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Jacobi Fields and Regularity of Functions of Least Gradient (*).

HAROLD R. PARKS - WILLIAM P. ZIEMER

0. – Introduction.

Let Ω be a bounded open subset of $\boldsymbol{R}^n$ and suppose $u \in BV(\Omega)$. We say that u is of *least gradient* (*with respect to* Ω) if, for every $v \in BV(\Omega)$ such that $v = u$ outside some compact subset of Ω,

$$\int_{\Omega} |Du| \leqslant \int_{\Omega} |Dv| .$$

A function of least gradient need not even be continuous. Indeed, for any subset A of Ω, the portion of the reduced boundary of A which lies in Ω is area minimizing if and only if the characteristic function of A is of least gradient. Because of this fact, functions of least gradient have been used for over two decades to study area minimizing oriented hypersurfaces. In this paper, we investigate functions of least gradient themselves as objects of interest. Of course, we will be using as tools many facts now known about oriented area minimizing hypersurfaces, since the level sets of a function of least gradient are (or are bounded by) such minimizing hypersurfaces.

As mentioned above, a function of least gradient need not be continuous. However, in [PH1] it was shown that if Ω is strictly convex and boundary values, $\varphi\colon \operatorname{Bdry}\Omega \to \boldsymbol{R}$, satisfying the bounded slope condition are prescribed, then the Dirichlet problem of finding a continuous function $u\colon \operatorname{Clos}\Omega \to \boldsymbol{R}$ with $u|\operatorname{Bdry}\Omega = \varphi$ which is of least gradient with respect to Ω admits a Lipschitzian solution; later in [PH2] it was shown that the solution is unique. With such an existence result in hand, it is natural to investigate the regularity of the extremal. The Euler-Lagrange partial

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differential equation associated with finding a function, u, of least gradient is

$$\operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 .$$

This differential equation falls outside the scope of existing regularity theory, and therefore it is not possible to employ the usual methods.

It quickly becomes apparent that one cannot obtain, for the least gradient problem, the type of regularity results which hold for the quasi-linear elliptic or parabolic problems usually studied. Examples show us some of the limitations. First, the least gradient problem admits solutions that are not smooth.

EXAMPLE. Define $u: \boldsymbol{R}^2 \to \boldsymbol{R}$ by setting

$$u(x, y) = \begin{cases} y & \text{if } y \leqslant 0 , \\ 2y & \text{if } y \geqslant 0 . \end{cases}$$

Then u is Lipschitzian and of least gradient with respect to any open subset of $\boldsymbol{R}^2$. Second, smoothness can even be lost as the following more sophisticated example due to John Brothers shows.

EXAMPLE. Define $u: \boldsymbol{R}^2 \cap \{(x, y): x^2 + y^2 \leqslant 1\} \to \boldsymbol{R}$ by setting

$$u(x, y) = \begin{cases} 2x^2 - 1 & \text{if } |x| \geqslant 1/\sqrt{2},\ |y| \leqslant 1/\sqrt{2} , \\ 0 & \text{if } |x| \leqslant 1/\sqrt{2},\ |y| \leqslant 1/\sqrt{2} , \\ 1 - 2y^2 & \text{if } |x| \leqslant 1/\sqrt{2},\ |y| \geqslant 1/\sqrt{2} . \end{cases}$$

Then u is Lipschitzian and of least gradient with respect to

$$\Omega = \boldsymbol{R}^2 \cap \{(x, y): x^2 + y^2 < 1\} .$$

For

$$(\cos\theta, \sin\theta) \in \operatorname{Bdry} \Omega$$

we have

$$u(\cos\theta, \sin\theta) = \cos(2\theta) ,$$

so $u|\operatorname{Bdry}\Omega$ is real analytic. Note that Du does not exist on

$$S = \Omega \cap \{(x, y): |x| = 1/\sqrt{2} \text{ or } |y| = 1/\sqrt{2}\}$$

and Du is discontinuous across S.

With those examples in mind, we now state our main result.

THEOREM. *Suppose* $2 \leqslant n \leqslant 7$. *If* $\Omega \subset \boldsymbol{R}^n$ *is bounded and open* Bdry Ω *is a class* $n-1$ *submanifold of* $\boldsymbol{R}^n$, φ: Bdry $\Omega \to \boldsymbol{R}$ *is of class* $n-1$, *and* u: Clos $\Omega \to \boldsymbol{R}^n$, *is Lipschitzian and of least gradient with* $u|$Bdry $\Omega = \varphi$ *then there exists an open dense set* $W \subset \Omega$ *such that* $u|W$ *is of class* $(n-3)$.

In the second example above, the discontinuity in Du is caused by the fact that the problem of finding a one dimensional integral current of least mass whose boundary is the alternatingly signed set of corners of a square admits two distinct solutions. Such non-uniqueness must be expected to result in discontinuities of Du. On the other hand, the discontinuities are confined to two area minimizing hypersurfaces, and there can be at most countably many such non-uniquenesses for any given u, because, for each non-uniqueness, an open set where u is constant is enclosed between two area minimizing surfaces. The dimension restriction plays no role here.

A more vexing problem is the possibility that $Du = 0$ at some point not associated with the non-uniquenesses discussed above. From now on restrict n to $2 \leqslant n \leqslant 7$. For simplicity, consider each Clos $\Omega \cap u^{-1}(t)$ to be an area minimizing surface with boundary and assume it is the unique area minimizing surface with that boundary. The point of Section 3 of this paper and an essential fact underlying our proof of the main theorem above is that (for $2 \leqslant n \leqslant 7$, the range where $\Omega \cap u^{-1}(t)$ must be regular) the behavior of Du at one point of $\Omega \cap u^{-1}(t)$ determines the behavior of Du on all of $\Omega \cap u^{-1}(t)$. Indeed, if $Du(x_0) = 0$ for some $x_0 \in \Omega \cap u^{-1}(t)$, then the methods of Section 3 can be used to show that $Du(x) = 0$ for every $x \in \Omega \cap u^{-1}(t)$; in this case, we do not know any way to prove smoothness of u near $\Omega \cap u^{-1}(t)$. If, instead, $Du(x_0) = 0$ is not true (i.e. if either $Du(x_0) \neq 0$ or $Du(x_0)$ does not exist) for some $x_0 \in \Omega \cap u^{-1}(t)$, hence for every $x \in \Omega \cap u^{-1}(t)$, then, in Section 4 using estimates from Section 3, we show how to construct a solution of Jacobi's equation on $\Omega \cap u^{-1}(t)$ which has a positive lower bound. (Jacobi's equation is the partial differential equation which a flow of minimal surfaces starting at $\Omega \cap u^{-1}(t)$ must initially satisfy.) A simple argument then shows there are no non-trivial solutions of Jacobi's equation on $\Omega \cap u^{-1}(t)$ which vanish at the boundary of $\Omega \cap u^{-1}(t)$ (see 4.3). This situation was considered by Brian White in his Princeton University Ph.D. Thesis. His result is that the minimal surfaces near $\Omega \cap u^{-1}(t)$ vary smoothly as a function of their boundaries. But the value of $u(x)$ is determined by the minimal surface x is on, so u is smooth near $\Omega \cap u^{-1}(t)$ (see 4.4). The major thrust of this paper is devoted to making this paragraph precise. The dimension restriction $2 \leqslant n \leqslant 7$ is used because then curvature estimates for $\Omega \cap u^{-1}(t)$ are available and all nearby area minimizing hypersurfaces can be essentially obtained from $\Omega \cap u^{-1}(t)$ by deformation in the normal direction.

We wish to acknowledge helpful conversations with John Brothers and Brian White.

1. – Preliminaries.

1.1. *Notation, terminology, and assumptions.*

Except as otherwise noted we will follow the notation and terminology of [FH].

(1) We will denote by Ω a bounded, open, connected subset of $\boldsymbol{R}^n$, where $2 \leqslant n \leqslant 7$. Set

$$\bar{\Omega} = \operatorname{Clos} \Omega ,$$

$$\Gamma = \operatorname{Bdry} \Omega .$$

We will assume that Γ is a class q, $4 \leqslant q < \infty$, submanifold of $\boldsymbol{R}^n$.
(2) Let $\varphi\colon \Gamma \to R$ be of class q. Set

$$a = \inf \{\varphi(x) : x \in \Gamma\} ,$$

$$b = \sup \{\varphi(x) : x \in \Gamma\} .$$

Let $u\colon \bar{\Omega} \to \boldsymbol{R}$ be a Lipschitzian function of least gradient satisfying

$$u|\Gamma = \varphi .$$

set

$$M = \operatorname{Lip}(u) ,$$

and note that

$$a \leqslant u(x) \leqslant b$$

holds for all $x \in \bar{\Omega}$.

We remark that, by [MM; proposition 6.2], [PH1; 4], and [PH2; 3.2.2], if φ is given and Ω is also uniformly convex, then such a function u exists and is unique.

(3) Set

$$\Omega_0 = \boldsymbol{E}^n \llcorner \Omega ,$$

$$\Gamma_0 = \partial \Omega_0 .$$

For each r with $a < r < b$ set

$$Q_r = -\,\partial\big(\Gamma_0 \llcorner \{x\colon \varphi(x) \leqslant r\}\big)\,,$$

$$R_r = \partial\big(\Gamma_0 \llcorner \{x\colon \varphi(x) \geqslant r\}\big)\,,$$

$$S_r = \partial\big(\Omega_0 \llcorner \{x\colon u(x) \leqslant r\}\big) - \Gamma_0 \llcorner \{x\colon \varphi(x) \leqslant r\}\,,$$

$$T_r = \Gamma_0 \llcorner \{x\colon \varphi(x) \geqslant r\} - \partial\big(\Omega_0 \llcorner \{x\colon u(x) \geqslant r\}\big)\,.$$

By [BDG; Theorem 1] the currents S_r and T_r are area minimizing. Set

$$\mathcal{U} = \{r\colon a < r < b,\ \mathcal{L}^n[\bar{\Omega} \cap u^{-1}(r)] = 0\,,$$

$$H^{n-1}[\Gamma \cap \varphi^{-1}(r)] = 0\,, \qquad H^{n-1}[\bar{\Omega} \cap u^{-1}(r)] > 0\,.$$

By [PH2; 3.2.1], if $r \in \mathcal{U}$, then $S_r = T_r$ is the unique rectifiable current with boundary $Q_r = R_r$ which is absolutely area minimizing with respect to $(\bar{\Omega}, \varphi)$ and, further,

$$\Omega \cap u^{-1}(r) = \Omega \cap \operatorname{spt}(S_r) = \Omega \cap \operatorname{spt}(T_r)\,.$$

(4) For $i = 1, 2, \dots, n$, $\tilde{D}_i$ will be the tangential gradient operator on Γ defined by setting

$$\tilde{D}_i = D_i - \mathcal{N}_i \sum_{k=1}^{n} \mathcal{N}_k D_k\,,$$

Where $\big(\mathcal{N}_1(x), \mathcal{N}_2(x), \dots, \mathcal{N}_n(x)\big)$ is the outward unit normal to Γ at x. By the Boundary Regularity Theory of Allard (see [AW; 5.2]) and [FH; 5.4.15], if $a < r < b$ and

$$0 < \inf\Big\{\sum_{i=1}^{n} (\tilde{D}_i \varphi(x))^2\colon x \in \Gamma \cap \varphi^{-1}(r)\Big\}\,,$$

then $\operatorname{spt}(T_r) \cap \Omega$ [resp., $\operatorname{spt}(S_r) \cap \Omega$] is a real analytic minimal submanifold of $\mathbf{R}^n$ and $\operatorname{spt}(T_r)$ [resp., $\operatorname{spt}(S_r)$] is a class q manifold with boundary; for any such r define

$$NT_r\colon \operatorname{spt}(T_r) \to \mathbf{G}(n, 1)$$

[resp., $NS_r\colon \operatorname{spt}(S_r) \to \mathbf{G}(n,1)$] by setting $NT_r(x)$ [resp., $NS_r(x)$] equal to the orthogonal complement of the linear span of $\operatorname{Tan}\big(\operatorname{spt}(T_r), x\big)$ [resp., $\operatorname{Tan}\big(\operatorname{spt}(S_r), x\big)$] for $x \in \operatorname{spt}(T_r)$ [resp., $x \in \operatorname{spt}(S_r)$].

(5) We will assume

(i) $a < 0 < b$,

(ii) $0 \in \mathcal{U}$,

(iii) $0 < \inf\left\{ \sum_{i=1}^{n} (\tilde{D}_i \varphi(x))^2 \colon x \in \Gamma \cap \varphi^{-1}(0) \right\}$,

(iv) $\Omega \cap u^{-1}(0)$ is connected,

(v) there exist $x^* \in \Omega \cap u^{-1}(0)$, a sequence $r^*(1), r^*(2), \ldots$, with $r^*(i) \neq 0$ and $\lim_i r^*(i) = 0$, and a sequence $x^*(1), x^*(2), \ldots$, with $x^*(i) \in \Omega \cap u^{-1}(r^*(i))$ and $\lim_i x^*(i) = x^*$, such that

$$0 < \liminf_{i \to \infty} \frac{|u(x^*(i)) - u(x^*)|}{|x^*(i) - x^*|}.$$

Without loss of generality we may also assume

(vi) $r^*(i) > 0$ for $i = 1, 2, \ldots$,

(vii) $x^*(i) \in \operatorname{spt}(T_{r^*(i)})$ for $i = 1, 2, \ldots$.

Let $0 < m^*$ be such that

$$m^* \leqslant \frac{|u(x^*(i)) - u(x^*)|}{|x^*(i) - x \;|},$$

for $i = 1, 2, \ldots$. Note that there exist $r_1 > 0$, $m_1 > 0$ such that

$$(m_1)^2 \leqslant \sum_{i=1}^{n} (\tilde{D}_i \varphi(x))^2$$

holds for each $x \in \Gamma$ with

$$-r_1 \leqslant \varphi(x) \leqslant r_1,$$

there exists a neighborhood, $\mathcal{N}$, of $\bar{\Omega} \cap u^{-1}(0)$ such that, Π, the nearest point retraction onto $\bar{\Omega} \cap u^{-1}(0)$, is defined on $\mathcal{N}$, and there exists $\varrho_1 > 0$ such that for each ϱ, with $0 < \varrho \leqslant \varrho_1$,

$$u^{-1}(0) \cap \{x \colon \operatorname{dist}(x, \mathbf{R}^n \sim \Omega) \geqslant \varrho\}$$

is connected. Also note that (ii) and (iii) imply

$$\bar{\Omega} \cap u^{-1}(0) = \operatorname{spt}(S_0) = \operatorname{spt}(T_0).$$

We will let $N\colon \bar{\Omega} \cap u^{-1}(0) \to \boldsymbol{S}^{n-1}$ be a continuous, and hence class $(q-1)$, function such that

$$N(x) \in NT_0(x)$$

for each $x \in \bar{\Omega} \cap u^{-1}(0)$.

(6) Let $\mathrm{HD}(E, F)$ denote the Hausdorff distance between the non-empty compact sets E and F.

(7) For k an integer with $1 \leqslant k \leqslant n-1$ and $V, W \in \boldsymbol{G}(n, k)$ set

$$\begin{aligned} d_k(V, W) = \inf\{&\left|v|v|^{-1} - w|w|^{-1}\right| : 0 \neq v, w \in \Lambda_k \boldsymbol{R}^n, \\ &v, w \text{ are simple, } V \text{ is the associated subspace of} \\ &v, W \text{ is the associated subspace of } w\}. \end{aligned}$$

It is easy to show that d_k is a metric on $\boldsymbol{G}(n, k)$ and to show that if $A, B \in \boldsymbol{G}(n, n-k)$, A is orthogonal to $V \in \boldsymbol{G}(n, k)$, and B is orthogonal to $W \in \boldsymbol{G}(n, k)$, then

$$d_{n-k}(A, B) = d_k(V, W).$$

(8) For k a positive integer, $0 < r < \infty$, and $x \in \boldsymbol{R}^k$, set

$$\boldsymbol{B}^k(x, r) = \boldsymbol{R}^k \cap \{z\colon |z - x| \leqslant r\},$$

$$\boldsymbol{U}^k(x, r) = \boldsymbol{R}^k \cap \{z\colon |z - x| < r\}.$$

1.2. LEMMA

$$\lim_{r \to 0} \mathrm{HD}[\bar{\Omega} \cap u^{-1}(r), \Omega \cap u^{-1}(0)] = 0.$$

PROOF. Suppose not. Then there exists $\varepsilon > 0$, a sequence $r(1), r(2), \ldots$ of non-zero real numbers with $\lim_i r(i) = 0$, and a sequence $x(1), x(2), \ldots$ such that either

(1) $x(i) \in \bar{\Omega} \cap u^{-1}(0)$ and $\boldsymbol{B}^n(x(i), \varepsilon) \cap \bar{\Omega} \cap u^{-1}(r(i)) = \emptyset$ for each $i = 1, 2, \ldots$

or

(2) $x(i) \in \bar{\Omega} \cap u^{-1}(r(i))$ and $\boldsymbol{B}^n(x(i), \varepsilon) \cap \bar{\Omega} \cap u^{-1}(0) = \emptyset$ for each $i = 1, 2, \ldots$.

Passing to a subsequence if necessary, we may assume that $x(i)$ converges to x. By the continuity of u we have

$$x \in \bar{\Omega} \cap u^{-1}(0),$$

which shows (2) is impossible. Assuming now that (1) holds, we note that, for all sufficiently large i,

$$\boldsymbol{B}^n(x, \varepsilon/2) \cap u^{-1}(r(i)) = \emptyset$$

holds. We conclude that either

$$\bar{\Omega} \cap \boldsymbol{B}^n(x, \varepsilon/2) \subset \bar{\Omega} \cap \{x\colon u(x) \leqslant r\}$$

or

$$\bar{\Omega} \cap \boldsymbol{B}^n(x, \varepsilon/2) \subset \bar{\Omega} \cap \{x\colon u(x) \geqslant r\}.$$

The former implies $x \notin \operatorname{spt}(S_0)$ and the latter implies $x \notin \operatorname{spt} T_0$, which are both contradictions. ⊠

1.3. *Notation.*

(9) By 1.2 there exists r_2 with $0 < r_2 \leqslant r_1$ such that $-r_2 \leqslant r \leqslant r_2$ implies $\bar{\Omega} \cap u^{-1}(r) \subset \mathcal{N}$.

(10) For r with $-r_2 \leqslant r \leqslant r_2$ set $H(r)$ equal to the supremum of the numbers

$$d_1[NT_r(x), NT_0(\Pi(x))], \qquad x \in \operatorname{spt}(T_r),$$

and the numbers

$$d_1[NS_r(x), NS_0(\Pi(x))], \qquad x \in \operatorname{spt}(S_r).$$

(11) It is easy to see, because of 1.1(5*iii*), that there exist r_3 and c_1 with

$$0 < r_3 \leqslant r_2 \quad \text{and} \quad 0 < c_1 < \infty$$

such that if

$$-r_3 \leqslant r \leqslant r_3$$

and $y, z \in \operatorname{spt}(R_r)$, then

$$d_2[\operatorname{Nor}(\operatorname{spt} R_r, y), \operatorname{Nor}(\operatorname{spt} R_r, z)] \leqslant c_1|y - z|.$$

2. – Convergence.

2.1. LEMMA. *For each* $\delta > 0$ *there exist* $r_4 = r_4(\delta)$ *and* $\varrho_2 = \varrho_2(\delta)$ *with*

$$0 < r_4 \leqslant r_3 \quad \textit{and} \quad 0 < \varrho_2 \leqslant \inf\{\delta, \varrho_1\}$$

such that if $x \in \Gamma$ *with*

$$-r_4 \leqslant \varphi(x) \leqslant r_4,$$

then

$$\|S_{\varphi(x)}\| \boldsymbol{B}^n(x, \varrho_2) \leqslant \tfrac{1}{2}\big(\alpha(n-1) + \delta\big)(\varrho_2)^{n-1},$$

$$\|T_{\varphi(x)}\| \boldsymbol{B}^n(x, \varrho_2) \leqslant \tfrac{1}{2}\big(\alpha(n-1) + \delta\big)(\varrho_2)^{n-1}.$$

PROOF. The lemma follows easily from the boundary regularity of $S_0 = T_0$ and the lower-semi-continuity of area. ⊠

2.2. LEMMA. *There exist* r_5 *and* c_2 *with*

$$0 < r_5 \leqslant r_3 \quad \text{and} \quad c_1 \leqslant c_2 < \infty$$

such that if

$$-r_5 \leqslant r \leqslant r_5$$

and $y, z \in \operatorname{spt}(T_r)$ [resp., $y, z \in \operatorname{spt}(S_r)$], *then*

$$d_1[NS_r(y), NS_r(z)] \leqslant c_2|y-z|^{1/n}$$

[resp.,

$$d_1[NS_r(y), NS_r(z)] \leqslant c_2|y-z|^{1/n}].$$

PROOF. First, we take $\varepsilon = 2^{-2}$ in [AW; 4] and obtain $\delta > 0$ by that theorem and 1.3(11). We then apply 2.1 to obtain $r_4(\delta)$ and $\varrho_2(\delta)$. Setting

$$r_5 = r_4(\delta), \qquad \varrho_3 = \tfrac{3}{4}\varrho_2(\delta),$$

we see that the lemma follows from [AW; 4] and the interior curvature estimate of [SL]. ⊠

2.3. LEMMA. *Fix* $\varrho > 0$. *If*

$$-r_2 \leqslant r \leqslant r_2, \qquad x \in \operatorname{spt}(T_r)$$

[resp., $x \in \operatorname{spt}(S_r)$],

$$\operatorname{dist}(x, \boldsymbol{R}^n \sim \Omega) \geqslant \varrho\,,$$

$$\operatorname{dist}\big(\Pi(x), \boldsymbol{R}^n \sim \Omega\big) \geqslant \varrho\,,$$

then

$$d_1\big[NT_r(x), NT_0\big(\Pi(x)\big)\big] \leqslant c_3 \varrho^{-\frac{1}{2}} |x - \Pi(x)|^{\frac{1}{2}}$$

[resp., $d\big[NS_r(x), NT_0\big(\Pi(x)\big)\big]$

$$c_3 \varrho^{-\frac{1}{2}} |x - \Pi(x)|^{\frac{1}{2}}\,,$$

where

$$c_3 = 2^2 (3n)^{\frac{1}{2}} (n-1)^{\frac{1}{4}} \gamma_8^{-\frac{1}{2}}\,,$$

with γ_8 as in [PH2].

REMARK. In [PH2; § 4] there is the dimension restriction $n \leqslant 6$, but if the explicitly computable constant γ_7, which is not explicitly computable, obtained from [SL], then all the arguments go through as before when $n = 7$.

PROOF. Applying [PH2; 4.2] with α to be specified later, we see there are orthonormal bases $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ for $\boldsymbol{R}^n$ and functions

$$f, g \colon \boldsymbol{U}^{n-1}(0, \gamma_8 \alpha \varrho) \to \boldsymbol{R}$$

such that

$$v_n \in NT_r(x)\,, \qquad w_n \in NT_0\big(\Pi(x)\big)\,,$$

$$|v_n - w_n| = d\big[NT_r(x), NT_0\big(\Pi(x)\big)\big]\,,$$

$$w_i = v_i\,, \qquad \text{for } i = 1, 2, \ldots, n-2\,,$$

$$w_{n-1} = \cos\theta\, v_{n-1} + \sin\theta\, v_n\,,$$

$$w_n = -\sin\theta\, v_{n-1} + \cos\theta\, v_n\,,$$

$$f(0) = g(0) = 0\,, \qquad Df(0) = Dg(0) = 0\,,$$

$$\operatorname{Lip}(f) \leqslant \alpha\,, \qquad \operatorname{Lip}(g) \leqslant \alpha\,,$$

for each $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{n-1}) \in \boldsymbol{U}^{n-1}(0, \gamma_8 \alpha \varrho)$

$$x + \sum_{j=1}^{n-1} \zeta_j v_j + f(\zeta) v_n \in \operatorname{spt}(T_r)$$

and

$$\Pi(x) + \sum_{j=1}^{n-1} \zeta_j w_j + g(\zeta) w_n \in \operatorname{spt}(T)\,.$$

We may assume $r \neq 0$ and $\sin\theta \neq 0$; also, by comparing $|v_n - w_n|$ and $|v_n + w_n|$ we see that $0 < |\theta| \leqslant \pi/2$ may be assumed, and consequently,

$$|v_n - w_n| \leqslant 2|\sin\theta|.$$

Since $\operatorname{spt}(T_r) \cap \operatorname{spt}(T_0) = \emptyset$, there cannot be any

$$(\zeta^1, \zeta^2) \in \boldsymbol{U}^{n-1}(0, \gamma_8\alpha\varrho) \times \boldsymbol{U}^{n-1}(0, \gamma_8\alpha\varrho)$$

which solves

$$x + \sum_{j=1}^{n-1} \zeta_j^1 v_j + f(\zeta^1) v_n = \Pi(x) + \sum_{j=1}^{n-1} \zeta_j^1 w_j + g(\zeta^2) w_n, \tag{12}$$

where $\zeta^i = (\zeta^i, \zeta_2^i, \ldots, \zeta_{n-1}^i)$, $i = 1, 2$. Writing

$$x_j = x \cdot v_j, \quad y_j = \Pi(x) \cdot v_j, \qquad j = 1, 2, \ldots, n,$$

we see that (12) is equivalent to

$$\left\{\begin{array}{l} x_j - y_j = \zeta_j^2 - \zeta_j^1, \quad j = 1, 2, \ldots, n-2, \\ x_{n-1} - y_{n-1} = \cos\theta\, \zeta_{n-1}^2 - \sin\theta\, g(\zeta^2) - \zeta_{n-1}^1, \\ x_n - y_n = \sin\theta\, \zeta_{n-1}^2 + \cos\theta\, g(\zeta^2) - f(\zeta^1). \end{array}\right. \tag{13}$$

Define a function

$$F\colon \boldsymbol{U}^{n-1}(0, \gamma_8\alpha\varrho) \times \boldsymbol{U}^{n-1}(0, \gamma_8\alpha\varrho) \to \boldsymbol{R}^{2n-2}$$

by setting

$$F_j(\zeta^1, \zeta^2) = \zeta_j^1, \quad j = 1, 2, \ldots, n-2,$$

$$F_{n-1}(\zeta^1, \zeta^2) = \cos\theta\, \zeta_{n-}^2 - \sin\theta\, g(\zeta^2) - \zeta_{n-1}^1,$$

$$F_{n-1+j}(\zeta^1, \zeta^2) = \zeta_i^2 - \zeta_j^1, \quad j = 1, 2, \ldots, n-2,$$

$$F_{2n-2}(\zeta^1, \zeta^2) = \sin\theta\, \zeta_{n-1}^2 + \cos\theta\, g(\zeta^2) - f(\zeta^1).$$

One can easily compute DF and estimate

$$\|DF(0, 0)^{-1}\| \leqslant (3n)^{\frac{1}{2}} |\csc\theta|,$$

$$\|DF(\zeta^1, \zeta^2) - DF(0, 0)\| \leqslant (2n-2)^{\frac{1}{2}} \alpha.$$

Taking

$$\alpha = 2^{-2}(3n(n-1))^{-\frac{1}{2}}|\sin\theta|$$

and using the proof of the Inverse Function Theorem in [RW; 9.17], we see that

$$\boldsymbol{U}^{2n-2}(0, \lambda\gamma_8\alpha\varrho) \subset F[\boldsymbol{U}^{2n-2}(0, \gamma_8\alpha\varrho)],$$

where

$$\lambda = 1/(4\|DF(0,0)^{-1}\|).$$

We can now conclude that

$$|x-y| \geqslant \lambda\gamma_8\alpha\varrho; \tag{14}$$

for the contrary inequality would imply that

$$(0, 0, \ldots, 0, x_{n-1}-y_{n-1}, x_1-y_1, x_2-y_2, \ldots, x_{n-2}-y_{n-2}, x_n-y_n)$$

is the image under F of some $(\zeta^1, \zeta^2) \in \boldsymbol{U}^{2n-2}(0, \gamma_8\alpha\varrho)$ which would give us a solution of (13). Substituting α into (14) and our estimate for $\|DF(0,0)^{-1}\|$ and then replacing $|\sin\theta|$ by $\frac{1}{2}|v_n - w_n|$, we obtain the desired inequality. ⊠

2.4. Theorem.

$$\lim_{r\to 0} H(r) = 0.$$

Proof. The conclusion of the theorem follows readily from 2.2 and 2.3 with the aid of 1.2 and 1.3(11). ⊠

2.5. Corollary. *For each $\varrho > 0$ there exist $r_6 = r_6(\varrho)$, with*

$$0 < r_6 \leqslant r_5,$$

and an open $U = U(\varrho)$, with

$$U \subset \Omega,$$

such that for each r with

$$-r_6 \leqslant r \leqslant r_6$$

we have

$$\operatorname{spt}(T_r) \cap \{x : \operatorname{dist}(x, \boldsymbol{R}^n \sim \Omega) > \varrho\} \subset \operatorname{spt}(T_r) \cap U$$

and $\Pi|\operatorname{spt}(T_r) \cap U$ *is one-to-one onto a set containing*

$$\operatorname{spt}(T_0) \cap \{x\colon \operatorname{dist}(x, \boldsymbol{R}^n \sim \Omega) > \varrho\}\,.$$

PROOF. We set

$$U = \{x\colon \operatorname{dist}(x, \boldsymbol{R}^n \sim \Omega) > \varrho/2\}\,.$$

By 1.2 and 2.4, we see that for all r, with sufficiently small absolute value, $\Pi|\operatorname{spt}(T_r) \cap U$ will be k-to-one onto a set containing

$$\operatorname{spt}(T_0) \cap \{x\colon \operatorname{dist}(x, \boldsymbol{R}^n \sim \Omega) > \varrho\}\,,$$

with k a positive integer.

Since we need only consider all sufficiently small $\varrho > 0$, we can arrange that $\|T_r\|(U)$ be approximately equal to $\boldsymbol{M}(T_0)$ and thus conclude that $k = 1$ when $|r|$ is small enough that the $(n-1)$ dimensional Jacobian of $\Pi|\operatorname{spt}(T_r) \cap U$ is nearly 1. ⊠

2.6. NOTATION. For $\varrho > 0$ and r with

$$-r_6(\varrho) \leqslant r \leqslant r_6(\varrho)$$

we denote by $w(\cdot, r)$ the function on

$$\operatorname{spt}(T_0) \cap \{x\colon \operatorname{dist}(x, \boldsymbol{R}^n \sim \Omega) > \varrho\}$$

defined by requiring

$$w(\Pi(x), r) = \operatorname{sgn}(r)|x - \Pi(x)|\,,$$

for $x \in \operatorname{spt}(T_r) \cap U(\varrho)$ with

$$\Pi(x) \in \operatorname{spt}(T_0) \cap \{x\colon \operatorname{dist}(x, \boldsymbol{R}^n \sim \Omega) > \varrho\}\,.$$

3. – Application of Harnack's inequality.

3.1. THEOREM. *Fix* $\varrho > 0$. *For* $x_0 \in \Omega \cap u^{-1}(0)$ *and*

$$\operatorname{dist}(x_0, \boldsymbol{R}^n \sim \Omega) > \varrho\,,$$

there exist an open set $V = V(x_0, \varrho)$, *with*

$$x_0 \in V \subset \{z: \operatorname{dist}(z, \boldsymbol{R}^n \sim \Omega) > \varrho\},$$

and $0 < c_4 = c_4(x_0, \varrho) < \infty$ *such that*

$$\sup\{w(z, r): z \in V \cap u^{-1}(0)\} \leqslant c_4 \inf\{w(z, r): z \in V \cap u^{-1}(0)\},$$

for $0 < r \leqslant r_6(\varrho)$.

PROOF. Fix ϱ, x_0, r as in the statement of the iemma. Since $\Omega \cap u^{-1}(0)$ is a submanifold of $\boldsymbol{R}^n$, there exists a coordinate patch

$$x(\xi_1, \xi_2, \ldots, \xi_{n-1})$$

about x_0 completely contained in $\{z: \operatorname{dist}(z, \boldsymbol{R}^n \sim \Omega) > \varrho\}$. In this proof we will write

$$w(\xi_1, \xi_2, \ldots, \xi_{n-1}) = w\big(x(\xi_1, \xi_2, \ldots, \xi_{n-1}), r\big),$$

$$x_i = \frac{\partial x}{\partial \xi_i}, \quad i = 1, 2, \ldots, n-1,$$

$$N_i = \frac{\partial}{\partial \xi_i} N[x(\xi_1, \xi_2, \ldots, \xi_{n-1})], \; i = 1, 2, \ldots, n-1$$

(recall $N = NT_0$).

Part of the area minimizing hypersurface $\operatorname{spt}(T_r)$ is parametrized by

$$x + wN.$$

The area of the surface parametrized by

$$x + (w + t_\eta)N,$$

where $t \in \boldsymbol{R}$ and $\eta(\xi_1, \xi_2, \ldots, \xi_{n-1})$ is a test function, is minimized when $t = 0$. Therefore, a straight forward calculation of the first variation yields

$$\int\Big(B\eta + \sum_{i=1}^{n-1} A^i \eta_i\Big) d\mathfrak{L}^{n-1} = 0 \tag{15}$$

where

$$A^i = 2[\det(\tilde{g}_{ij})]^{-\frac{1}{2}}\left[\sum_{j=1}^{n-1}(-1)^{i+j}\tilde{M}^{ij}w_j\right],$$

$$B = [\det(\tilde{g}_{ij})]^{-\frac{1}{2}}\left[\sum_{j=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\tilde{M}^{ij}\big((x_i\cdot N_j + x_j\cdot N_i) + 2wN_i\cdot N_j\big)\right],$$

$$\tilde{g}_{ij} = x_i\cdot x_j + w(x_i\cdot N_j + x_j\cdot N_i) + wN_i\cdot N_j + w_i w_j,$$

$$\tilde{M}^{ij} = \det_{k\neq i, l\neq j}(\tilde{g}_{kl}),$$

$$w_i = \frac{\partial w}{\partial \xi_i},$$

$$\eta_i = \frac{\partial \eta}{\partial \xi_i}.$$

We may write

$$\tilde{M}_{ij} = \det_{k\neq i, l\neq j}(x_k\cdot x_l + w(x_r\cdot N_l + x_l\cdot N_r) + wN_k\cdot N_l + w_r w_l)$$
$$= M_{ij} = w\tilde{Q}_{ij} + \sum_{k\neq i, l\neq j} w_k w_l \tilde{R}^{kl}_{ij}$$

where

$$M_{ij} = \det_{k\neq i\ l\neq j}(x_k\cdot x_l).$$

We now write

$$B = B_1 + B_2 + B_3$$

where

$$B_1 = [\det(\tilde{g}_{ij})]^{-\frac{1}{2}}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}M_{ij}(x_i\cdot N_j + x_j\cdot N_i)\right]$$

$$B_2 = w[\det(\tilde{g}_{ij})]^{-\frac{1}{2}}$$
$$\cdot\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}\{2M_{ij}N_i\cdot N_j + \tilde{Q}_{ij}((x_i\cdot N_j + x_jN_i) + 2wN_i\cdot N_j)\right.$$
$$\left.+ 2N_i\cdot N_j\sum w_k w_l \tilde{R}^{kl}_{ij}\}\right],$$

and

$$B_3 = [\det(\tilde{g}_{ij})]^{-\frac{1}{2}}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i\cdot N_j + x_j\cdot N_i)\left(\sum_{k\neq i, l\neq j} w_k w_l \tilde{R}^{kl}_{ij}\right)\right].$$

Since x parametrizes a minimal surface, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}M_{ij}(x_i\cdot N_j + x_j\cdot N_i) = 0$$

and therefore $B_1 = 0$.

Note that

$$\tilde{M}^{ij} = \det(\tilde{g}_{k1})(-1)^{i+j}\tilde{g}_{ij}$$

where $(\tilde{g}^{ij})$ is the matrix inverse to $(\tilde{g}^{ij})$, $i, j = 1, 2, \ldots, n-1$. Thus, we may regard (15) as a uniformly elliptic equation of the form

$$\operatorname{div} A = B$$

ewher

(16) $$A = A(\xi_1, \ldots, \xi_{n-1}, \nabla w) \quad \text{and} \quad B = B(\xi_1, \ldots, \xi_{n-1}, w, \nabla w)$$

satisfy the growth conditions

(17) $$|A| \leqslant a_0|\nabla w|, \qquad |B| \leqslant b_1|\nabla w| + b_2|w|.$$

Here, because of 1.2 and 2.4, a_0, b_1, and b_2 may be regarded as uniformly bounded functions of $\xi_1, \xi_2, \ldots, \xi_{n-1}$ for $0 < \gamma \leqslant \gamma_6(\varrho)$. Therefore, we apply Harnack's inequality [GT, § 8.8] to obtain the conclusion. ⊠

3.2. Corollary. *For each compact* $K \subset \Omega \cap u^{-1}(0)$, *there exist* $0 < r_7 = r_7(K)$ *and* $0 < c_5(K) < \infty$ *such that for* r *with*

$$0 < r \leqslant r_7$$

we have

$$\sup\{w(x, r) : z \in K \cap u^{-1}(0)\} \leqslant c_5 \inf\{w(x, r) : z \in K \cap u^{-1}(0)\}.$$

Proof. The corollary follows readily by choosing ϱ with $0 < \varrho \leqslant \varrho_1$ so that

$$K \subset u^{-1}(0) \cap \{z : \operatorname{dist}(z, \boldsymbol{R}^n \sim \Omega) \geqslant \varrho\} = K',$$

covering K' with finitely many open sets V as in 3.1, and using the fact that K' is connected (see 1.1(5)). ⊠

4. – Jacobi's equation.

4.1. Definition. Let S be an oriented $(n-1)$-dimensional minimal submanifold of $\boldsymbol{R}^n$ with unit normal field $\mathcal{N}: S \to \boldsymbol{S}^{n-1}$. We say that $\zeta: S \to \boldsymbol{R}$ is a *solution of Jacobi's equation* if for each coordinate patch

$$x: W \subset \boldsymbol{R}^{n-1} \to S$$

we have

$$\sum_{j=1}^{n-1} \frac{\partial}{\partial \xi_i}(A^i) - B = 0\,, \tag{18}$$

where

$$A^i = 2[\det(g_{ij})]^{-\frac{1}{2}} \left[\sum_{j=1}^{n-1}(-1)^{i+j} M^{ij} \zeta_j \circ x\right],$$

$$\begin{aligned} B &= [\det(g_{ij})]^{-\frac{1}{2}} \\ &\cdot \left[\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\left\{\sum_{\alpha\neq i,\beta\neq j} \tau(i, j; \alpha, \beta)\, T^{ij,\alpha\beta}(x_k\cdot \mathcal{M}_1 + x_1\cdot \mathcal{M}_k)(x_i\cdot \mathcal{M}_j + x_j \cdot \mathcal{M}_i)\right.\right. \\ &\left. + 2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j} M^{ij} \mathcal{M}_i \cdot \mathcal{M}_j\right] \zeta\circ x\,, \end{aligned}$$

$$g_{ij} = x_i \cdot x_j\,,$$

$$M^{ij} = \det_{k\neq i,\, l\neq i} (x_k \cdot x_l)\,,$$

$$T^{ij,\alpha\beta} = \det_{\substack{r\neq i,\, s\neq i \\ r\neq\alpha,\, s\neq\beta}} (x_r \cdot x_5)\,,$$

$$\tau(i, j; \alpha, \beta) = (-1)^{\alpha+\beta+1+\frac{1}{2}[\operatorname{sgn}(\alpha-i)+\operatorname{sgn}(\beta-j)]}\,,$$

$$x_i = \frac{\partial x}{\partial \xi_i}\,,$$

$$\mathcal{M}_i = \frac{\partial}{\partial \xi_i} \mathcal{M}\circ x\,.$$

Of course, as is well known, (18) is the Euler equation of the second variation or, equivalently, the equation of variation of (15). It is a routine exercise to check that a change of coordinates results in multiplying the above differential equation by a non-zero function, so we need only consider one coordinate patch about each point of S to verify that ζ is a solution of Jacobi's equation.

4.2. THEOREM. *There exists a subsequence* $k(1), k(2), \ldots$ *of* $1, 2, \ldots$ *such that*

$$\lim_{l\leftarrow\infty} \frac{w[x, r^*(k(l))]}{r^*(k(l))} = \zeta(x)$$

where

$$\zeta\colon \Omega \cap u^{-1}(0) \to \boldsymbol{R}$$

is a solution of Jacobi's equation satisfying

$$1/M \leqslant \zeta(x) \leqslant c_5(K)/m^*$$

for each compact $K \subset \Omega \cap u^{-1}(0)$ *and each* $x \in K$.

PROOF. Fix a compact $K \subset \Omega \cap u^{-1}(0)$ and $x_0 \in K$. We consider a coordinate patch,

$$x\colon W \subset \boldsymbol{R}^{n-1} \to \Omega \cap u^{-1}(0)\,,$$

about x_0 and a compact K', containing the image of the coordinate patch, with

$$K \subset K' \subset \Omega \cap u^{-1}(0)\,.$$

For $k = 1, 2, \ldots$ set

$$F_k(\xi_1, \xi_2, \ldots, \xi_{n-1}) = \frac{w[x(\xi_1, \xi_2, \ldots, \xi_{n-1}), r^*(k)]}{r^*(k)}\,.$$

By 1.1(5) and 3.2 we have, for large enough k,

$$\sup_{K'} F_k \leqslant c_5(K')/m^*\,.$$

Referring to (16), it is clear that F_k is a solution of an elliptic equation of the form satisfied by w and with structure similar to (17). Therefore, F_k satisfies the Harnack inequality which implies that

$$F_k \in C^{0,\alpha}(W)\,.$$

That is, F_k is Hölder continuous of order α, where α is independent of k for large k. Because F_k is uniformly bounded for large k, it follows from elementary estimates [GT, (8.52)] and (16), (17) that $\{F_k\}$ is a bounded set in the Sobolev space $W^{1,2}(W)$. Accordingly, there exists a subsequence $k(1), k(2), \ldots$ such that $F_{k(l)}$ converges uniformly in W to $F = \zeta \circ x$ and $DF_{k(l)}$ converges weakly in L_2 to DF.

By 1.2 and 2.4, we see that

$$2[\det(\tilde{g}_{ij})]^{-\frac{1}{2}}(-1)^{i+j}\tilde{M}^{ij}\,,$$

where $\tilde{g}_{ij}$ and $\tilde{M}^{ij}$ are defined as in the proof of 3.1, but with

$$w(\xi_1, \xi_2, \ldots, \xi_{n-1}) = w[x(\xi_1, \xi_2, \ldots, \xi_{n-1}), r^*(k(l))]\,,$$

converges uniformly to

$$2[\det(g_{ij})]^{-\frac{1}{2}}(-1)^{i+j}M^{ij}$$

as $l \to \infty$. It follows that, for any test function η on W,

$$\lim_{l\to\infty}[r^*(k(l))]^{-1}\int\sum_{i=1}^{n-1}A^i\eta_i d\mathfrak{L}^{n-1}\xi = \int\sum_{i=1}^{n-1}A^i\cdot\eta_i d\mathfrak{L}^{n-1}\xi ,$$

where

$$\eta_i = \frac{\partial\eta}{\partial\xi_i}$$

and A^i is as in the proof of 3.1. It is also clear that

$$\lim_{l\to\infty}\int[\det(\tilde{g}_{ij})]^{-\frac{1}{2}}\Big[2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\tilde{M}^{ij}N_i\cdot N_j\Big]F_{k(l)}\eta\, dl^{n-1}\xi$$
$$= \int[\det(g_{ij})]^{-\frac{1}{2}}\Big[2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}M^{ij}N_i\cdot N_j\Big]F\eta\, dl^{n-1}\xi .$$

Finally, we must consider

$$\lim_{l\to\infty}\int[\det(\tilde{g}_{ij})]^{-\frac{1}{2}}\Big[\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\tilde{M}^{ij}[r^*(k(l))]^{-1}(x_i\cdot N_j + x_j\cdot N_i)\Big]\eta\, dl^{n-1}\xi .$$

This can be rewritten as

$$\lim_{l\to\infty}\Big\{\int[\det(\tilde{g}_{ij})]^{-\frac{1}{2}}\Big[\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\frac{\tilde{M}^{ij}-M^{ij}}{r^*(k(l))}(x_i\cdot N_j + x_j\cdot N_i)\Big]\eta\, d\mathfrak{L}^{n-1}\xi$$
$$+\int[\det(\tilde{g}_{ij})]^{-\frac{1}{2}}\cdot[r^*(k(l))]^{-1}\Big[\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}M^{ij}(x_i\cdot N_j + x_j\cdot N_i)\Big]\eta\, d\mathfrak{L}^{n-1}\xi\Big\}.$$

Since x parametrizes a minimal surface, we have

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}M^{ij}(x_i\cdot N_j + x_j\cdot N_i) = 0 ,$$

so the second integral vanishes. The first integral converges, by Lebesgue's Dominated Convergence Theorem, to

$$\int[\det(g_{ij})]^{-\frac{1}{2}}$$
$$\cdot\Big[\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(-1)^{i+j}\Big\{\sum_{\alpha\neq i,\beta\neq j}\tau(i, j; \alpha, \beta)T^{ij,\alpha\beta}(x_\alpha\cdot N_\beta + x_\beta\cdot N_\alpha)(x_i\cdot N_j + x_j\cdot N_i)\Big\}\Big]$$
$$\cdot\zeta\eta\, d\mathfrak{L}^{n-1}\xi .$$

Thus, since, for each large enough l,

$$[r^*(k(l))]^{-1}\int\Big(B\eta+\sum_{i=1}^{n-1}A^i\eta_i\Big)d\mathfrak{L}^{n-1}\xi=0$$

holds, we see that ζ is a solution of Jacobi's equation in the image of the coordinate patch.

The global existence of ζ is obtained by diagonalization and the estimate

$$1/M\leqslant\zeta(x_0)\leqslant c_5(K)/m^*$$

follows from 1.1(2, 5) and 3.2. ⊠

4.3. COROLLARY. *If*

$$\zeta\colon\bar{\Omega}\cap u^{-1}(0)\to\boldsymbol{R}$$

is continuous with

$$\zeta|\Gamma\cap u^{-1}(0)=0$$

and is a solution of Jacobi's equation on $\Omega\cap u^{-1}(0)$, *then*

$$\zeta\equiv 0\,.$$

PROOF. Suppose ζ_0 is the solution of Jacobi's equation from 4.2 and ζ is as above. Suppose there exists $x_1\in\Omega\cap u^{-1}(0)$ such that $\zeta(x_1)>0$. Then we can find $0<c<\infty$ such that

$$c\zeta(x)\leqslant\zeta_0(x)\,,$$

for all $x\in\Omega\cap u^{-1}(0)$, and there exists $x_2\in\Omega\cap u^{-1}(0)$ such that

$$c\zeta(x_2)=\zeta_0(x_2)\,.$$

But then

$$\zeta_0-c\zeta\geqslant 0$$

is a solution of Jacobi's equation which vanishes at x_2, so by Harnack's inequality

$$\zeta_0-c\zeta\equiv 0\,.$$

This is impossible since

$$\zeta_0\geqslant 1/M\qquad\text{and}\qquad\zeta|\Gamma\cap u^{-1}(0)=0\,.$$

Thus we have $\zeta\leqslant 0$, and we similarly see that $\zeta\geqslant 0$. ⊠

4.4. THEOREM. *There exists an open set* $W \subset \Omega$ *with*

$$\Omega \cap u^{-1}(0) \subset W$$

such that $u|W$ *is of class* $(q-2)$.

PROOF. There exists $r_8 > 0$ so that if $z \in \Gamma$ and $-r_8 < u(z) < r_8$, then there is a unique point $\Xi(z) \in \Gamma \cap \varphi^{-1}(0)$ which is nearest to z and $\Xi(\cdot)$ is of class $(q-1)$ on $\Gamma \cap \{z: -r_8 < u(z) < r_8\}$. Define

$$f: \Gamma \cap \{z: -r_8 < u(z) < r_8\} \to \boldsymbol{R} \times \big(\Gamma \cap \varphi^{-1}(0)\big)$$

by setting

$$f(z) = \big(\varphi(z), \Xi(z)\big) .$$

By the Inverse Function Theorem, we can find r_9 with $0 < r_9 \leqslant r_8$ such that f^{-1} is defined and of class $(q-1)$ on

$$\{t: -r_9 < t < r_9\} \times \big(\Gamma \cap \varphi^{-1}(0)\big) .$$

For each t with $-r_9 < t < r_9$ define $\gamma_t: \Gamma \cap \varphi^{-1}(0) \to \boldsymbol{R}^n$ by setting

$$\gamma_t(x) = f^{-1}(t, x) .$$

Now, fix any α with $0 < \alpha < 1$ and apply [WB; 3.1], as we can do by 4.3, with $\iota = \alpha_0$, to obtain

$$F: U \subset C^{q-2,\alpha}\big(\Gamma \cap \varphi^{-1}(0), \boldsymbol{R}^n\big) \to C^{q-2,\alpha}\big(\Omega \cap u^{-1}(0), \boldsymbol{R}^n\big) .$$

Then

$$F(\gamma_t)(x) = g(t, x)$$

is a class $(q-2)$ function of all small enough t and $x \in \Omega \cap u^{-1}(0)$. Fixing $x_0 \in \Omega \cap u^{-1}(0)$ we see that $Dg(0, x_0)$ is non-singular, so there is, again by the Inverse Function Theorem, an open $W_{x_0} \subset \boldsymbol{R}^n$ with $x_0 \in W_{x_0}$ on which g^{-1} is defined and of class $(q-2)$. Finally, by the uniqueness property of F $\big($here we also use $0 \in \mathfrak{U}$ $(1.1(5\text{ii}))\big)$, we have

$$u(z) = p \circ g^{-1}(z)$$

for $z \in W_{x_0}$ where

$$p: \boldsymbol{R} \times (\Omega \cap u^{-1}(0)) \to \boldsymbol{R}$$

is projection on the first factor. Thus $u|W_{x_0}$ is of class $(q-2)$. ⊠

4.5. THEOREM. *If* Bdry Ω *is a class* $(n-1)$ *submanifold of* $\boldsymbol{R}^n$ *and* φ: bdry $\Omega \to \boldsymbol{R}$ *is of class* $n-1$, *then there exists an open dense set* $U \subset \Omega$ *such that* $u|U$ *is of class* $(n-3)$.

PROOF. Let

$$N_1 = \text{Bdry}\, \Omega \cap \left\{x\colon \sum_{i=1}^{n} \tilde{D}_i \varphi(x) = 0\right\} \quad \text{and} \quad N_2 = \Omega \cap \{x\colon \nabla u(x) = 0\}.$$

From Sard's theorem we have that $\varphi(N_1)$ has Lebesgue measure 0. Also, because u is Lipschitzian, we may apply the co-area formula [FH, 3.2.12] to conclude that $H^{n-1}[u^{-1}(t) \cap N_2] = 0$ for $\mathfrak{L}^1$ almost every t.

Let $x \in \Omega$ and let $B \subset \Omega$ be an open ball containing x. If u is constant on B then, of course, u is smooth on B. If u is not constant on B, then $u(B)$ is an interval. Choose $t \in u(B)$ such that $t \notin \varphi(N_1)$ and $H^{n-1}[u^{-1}(t) \cap N_2] = 0$. Then we may apply Theorem 4.4 to conclude that there is an open set $W_t \subset \Omega \cap u^{-1}(t)$ such that $W_t \cap B \neq \emptyset$ and $u|W_t$ is of class $(n-3)$. The conclusion of the theorem follows if U is defined as the union of all such W_t and all open balls $B \subset \Omega$ such that $u|B$ is a constant.

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