(1.1.9)_j$ we shall use the following integral representation for $J_{\nu_0}(z)$ (see G. Watson [7, p. 163 (1)]):

$$(1.3.1)\quad J_{\nu_0}(z)=\frac{\Gamma(\frac{1}{2}-\nu_0)}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu_0}\int_L \exp[iz\sigma](\sigma^2-1)^{\nu_0-\frac{1}{2}}\,\check{d}\sigma,\ \left(\check{d}\sigma=\frac{1}{2\pi i}\,d\sigma\right),$$

where L is the contour shown in fig. 2 and the argument of $\sigma+1$ and $\sigma-1$ is chosen to be zero at the point A.

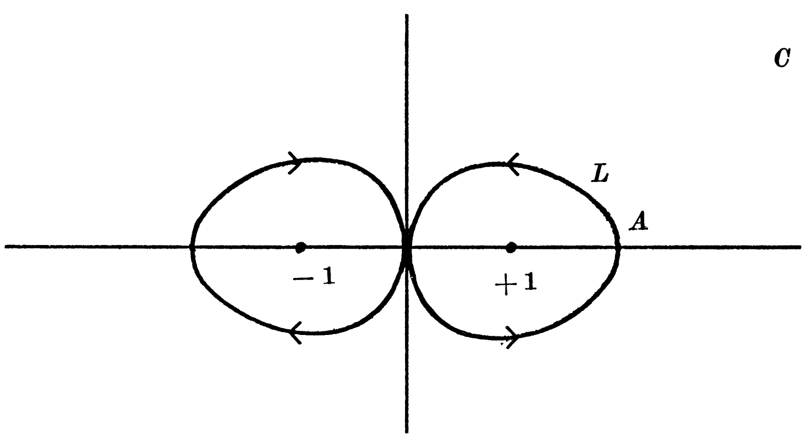

Fig. 2 – L is a contour symmetric with respect to the origin, enclosing the points ± 1.

The above representation makes sense provided

$$(1.3.2)\qquad \nu_0(x)-\tfrac{1}{2}\notin\{0,1,2,\dots\},\qquad x\in R^n,$$

which is a technical condition we shall assume from now on.

Putting

$$\hat{q}_0(z; x) = \int_L \exp[iz\sigma](\sigma^2 - 1)^{\nu_0(x)-\frac{1}{2}} \,\check{d}\sigma \,, \tag{1.3.3}$$

we have proved in Sect. 1.2 that

$$q_0(t, x, \xi) = \frac{\Gamma(\nu_0(x) + 1)\Gamma(\frac{1}{2} - \nu_0(x))}{\sqrt{\pi}} \hat{q}_0(2\sqrt{t}\, a(x, \xi); x) \,. \tag{1.3.4}$$

To state the main result of this Sect. we need some definitions.

DEF. 1.3.1. Let $p \in Z$, $q \in Z_+$. Define

$$\begin{cases} \varphi_{p,q}(z; x) = \int_L \exp[iz\sigma](\sigma^2 - 1)^{\nu_0(x)-p-\frac{1}{2}}(\log(\sigma^2 - 1))^q \,\check{d}\sigma \\ \tilde{\varphi}_{p,q}(z; x) = \int_L \exp[iz\sigma]\sigma(\sigma^2 - 1)^{\nu_0(x)-p-\frac{1}{2}}(\log(\sigma^2 - 1))^q \,\check{d}\sigma \,. \end{cases} \tag{1.3.5}$$

Let $\alpha, \beta, j \in Z_+$. By $U^{\alpha,\beta}_{-j}$ we denote the class of all functions g of the form

$$g(z; x, \xi) = \sum_{p=0}^{\alpha} \sum_{q=0}^{\beta} c_{p,q}(x, \xi)\varphi_{p,q}(z; x) \,, \tag{1.3.6}$$

where $c_{p,q} \in \mathcal{O}^{-j}$.

Note that the functions $\varphi_{p,q}$ and $\tilde{\varphi}_{p,q}$ are holomorphic with respect to the variable z and C^∞ in x; moreover, because of the symmetry of the contour L, the functions $\varphi_{p,q}$ are even functions of z.

From this remark it follows that given $g(z; x, \xi) \in U^{\alpha,\beta}_{-j}$ then $g(2\sqrt{t}a(x, \xi)); x, \xi) \in \Psi^{-j/2}$.

THEOREM 1.3.1. *For every $j \geq 0$ there exists a function $\hat{q}_{-j}(z; x, \xi) \in U^{2j,2j}_{-j}$ such that the functions*

$$q_{-j/2}(t, x, \xi) = \hat{q}_{-j}(2\sqrt{t}\, a(x, \xi); x, \xi) \in \Psi^{-j/2}$$

are solutions of the Cauchy problems (1.1.8), $(1.1.9)_j$.

PROOF. By induction on j. For $j = 0$ the assertion follows from the construction in Sect. 1.2 and from (1.3.3), (1.3.4).

Let us suppose that we have already found functions $\hat{q}_0, \hat{q}_{-1}, \dots, \hat{q}_{-(j-1)}$, $j \geqslant 1$, with $\hat{q}_{-h} \in U^{2h,2h}_{-h}$, such that $q_{-h/2}(t, x, \xi) = \hat{q}_{-h}(2\sqrt{t}\, a(x, \xi); x, \xi)$ satisfies (1.1.8), if $h = 0$, and $(1.1.9)_h$ for $h = 1, 2, \dots, j - 1$.

Equation $(1.1.9)_j$ can be rewritten as

$$(1.3.7)\qquad L_1 q_{-j/2} = -\sum_{\substack{k\geq 0\\ 2k+1\leq j}} L_{\frac{1}{2}-k} q_{-(j-2k-1)/2} - \sum_{\substack{k\geq 0\\ 2k+2\leq j}} L_{-k} q_{-(j-2k-2)/2}\,.$$

Moreover, using (1.1.5) we can write

$$(1.3.8)\qquad L_{\frac{1}{2}-k} = t^k\Big[\sum_{j=1}^{n} c_{j,k}(x,\xi)\,\partial_{z_j} + c_{0,k}(x,\xi)\Big],$$

for some $c_{j,k},\ c_{0,k}\in \mathcal{O}^1$.

Analogously:

$$(1.3.9)\qquad L_{-k} = t^k \sum_{|\alpha|\leq 2} c_{\alpha,k}(x)\,\partial_x^\alpha + t^{k+1}A_{k+1}(x,\xi) + t^{k+1}\nu_{k+1}(x)\,\partial_t\,,$$

where $A_{k+1}(x,\xi)$ is defined in (1.1.4) and $c_{\alpha,k}$ are smooth functions of x.

Since $t^k = z^{2k}/(2a(x,\xi))^{2k}$ and $t\partial_t = \frac{1}{2}z\partial_z$, (1.3.8) and the inductive hypothesis imply that

$$(1.3.10)\qquad -\sum_{\substack{k\geq 0\\ 2k+1\leq j}} L_{\frac{1}{2}-k} q_{-(j-2k-1)/2} = \sum_{\substack{k\geq 0\\ 2k+1\leq j}}\ \sum_{\substack{0\leq p\leq 2(j-2k-1)\\ 0\leq q\leq 2(j-2k-1)}} \cdot\Big[c_{p,q,k}(x,\xi)z^{2k}\varphi_{p,q}(z;x) + \sum_{l=1}^{n} d_{p,q,k,l}(x,\xi)\,z^{2k}\partial_{x_l}\varphi_{p,q}(z;x)\Big],$$

for some $c_{p,q,k},\ d_{p,q,k,l}\in \mathcal{O}^{-j+2}$.

In the same way, using (1.3.9) we get

$$(1.3.11)\qquad -\sum_{\substack{k\geq 0\\ 2k+2\leq j}} L_{-k} q_{-(j-2k-2)/2} = \sum_{\substack{k\geq 0\\ 2k+2\leq j}}\ \sum_{\substack{0\leq p\leq 2(j-2k-2)\\ 0\leq q\leq 2(j-2k-2)}}$$

$$\cdot\Big[c_{p,q,k}(x,\xi)z^{2k+2}\varphi_{p,q}(z;x) + \sum_{|\gamma|\leq 2} d_{p,q,k,\gamma}(x,\xi)z^{2k}\partial_x^\gamma\varphi_{p,q}(z;x) + z^{2k}\{e_{p,q,k}(x,\xi)\varphi_{p,q}(z;x)$$

$$+ f_{p,q,k}(x,\xi)\varphi_{p+1,q}(z;x) + g_{p,q,k}(x,\xi)\varphi_{p,(q-1)_+} + h_{p,q,k}(x,\xi)\varphi_{p+1,(q-1)_+}\}\Big],$$

where the coefficients belong to $\mathcal{O}^{-j+2}$.

Taking into account the formula

$$(1.3.12)\qquad \partial z/\partial x_j = \frac{\partial_{x_j}a(x,\xi)}{a(x,\xi)}\,z\,,\qquad j=1,\dots,n\,,$$

it is easy to recognize that $\partial_{x_j}\varphi_{p,q}$ is a linear combination of $\varphi_{p,q+1}$ and $z\tilde{\varphi}_{p,q}$ with coefficients in $\mathcal{O}^0$, while $\partial_{x_l}\partial_{x_j}\varphi_{p,q}$ is a linear combination, with coefficients in $\mathcal{O}^0$, of $\varphi_{p,q+1}$, $\varphi_{p,q+2}$, $z\tilde{\varphi}_{p,q}$, $z\tilde{\varphi}_{p,q+1}$, $z^2\varphi_{p,q}$, $z^2\varphi_{p-1,q}$.

Now we note that the operator L_1 written in the z variable becomes:

$$L_1 = A_0(x, \xi)\left[\partial_z^2 + \frac{2\nu_0(x)+1}{z}\,\partial_z + 1\right]. \tag{1.3.13}$$

Taking into account the preceding remark, formulas (1.3.10), (1.3.11) and (1.3.13), it is easily seen that to solve eq. $(1.1.9)_j$ it is enough to handle the following equations:

$$\left(z\partial_z^2 + (2\nu_0(x)+1)\,\partial_z + z\right)w(z) = \begin{cases} \begin{cases} z^{2k+1}\varphi_{p,q} \\ z^{2k+2}\varphi_{p,q} \end{cases}, & p, q \geqq 0 \\ z^{2k+2}\tilde{\varphi}_{p,q}\,, & p, q \geqq 0 \\ z^{2k+3}\varphi_{p,q}\,, & p \geqq -1\,,\ q \geqq 0\,. \end{cases} \tag{1.3.14}$$

We look for a solution of (1.3.14) of the form

$$w(z) = \int_L \exp\,[iz\sigma](\sigma^2 - 1)^{\nu_0(x)-\frac{1}{2}}\theta(\sigma)\,\check{d}\sigma\,. \tag{1.3.15}$$

Since

$$\left(z\partial_z^2 + (2\nu_0(x)+1)\,\partial_z + z\right)w(z) = \frac{1}{i}\int_L \exp\,[iz\sigma]\,(\sigma^2-1)^{\nu_0(x)+\frac{1}{2}}\,\frac{d\theta}{d\sigma}(\sigma)\,\check{d}\sigma\,,$$

we are reduced to solving the equations:

$$(\sigma^2-1)^{\nu_0(x)+\frac{1}{2}}\frac{d\theta}{d\sigma}(\sigma) \tag{1.3.16}$$

$$= \begin{cases} \partial_\sigma^{2s+1}(\sigma^2-1)^{\nu_0-p-\frac{1}{2}}(\log(\sigma^2-1))^q\,, & s = k\,,\ \ k+\frac{1}{2}\,,\ \ p, q \geqq 0 \\ \partial_\sigma^{2k+2}\sigma(\sigma^2-1)^{\nu_0-p-\frac{1}{2}}(\log(\sigma^2-1))^q\,, & p\,, q \geqq 0 \\ \partial_\sigma^{2k+3}(\sigma^2-1)^{\nu_0-p-\frac{1}{2}}(\log(\sigma^2-1))^q\,, & p \geqq -1\,,\ \ q \geqq 0 \end{cases}$$

As a preliminary remark we note that the integration of the second equation in (1.3.16) can be actually reduced to that of the first one in the above formula (with $s = k$ and $p, p+1, q, (q-1)_+$).

Now, by induction, the following formula can be easily proved

$$\partial_\sigma^{2h+1}\left[(\sigma^2-1)^{\nu_0(x)-\frac{1}{2}-l}(\log(\sigma^2-1))^m\right], \tag{1.3.17}$$

$$= \sum_{\substack{h+1\leqq j\leqq 2h+1 \\ 0\leqq i\leqq \min\{m,2h+1\}}} c_{ij}(x)2\sigma(\sigma^2-1)^{\nu_0(x)-\frac{1}{2}-l-j}(\log(\sigma^2-1))^{m-i} \quad m, h\in Z_+, \quad l\in Z\,,$$

for some smooth functions $c_{ij}(x)$.

Using (1.3.17) we conclude that eq. (1.3.16) can be reduced to the form:

$$\frac{d\theta}{d\sigma}(\sigma) = 2\sigma(\sigma^2-1)^{-r}(\log(\sigma^2-1))^s, \qquad r, s \in Z_+, \quad r \geqq 1. \tag{1.3.18}$$

Equation (1.3.18) is immediately solved:

$$\theta(\sigma) = \begin{cases} (\sigma^2-1)^{1-r}\sum_{j=0}^{s} c_j(\log(\sigma^2-1))^{s-j}, & \text{if } r>1 \\ \dfrac{(\log(\sigma^2-1))^{s+1}}{s+1}, & \text{if } r=1, \end{cases} \tag{1.3.19}$$

where the c_j are suitable constants.

From the above results it follows that eq. (1.3.7) has a solution which, when written in the z-variable, is a linear combination with coefficients in $\mathcal{O}^{-j}$ of functions $\varphi_{p,q}(z; x)$ with $p, q \leqq 2j$. Therefore we have proved that there exists a function $Q_{-j}(z; x, \xi) \in U_{-j}^{2j,2j}$ for which

$$L_1 Q_{-j}(2\sqrt{t}\,a(x,\xi); x, \xi) = -\sum_{h=1}^{j} L_{1-h/2}\, q_{-j/2+h/2}(t, x, \xi), \qquad t>0.$$

Since $L_1\varphi_{0,0}(2\sqrt{t}\,a(x,\xi); x) = 0$ and $Q_{-j}(0; x, \xi) \in \mathcal{O}^{-j}$, it is enough to put

$$\tilde{q}_{-j}(z; x, \xi) = Q_{-j}(z; x, \xi) - \frac{Q_{-j}(0; x, \xi)}{\varphi_{0,0}(0; x)}\varphi_{0,0}(z; x), \tag{1.3.20}$$

which proves the theorem. q.e.d.

REMARK. It is worthwhile to point out that if the coefficients a_{ij} in (0.1) are constants then Theorem 1.3.1 holds with $\tilde{q}_{-j}(z; x, \xi) \in U_{-j}^{j,2j}$, $j \geqq 0$. This is a consequence of two remarks:

a) in L_{-k} (see (1.1.5)) there is not the term $t^{k+1}A_{k+1}(x, \xi)$;

b) $\partial_{x_j}\varphi_{p,q} = (\partial_{x_j}\nu_0)\varphi_{p,q+1}$, $\partial^2_{x_i x_j}\varphi_{p,q} = (\partial^2_{x_i x_j}\nu_0)\varphi_{p,q+1} + (\partial_{x_i}\nu_0)(\partial_{x_j}\nu_0)\varphi_{p,q+2}$.

Then only the first equation in (1.3.14) must be solved.

1.4. – Formal parametrix in the region $\sqrt{t}\,|\xi| >$ const.

We shall use the notation

$$\begin{cases} A(t, x, \xi) = \sum_{i,j=1}^{n} a_{i,j}(t, x)\xi_i\xi_j \\ B(t, x, \xi) = \sum_{j=1}^{n} b_j(t, x)\xi_j. \end{cases} \tag{1.4.1}$$

Denote by $\Psi^{\pm}(s, x, \xi)$ the solution of the non-linear Cauchy problem

$$(1.4.2)\qquad \begin{cases} \dfrac{\partial \Psi^{\pm}}{\partial s}(s, x, \xi) = \pm\sqrt{A(s^2/4, x, d_x\Psi^{\pm}(s, x, \xi))} \\ \Psi^{\pm}(0, x, \xi) = x\cdot\xi\,. \end{cases}$$

It is well known that pb. (1.4.2) has a unique solution $\Psi^{\pm}(s, x, \xi) \in C^\infty([0, 2\sqrt{T}]\times R^n_x \times \dot{R}^n_\xi)$for a suitable $T > 0$.

Define

$$(1.4.3)\qquad \varphi^{\pm}(t, x, \xi) = \Psi^{\pm}(2\sqrt{t}, x, \xi)\,, \qquad t\in[0, T]\,.$$

Thus $\varphi^{\pm}$ solves the eikonal equation:

$$(1.4.4)\qquad \begin{cases} \sqrt{t}\,\dfrac{\partial \varphi^{\pm}}{\partial t}(t, x, \xi) = \pm\sqrt{A(t, x, d_x\varphi^{\pm}(t, x, \xi))}\,, \qquad 0 < t \leqq T \\ \varphi^{\pm}(0, x, \xi) = x\cdot\xi\,. \end{cases}$$

We explicitly note that $\varphi^{\pm}$ is not a smooth function of t at $t = 0$.

Writing the formal Taylor series of $\Psi^{\pm}$ with respect to the s variable and putting as in (1.2.1) $z = 2\sqrt{t}\,a(x, \xi)$, we can write

$$(1.4.5)\qquad \varphi^{\pm}(t, x, \xi) \sim x\cdot\xi \pm z + R^{\pm}(z; x, \xi)\,,$$

with

$$(1.4.6)\qquad R^{\pm}(z; x, \xi) \sim \sum_{k\geqq 2} \alpha^{\pm}_{1-k}(x, \xi) z^k\,, \qquad \alpha^{\pm}_{1-k}\in \mathcal{O}^{1-k}\,, \quad k \geqq 2\,.$$

The following definition will be convenient.

DEFINITION 1.4.1. *Let m be a real number.*

i) *By Ψ^m we denote the class of the functions $f(t, x, \xi)\in C^\infty(R^+_t\times R^n_x\times\dot{R}^n_\xi)$ such that*

$$f(t/\lambda, x, \sqrt{\lambda}\xi) = \lambda^m f(t, x, \xi)\,, \qquad \lambda > 0\,.$$

ii) *By Φ^m we denote the class of the functions $g(z, x, \xi)\in C^\infty(R^+_z\times R^n_x\times\dot{R}^n_\xi)$ such that*

$$g(z, x, \lambda\xi) = \lambda^m g(z, x, \xi)\,, \qquad \lambda > 0\,.$$

We note that the map

$$(1.4.7)\qquad \Psi^m \ni f(t, x, \xi) \to \tilde{f}(z, x, \xi) = f(z/4a(x, \xi)^2, x, \xi) \in \Phi^{2m}$$

is a bijection.

From now on we shall denote by $\varrho_{-k}(x, \xi)$ $\left(\text{or } \varrho^{\pm}_{-k}(x, \xi)\right)$ elements of $\mathcal{O}^{-k}$ which we do not need to specify.

We are looking for an operator formally defined by

$$(1.4.8) \qquad Eg(t, x) = \int \{\exp[i\varphi^{+}(t, x, \xi)]p^{+}(t, x, \xi) + \exp[i\varphi^{-}(t, x, \xi)]p^{-}(t, x, \xi)\} \cdot \hat{g}(\xi)\, \bar{d}\xi ,$$

where $p^{\pm}(t, x, \xi) \sim \sum_{j\geq 0} p^{\pm}_{-j/2}(t, x, \xi)$, $p^{\pm}_{-j/2} \in \Psi^{-j/2}$ $j \geq 0$, and such that $PEg = 0$ for $t > 0$.

As usual we require that

$$(1.4.9) \qquad \exp[-i\varphi^{\pm}]P\left(\exp[i\varphi^{\pm}]p^{\pm}\right) \sim 0 .$$

Using (1.4.4), (1.4.5), by a computation we obtain

$$\begin{aligned}(1.4.10) \qquad & \exp[-i\varphi^{\pm}]P\left(\exp[i\varphi^{\pm}]p^{\pm}\right) \\ & \sim [t\partial_t^2 + (\nu(t, x) + 1)\partial_t + 2i(t\partial_t\varphi^{\pm})\partial_t]p^{\pm} \pm i\frac{a(x, \xi)^2(2\nu(t, x) + 1)}{z}p^{\pm} \\ & + i\{[t\partial_t^2 + (\nu(t, x) + 1)\partial_t]R^{\pm} - A(t, x, \partial_x)\varphi^{\pm} + B(t, x, \partial_x)\varphi^{\pm}\}p^{\pm} \\ & + 2i\sum_{i,j=1}^{1} a_{ij}(t, x)\partial_{x_i}\varphi^{\pm}\partial_{x_j}p^{\pm} + [-A(t, x, \partial_x) + B(t, x, \partial_x) + b_0(t, x)]p^{\pm} ,\end{aligned}$$

where z is given by (1.2.1).

We can write

$$(1.4.11) \qquad \begin{cases} \nu(t, x) \sim \nu_0(x) + \sum_{k\geq 1} \varrho_{-2k} z^{2k} \\ 2it\partial_t\varphi^{\pm} \sim \pm iz + iz\partial_z R^{\pm} \sim \pm iz + \sum_{k\geq 2} \varrho^{\pm}_{1-k}(x, \xi) z^k \\ \pm i\dfrac{a(x, \xi)^2(2\nu(t, x) + 1)}{z} \sim \pm i\dfrac{a(x, \xi)^2(2\nu_0(x) + 1)}{z} \\ \qquad\qquad \pm \sum_{k\geq 1} \varrho_{2-2k}(x, \xi) z^{2k-1} \\ i[t\partial_t^2 + (\nu(t, x) + 1)\partial_t]R^{\pm} \sim \sum_{k\geq 1} \varrho^{\pm}_{1-k}(x, \xi) z^k \\ -iA(t, x, \partial_x)\varphi^{\pm} \sim \sum_{k\geq 1} \varrho^{\pm}_{1-k}(x, \xi) z^k \end{cases}$$

$$
(1.4.11)\quad \begin{cases}
iB(t, x, \partial_x)\varphi^\pm \sim \sum_{k\geq 1} \varrho^\pm_{1-k}(x, \xi) z^k \\
2i \sum_{i,j=1}^n a_{ij}(t, x)\partial_{x_i}\varphi^\pm \partial_{x_j} \sim \sum_{j=1}^n \sum_{k\geq 0} \varrho^\pm_{1-k}(x, \xi) z^k \partial_{x_j} \\
A(t, x, \partial_x) \sim \sum_{i,j=1}^n \sum_{k\geq 0} \varrho_{-2k}(x, \xi) z^{2k} \partial^2_{x_i x_j} \\
B(t, x, \partial_x) \sim \sum_{j=1}^n \sum_{k\geq 0} \varrho_{-2k}(x, \xi) z^{2k} \partial_{x_j} \\
b_0(t, x) \sim \sum_{k\geq 0} \varrho_{-2k}(x, \xi) z^{2k} .
\end{cases}
$$

Substituting (1.4.11) into (1.4.10) and replacing $p^\pm(t, x, \xi)$ by $\tilde{p}^\pm(z, x, \xi)$ according to (1.4.7), yields

$$
\begin{aligned}
(1.4.12)\quad & \exp[-i\varphi^\pm] P(\exp[i\varphi^\pm] p^\pm) \\
& \sim a(x, \xi)^2 \left[\partial_z^2 + \frac{2\nu_0(x) + 1 \pm iz}{z}\partial_z \pm i\,\frac{2\nu_0(x) + 1}{z}\right]\tilde{p}^\pm \\
& + \sum_{k\geq 0} \varrho_{-2k}(x, \xi)\, z^{2k+1}\, \partial_z \tilde{p}^\pm + \sum_{k\geq 0} \varrho^\pm_{1-k}(x, \xi)\, z^{k+1}\, \partial_z \tilde{p}^\pm \\
& + \sum_{k\geq 0} \varrho^\pm_{1-k}(x, \xi) z^k \tilde{p}^\pm + \sum_{k\geq 0} \varrho_{-2k}(x, \xi) z^{2k}\tilde{p}^\pm + \sum_{k\geq 0} \varrho^\pm_{-k}(x, \xi) z^{k+1}\tilde{p}^\pm \\
& + \sum_{k\geq 0}\left(\sum_{l=1}^n \varrho^{\pm,(l)}_{1-k}(x, \xi) z^k \partial_{x_l}\right)\tilde{p}^\pm + \sum_{k\geq 0}\left(\sum_{l=1}^n \varrho^{(l)}_{-2k}(x, \xi) z^{2k}\partial_{x_l}\right)\tilde{p}^\pm \\
& + \sum_{k\geq 0} \varrho_{-2k}(x, \xi) z^{2k+2}\partial_z^2 \tilde{p}^\pm + \sum_{k\geq 0}\left(\sum_{l=1}^n \varrho^{(l)}_{-2k}(x, \xi) z^{2k+1}\partial^1_{x_l z}\right)\tilde{p}^\pm \\
& + \sum_{k\geq 0}\left(\sum_{i,j=1}^n \varrho^{(i,j)}_{-2k}(x, \xi) z^{2k}\partial^2_{x_i x_j}\right)\tilde{p}^\pm .
\end{aligned}
$$

Let us define the following operators:

$$
(1.4.13)\quad \mathfrak{L}^\pm_1(z, x, \xi; \partial_z) = a(x, \xi)^2\left\{\partial_z^2 + \frac{2\nu_0(x) + 1 \pm 2iz}{z}\partial_z \pm i\,\frac{2\nu_0(x) + 1}{z}\right\}
$$

$$
\begin{aligned}
(1.4.14)\quad \mathfrak{L}^{(k),\pm}_{1/2}(z, x, \xi; \partial_z, \partial_x) = {} & \varrho^\pm_{1-k}(x, \xi) z^{k+1}\partial_z + \sum_{l=1}^n \varrho^{\pm,(l)}_{1-k}(x, \xi) z^k \partial_{x_l} + \\
& + \varrho^\pm_{1-k}(x, \xi) z^k , \qquad k = 0, 1, \dots
\end{aligned}
$$

$$
\begin{aligned}
(1.4.15)\quad & \mathfrak{L}^{(k),\pm}_0(z, x, \xi; \partial_x, \partial_z) = \varrho_{-k}(x, \xi) z^{k+2}\partial_z^2 + \sum_{l=1}^n \varrho^{(l)}_{-k}(x, \xi) z^{k+1}\partial^2_{x_l z} \\
& + \sum_{i,j=1}^n \varrho^{(i,j)}_{-k}(x, \xi) z^k \partial^2_{x_i x_j} + \varrho_{-k}(x, \xi) z^{k+1}\partial_z \\
& + \sum_{l=1}^n \varrho^{(l)}_{-k}(x, \xi) z^k \partial_{x_l} + \varrho^\pm_{-k}(x, \xi) z^k + \varrho^\pm_{-k}(x, \xi) z^{k+1} , \qquad k = 0, 1, 2, \dots .
\end{aligned}
$$

We explicitly note that when k is odd, in $\mathcal{L}_0^{(k),\pm}$ all the coefficients vanish except for $\varrho_{-k}^{\pm}(x,\xi)z^{k+1}$.

It is worth remarking that

$$\mathcal{L}_1^{\pm}\colon \Phi^m \to \Phi^{m+2}\,,\quad \mathcal{L}_{\frac{1}{2}}^{(k),\pm}\colon \Phi^m \to \Phi^{m+1-k}\,,\quad \mathcal{L}_0^{(k),\pm}\colon \Phi^m \to \Phi^{m-k}\,,\qquad k\geqq 0\,.$$

Using (1.4.13)-(1.4.15) and writing $\tilde{p}^{\pm}\sim \sum_{j\geqq 0}\tilde{p}_{-j/2}^{\pm}$, we can put (1.4.12) in the final form

$$\text{(1.4.16)}\qquad \exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) \sim \sum_{k\geqq 0}\Big[\mathcal{L}_1^{\pm}\tilde{p}_{-k/2}^{\pm} + \sum_{\substack{j,h\geqq 0\\ j+h=k-1}}\mathcal{L}_{\frac{1}{2}}^{(j),\pm}\tilde{p}_{-h/2}^{\pm} + \sum_{\substack{j,h\geqq 0\\ j+h=k-2}}\mathcal{L}_0^{(j),\pm}\tilde{p}_{-h/2}^{\pm}\Big]\,.$$

In (1.4.16) we use the convention that a sum over negative integers is zero.

To implement (1.4.9) we are forced to solve the following sequence of transport equations:

$$\text{(1.4.17)}\qquad \mathcal{L}_1^{\pm}\tilde{p}_0^{\pm} = 0\,,\quad z>0\,,$$

$$\text{(1.4.18)}\qquad \mathcal{L}_1^{\pm}\tilde{p}_{-\frac{1}{2}}^{\pm} = -\mathcal{L}_{\frac{1}{2}}^{(0),\pm}\tilde{p}_0^{\pm}\,,\quad z>0$$

$$\text{(1.4.19)}_k\qquad \mathcal{L}_1^{\pm}\tilde{p}_{-k/2}^{\pm} = -\Big(\sum_{\substack{j,h\geqq 0\\ j+h=k+1}}\mathcal{L}_{\frac{1}{2}}^{(j),\pm}\tilde{p}_{-h/2}^{\pm} + \sum_{\substack{j,h\geqq 0\\ j+h=k-2}}\mathcal{L}_0^{(j),\pm}\tilde{p}_{-h/2}^{\pm}\Big)\,,\qquad z>0\,,\quad k\geqq 2\,.$$

1.5. – The first transport equation $\mathcal{L}_1^{\pm}\tilde{p}_0^{\pm} = 0$.

The following transmutation formula will play a crucial role in the sequel:

$$\text{(1.5.1)}\qquad \exp[\pm iz]\mathcal{L}_1^{\pm}(\exp[\mp iz]G(z)) = \frac{a(x,\xi)^2}{z}M(z,x;\partial_z)G(z)\,,$$

where

$$\text{(1.5.2)}\qquad M(z,x;\partial_z) = z\partial_z^2 + \big(2\nu_0(x)+1\big)\partial_z + z\,.$$

Under the hypothesis (1.3.2) the equation $MG(z) = 0$, $z>0$, has two independent solutions given by:

$$\text{(1.5.3)}\qquad I_{\nu_0}^{\pm}(x;z) = \frac{\Gamma(\frac{1}{2}-\nu_0(x))}{\sqrt{\pi}}\int_{L^{\pm}}\exp[iz\sigma](\sigma^2-1)^{\nu_0(x)-\frac{1}{2}}\,\bar{d}\sigma\,,$$

where $L^{\pm}$ are the contours shown in fig. 3.

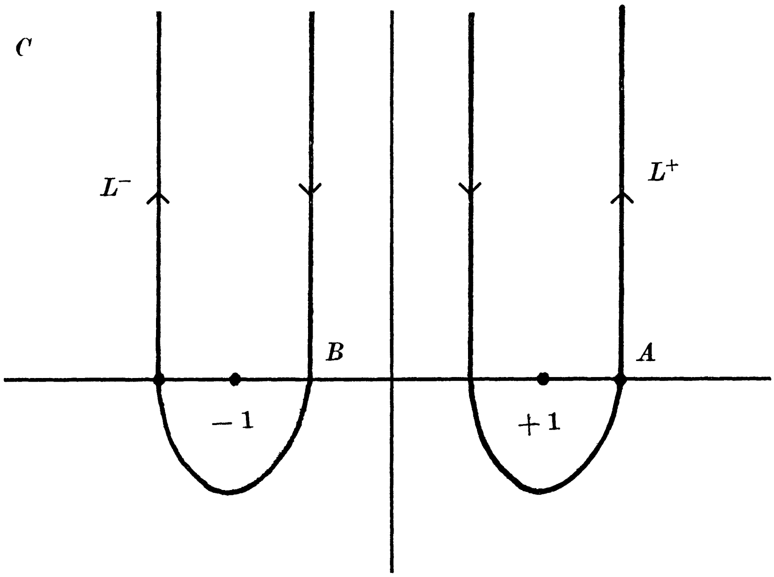

Fig. 3 – The contours $L^{\pm}$; $\arg(\sigma+1)$ and $\arg(\sigma-1)$ is chosen to be 0 in A and $-\pi$ in B.

From G. Watson [7, p. 167, (6), (7)] it follows that

(1.5.4)
$$\begin{cases} I^+_{\nu_0}(x;\, z) = (z/2)^{-\nu_0(x)} H^{(1)}_{\nu_0(x)}(z) \\ I^-_{\nu_0}(x;\, z) = (z/2)^{-\nu_0(x)} H^{(2)}_{\nu_0(x)}(z) \\ I^+_{\nu_0} + I^-_{\nu_0} = (z/2)^{-\nu_0(x)} J(z)_{\nu_0(x)}\,, \end{cases}$$

where $H^{(1)}_{\nu_0}, H^{(2)}_{\nu_0}$ are the Hankel functions and $J_{\nu_0(x)}(z)$ is given by (1.2.5). Using (1.5.1) we solve the first transport equation (1.4.17) putting:

(1.5.5)
$$\begin{cases} \tilde{p}^+_0(z, x, \xi) = \Gamma(\nu_0(x)+1)\exp[-iz] I^+_{\nu_0}(x;\, z) \\ \tilde{p}^-_0(z, x, \xi) = \Gamma(\nu_0(x)+1)\exp[iz] I^-_{\nu_0}(x;\, z)\,. \end{cases}$$

We point out that $\tilde{p}^{\pm}_0 \in \Phi^0$, i.e. $p^{\pm}_0(t, x, \xi) = \tilde{p}^{\pm}_0(2\sqrt{t}\,a(x, \xi), x, \xi) \in \Psi^0$, and $\tilde{p}^+(0, x, \xi) + \tilde{p}^-(0, x, \xi) = 1$.

1.6. – The other transport equations.

To solve eq. (1.4.18), $(1.4.19)_k$, we shall need the following definitions.
DEFINITION 1.6.1. *Let*

$$S^0 = \{\sigma \in C| \ |\mathrm{Re}\,\sigma| < 2, \sigma \neq iy, y \geqq 0\}$$
$$S^{\pm} = S_0 \pm 1$$
$$\hat{S}_0 = \{\sigma \in C| \ |\mathrm{Re}\,\sigma| < 2, \sigma \neq 0\}\,.$$

By $\mathcal{A}^0_{-k,j}$, $j, k \in Z_+$, *we denote the class of functions* $\psi(\sigma, x, \xi) \in C^\infty(S^0 \times R^n_x \times \dot{R}^n_\xi)$ *such that:*

i) $\psi(\sigma, x, \lambda\xi) = \lambda^{-k}\psi(\sigma, x, \xi)$, $\lambda > 0$.

ii) *For every* $\alpha, \beta \in Z^n_+$ $\partial^\alpha_x \partial^\beta_\xi \psi$ *can be written in the form*

$$\partial^\alpha_x \partial^\beta_\xi \psi = \sum_{l=0}^{m_{\alpha\beta}} \psi^{\alpha\beta}_l(\sigma, x, \xi)(\log\sigma)^l\,,$$

where the $\psi^{\alpha\beta}_l$ *are holomorphic functions of* σ *in* $\hat{S}_0$ *having a pole of order at most* j *at* $\sigma = 0$. *Furthermore, for every* $K \subset\subset R^n_x$ *and for every* $\delta \in]0, 1[$ *there exists a non-negative integer* $N = N(\alpha, \beta, l, K, \delta)$ *such that*

$$\sup_{\substack{x\in K, |\xi|=1 \\ |\mathrm{Re}\,\sigma|\leqq 1-\delta \\ |\mathrm{Im}\,\sigma|\geqq 1}} |\mathrm{Im}\,\sigma|^{-N}|\psi^{\alpha\beta}_l(\sigma, x, \xi)| < +\infty \tag{1.6.1}$$

(*here* $\log\sigma$ *is defined cutting* C *along the positive imaginary axis*). *By* $\mathcal{A}^{\pm}_{-k,j}$ *we denote che class of functions* $\psi^{\pm}(\sigma, x, \xi)$ *defined in* $S^{\pm} \times R^n_x \times \dot{R}^n_\xi$ *such that* $\psi^{\pm}(\sigma \pm 1, x, \xi) \in \mathcal{A}^0_{-k,j}$.

DEFINITION 1.6.2. *Let* $\psi^{\pm}(\sigma, x, \xi) \in \mathcal{A}^{\pm}_{-k,j}$. *We define*

$$I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm}) = \int_{L^{\pm}} \exp[iz\sigma]\,(\sigma \mp 1)^{\nu_0(x)-\frac{1}{2}}\psi^{\pm}(\sigma; x, \xi)\, \check{d}\sigma\,, \tag{1.6.2}$$

where $L^{\pm}$ *are the contours described in fig.* 3.

We remark that, for $\psi^{\pm} \in \mathcal{A}^{\pm}_{-k,j}$, $I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm}) \in \Phi^{-k}$.
Furthermore (1.5.3) can be rewritten as $I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm})$ with $\psi^{\pm}(\sigma, x, \xi) = \Gamma(\frac{1}{2} - \nu_0(x))\pi^{-\frac{1}{2}}(\sigma \pm 1)^{\nu_0(x)-\frac{1}{2}} \in \mathcal{A}^{\pm}_{0,0}$.
We have the following result.

THEOREM 1.6.1. *Let $\tilde{p}_0^{\pm}$ be defined as in* (1.5.5). *Then for every $k \geqq 1$ the transport equations* (1.4.18), $(1.4.19)_k$ *have a solution of the form*

$$\tilde{p}^{\pm}_{-k/2}(z, x, \xi) = \exp[\mp iz] I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm}_{-k}) \tag{1.6.3}$$

for suitable $\psi^{\pm}_{-k} \in \mathcal{A}^{\pm}_{-k,k}$.

PROOF. As we noted above the first transport equation (1.4.17) has already been solved by a function of the form (1.6.3); thus we can proceed by induction on k. Suppose we have already constructed $\tilde{p}_0^{\pm}, \ldots, \tilde{p}^{\pm}_{-(k-1)/2}$ of the form (1.6.3); let us now try to find $\tilde{p}^{+}_{-k/2}$ (the case $\tilde{p}^{-}_{-k/2}$ is quite analogous).

We look for $\tilde{p}^{+}_{-k/2}$ of the form $\exp[-iz]G(z, x, \xi)$. Using (1.5.1) we obtain

$$\begin{aligned} \mathfrak{L}_1^{+}(\exp[-iz]G) &= a(x, \xi)^2 \exp[-iz] \frac{1}{z} M(z, x, \partial_z) G(z) = \\ &= -\sum_{\substack{j,h \geqq 0 \\ j+h=k-1}} \mathfrak{L}_{1/2}^{(j),+}(\exp[-iz] I^{+}(z, x, \xi; \nu_0, \psi^{+}_{-h})) \\ &\quad - \sum_{\substack{j,h \geqq 0 \\ j+h=k-2}} \mathfrak{L}_{0}^{(j),+}(\exp[-iz] I^{+}(z, x, \xi; \nu_0, \psi^{+}_{-h})), \end{aligned} \tag{1.6.4}$$

where $\psi^{+}_{-h} \in \mathcal{A}^{+}_{-h,h}$ are the functions appearing in $\tilde{p}^{+}_{-h/2}$, $h = 0, 1, \ldots, k-1$.

The last sum in (1.6.4) vanishes if $k = 1$.

A straightforward computation shows that the r.h.s. in (1.6.4) can be written in the form $\exp[-iz]\big(I^{+}(z, x, \xi; \nu_0, \chi) + I^{+}(z, x, \xi; \nu_0, \mu)\big)$ for some $\chi \in \mathcal{A}^{+}_{2-k,k-1}$, $\mu \in \mathcal{A}^{+}_{2-k,k-2} \subset \mathcal{A}^{+}_{2-k,k-1}$.

We are thus reduced to solve the equation

$$M(z, x; \partial_z) G(z) = I^{+}(z, x, \xi; \nu_0, \psi), \tag{1.6.5}$$

where $\psi \in \mathcal{A}^{+}_{-k,k}$.

We look for a G in the form

$$G(z, x, \xi) = \int_{L^+} \exp[iz\sigma](\sigma^2 - 1)^{\nu_0(x) - \frac{1}{2}} \Phi(\sigma, x, \xi)\, \bar{d}\sigma. \tag{1.6.6}$$

For Φ we obtain the equation

$$\begin{aligned} \frac{d\Phi}{d\sigma}(\sigma, x, \xi) &= i(\sigma^2 - 1)^{-\nu_0(x) - \frac{1}{2}} (\sigma - 1)^{\nu_0(x) - \frac{1}{2}} \psi(\sigma, x, \xi) \\ &= f(\sigma, x, \xi) \in \mathcal{A}^{+}_{-k,k+1}. \end{aligned} \tag{1.6.7}$$

By Def. 1.6.1, $f(\sigma, x, \xi) = g(\sigma - 1, x, \xi)$ with $g \in \mathcal{A}^0_{-k,+1}$; thus we can write

$$g(\sigma-1, x, \xi) = \sum_{l=0}^{m} g_l(\sigma-1, x, \xi)(\log(\sigma-1))^l = \tag{1.6.8}$$

$$= \sum_{l=0}^{m} \sum_{j=0}^{k+1} \frac{c_{jl}(x, \xi)}{(\sigma-1)^j} (\log(\sigma-1))^l + \sum_{l=0}^{m} \theta_l(\sigma-1, x, \xi)(\log(\sigma-1))^l,$$

for some $c_{jl} \in \mathcal{O}^{-k}$ and some functions $\theta_l(\zeta, x, \xi)$ holomorphic in the strip $|\mathrm{Re}\,\zeta| < 2$ and vanishing at $\zeta = 0$.

It is now a trivial fact to recognize that eq. (1.6.7) can be solved within the class $\mathcal{A}^+_{-k,k}$. q.e.d.

1.7. – Asymptotic expansions of some integrals.

In this Sect. we study the asymptotic expansion for $z \to +\infty$ of integrals of the following type:

$$I(z, x, \xi; \nu_0, \psi) = \int_{L_0} \exp[iz\sigma]\sigma^{\nu_0(x)-\frac{1}{2}}\psi(\sigma, x, \xi)\, \check{d}\sigma\,, \qquad z > 0\,, \tag{1.7.1}$$

where $\psi \in \mathcal{A}^0_{-k,j}$ and L_0 is the contour shown in fig. 4.

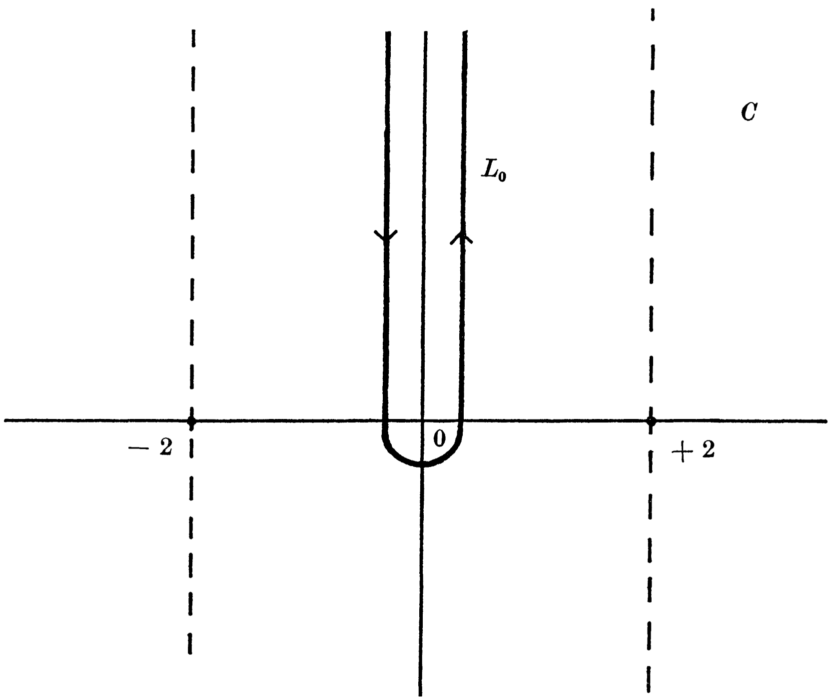

Fig. 4 – The contour L_0.

Here and in the sequel we shall always suppose that condition (1.3.2) is satisfied.

Performing the change of variables $\sigma = iu/z$, $d\sigma = i/z\, du$, since for $\sigma \in \mathcal{S}^0$ $\log \sigma = \log(iu) - \log z$, we obtain

$$(1.7.2) \qquad I(z, x, \xi; \nu_0, \psi) = iz^{-\nu_0(x)-\frac{1}{2}} \int_\gamma \exp[-u](iu)^{\nu_0(x)-\frac{1}{2}} \psi\left(\frac{iu}{z}, x, \xi\right) \check{d}u\,,$$

where γ is the contour shown in fig. 5.

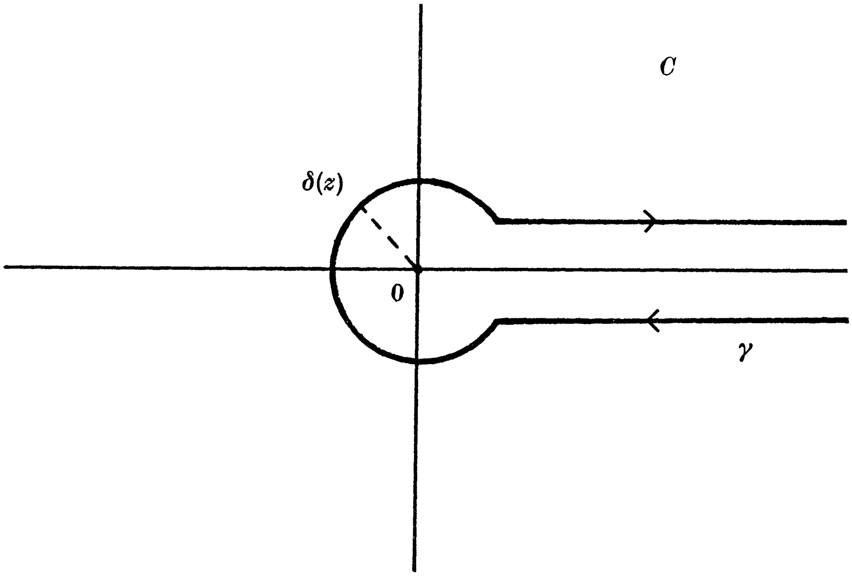

Fig. 5 – The contour γ; the radius $\delta(z)$ is chosen such that $0 < \delta(z) < z/2$.

Let us prove the following lemmas.

LEMMA 1.7.1. *Let* $\psi \in \mathcal{A}^0_{-k,0}$. *Then, for every* $K \subset\subset R^n_x$, $M \in Z_+$, $\varepsilon > 0$, *there exists a constant* $C = C(M, \varepsilon, K, \psi) > 0$ *such that*

$$(1.7.3) \qquad |I(z, x, \xi; \nu_0, \sigma^M \psi(\sigma, x, \xi))| \leq C z^{-\mathrm{Re}\,\nu_0(x)-\frac{1}{2}-M+\varepsilon}, \quad (z, x, \xi) \in [1, +\infty[\times K \times S^{n-1}.$$

PROOF. – For $z \geq 1$ we choose $\delta = \delta(z) = \frac{1}{2}$; then, by Def. 1.6.1, $I(z, x, \xi; \nu_0, \sigma^M \psi)$ is linear combination of integrals of the type

$$iz^{-\nu_0(x)-\frac{1}{2}-M} \int_\gamma \exp[-u](iu)^{\nu_0(x)+M-\frac{1}{2}} \chi\left(\frac{iu}{z}, x, \xi\right)\left(\log\left(\frac{iu}{z}\right)\right)^l \check{d}u\,,$$

where $\chi(\sigma, \cdot)$ is holomorphic in $\mathcal{S}^0$.

From the estimates

$$|\chi(iu/z, x, \xi)| \leqq \text{const}\,(1 + |u/z|)^N$$
$$|\log(iu/z)|^l \leqq \text{const.}\,(\log|z|)^l(|\log|iu|| + 1)^l\,,$$

which hold with a suitable N for $z \geqq 1$, $(x, \xi) \in K \times S^{n-1}$, it follows

$$|z|^{-\varepsilon}|\log(iu/z)|^l \leqq \text{const.}\,(|\log|iu|| + 1)^l\,.$$

Hence the lemma is proved. q.e.d.

LEMMA 1.7.2. *For every $\nu \in C$ and every $j \in Z^+$ we have*

$$\int_{L_0} \exp[iz\sigma]\sigma^{\nu-\frac{1}{2}-j}\,\check{d}\sigma = z^{-\nu-\frac{1}{2}+j}C(\nu - j)\,. \quad z > 0\,,$$

where

$$C(\zeta) = \frac{1}{2\pi i}\exp\left[i(\zeta - \tfrac{1}{2})\frac{\pi}{2}\right]\Gamma(\zeta + \tfrac{1}{2})[1 - \exp[-2\pi i(\zeta - \tfrac{1}{2})]]\,.$$

PROOF. Both sides of the above relation are entire functions of $\zeta = \nu - j \in C$. The equality is trivially proved when $\operatorname{Re}\zeta - \frac{1}{2} > -1$; hence the lemma. q.e.d.

LEMMA 1.7.3. *For every $\nu \in C$ and $l \in Z_+$ we have*

$$\int_{L_0} \exp[iz\sigma]\sigma^{\nu-\frac{1}{2}}(\log\sigma)^l\,\check{d}\sigma = z^{-\nu-\frac{1}{2}}(\partial_\nu - \log z)^l C(\nu) =$$
$$= z^{-\nu-\frac{1}{2}}\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(\log z)^{l-j}\partial_\nu^j C(\nu)\,,$$

with the same $C(\nu)$ as in the preceding lemma.

PROOF. Straightforward. q.e.d.

We state now the main results of this Sections.

THEOREM 1.7.1. *Let $\psi \in \mathcal{A}^0_{-k,j}$ containing powers of $\log\sigma$ of order at most L. For every pair of integers (m, l), $m \geqq -j$, $l \in \{0, \ldots, L\}$, there exist functions $\varrho^{(m,l)}_{-k}(x, \xi) \in \mathcal{O}^{-k}$ such that for every $K \subset\subset R^n_x$, $M \geqq 0$, $\varepsilon > 0$, there is a constant $C > 0$ for which:*

$$|I(z, x, \xi; \nu_0, \psi) - \sum_{m=-j}^{M}\sum_{l=0}^{L}\varrho^{(m,l)}_{-k}(x, \xi)z^{-\nu_0(x)-\frac{1}{2}-m}(\log z)^l|$$
$$\leqq Cz^{-\operatorname{Re}\nu_0(x)-\frac{1}{2}-M-1+\varepsilon}\,, \qquad z \geqq 1\,, \quad (x, \xi) \in K \times S^{n-1}\,.$$

PROOF. – Using the representation

$$\psi(\sigma, x, \xi) = \sum_{l=0}^{L} \sum_{m=-j}^{M} a_{-k}^{(m,l)}(x, \xi)\sigma^m(\log\sigma)^l + \\ + \sigma^{M+1}\sum_{l=0}^{L} b_{-k}^{(l)}(\sigma, x, \xi)(\log\sigma)^l , \qquad a_{-k}^{(m,l)} \in \mathcal{O}^{-k} , \quad b_{-k}^{(l)} \in \mathcal{A}^0_{-k,0} ,$$

The Theorem easily follows from Lemmas 1.7.1.-1.7.3. q.e.d.

THEOREM 1.7.2. *Let* $\tilde{p}^{\pm}_{-j/2}(z, x, \xi) \in \Phi^{-j}$ *be the functions constructed in Theorem* 1.6.1. *and* $L^{\pm}(j)$ *the maximum order of powers of* $\log\sigma$ *appearing in the integral representation of* $\tilde{p}^{\pm}_{-j/2}$. *Then for every pair of integers* (m, l), $m \geq -j$, $0 \leq l \leq L^{\pm}(j)$, *there exist functions* $\varrho_{-j}^{(m,l),\pm}(x, \xi) \in \mathcal{O}^{-j}$ *such that for every* $K \subset\subset R^n_x$, $M \geq 0$, $\varepsilon > 0$, $\alpha, \beta \in Z^n_+$, $r \in Z_+$, *there is a constant* $C_{\alpha,\beta,K,r} > 0$ *for which*

$$(1.7.4) \qquad \left|\partial_x^\alpha \partial_\xi^\beta \partial_z^r \tilde{p}^{\pm}_{-j/2}(z, x, \xi) - \sum_{m=-j}^{M} \sum_{l=0}^{L^{\pm}(j)} z^{-\nu_0(x)-\frac{1}{2}-m}(\log z)^l \varrho_{-j}^{(m,l),\pm}(x, \xi)\right| \\ \leq C_{\alpha,\beta,K,r} z^{-\operatorname{Re}\nu_0(x)-\frac{1}{2}-r-M-1+\varepsilon}|\xi|^{-j-|\beta|} ,$$

for $z \geq 1$, $x \in K$, $\xi \neq 0$.

(1.7.4) *will be written briefly*

$$(1.7.5) \qquad \tilde{p}^{\pm}_{-j/2}(z, x, \xi) \sim \sum_{m \geq -j} \sum_{l=0}^{L^{\pm}(j)} z^{-\nu_0(x)-\frac{1}{2}-m}(\log z)^l \varrho_{-j}^{(m,l),\pm}(x, \xi) .$$

PROOF A trivial consequence of Theorems 1.6.1. 1.7.1. q.e.d.

In the next theorem we prove some kind of converse of the preceding result.

THEOREM 1.7.3. *Let* $\tilde{p}^{\pm}_{-j/2}(z, x, \xi) \in \Phi^{-j}$, $j \geq 0$, *be such that*:

i) $\tilde{p}^{\pm}_0(z, x, \xi) = I^{\pm}(z, x, \xi; \nu_0, \zeta)$, *for some* $\zeta \in \mathcal{A}^{\pm}_{0,0}$.

ii) *For* $j \geq 1$, $\tilde{p}^{\pm}_{-j/2}(z, x, \xi)$ *are solutions of the equations* (1.4.18), $(1.4.19)_j$:

iii) *For every* $j \geq 0$ *there exist* $J^{\pm}(j) \in Z$, $L^{\pm}(j) \in Z^+$ *and a sequence of functions* $\varrho_{-j}^{(m,l),\pm}(x, \xi) \in \mathcal{O}^{-j}$, $m \geq -J^{\pm}(j)$, $0 \leq l \leq L^{\pm}(j)$, *such that*

$$(1.7.6)_j \qquad \tilde{p}^{\pm}_{-j/2}(z, x, \xi) \sim \sum_{m \geq -J^{\pm}(j)} \sum_{l=0}^{L^{\pm}(j)} z^{-\nu_0(x)-\frac{1}{2}-m}(\log z)^l \varrho_{-j}^{(m,l),\pm}(x, \xi) ,$$

where the $\sim$ *has the same meaning as in* (1.7.5).

Then there exist functions $\psi_j^{\pm}(\sigma, x, \xi) \in \mathcal{A}^{\pm}_{-j,j}$, $j \geqq 0$, *for which*

$$\tilde{p}^{\pm}_{-j/2}(z, x, \xi) = \exp[\mp iz] I^{\pm}(z, x, \xi; \nu_0, \psi_j^{\pm}) , \tag{1.7.7$_j$}$$

where $I^{\pm}$ *are the integrals defined in* (1.6.2).

PROOF. – By induction on j. When $j = 0$ there is nothing to prove since by assumption i),

$$\tilde{p}_0^{\pm}(z, x, \xi) = \exp[\pm iz] I^{\pm}\big(z, x, \xi; \nu_0, \psi_0^{\pm}\big) = \frac{\Gamma(\nu_0(x)+1)\Gamma(\frac{1}{2}-\nu_0(x))}{\sqrt{\pi}} (\sigma \pm 1)^{\nu_0(x)-\frac{1}{2}}.$$

Suppose that the assertion holds up to $j-1$, $j \geqslant 1$, and let us prove it for $\tilde{p}^{+}_{-j/2}$ (the case $\tilde{p}^{-}_{-j/2}$ is quite analogous).

Write $\tilde{p}^{+}_{-j/2}(z, x, \xi) = \exp[-iz] G(z, x, \xi)$. By the inductive hypothesis we have

$$MG(z) = \big(z\partial_z^2 + (2\nu_0(x)+1)\partial_z + z\big) G(z, x, \xi) = I^{+}(z, x, \xi; \nu_0, \chi) , \tag{1.7.8}$$

with a suitable $\chi \in \mathcal{A}^{+}_{-j,j}$.

Since two independent solutions of the homogeneous equation $MG(z) = 0$, $z > 0$, are given by $I^{\pm}_{\nu_0}(x; z)$ (see (1.5.4)), by the proof of Theorem 1.6.1 there exists a function $\psi^{+}(\sigma, x, \xi) \in \mathcal{A}^{+}_{-j,j}$ such that

$$G(z, x, \xi) = I^{+}(z, x, \xi; \nu_0, \psi^{+}) + c^{+}_{-j}(x, \xi) I^{+}_{\nu_0}(x; z) + c^{-}_{-j}(x, \xi) I^{-}_{\nu_0}(x; z) , \tag{1.7.9}$$

for some functions $c^{\pm}_{-j} \in \mathcal{O}^{-j}$.

From (1.7.6)$_j$ and Lemma 1.7.1 we get

$$\exp[-iz] I^{+}(z, x, \xi; \nu_0, \psi^{+}) + c^{+}_{-j}(x, \xi) I^{+}_{\nu_0}(x; z) \sim \sum_{m \geqq \min(0, -J^{+}(j))} \sum_{l=0}^{L^{+}(j)} z^{-\nu_0(x)-\frac{1}{2}-m} (\log z)^l \tilde{\varrho}^{(m,l),+}_{-j}(x, \xi) , \tag{1.7.10}$$

with some new $\tilde{\varrho}^{(m,l),+}_{-j} \in \mathcal{O}^{-j}$.

On the other hand

$$\exp[iz] c^{-}_{-j}(x, \xi) I^{-}_{\nu_0}(x; z) \sim \sum_{m \geqq 0} z^{-\nu_0(x)-\frac{1}{2}-m} c^{-}_{-j}(x, \xi) b_m(\nu_0(x)) , \tag{1.7.11}$$

for some suitable functions b_m, with $b_0(\nu_0(x)) \neq 0$ (see W. Magnus - F. Oberhettinger - R. P. Soni [6], p. 139).

Comparing (1.7.10), (1.7.11) with $(1.7.6)_j$ we conclude that $c^-_{-j} = 0$. Choosing in (1.7.9) $\psi_j^+ = \psi^+ + c^+_{-j}\Gamma(\frac{1}{2} - \nu_0(x))\pi^{-\frac{1}{2}}(\sigma + 1)^{\nu_0(x)-\frac{1}{2}}$ we prove the theorem. q.e.d.

Chapter 2

THE RIGOROUS DISCUSSION

2.1. – Symbol classes and oscillatory integrals.

To put the formal series $\Sigma q_{-j/2}$ and $\Sigma p^{\pm}_{-j/2}$ constructed in Ch. 1 in a rigorous framework we need to define some classes of symbols which are closely connected with those considered by L. Boutet de Monvel [2]. In the sequel by a cutoff function we mean any function $\chi \in C_0^\infty(R)$ which is identically 1 in a neighborhood of the origin.

Definition 2.1.1. *By* $S^{m,k}(0, T)$, $m, k, T \in R$, $0 < T \leqq +\infty$, *we denote the class of all functions* $p(t, x, \xi) \in C^\infty([0, T] \times R^n_x \times R^n_\xi)$ *such that for every* $K \subset\subset R^n_x$, $\alpha, \beta \in Z^n_+$, $r \in Z_+$, *there exists a constant* $C_{\alpha,\beta,r,K} > 0$ *for which*

$$|\partial_t^r \partial_x^\alpha \partial_\xi^\beta p(t, x, \xi)| \leqq C_{\alpha,\beta,r,K} |\xi|^{m-|\beta|} \left(\sqrt{t} + \frac{1}{|\xi|}\right)^{k-2r}, \tag{2.1.1}$$

for $|\xi| \geqq 2$, $x \in K$, $0 \leqq t < \min\{\frac{1}{2}, T\}$.

We put $S^{-\infty,k}(0, T) = \bigcap_m S^{m,k}(0, T)$, $S^{m,\infty}(0, T) = \bigcup_k S^{m,k}(0, T)$.

By $\tilde{S}^{m,k}(0, T)$ *we denote the intersection* $\bigcap_{\varepsilon>0} S^{m+\varepsilon,k+\varepsilon}(0, T)$.

$S_0^{m,k}(0, T)$ *will denote the space of all symbols* $p(t, x, \xi)$ *such that* $\chi(t|\xi|^2)p(t, x, \xi) \in S^{m,k}$ *for every cutoff function* χ. $\tilde{S}_\infty^{m,k}(0, T)$ *will denote the space of all symbols* $p(t, x, \xi)$ *such that* $\big(1 - \chi(t|\xi|^2)\big)p(t, x, \xi) \in \tilde{S}^{m,k}$ *for every cutoff function* χ. *All these spaces are equipped with their natural topology.*

Examples. 1) Let $q(z, x, \xi) \in C^\infty(R_z \times R^n_x \times \dot{R}^n_\xi)$ satisfy:

i) $q(z, x, \lambda\xi) = \lambda^m q(z, x, \xi)$, $\lambda > 0$;

ii) $z \to q(z, \cdot)$ in an even analytic function of z. Then for every fixed cutoff χ_0 the symbol

$$q_1(t, x, \xi) = \big(1 - \chi_0(|\xi|^2)\big) q(2\sqrt{t}\, a(x, \xi), x, \xi) \in S_0^{m,0}(0, +\infty).$$

2) Let $p(z, x, \xi) \in C^\infty(R_z^+ \times R_x^n \times \dot{R}_\xi^n)$ satisfy:

i) $p(z, x, \lambda\xi) = \lambda^m p(z, x, \xi)$, $\lambda > 0$;

ii) for some $\mu \in R$ and for every $K \subset\subset R_x^n$, $\alpha, \beta \in Z_+^n$, $r \in Z_+$, $\varepsilon > 0$ there is a constant $C > 0$ for which $|(z\partial_z)^r \partial_x^\alpha \partial_\xi^\beta p(z, x, \xi)| \leqq c|\xi|^{m-|\beta|} z^{\mu+\varepsilon}$ for $x \in K$, $\xi \neq 0$, $z \geqq 1$. Then for every fixed cutoff χ_0 the symbol $p_1(t, x, \xi) = (1 - \chi_0(|\xi|^2)) p(2\sqrt{t}\, a(x, \xi), x, \xi) \in \tilde{S}_\infty^{m+\mu,\mu}(0, +\infty)$.

In the next lemma some properties of the classes of symbols defined above are collected. The proof, which follows along standard arguments will be omitted (see e.g. L. Boutet de Monvel [2]).

Lemma 2.1.1.

i) *$S^{m,k}(0, T) \hookrightarrow S^{m',k'}(0, T)$ iff $m \leqq m'$ and $m - k \leqq m' - k'$.*

ii) *Let χ be a cutoff function and $\lambda \geqq 1$. Define*

$$\varphi_\lambda^1(t, x, \xi) = 1 - \chi(|\xi|/\lambda), \qquad \varphi_\lambda^2(t, x, \xi) = \varphi_\lambda^1(t, x, \xi)\chi(\lambda^2 t).$$

Then:

a) $\varphi_\lambda^1 \in S^{0,0}$, $1 - \varphi_\lambda^1 \in S^{-\infty,0}$, $\{\lambda\varphi_\lambda^1 | \lambda \geqq 1\}$ is a bonnded subset of $S^{1,0}$.

b) $\varphi_\lambda^2 \in S^{0,0}$, $1 - \varphi_\lambda^2 \in S^{0,\infty}$, $\{\lambda\varphi_\lambda^2 | \lambda \geqq 1\}$ is a bounded subset of $S^{0,-1}$.

iii) *Let $p_j \in S_0^{m-j,k}(0, T)$, $j = 0, 1, \ldots$; then there exists a symbol $p \in S_0^{m,k}(0, T)$ such that $p \sim \sum_{j \geqq 0} p_j$, i.e.*

$$p - \sum_{j=0}^{M-1} p_j \in S_0^{m-M,k}(0, T), \qquad \forall M \geqq 1.$$

iv) *Let $p_j \in \tilde{S}_\infty^{m,k+j}(0, T)$, $j = 0, 1, \ldots$; then there exists a symbol $p \in \tilde{S}_\infty^{m,k}(0, T)$ such that $p \sim \sum_{j \geqq 0} p_j$, i.e.*

$$p - \sum_{j=0}^{M-1} p_j \in \tilde{S}_\infty^{m,k+M}(0, T), \qquad \forall M \geqq 1.$$

True symbols can be recovered from the «formal» symbols of Chp. 1 using the following lemma.

Lemma 2.1.2.

i) *Let $f(t, x, \xi) \in \Psi^{-j/2}$, then for every cutoff χ*

$$(1 - \chi(|\xi|)) f(t, x, \xi) \in S_0^{-j,0}(0, +\infty)$$

ii) *Let* $\psi^{\pm}(\sigma, x, \xi) \in \mathcal{A}^{\pm}_{-k,j}$ *and define*

$$p^{\pm}(t, x, \xi) = \exp[\mp iz] I^{\pm}(z, x, \xi; \nu_0, \psi^{\pm})|_{z=2\sqrt{t}a(x,\xi)}$$

Then for every cutoff χ

$$\big(1-\chi(|\xi|)\big)p^{\pm}(t, x, \xi) \in \tilde{S}^{\mu+j-k,\mu+j}_{\infty}(0, +\infty),$$

where $\mu = \sup_{x\in R^n} (-\mathrm{Re}\,\nu_0(x) - \frac{1}{2})$.

iii) *Let* $f \in S^{m,k}_0(0, T)$ *and* χ *any cutoff. Then, for every* $l \in Z_+$

$$\partial_t^l\big(\chi(t|\xi|^2) f(t, x, \xi)\big) \in C^0([0, T]; S^{m-k+2l+2}_{1,0}(R^n_x \times R^n_\xi)).$$

iv) *Let* $f \in \tilde{S}^{m,k}_{\infty}(0, T)$ *and* χ *any cutoff. Then, for every* $l \in Z_+$

$$\partial_t^l\big((1-\chi(t|\xi|^2)) f(t, x, \xi)\big) \in \begin{cases} C^{(k-2l+\varepsilon)/2}_{-}([0, T]; S^{m+\varepsilon}_{1,0}(R^n_x \times R^n_\xi)), & \varepsilon > 0, \quad \text{if } k-2l \geq 0 \\ C^{\frac{1}{2}}([0, T]; S^{m+2l-k+1}_{1,0}(R^n_x \times R^n_\xi)), & \text{if } k-2l < 0. \end{cases}$$

PROOF. i) By Def. 1.1.1 $f(t, x, \xi) = |\xi|^{-j} f(t|\xi|^2, x, \xi/|\xi|)$, thus the conclusion follows taking into account that $\sqrt{t} + 1/|\xi| \sim 1/|\xi|$ on the support of $\chi(t|\xi|^2)$.

ii) Is a trivial consequence of Theorem 1.7.1, of Example 2) and Lemma 2.1.1 iv).

iii) Since $\partial_t\colon S^{m,k}_0 \to S^{m+2,k}_0$ it is enough to prove the assertion in the case $l = 0$. Now, locally in x we have

$$|\partial_x^\alpha \partial_\xi^\beta[\chi(t|\xi|^2) f(t, x, \xi)]| \leq C|\xi|^{m-|\beta|}\left(\sqrt{t} + \frac{1}{|\xi|}\right)^k \leq C|\xi|^{m-k-|\beta|}, \qquad |\xi| \geq 2, \ t \in [0, T].$$

Moreover, locally in x we have

$$\left|\partial_x^\alpha \partial_\xi^\beta\left[\partial_t\big(\chi(t|\xi|^2) f(t, x, \xi)\big)\right]\right| \leq C|\xi|^{m-k+2-|\beta|}, \qquad |\xi| \geq 2, \qquad t \in [0, T].$$

Thus the claim follows.

iv) Suppose first $k-2l \geqq 0$. Then for $j=0,1$ locally in x we have

$$\left|\partial_t^{j+l}\partial_x^\alpha\partial_\xi^\beta\Big[\big(1-\chi(t|\xi|^2)f(t,x,\xi)\big)\Big]\right| \leqq$$
$$\leqq C|\xi|^{m-|\beta|+\varepsilon}(\sqrt{t})^{k-2l+\varepsilon-2j}, \qquad \forall\varepsilon>0, \quad |\xi|\geqq 2, \quad t\in[0,T].$$

Hence the first assertion follows. If $k-2l<0$, take $\varepsilon>0$ such that $1+2l-k-\varepsilon>0$. Then for $j=0,1$ locally in x we have

$$\left|\partial_t^{j+l}\partial_x^\alpha\partial_\xi^\beta\Big[\big(1-\chi(t|\xi|^2)\big)f(t,x,\xi)\Big]\right| \leqq$$
$$\leqq C|\xi|^{m-|\beta|+\varepsilon}t^{\frac12-j}t^{(k-2l-1+\varepsilon)/2} \leqq C|\xi|^{m+2l-k+1-|\beta|}t^{\frac12-j}, \qquad |\xi|\geqq 2, \quad t\in[0,T].$$

Hence the second assertion follows. q.e.d.

We now turn to the discussion of some oscillatory integrals. Let $q(t,x,\xi)\in S_0^{m,k}(0,T)$ and let χ be any cutoff function. We consider the following operator:

$$\text{(2.1.2)} \qquad \begin{cases} E\colon C_0^\infty(R^n)\to C^\infty([0,T]\times R^n) \\ Eg(t,x)=\displaystyle\int \exp[ix\cdot\xi]\chi(t|\xi|^2)\,q(t,x,\xi)\hat{g}(\xi)\,\check{d}\xi\,. \end{cases}$$

The continuity of E follows from Lemma 2.1.2 iii). We now show that E can be continuously extended to an operator, still denoted by E, from $\mathcal{E}'(R^n)$ into $C^\infty([0,T];\mathcal{D}'(R^n))$.

Take $g\in C_0^\infty(R^n)$ and $f(t,x)\in C_0^\infty([0,T]\times R^n)$; then

$$\int_0^T\!\!\int (Eg)(t,x)f(t,x)\,dt\,dx=\int\hat{g}(\xi)(\mathcal{E}f)(\xi)\,\check{d}\xi\,,$$

with

$$(\mathcal{E}f)(\xi)=\int_0^T\!\!\int\exp[ix\cdot\xi]\,\chi(t|\xi|^2)\,q(t,x,\xi)\,f(t,x)\,dt\,dx\,.$$

Integration by parts with respect to x shows that $\mathcal{E}f(\xi)$ is C^∞ and rapidly decreasing for $\xi\to\infty$.

Therefore, by the Paley-Wiener theorem we can define $\mathcal{E}'(R^n)\ni g\to Eg$ by the relation $\langle Eg,f\rangle=\int\hat{g}(\xi)(\mathcal{E}f)(\xi)\,d\xi$.

By an application of Lemma 2.1.2 iii) it follows that $Eg\in C^\infty([0,T];\mathcal{D}'(R^n))$ and the map $g\to Eg$ is continuous.

It is worthwhile to observe that for every $j \geqq 0$ and for every $s \in [0, T]$ the operator

$$\mathcal{E}'(R^n) \ni g \to \partial_t^j Eg|_{t=s} \in \mathcal{D}'(R^n)$$

is a pseudo differential operator of order $m - k + 2j$.

Operators of the form (2.1.2) will take care of the formal parametrix in the region $t|\xi|^2 \leqq$ const. constructed in Sects. 1.1-1.3.

To give meaning to the formal operators introduced in Sect. 1.4 let $\varphi(t, x, \xi)$ denote any one of the two phase functions $\varphi^{\pm}(t, x, \xi)$ defined in (1.4.3).

Let $p(t, x, \xi) \in \tilde{S}^{m,k}_{\infty}(0, T)$ and let χ be any cutoff function.

Consider the operator

$$(2.1.3) \qquad Eg(t, x) = \int \exp[i\varphi(t, x, \xi)]\big(1 - \chi(t|\xi|^2)\big)p(t, x, \xi)\hat{g}(\xi)\, \bar{d}\xi\,, \qquad g \in C_0^\infty(R^n)\,.$$

We now show that E maps continuously $C_0^\infty(R^n)$ into $C^\infty_{\text{flat}}([0, T] \times R^n)$ where the latter denotes the subspace of $C^\infty([0, T] \times R^n)$ whose elements are flat functions at $t = 0$.

It is easy to recognize that $\partial_t^j \partial_x^\alpha Eg(t, x)$ can be written as an sum of integrals like (2.1.3) with new amplitudes in $\tilde{S}^{m+2j,k}_{\infty}(0, T)$ and new cutoffs. This proves that $Eg \in C^\infty([0, T] \times R^n)$. To show that Eg is flat at $t = 0$ consider $t^{-N}Eg(t, x)$, $N \geqq 0$. Locally in x we have the estimate

$$\begin{aligned}\left|t^{-N}\big(1 - \chi(t|\xi|^2)\big)p(t, x, \xi)\right| &\leqq Ct^{-N+k/2+\varepsilon/2}(1 + |\xi|)^{m+\varepsilon} \\ &\leqq C\big(t|\xi|^2\big)^{-N+k/2+\varepsilon/2}(1 + |\xi|)^{m+2N-k} \leqq C(1 + |\xi|)^{m+2N-k}\,,\end{aligned}$$

if $2N - k > 0$.

Therefore $t^{-N}Eg(t, x) \to 0$, $t \to 0+$, for N large enough.

Let us now show that the operator (2.1.3) can be continuously extended as an operator from $\mathcal{E}'(R^n)$ into $C^\infty_{\text{flat}}([0, T]; \mathcal{D}'(R^n))$.

Take $g \in C_0^\infty(R^n)$ and $f \in C_0^\infty([0, T] \times R^n)$; then

$$\int_0^T\!\!\int Eg(t, x) f(t, x)\, dt\, dx = \int \hat{g}(\xi)(\mathcal{E}f)(\xi)\, \bar{d}\xi\,,$$

where

$$(\mathcal{E}f)(\xi) = \int_0^T\!\!\int \exp[i\varphi(t, x, \xi)]\big(1 - \chi(t|\xi|^2)\big)p(t, x, \xi) f(t, x)\, dt\, dx\,.$$

As a consequence of (1.4.3) the following estimate holds for x in a compact set:

$$|d_x\varphi(t, x, \xi)| \geqq C|\xi|\,, \qquad t \in [0, T]\,, \quad \xi \neq 0\,.$$

Consider the operator

$$L = \sum_{j=1}^{n} \frac{1}{i}\,(1-\theta(\xi))\,\frac{d_{x_j}\varphi(t, x, \xi)}{|d_x\varphi(t, x, \xi)|^2}\,d_{x_j} + \theta(\xi)\,,$$

where θ is a cutoff function.

Integrating by parts we get, for every $N \geqq 0$:

$$(\mathcal{E}f)(\xi) = \int\!\!\int_0^T \exp\,[i\varphi(t, x, \xi)]\,\big(1-\chi(t|\xi|^2)\big)^t L^N[p(t, x, \xi)\,f(t, x)]\,dt\,dx\,.$$

It is easily verified that, for every $\varepsilon > 0$ we have the estimate

$$\big|\big(1-\chi(t|\xi|^2)^t L^N[p(t, x, \xi)\,f(t, x)]\big| \leqq C_N(1+|\xi|)^{m+\varepsilon-N+\max(0,-k-\varepsilon)}\,.$$

The rapid decrease of $(\mathcal{E}f)(\xi)$ allows to define Eg, when $g \in \mathcal{E}'(R^n)$, according to the formula

$$\langle Eg, f\rangle = \int \hat{g}(\xi)(\mathcal{E}f)(\xi)\,\check{d}\xi\,.$$

One can easily see that $Eg \in C^\infty([0, T]; \mathcal{D}'(R^n))$. Moreover, arguing as above, one can verify that $\partial_t^j Eg|_{t=0} = 0$, $\forall j \geqq 0$. Of course, for every $j \geqq 0$ and for every $s \in\,]0, T[$, the operator

$$\mathcal{E}'(R^n) \ni g \to \partial_t^j Eg|_{t=s} \in \mathcal{D}'(R^n)$$

is a Fourier integral operator with phase $\varphi(s, x, \xi)$ and amplitude in $S_{1,0}^{m+2j-k+\varepsilon}$, $\forall \varepsilon > 0$.

2.2. – Construction of the true parametrix.

Our first attempt to construct a parametrix for pb. (0.2) will be to consider an operator of the form:

$$\begin{aligned}(2.2.1)\qquad Eg(t, x) &= \int \exp\,[ix\cdot\xi]\,\chi(t|\xi|^2)\,q(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\, + \\ &+ \int \exp\,[i\varphi^+(t, x, \xi)]\big(1-\chi(t|\xi|^2)\big)p^+(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\, + \\ &+ \int \exp\,[i\varphi^-(t, x, \xi)]\,\big(1-\chi(t|\xi|^2)\big)p^-(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\,, \qquad g \in C_0^\infty(R^n)\,,\end{aligned}$$

where:

i) χ is any cutoff function.

ii) $q \in S_0^{0,0}(0, +\infty)$ with $q \sim \sum_{j\geq 0} q_{-j/2}$ and the $q_{-j/2} \in \Psi^{-j/2}$, $j \geq 0$, are the functions constructed in Sects. 1.2, 1.3. We recall that, for any cutoff χ_0, $\big(1 - \chi_0(|\xi|)\big) q_{-j/2}(t, x, \xi) \in S_0^{-j,0}(0, +\infty)$ according to Lemma 1.1.2 i).

iii) $p^{\pm} \in \tilde{S}_{\infty}^{\mu,\mu}(0, T)$, $\mu = \sup_{x\in R^n} (-\operatorname{Re} \nu_0(x) - \frac{1}{2})$, with $p^{\pm} \sim \sum_{j\geq 0} p^{\pm}_{-j/2}$ and the $p^{\pm}_{-j/2}$ are those constructed in Sects. 1.5, 1.6.

We recall that, for any cutoff χ_0, $\big(1 - \chi_0(|\xi|)\big) p^{\pm}_{-j/2} \in \tilde{S}_{\infty}^{\mu,\mu+j}(0, T)$ according to Lemma 2.1.2 ii).

iv) $\varphi^{\pm}(t, x, \xi)$ are the phases (defined for $t \in [0, T]$) constructed in (1.4.3).

First we observe that

$$Eg|_{t=0} - g = \int \exp[ix\cdot\xi](1 - q(0, x, \xi))\hat{g}(\xi)\, \check{d}\xi\, .$$

Since $q(0, x, \xi) \in S_{1,0}^{0}(R^n)$ and

$$q_{-j/2}(0, x, \xi) = \begin{cases} 1\,, & j = 0 \\ 0\,, & j > 0\,, \end{cases}$$

we conclude that $1 - q(0, x, \xi) \in S_{1,0}^{-\infty}$ and thus the second condition in (0.3) is fulfilled.

Now

$$\begin{aligned}(2.2.2) \qquad PEg(t, x) &= \int \exp[ix\cdot\xi]\chi(t|\xi|^2)\big[\exp[-ix\cdot\xi]P\big(\exp[ix\cdot\xi]q\big)\big]\hat{g}(\xi)\, \check{d}\xi \\ &+ \int \exp[i\varphi^+(t, x, \xi)]\big(1 - \chi(t|\xi|^2)\big)\big[\exp[-i\varphi^+]P\big(\exp[i\varphi^+]p^+\big)\big]\hat{g}(\xi)\, \check{d}\xi \\ &+ \int \exp[i\varphi^-(t, x, \xi)]\big(1 - \chi(t|\xi|^2)\big)\big[\exp[-i\varphi^-]P\big(\exp[i\varphi^-]p^-\big)\big]\hat{g}(\xi)\, \check{d}\xi \\ &+ \int [P, \chi(t|\xi|^2)]\{\exp[ix\cdot\xi]q(t, x, \xi)\}\hat{g}(\xi)\, \check{d}\xi \\ &- \int [P, \chi(t|\xi|^2)]\{\exp[i\varphi^+(t, x, \xi)]p^+(t, x, \xi) + \exp[i\varphi^-(t, x, \xi)]p^-(t, x, \xi)\}\hat{g}(\xi)\, \check{d}\xi.\end{aligned}$$

Now the following crucial remarks are in order:

I) By construction $q - \sum_{j=0}^{N-1} q_{-j/2} \in S_0^{-N,0}$, for large ξ, for every $N \geq 1$.

Moreover, by the construction performed in Sect. 1.3 $\exp[-ix\cdot\xi]P\cdot(\exp[ix\cdot\xi]q)\in S_0^{1,0}$ has, for large ξ, the asymptotic expansion

$$\exp[-ix\cdot\xi]P(\exp[ix\cdot\xi]q)\sim\sum_{j\geq 0}\tilde{q}_{-j/2}\,,\qquad \text{with}$$

$$\tilde{q}_{-j/2}=\sum_{h=0}^{j}L_{1-h/2}q_{-j/2+h/2}\,,\qquad j\geq 0\,,$$

and the operators $L_{1-h/2}$ are defined in (1.1.5).

From Theorem 1.3.1 it follows that $\exp[-ix\cdot\xi]P(\exp[ix\cdot\xi]q)\sim 0$. As a consequence the operator

$$g\to\int\exp[ix\cdot\xi]\chi(t|\xi|^2)[\exp[-ix\cdot\xi]P(\exp[ix\cdot\xi]q)]\hat{g}(\xi)\,\check{d}\xi$$

is smoothing.

II) By construction $p^{\pm}-\sum_{j=0}^{N-1}p^{\pm}_{-j/2}\in\tilde{S}_{\infty}^{\mu,\mu+N}(0,T)$, for large ξ, for every $N\geq 1$.

Now we claim that $\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm})\in\tilde{S}_{\infty}^{\mu+1,\mu}(0,T)$ with asymptotic expansion (1.4.15), for large ξ, computed for $z=2\sqrt{t}\,a(x,\xi)$.

To prove our claim, i.e. to show that the formal computations performed in Sect. 1.4 have a meaning within the classes $\tilde{S}_{\infty}$ we only need to show that for large ξ, $\varphi^{\pm}(t,x,\xi)\in\tilde{S}_{\infty}^{1,0}(0,T)$ with the asymptotic expansion (1.4.4) (computed for $z=2\sqrt{t}\,a(x,\xi)$).

To prove this fact we recall (1.4.2); from the Taylor expansion

$$\psi^{\pm}(s,x,\xi)\sim x\cdot\xi\pm a(x,\xi)s+\sum_{k\geq 2}\frac{1}{k!}(\partial_s^k\psi^{\pm})(0,x,\xi)s^k$$

we get the estimate:

$$\partial_t^l\partial_x^{\alpha}\partial_{\xi}^{\beta}\left[\psi^{\pm}(s,x,\xi)-(x\cdot\xi\pm a(x,\xi)s+\sum_{k\geq 2}^{N}\frac{\partial_s^k\psi^{\pm}}{k!}(0,x,\xi)s^k\right]$$
$$=0(|\xi|^{1-|\beta|}s^{N+1-l})\,,\qquad |\xi|\geq 1\,,\quad 0\leq s<\min\{\tfrac{1}{2},2\sqrt{T}\}\,.$$

Hence the estimate:

$$\partial_t^l\partial_x^{\alpha}\partial_{\xi}^{\beta}\left[\varphi^{\pm}(t,x,\xi)-\left(x\cdot\xi\pm 2\sqrt{t}\,a(x,\xi)+\sum_{k=2}^{N}\alpha_{1-k}(x,\xi)(2\sqrt{t}\,a(x,\xi))^k\right)\right]$$
$$=O\left(|\xi|^{1-|\beta|}(\sqrt{t})^{N+1-2l}\right)=O\left(|\xi|^{1-|\beta|}(\sqrt{t}+1/|\xi|)^{N+1-2l}\right),$$

in any region $t|\xi|^2\geq$ const., $|\xi|\geq 1$, $0\leq t<\min\{\frac{1}{2},T\}$ and locally in x.

The claim on $\varphi^{\pm}$ being proved, from the construction performed in Theorem 1.6.1 it follows that $\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) \in \tilde{S}_{\infty}^{\mu+1,\infty}(0, T)$.

As a consequence, it is easily verified that

$$(2.2.3) \qquad \big(1-\chi(t|\xi|^2)\big)\exp[-i\varphi^{\pm}]P(\exp[i\varphi^{\pm}]p^{\pm}) = b^{\pm}(t, x, \xi) \in C^{\infty}_{\text{flat}}\big([0, T];\, S^{\mu+1}_{1,0}(R^n_x\times R^n_{\xi})\big)\,.$$

III) To control the symbol

$$[P, \chi(t|\xi|^2)]\{\exp[ix\cdot\xi]q(t, x, \xi) - \exp[i\varphi^+(t, x, \xi)]p^+(t, x, \xi) - \exp[i\varphi^-(t, x, \xi)]p^-(t, x, \xi)\}$$

we need to prove the following assertion:

If $R(t, x, \xi) \in S^{1,2}_{\infty}(0, T)$ and $\tilde{\chi} \in C_0^{\infty}(R^+)$, $\tilde{\chi} \equiv 1$ on some interval, then:

i) $\tilde{\chi}(t|\xi|^2)R(t, x, \xi) \in S^{-1,0}(0, T)$.

ii) For every $N \geqslant 1$

$$\tilde{\chi}(t|\xi|^2)\left[\exp[iR(t, x, \xi)] - \sum_{j=0}^{N-1}\frac{(iR(t, x, \xi))^j}{j!}\right] \in S^{-N,0}(0, T)\,.$$

The proof of i) is obvious since $\sqrt{t} \sim 1/|\xi|$ on the support of χ. To prove ii) we write

$$\exp[iR] - \sum_{j=0}^{N-1}\frac{(iR)^j}{j!} = \frac{(iR)^N}{(N-1)!}\int_0^1 (1-\sigma)^{N-1}\exp[i\sigma R]\,d\sigma\,,$$

and note that on the support of $\tilde{\chi}$ and locally in x we have the estimates:

$$\partial_t^l\partial_x^{\alpha}\partial_{\xi}^{\beta}R = O(|\xi|)^{-1-|\beta|-2l}\,, \qquad \partial_t^l\partial_x^{\alpha}\partial_{\xi}^{\beta}\exp[i\sigma R] = O(1)$$

if $|\alpha| + |\beta| + 1 = 0$ and $= O(|\xi|^{-1-|\beta|-2l})$ if $|\alpha| + |\beta| + l > 0$.

Hence $\tilde{\chi}R^N \in S^{-N,0}(0, T)$ and $\tilde{\chi}\int_0^1 (1-\sigma)^{N-1}\exp[i\sigma R]\,d\sigma \in S^{0,0}(0, T)$. This proves our assertion.

The commutator $[P, \chi(t|\xi|^2)]$ can be written as

$$\tilde{\chi}_1(t|\xi|^2)\alpha(t, x)\,\partial_t + \tilde{\chi}_2(t|\xi|^2)\beta(t, x) \qquad \text{for suitable functions} \qquad \tilde{\chi}_1, \tilde{\chi}_2 \in C_0^{\infty}(R^+) \quad \text{and } \alpha, \beta \in C^{\infty}\,.$$

Let us consider the symbol

$$\tilde{\chi}_2(t|\xi|^2)\big[\exp[ix\cdot\xi]q(t,x,\xi) - \exp[i\varphi^+(t,x,\xi)]p^+(t,x,\xi) - \exp[i\varphi^-(t,x,\xi)]p^-(t,x,\xi)\big]\,. \tag{2.2.4}$$

We recall that in Theorem 1.3.1 the symbol $q_{-j/2}(t,x,\xi)$ was obtained as $q_{-j/2}(t,x,\xi) = \hat{q}_{-j}(z;x,\xi)|_{z=2\sqrt{t}a(x,\xi)}$, where $\hat{q}_{-j}$ has the form

$$\hat{q}_{-j}(z,x,\xi) = \sum_{h,k\leqq 2j} \varrho_{-j}^{(h,k)}(x,\xi)\varphi_{h,k}(z;x)\,, \qquad \varrho_{-j}^{(h,k)}\in \mathcal{O}^{-j}\,,$$

with the $\varphi_{h,k}$ given in (1.3.5).

Putting

$$\varphi_{h,k}^{\pm}(z;x) = \int_{L^\pm} \exp[iz(\sigma\mp 1)](\sigma^2-1)^{\nu_0(x)-\frac{1}{2}-h}(\log(\sigma^2-1))^k\, \check{d}\sigma\,,$$

the contours $L^\pm$ being those of fig. 3, we define accordingly

$$\begin{cases} \hat{q}_{-j}^{\pm}(z;x,\xi) = \sum\limits_{h,k\leqq 2j} \varrho_{-j}^{(h,k)}(x,\xi)\varphi_{h,k}^{\pm}(z;x) \\ q_{-j/2}^{\pm}(t,x,\xi) = \hat{q}_{-j}^{\pm}(z;x,\xi)\big|_{z=2\sqrt{t}a(x,\xi)}\,. \end{cases}$$

Thus

$$q_{-j/2}(t,x,\xi) = \exp[iz]q_{-j/2}^{+}(t,x,\xi) + \exp[-iz]q_{-j/2}^{-}(t,x,\xi)\,, \qquad z = 2\sqrt{t}\,a(x,\xi)\,.$$

Let us fix a function $\tilde{\chi}\in C_0^\infty(R^+)$ with $\tilde{\chi}\equiv 1$ on $\operatorname{supp}\tilde{\chi}_1 \cup \operatorname{supp}\tilde{\chi}_2$. By Lemma 2.1.1 iii) we can construct two symbols $q^\pm(t,x,\xi)\in S_0^{0,0}(0,+\infty)$ with $q^\pm \sim \sum_{j\geqq 0}\tilde{\chi}(t|\xi|^2)q_{-j/2}^{\pm}$.

It follows that (2.2.4) can be rewritten as

$$\begin{aligned} \tilde{\chi}_2(t|\xi|^2)\big[&\exp[i(x\cdot\xi + z)]q^+ + \exp[i(x\cdot\xi - z)]q^- \\ &- \exp[i\varphi^+]p^+ - \exp[i\varphi^-]p^-\big] = \exp[i\varphi^+]\tilde{\chi}_2(t|\xi|^2) \\ &\cdot\{\exp[-iR^+]q^+ - p^+\} + \exp[i\varphi^-]\tilde{\chi}_2(t|\xi|^2)\{\exp[-iR^-]q^- - p^-\}\,, \end{aligned} \tag{2.2.5}$$

where $R^\pm \in S_\infty^{1,2}(0,T)$ are defined in (1.4.4).

Now we claim that we can modify the symbols $p_{-j/2}^{\pm}$, $j\geqq 1$, constructed in Theorem 1.6.1. in such a way that the new $p_{-j/2}^{\pm}$, $j\geqq 1$, satisfy the transport equations (1.4.17), (1.4.18), keep the structure (1.6.3) and be such that

$$\tilde{\chi}_2(t|\xi|^2)\{\exp[-iR^\pm]q^\pm - p^\pm\}\in S^{-\infty,0}(0,T)\,. \tag{2.2.6}$$

The proof of our claim is based on the remark that while $p_0^+ + p_0^-$ satisfies the initial condition $p_0^+ + p_0^-|_{t=0} = 1$, no initial condition has ever been imposed until now on $p_{-j/2}^+ + p_{-j/2}^-$ for $j \geqq 1$.

We deal only with the sign $+$ and make a preliminary formal computation. From the definition of $q_{-j/2}^{\pm}$ and the construction of Sect. 1.3, we have

$$\exp[-ix\cdot\xi]P\Big(\exp[ix\cdot\xi]\exp[iz]\sum_{j\geqq 0} q_{-j/2}^+\Big) \sim 0\,,$$

so that

$$\exp[-i\varphi^+]P\Big[\exp[i\varphi^+]\Big(\exp[-iR^+]\sum_{j\geqq 0} q_{-j/2}^+\Big)\Big] \sim 0\,. \tag{2.2.7}$$

On the other hand the construction in Sect. 1.6 yields

$$\exp[-i\varphi^+]P\Big[\exp[i\varphi^+]\sum_{j\geqq 0} p_{-j/2}^+\Big] \sim 0\,. \tag{2.2.8}$$

Hence

$$\exp[-i\varphi^+]P\Big[\exp[i\varphi^+]\Big(\exp[-iR^+]\sum_{j\geqq 0} q_{-j/2}^+ - \sum_{j\geqq 0} p_{-j/2}^+\Big)\Big] \sim 0\,. \tag{2.2.9}$$

Expanding $\exp[-iR^+]$ as a sum of terms of decreasing homogeneity, we write

$$\exp[-iR^+]\sum_{j\geqq 0} q_{-j/2}^+ - \sum_{j\geqq 0} p_{-j/2}^+ = \sum_{j\geqq 0} \Phi_{-j/2}\,,\ \Phi_{-j/2} \in \Psi^{-j/2}\,, \qquad j \geqq 0\,. \tag{2.2.10}$$

The symbols $\Phi_{-j/2}$ satisfy the transport equations (1.4.17), (1.4.18) and have an asymptotic expansion as in Theorem 1.7.3; moreover, $\Phi_0 = 0$, so that we can apply Theorem 1.7.3 and conclude that there exists a function $\psi_j^+ \in \mathcal{A}_{-j,j}^+$ for which

$$\Phi_{-j/2} = I^+(z, x, \xi; \nu_0, \psi_j^+)|_{z=2\sqrt{ia(x,\xi)}}\,, \qquad j \geqq 1\,. \tag{2.2.11}$$

We define new symbols $p_{-j/2}^{\#,+}$ by

$$p_{-j/2}^{\#,+} = p_{-j/2}^+ + \Phi_{-j/2}\,, \qquad j \geqq 1\,. \tag{2.2.12}$$

For convenience we shall continue to denote by $p_{-j/2}^+$ the modified symbols $p_{-j/2}^{\#,+}$. We emphasize that no modification is needed for p_0^+.

Let us now turn to the claim (2.2.6); we have

$$(2.2.13)\qquad \tilde{\chi}_2(t|\xi|^2)\{\exp[-iR^+]q^+ - p^+\} = \tilde{\chi}_2(t|\xi|^2)\Big\{\exp[-iR^+]$$
$$\cdot\Big[q^+ - \sum_{j=0}^{N-1} q^+_{-j/2}\Big] - \Big[p^+ - \sum_{j=0}^{N-1} p^+_{-j/2}\Big]\Big\} + \tilde{\chi}_2(t|\xi|^2)\Big\{\exp[-iR^+]\sum_{j=0}^{N-1} q^+_{-j/2} - \sum_{j=0}^{N-1} p^+_{-j/2}\Big\}.$$

Now

$$(2.2.14)\qquad \begin{cases} \tilde{\chi}(t|\xi|^2)\Big[q^+ - \sum_{j=0}^{N-1} q^+_{-j/2}\Big] \in S^{-N,0}(0, +\infty) \\ \tilde{\chi}_2(t|\xi|)^2 \exp[-iR^+] \in S^{0,0}(0, T)\,. \end{cases}$$

Hence, since $\tilde{\chi}\tilde{\chi}_2 = \tilde{\chi}_2$,

$$(2.2.15)\qquad \tilde{\chi}_2(t|\xi|^2)\Big\{\exp[-iR^+]\Big[q^+ - \sum_{j=0}^{N-1} q^+_{-j/2}\Big]\Big\} \in S^{-N,0}(0, T)\,.$$

Furthermore, by definition:

$$(2.2.16)\qquad p^+ - \sum_{j=0}^{N-1} p^+_{-j/2} \in \tilde{S}^{\mu,\mu+N}_{\infty}(0, T)\,.$$

Hence, since $\sqrt{t} \sim 1/|\xi|$ on supp $\tilde{\chi}_2$,

$$(2.2.17)\qquad \tilde{\chi}_2(t|\xi|^2)\Big[p^+ - \sum_{j=0}^{N-1} p^+_{-j/2}\Big] \in S^{-N,0}(0, T)\,.$$

Now

$$\tilde{\chi}_2(t|\xi|^2)\Big\{\exp[-iR^+]\sum_{j=0}^{N-1} q^+_{-j/2} - \sum_{j=0}^{N-1} p^+_{-j/2}\Big\}$$
$$= \tilde{\chi}_2(t|\xi|^2)\Big\{\Big(\sum_{l=0}^{N-1}\frac{(-iR^+)^l}{l!}\Big)\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big) - \sum_{j=0}^{N-1} p^+_{-j/2}\Big\}$$
$$+ \tilde{\chi}_2(t|\xi|^2)\Big(\exp[-iR^+] - \sum_{l=0}^{N-1}\frac{(iR^+)^l}{l!}\Big)\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big).$$

By Remark III, ii)

$$(2.2.18)\quad \tilde{\chi}_2(t|\xi|^2)\Big(\exp[-iR^+] - \sum_{j=0}^{N-1}\frac{(-iR^+)^l}{l!}\Big)\Big(\sum_{j=0}^{N-1} q^+_{-j/2}\Big) \in S^{-N,0}(0, T)\,.$$

Since $R^+ \in S^{1,2}_\infty$ with asymptotic expansion $R^+ \sim \sum_{k\geqq 2} \alpha^+_{1-k}(x, \xi)z^k$ (see (1.4.5)), we have

$$(2.2.19) \qquad \tilde{\chi}_2(t|\xi|^2)\Big(R^+ - \sum_{k=2}^{N} \alpha^+_{1-k} z^k\Big) \in S^{-N,0}(0, T)\,.$$

Then

$$\tilde{\chi}_2(t|\xi|^2)\left(\sum_{l=0}^{N-1} \frac{(-iR^+)^l}{l!}\right)\left(\sum_{j=0}^{N-1} q^+_{-j/2}\right)$$

$$= \sum_{l=0}^{N-1} \frac{(-i)^l}{l!}\left[\tilde{\chi}_2(t|\xi|^2)\Big(\sum_{k=2}^{N} \alpha^+_{1-k} z^k\Big)^l + \text{a symbol of } S^{-N,0}\right]\left(\sum_{j=0}^{N-1} q^+_{-j/2}\right)$$

$$= (\text{mod. } S^{-N,0}(0, T)) = \tilde{\chi}_2(t|\xi|^2)\sum_{j=0}^{N-1}\Big(\exp[-iR^+]\sum_{k\geqq 0} q^+_{-j/2}\Big)_{-j/2}\,.$$

By the modification of the $p^+_{-j/2}$ performed above, we can conclude that

$$(2.2.20) \quad \tilde{\chi}_2(t|\xi|^2)\left\{\left(\sum_{l=0}^{N-1} \frac{(-iR^+)^l}{l!}\right)\left(\sum_{j=0}^{N-1} q^+_{-j/2}\right) - \sum_{j=0}^{N-1} p^+_{-j/2}\right\} \in S^{-N,0}(0, T)\,.$$

As a consequence we have that claim (2.2.6) is proved. In the same way one can prove that

$$(2.2.21) \qquad \tilde{\chi}_1(t|\xi|^2)\,\partial_t\{\exp[-iR^\pm]q^\pm - p^\pm\} \in S^{-\infty,0}(0, T)\,.$$

We summarize all the preceding remarks in the following theorem.

THEOREM 2.2.1. *There exist a symbol* $q(t, x, \xi) \in \tilde{S}^{0,0}_0(0, +\infty)$ *and two symbols* $p^\pm(t, x, \xi) \in \tilde{S}^{\mu,\mu}_\infty(0, T)$, $\mu = \sup_{x\in R^n}(-\operatorname{Re}\nu_0(x) - \frac{1}{2})$, *such that the operator E defined in* (2.2.1) *has the following properties:*

i) $PE = C + B^+ + B^-$,

where

$$(2.2.22) \qquad Cg(t, x) = \int \exp[ix\cdot\xi]c(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\,,$$

with a symbol $c(t, x, \xi) \in C^\infty(\overline{R^+_t}; S^{-\infty}_{1,0}(R^n_x \times R^n_\xi))$, *and*

$$(2.2.23) \qquad B^\pm g(t, x) = \int \exp[i\varphi^\pm(t, x, \xi)]b^\pm(t, x, \xi)\hat{g}(\xi)\,\check{d}\xi\,,$$

with symbols $b^\pm(t, x, \xi) \in C^\infty_{\text{flat}}([0, T]; S^{\mu+1}_{1,0}(R^n_x \times R^n_\xi))$.

ii) $\gamma E - I$ *is a smoothing operator.*

The operator C is obviously a smoothing operator, precisely $C\colon \mathcal{E}'(R^n) \to C^\infty(\overline{R_t^+}\times R_x^n)$. Therefore, to obtain a parametrix we need to exorcise the terms $B^\pm$. This will be done in the following theorem.

THEOREM 2.2.2. *There exist two symbols*

$$r^\pm(t, x, \xi) \in C^\infty_{\text{flat}}\big([0, T];\, S^\mu_{1,0}(R^n_x \times R^n_\xi)\big)$$

such that

$$\exp[-i\varphi^\pm]P\big(\exp[i\varphi^\pm]r^\pm\big) + b^\pm \in C^\infty_{\text{flat}}\big([0, T];\, S^{-\infty}_{1,0}(R^n_x \times R^n_\xi)\big)\,. \tag{2.2.24}$$

PROOF. We prove the theorem in the case of the sign $+$, dropping for simplicity the superscript. Putting $s = 2\sqrt{t}$ and $T' = 2\sqrt{T}$ we need to prove that for a given symbol $\tilde{b}(s, x, \xi) = b(s^2/4, x, \xi) \in C^\infty_{\text{flat}}([0, T'];\, S^{\mu+1}_{1,0})$ there exists a symbol $\tilde{r}(s, x, \xi)$ belonging to $C^\infty_{\text{flat}}([0, T'];\, S^\mu_{1,0})$ for which

$$\begin{aligned}\exp[-i\psi(s, x, \xi)]\tilde{P}[\exp[i\psi(s, x, \xi)]r(s, x, \xi)]&\\ + \tilde{b}(s, x, \xi) \in C^\infty_{\text{flat}}\big([0, T'];\, S^{-\infty}_{1,0}\big)&\,,\end{aligned} \tag{2.2.25}$$

where $\psi = \psi^+(s, x, \xi)$ has been defined in (1.4.1) and $\tilde{P}$ is the operator (0.1) written in the new variables (s, x), i.e.

$$\begin{aligned}\tilde{P} = \partial_s^2 + \frac{2\nu(s^2/4, x) + 1}{s}\,\partial_s - A(s^2/4, x, \partial_x)&\\ + B(s^2/4, x, \partial_x) + b_0(s^2/4, x)&\,.\end{aligned} \tag{2.2.26}$$

A computation yields:

$$\begin{aligned}\exp[-i\psi]\tilde{P}\big(\exp[i\psi]\tilde{r}\big) &= 2i\Big[\partial_s\psi\partial_s + \sum_{j,i=1}^n a_{ij}(s^2/4, x)\,\partial_{x_i}\psi\partial_{x_j}\Big]\,\tilde{r}\\ &\quad + i\left(\partial_s^2\psi - A(s^2/4, x, \partial_x)\psi + \frac{2\nu(s^2/4, x) + 1}{s}\,\partial_s\psi\right)\tilde{r}\\ &\quad + \tilde{P}\tilde{r} + \frac{2\nu(s^2/4, x) + 1}{s}\,\partial_s\tilde{r}\,.\end{aligned} \tag{2.2.27}$$

Since $\partial_s\psi = \sqrt{A(s^2/4, x, d_x\psi)} \neq 0$ we divide both sides of (2.2.27) by $(2i\partial_s\psi)/s$ and obtain the condition

$$\begin{aligned}&s\Big(\partial_s + \sum_{j=1}^n c_{j0}(x, x;\, \xi)\,\partial_{x_j}\Big)\tilde{r}\, +\\ &\quad + d_0(s, x;\, \xi)\tilde{r} + Q(s, x, \xi;\, \partial_s, \partial_x)\tilde{r} + \tilde{g} \in C^\infty_{\text{flat}}\big([0, T'];\, S^{-\infty}_{1,0}\big)\,,\end{aligned} \tag{2.2.28}$$

where $\tilde{g}(s, x, \xi) = (s/2i\partial_s\psi)\tilde{b} \in C^\infty_{\text{flat}}([0, T']; S^\mu_{1,0})$, $c_{j0}(s)$, $d_0(s) \in \mathcal{O}^0$ and Q is a second order operator with smooth coefficients homogeneous of degree -1 with respect to ξ; note that

$$c_{j0}(s, x; \xi) = \sum_{i=1}^{n} a_{ij}(s^2/4, x)\,\partial_{x_i}\psi(s, x, \xi)\big(2i\partial_s\psi(s, x, \xi)\big)^{-1}, \qquad j = 1, \ldots, n.$$

As a consequence, the following system:

$$\text{(2.2.29)} \qquad \begin{cases} \dfrac{d}{ds}\, x_j(s; y) = c_{j0}(s, x(s; y), \xi)\,, & j = 1, \ldots, n \\ x_j(0; y) = y_j \end{cases}$$

is a part of the Hamiltonian system for ψ so that the map $[0, T']\times R^n \in (s, y) \to \big(s, x(s; y)\big)$ is a global diffeomorphism. Writing (2.2.28) in the new variables (s, y) we obtain:

$$\text{(2.2.30)} \qquad [s\partial_s + d_0(s, y; \xi) + Q(s, y, \xi; \partial_s, \partial_y)]\tilde{r}(s, y, \xi) + \tilde{g}(s, y, \xi) \in C^\infty_{\text{flat}}([0, T']; S^{-\infty}_{1,0})\,,$$

where, for simplicity, we continue to denote with the same notation the functions written in the new variables.

Since $Q(s, y, \xi; \partial_s, \partial_y)$ maps $C^\infty_{\text{flat}}([0, T']; S^\mu_{1,0})$ into $C^\infty_{\text{flat}}([0, T']; S^{\mu-1}_{1,0})$, to prove the existence of $\tilde{r} \in C^\infty_{\text{flat}}([0, T']; S^\mu_{1,0})$ satisfying (2.2.30) it will be enough to show that for every $m \in R$ and every $G(s, y, \xi) \in C^\infty_{\text{flat}}([0, T']; S^m_{1,0})$ there exists a symbol $h(s, y, \xi) \in C^\infty_{\text{flat}}([0, T']; S^m_{1,0})$ such that

$$\text{(2.2.31)} \qquad \big(s\partial_s + d_0(s, y, \xi)\big)h = G\,, \qquad s \in [0, T']\,.$$

To prove this assertion consider the operator

$$HG(s, y, \xi) = \int_0^1 G(\sigma, y, \xi)\,\frac{d\sigma}{\sigma}\,,$$

which maps C^∞_{flat} into itself and satisfies the equation $s\partial_s HG = G$. To solve (2.2.31) we take $h = H\Phi$ and obtain the equation $\Phi + d_0(s, x, \xi)H\Phi = G$ which can be solved in $C^\infty_{\text{flat}}([0, T']; S^m_{1,0})$ by the standard Picard's approximation procedure.

Let us now turn to (2.2.30). Using the preceding result we can construct a formal series $\sum_{j\geq 0} \tilde{r}_j$ with:

i) $\tilde{r}_j(s, y, \xi) \in C^\infty_{\text{flat}}([0, T']; S^{\mu-j}_{1,0})$, $j \geq 0$.

ii) $[s\partial_s + d_0 + Q]\left(\sum_{j=0}^{N-1} \tilde{r}_j\right) + \tilde{g} \in C^\infty_{\text{flat}}([0, T']; S^{\mu-N}_{1,0})$, $\forall N \geq 1$.

By a standard argument one can find $\tilde{r} \in C^\infty_{\text{flat}}([0, T']; S^{\mu}_{1,0})$ such that $\tilde{r} - \sum_{j=0}^{N-1} \tilde{r}_j \in C^\infty_{\text{flat}}([0, T']; S^{\mu-N}_{1,0})$, $\forall N \geq 1$.

This completes the proof of the theorem. q.e.d.

As a final consequence of Theorems 2.2.1, 2.2.2 we have

COROLLARY 2.2.1. *A parametrix for the Cauchy problem* (0.2) *is given by* $E + \mathcal{R}^+ + \mathcal{R}^-$ *where* E *is given by* (2.2.1) *and*

$$\mathcal{R}^\pm g(t, x) = \int \exp[i\varphi^\pm(t, x, \xi)] r^\pm(t, x, \xi) \hat{g}(\xi) \, d\!\!{}^-\xi\,, \qquad t \in [0, T]\,, \tag{2.2.32}$$

with the symbols $r^\pm$ *given by Theorem* 2.2.2.

In the next theorem we list some microlocal properties of the constructed parameterix $E + \mathcal{R}^+ + \mathcal{R}^- = \mathcal{Q}$.

THEOREM 2.2.3. *For every* $g \in \mathcal{E}'(R^n_x)$ *we have*:

i) $WF(\partial_t^k \mathcal{Q} g|_{t=0}) \subset WF(g)$, $k = 0, 1, \ldots$

ii) *For every* $s \in [0, T[$:

$$WF(\mathcal{Q} g|_{t=s}) = (\Lambda^+_{2\sqrt{s}} \cup \Lambda^-_{2\sqrt{s}}) \circ WF(g)\,, \tag{2.2.33}$$

where $\Lambda^\pm_t$ *have been defined in* (0.6).

PROOF. We split E into a sum $E = E_0 + E^+ + E^-$ corresponding to the three terms in (2.2.1). Then $\partial_t^k \mathcal{Q} g|_{t=0} = \partial_t^k E_0 g|_{t=0}$. As we have already remarked the operator $g \to \partial_t^k E_0 g|_{t=0}$ is a pseudo-differential operator, and this proves i).

To prove ii) we observe that (2.2.33) is obvious when $s = 0$ since $\Lambda^\pm_0 = \Delta(T^*R^n \setminus 0)$, the diagonal of $T^*R^n \setminus 0 \times T^*R^n \setminus 0$. For $s > 0$ we have $WF(\mathcal{Q} g|_{t=s}) = WF[(E^+ + \mathcal{R}^+) g|_{t=s} + (E^- + \mathcal{R}^-) g|_{t=s}]$ since the operator $g \to E_0\, g|_{t=s}$ is smoothing. As we have already remarked $g \to (E^\pm + \mathcal{R}^\pm) g|_{t=s}$ are Fourier integral operators with phases $\varphi^\pm(s, x, \xi)$ and amplitudes $p^\pm(s, x, \xi) + r^\pm(s, x, \xi) \in S^\mu_{1,0}(R^n_x \times R^n_\xi)$; therefore by well known results on the calculus of WF (see L. Hörmander [5]) we have

$$WF((E^\pm + \mathcal{R}^\pm) g|_{t=s}) \subset \Lambda^\pm_{2\sqrt{s}} \circ WF(g)\,.$$

To prove the converse inclusion we recall that for every s $\Lambda^{\pm}_{2\sqrt{s}}$ are the graphs of the symplectomorphism

$$T^*R^n\setminus 0 \ni (y,\eta) \to \big(x^{\pm}(2s;y,\eta),\xi^{\pm}(2s;y,\eta)\big) \in T^*R^n\setminus 0$$

(see (0.6)). Moreover, $p^{\pm}(s,x,\xi)+r^{\pm}(s,x,\xi)=p_0^{\pm}(s,x,\xi)+r^{\pm}(s,x,\xi)$ modulo $S^{\mu-1}_{1,0}$. Now $p_0^{\pm}(s,x,\xi)$ is an elliptic symbol as follows from (1.5.6), i.e. $p_0^{\pm}(s,x,\xi)=\Gamma(\nu_0(x)+1)\exp[iz_n](z/2)^{-\nu_0(x)}H^{(1),(2)}_{\nu_0(x)}(z)|_{z=2\sqrt{s}a(x,\xi)}$. Since $r^{\pm}(s,x,\xi)$ is flat at $s=0$, we can conclude that for some $\varepsilon>0$ the symbol $p^{\pm}(s,x,\xi)+r^{\pm}(s,x,\xi)$ is invertible in $S^{-\mu}_{1,0}$ for $s\leqq\varepsilon$. As a consequence we obtain

$$WF\big((E^{\pm}+\mathfrak{R}^{\pm})g_{t=s}\big)=\Lambda^{\pm}_{2\sqrt{s}}\circ WF(g)\,,\qquad 0\leqq s\leqq\varepsilon\,.$$

To finish we observe that

$$\big(\Lambda^{+}_{2\sqrt{s}}\circ WF(g)\big)\cap\big(\Lambda^{-}_{2\sqrt{s}}\circ WF(g)\big)=\emptyset\,,\qquad s>0\,,$$

so that

$$WF(\mathfrak{Q}g|_{t=s})=WF\big((E^{+}+\mathfrak{R}^{+})g|_{t=s}\big)\cup WF\big((E^{-}+\mathfrak{R}^{-})g|_{t=s}\big)\,,\qquad 0\leqq s\leqq\varepsilon\,.$$

The above equality holds then for all $s\in[0,T[$. To see this we observe that $WF(\partial_t\mathfrak{Q}g|_{t=s})\subset WF(\mathfrak{Q}g|_{t=s})$; and that for $t\geqslant s>0$, $g\to\mathfrak{Q}g$ solves a Cauchy problem for the strictly hyperbolic operator P, with $P\mathfrak{Q}^{+}\in C^{\infty}$.

Known results on the propagation of singularities for strictly hyperbolic Cauchy problems yield our thesis (see e.g. J. J. Duistermaat [3]). q.e.d.

REMARKS. 1) The construction of the parametrix $\mathfrak{Q}$ for pb. (0.2) has been performed under the hypotheses $\nu_0(x)+1\notin\{0,-1,-2,\ldots\}$, $\nu_0(x)-\frac{1}{2}\notin\{0,1,2,\ldots\}$. While the first condition on ν_0 is natural because of its necessity for C^{∞}-well posedness of the Cauchy problem (0.2), the second one is, in our opinion, only technical. We believe that by changing the integral representation for Bessel functions one should provide a way to drop the condition $\nu_0(x)-\frac{1}{2}\notin\{0,1,2,\ldots\}$.

2) According to Theorem 2.2.3, the parametrix $\mathfrak{Q}$ allows to describe the singularities of solutions of the equation $Pu\in C^{\infty}(\overline{R^{+}_t}\times R^n_x)$ which are normally regular, i.e. $u\in C^{\infty}\big(\overline{R^{+}_t};\mathcal{D}'(R^n_x)\big)$ (at least when $\nu_0(x)$ satisfies condition (1.2.7)). However, since $t=0$ is characteristic for P, one can find solutions of the equation $Pu=0$, $t>0$, which are not normally regular

distributions and with $WF(u|_{t=s}) = \Lambda^+_{2\sqrt{s}} \circ WF(g)$ or $WF(u|_{t=s}) = \Lambda^-_{2\sqrt{s}} \circ WF(g)$. Typical examples are the following ones;

$$u^{\pm}(t, x) = \frac{1}{\sqrt{\pi/2}} \int \exp\left[i[x\cdot\xi \pm 2\sqrt{t}\,|\xi|\right] \frac{1}{2\sqrt{t}\,|\xi|}\,\hat{g}(\xi)\,\bar{d}\xi$$

which solve $(t\partial_t^2 - \frac{1}{2}\partial_t - \Delta)u^{\pm}(t, x) = 0,\ t > 0$.

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