ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

JOSÉ-MARÍA ISIDRO
JEAN-PIERRE VIGUÉ

The group of biholomorphic automorphisms of symmetric Siegel domains and its topology

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 11, nº 3 (1984), p. 343-351
<http://www.numdam.org/item?id=ASNSP_1984_4_11_3_343_0>

The Group of Biholomorphic Automorphisms of Symmetric Siegel Domains and Its Topology.

JOSÉ-MARÍA ISIDRO (*) - JEAN-PIERRE VIGUÉ

Bounded symmetric domains in complex Banach spaces have been carefully studied during the last years (see for example [5], [10], [11] and [12]). As noticed by Harris [5], many bounded symmetric domains may be realized as Siegel domains and Kaup-Upmeier [8] have given a necessary and sufficient condition for the existence of such realizations. As in the finite dimensional case, it is important to study the Siegel domain realization of a bounded symmetric domain.

More precisely, let Δ be a bounded symmetric domain in a complex Banach space E. By [10], we may suppose that Δ is circular. Let us suppose that Δ admits a Siegel domain realization D and let $\tau\colon \Delta \to D$ be the Cayley transformation. We shall first study τ and τ^{-1} and prove that τ (respectively τ^{-1}) maps subsets lying strictly inside Δ (respectively D) onto subsets lying strictly inside D (respectively Δ).

On the group $G(\Delta)$ of biholomorphic automorphisms of Δ, we have defined in [10] the topology of local uniform convergence and, in [12], we have proved that this topology coincides with the topology of uniform convergence on Δ. By means of τ, we can transfer this topology to $G(D)$ and we shall prove that the result is the topology of uniform convergence on subsets lying strictly inside D.

To conclude this paper, we shall study the group of biholomorphic automorphisms of a symmetric Siegel domain in some special cases. In these cases, we shall describe completely the group $G(D)$.

Let us begin with some definitions and properties of bounded symmetric domains and their Siegel domain realizations.

(*) Research supported by the Deutscher Akademischer Austauschdienst grant 313-Südeuropa (1982).

Pervenuto alla Redazione il 6 Giugno 1983.

1. – Bounded symmetric domains and their Siegel domain realizations.

Let Δ be a domain in a complex Banach space E. A subset A of Δ is said to lie strictly inside D (and we note it $A \subset\subset D$) if A is bounded and the distance $d(A, C_E D)$ is strictly positive.

Let Δ be a bounded domain in E and let $G(\Delta)$ denote the group of biholomorphic automorphisms of Δ. On $G(\Delta)$, the topology of uniform convergence on a ball $B \subset\subset \Delta$ does not depend on the choice of B and it is compatible with the group structure of $G(\Delta)$; we call it [10] the topology of local uniform convergence. Let $s \in G(\Delta)$ be an analytic automorphism of Δ. We say that s is a symmetry at $a \in \Delta$ if $s^2 = id$ and a is an isolated fixed point for s. Such an s is unique and is denoted by s_a. The domain Δ is said to be symmetric if, for every $a \in \Delta$, there exists a symmetry s_a at a.

Let Δ be a bounded symmetric domain. We have proved in [10] that Δ is homogeneous and that Δ is isomorphic to a bounded circular domain. Moreover, by [10] and [7], $G(\Delta)$ has a real Lie-group structure (whose underlyng topology is the topology of local uniform convergence) such that the mapping

$$G(\Delta) \times \Delta \to \Delta$$

$$(f, x) \mapsto f(x)$$

is real-analytic. If Δ is realized as a bounded circular domain, the topology of local uniform convergence on $G(\Delta)$ coincides with the topology of uniform convergence on Δ [12].

Kaup and Upmeier [8] define Siegel domains in the following way: Let V be a real Banach space, $V^{\mathbf{C}} = V + iV$ its complexification and denote by W a complex Banach space. Let $\Omega \subset V$ be an open cone and $F\colon W \times W \to V^{\mathbf{C}}$ a continuous hermitian mapping. We say that

$$D = \{(v, w) \in V^{\mathbf{C}} \oplus W\,; \quad \operatorname{Im} v - F(w, w) \in \Omega\}$$

is a Siegel domain if D is biholomorphically equivalent to a bounded domain If D is a symmetric Siegel domain, then D is convex [8].

If Δ is a bounded symmetric circular domain, Δ admits a Siegel domain realization D if and only if its closure $\bar{\Delta}$ has a complex extreme point [8]. The biholomorphic isomorphism $\tau\colon \Delta \to D$ is called the Cayley transformation.

2. – The Cayley transformations.

Let Δ be a bounded circular symmetric domain in a complex Banach space E and let $\tau: \Delta \to D$ be its Siegel domain realization.

We denote by C^i_D (respectively C^i_Δ) the Carathéodory integrated pseudo-distance on D (respectively Δ) [3]. As D is isomorphic to a bounded domain, C^i_D (respectively C^i_Δ) is a distance which defines the topology of D (respectively Δ).

We have the following result

PROPOSITION 2.1. *Let A be a subset of D. The following propositions are equivalent*

i) *A is C^i_D-bounded;*

ii) *A is $\subset\subset D$.*

PROOF. It is clear that D is hyperbolic in the sense of [6], and D is convex. Thus we can apply proposition 2.3 of [6] whence the result follows.

Concerning bounded symmetric circular domains, we get the following proposition.

PROPOSITION 2.2. *Let Δ be a bounded symmetric circular domain and assume that A is a subset of Δ. The following propositions are equivalent:*

i) *A is C^i_Δ-bounded;*

ii) *A is $\subset\subset \Delta$.*

PROOF. Let us suppose that A is lying strictly inside Δ. By Cauchy's inequalities, we get upper bounds for the Carathéodory infinitesimal metric γ_Δ on A. As A is bounded, it is easy to prove that A is C^i_Δ-bounded.

To prove that (i) implies (ii), it is sufficient to show that every ball $B_{C^i_\Delta}(0, r)$ (for the Carathéodory distance C^i_Δ) with center 0 and radius r lies completely inside Δ. First, we can find a number $r_0 > 0$ such that $B_{C^i_\Delta}(0, r_0)$ is $\subset\subset \Delta$. As Δ is homogeneous and C^i_Δ is an intrinsic integrated distance, we get

$$B_{C^i_\Delta}(0, 2r_0) = \bigcup_{f \in I} f[B_{C^i_\Delta}(0, r_0)]$$

where

$$I = \{f \in G(\Delta)\,; \quad f(0) \in B_{C^i_\Delta}(0, r_0)\}\,.$$

We have proved in [11] that, for every $A \subset\subset \Delta$, there exist $\varepsilon > 0$ and M

sufficiently large such that every f, with $f(0) \in A$, admits a holomorphic extension to

$$\Delta_\varepsilon = \{x \in E\,; \quad d(x, \Delta) < \varepsilon\}$$

with values in $B(0, M)$. By Cauchy's inequalities, we prove that $B_{C^i_\Delta}(0, 2r_0)$ lies strictly inside Δ. By recurrence we deduce that, for every $n \in \mathbf{N}$, $B_{C^i_\Delta}(0, nr_0) \subset\subset \Delta$, whence the proposition follows.

Then, we can prove the following theorem

THEOREM 2.3. *Let Δ be a bounded symmetric circular domain and assume that Δ admits a Siegel domain realization D. Let $\tau: \Delta \to D$ be the Cayley transformation. Then, for every $A \subset\subset \Delta$ (respectively $B \subset\subset D$), we have $\tau(A) \subset\subset D$ (respectively $\tau^{-1}(B) \subset\subset \Delta$).*

PROOF. Since τ is a biholomorphic isomorphism, τ is an isometry for the distances C^i_Δ and C^i_D. Thus, it is clear that τ and τ^{-1} map C^i_D-bounded subsets and C^i_Δ-bounded subsets onto each other. Propositions 2.1 and 2.2 conclude the proof.

From propositions 2.1 and 2.2 we deduce the following corollary which answers, in the case of symmetric domains and symmetric Siegel domains, a question of [10], remark, page 211.

COROLLARY 2.4. *Let $f \in G(D)$ (respectively $f \in G(\Delta)$). For every $A \subset\subset D$ (respectively $A \subset\subset \Delta$), we have $f(A) \subset\subset D$ (respectively $f(A) \subset\subset \Delta$).*

PROOF. It is clear that f maps C^i_D (respectively C^i_Δ) bounded subsets onto themselves, which proves the result.

3. – The topology of local uniform convergence on $G(D)$.

Let Δ be a bounded symmetric circular domain in a complex Banach space and suppose that Δ admits a Siegel domain realization D. Let us denote by $\tau: \Delta \to D$ the Cayley transformation. The mapping τ induces an isomorphism $\tau_\#: G(\Delta) \to G(D)$. Thus, if we denote by $\mathcal{T}_\Delta$ the topology on $G(\Delta)$ of local uniform convergence over Δ, we can transfer $\mathcal{T}_\Delta$ to $G(D)$ by means of $\tau_\#$. We call $\mathcal{T}_D = \tau_\# \mathcal{T}_\Delta$ the topology on $G(D)$ of local uniform convergence over D. Now, we have the following theorem

THEOREM 3.1. *On $G(D)$, the topolovy $\mathcal{T}_D$ of local uniform convergence over D coincides with the topology of uniform convergence on subsets lying strictly inside D.*

PROOF. To prove the theorem we have to show the following result: Let $(f_n)_{n\in\mathbf{N}}$ be a sequence in $G(D)$ and assume that f_n converges to $f\in G(D)$ with regard to the topology of local uniform convergence. Then $f_n\to f$ uniformly on every subset lying strictly inside D.

So, let $A\subset\subset D$ be given and write $B=\tau^{-1}(A)\subset\subset \Delta$. Let us put $g_n = \tau^{-1}\circ f_n\circ\tau$ and $g=\tau^{-1}\circ f\circ\tau$. By definition, the sequence g_n converges to g in the topology of local uniform convergence, and by ([12], th. 4.3), g_n converges to g uniformly on Δ. We get

$$\|f_n-f\|_A=\|\tau\circ g_n\circ\tau^{-1}-\tau\circ g\circ\tau^{-1}\|_A=\|\tau\circ g_n-\tau\circ g\|_{\tau^{-1}(A)}\,.$$

By corollary 2.4 we have $g(B)\subset\subset\Delta$. Thus, there exist $n_0\in\mathbf{N}$ and $C\subset\subset\Delta$ such that we have $g_n(B)\subset\subset C$ for all $n\in\mathbf{N}$, $n\geqslant n_0$. We can find $r_0>0$ such that $C+B(0,r_0)\subset\subset\Delta$ and, by theorem 2.3, $\tau[C+B(0,r_0)]\subset\subset D$. So, τ is bounded on $C+B(0,r_0)$. By Cauchy's inequalities, there is a constant $K>0$ such that τ is K-lipschitzian on C. Therefore, for $n\geqslant n_0$ we have

$$\|f_n-f\|_A\leqslant K\|g_n-g\|_B$$

whence the theorem follows.

REMARK 3.2. *It is easy to check that, on $G(D)$, the topology of local uniform convergence is not equal to the topolovy of uniform convergence over D (compare* [12]*).*

To conclude this paper, we shall study two examples.

4. – The upper half-plane of $\mathcal{C}(\Omega)$.

Let Ω be a compact Hausdorff space and denote by $\mathcal{C}(\Omega)$ the Banach space of complex-valued continuous functions on Ω. The open unit ball Δ of $\mathcal{C}(\Omega)$ is a bounded symmetric circular domain. By [1] and [5], the group of biholomorphic automorphisms of Δ admits the following representation. Let $F\in G(\Delta)$; then, there exists a homeomorphism ψ of Ω, a continuous mapping $\theta\colon\Omega\to\mathbf{C}$ with $|\theta(x)|=1$ for all $x\in\Omega$, and a function $a\in\Delta$ such that we have

$$(Ff)(x)=\theta(x)\,\frac{f[\psi(x)]+a(x)}{1+\overline{a(x)}f[\psi(x)]}$$

for all $f\in\Delta$ and $x\in\Omega$.

The domain Δ admits a Siegel domain realization D, where

$$D = \{f \in \mathcal{C}(\Omega);\ \operatorname{Im} f(x) > 0 \text{ for all } x \in \Omega\}.$$

The Cayley transformation $\tau\colon \Delta \to D$ is given by

$$\tau(f) = \frac{f+i}{if+1}.$$

Now, we shall study the group $G(D)$. Let $SL_2(\mathbb{R})$ be the multiplicative group of all $(2,2)$-real matrices of determinant 1 and let $SL_2(\mathbb{R})/(+I, -I)$ denote the quotient of $SL_2(\mathbb{R})$ by the subgroup $(+I, -I)$. This group admits a natural group topology (in fact, it is a real Lie group) and $SL_2(\mathbb{R})/(+I, -I)$ is isomorphic to the group of biholomorphic automorphisms of the upper-half plane P of $\mathbb{C}$ in the following way: Let $A \in SL_2(\mathbb{R})/(+I, -I)$ and $z \in P$; let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a representant of A in $SL_2(\mathbb{R})/(+I, -I)$ and define Az by

$$Az = \frac{az+b}{cz+d}.$$

It is easy to check that Az does not depend on the choice of the representant. Now, from the previous considerations one can easily prove the following theorem.

THEOREM 4.1. *Let $F \in G(D)$. There exists a homeomorphism ψ of Ω and a continuous mapping $A\colon \Omega \to SL_2(\mathbb{R})/(+I, -I)$ such that we have*

$$(Ff)(x) = A(x)\, f[\psi(x)]$$

for all $f \in D$ and all $x \in \Omega$.

The following question raises in a natural way: Given $F \in G(D)$, does there exist continuous functions a, b, c, d on Ω such that

$$(Ff)(x) = \frac{a(x) f[\psi(x)] + b(x)}{c(x) f[\psi(x)] + d(x)}.$$

This question can be viewed in the following way. We have two continuous mappings

$$\begin{array}{ccc} & & SL_2(\mathbb{R}) \\ & & \big\downarrow p \\ \Omega & \xrightarrow{A} & SL_2(\mathbb{R})/(+I, -I) \end{array}$$

where the canonical projection p is a covering projection [4], and we want to find a lifting of A. We can find it in several cases; for example, if Ω is sufficiently discontinuous, we have.

PROPOSITION 4.2. *Let Ω be a compact Stonean space. Then every continuous mapping $A\colon \Omega \to SL_2(\mathbb{R})/(+I, -I)$ can be lifted to a continuous mapping $B\colon \Omega \to SL_2(\mathbb{R})$.*

PROOF. Since Ω is Stonean [9], we can find a covering by closed and open sets. Then it is very easy to define the lifting.

On the other hand, if Ω is pathwise connected, we get the following proposition [4].

PROPOSITION 4.3. *Let Ω be a pathwise connected space. Then, the following statements are equivalent:*

i) *A can be lifted.*

ii) *We have $A_*\pi_1(\Omega) \subset p_*\pi_1[SL_2(\mathbb{R})]$ (which is a subgroup of order 2 of $\pi_1[SL_2(\mathbb{R})/(+I, -I)]$).*

We get the following corollaries.

COROLLARY 4.4. *If F is sufficiently near to the identity, then A can be lifted.*

COROLLARY 4.5. *If Ω is simply connected, then A can be lifted.*

Now we transfer these results to $G(D)$:

COROLLARY 4.6. *Let $F \in G(D)$ be given. If F is sufficiently near to the identity, or if Ω is simply connected, there are continuous functions a, b, c, d on Ω such that we have*

$$(Ff)(x) = \frac{a(x)f[\psi(x)] + b(x)}{c(x)f[\psi(x)] + d(x)}$$

for all $f \in D$ and all $x \in \Omega$.

5. – The upper half-plane of commutative $\mathcal{C}^*$-algebras.

Let $\mathfrak{A}$ be a C^*-algebra of bounded linear operators on a complex Hilbert space H. By [5], the open unit ball Δ of $\mathfrak{A}$ is a bounded symmetric circular domain and the group $G(\Delta)$ of biholomorphic automorphisms of Δ admits the following representation: Let $F \in G(\Delta)$ be given; then, there are a surjective linear isometry $L: \mathfrak{A} \to \mathfrak{A}$ and a Moebius transformation $M_A: \Delta \to \Delta$ of Δ,

$$M_A(X) = (I - AA^*)^{-\frac{1}{2}}(X + A)(I + A^*X)^{-1}(I - A^*A)^{\frac{1}{2}}$$

with $A = F^{-1}(0) \in \Delta$ such that we have

$$F(X) = L \circ M_A(X)$$

for all $X \in \Delta$.

Moreover, if $\mathfrak{A}$ contains the identity operator I, then Δ admits a Siegel domain realization $\mathfrak{D}$, where

$$\mathfrak{D} =: \{X \in \mathfrak{A};\ \operatorname{Im} X > 0\} .$$

The Cayley transformation $\tau: \Delta \to \mathfrak{D}$ is given by

$$\tau(X) = (X + iI)(iX + I)^{-1} .$$

Now we study the group $G(\mathfrak{D})$ for some C^*-algebras. We have the following theorem.

THEOREM 5.1. *Let $\mathfrak{A}$ be a maximal commutative subalgebra of a W^*-algebra, with a unit. Then, for every $F \in G(\mathfrak{D})$, there are hermitian operators A, B, C, $D \in \mathfrak{A}$ with $AD\text{-}BC > 0$ and there is an order-preserving surjective linear isometry $L: \mathfrak{A} \to \mathfrak{A}$ such that we have*

$$F(X) = [AL(X) + B][CL(X) + D]^{-1}$$

for all $X \in \mathfrak{D}$.

PROOF. Due to the assumptions on $\mathfrak{A}$ [9], there exists a surjective iso metric $*$-isomorphism $\varphi: \mathfrak{A} \to \mathcal{C}(\Omega)$ for a suitable Stonean space Ω. Let us denote by $\mathfrak{D}_{\mathfrak{A}}$ and $\mathfrak{D}_{\Omega}$ the corresponding upper-half planes of $\mathfrak{A}$ and $\mathcal{C}(\Omega)$. Since φ is order-preserving, it induces an analytic isomorphism $\Psi: \mathfrak{D}_{\mathfrak{A}} \to \mathfrak{D}_{\Omega}$ and the adjoint map $\Psi_{\#} =: F \mapsto \Psi F \Psi^{-1}$ is an isomorphism of $G(\mathfrak{D}_{\mathfrak{A}})$ onto $G(\mathfrak{D}_{\Omega})$. Now the theorem follows immediately from proposition 4.2.

REFERENCES

[1] N. Dunford - J. Schwartz, *Linear operators, part I.* Interscience publishers, Inc., New York (1957).
[2] T. Franzoni, *The group of biholomorphic automorphisms in certain J^*-algebras,* Ann. Mat. Pura Appl. (4), **127** (1981), pp. 51-66.
[3] T. Franzoni - E. Visentini, *Holomorphic maps and invariant distances,* North-Holland, Amsterdam (1980).
[4] M. Greenberg, *Lectures on algebraic topology,* Benjamin, Inc., New York (1967).
[5] L. Harris, *Bounded symmetric homogeneous domains in ifnite dimensional spaces,* Lecture Notes nº 364 (1974), pp. 13-40, Springer-Verlag, Berlin.
[6] L. Harris, *Schwarz-Pick system of pseudometrics for domains in normed linear spaces,* in *Advances in Holomorphy,* J. Barroso editor, North-Holland, Amsterdam (1979), pp. 345-406.
[7] L. Harris - W. Kaup, *Linear algebraic groups in infinite dimensions,* Illinois J. Math., **21** (1977), pp. 666-674.
[8] W. Kaup - H. Upmeier, *Jordan Algebras and symmetric Siegel domains in Banach spaces,* Math. Z., **157** (1977) pp. 179-200.
[9] S. Sakai, *C^*-algebras and W^*-algebras,* Ergebnisse der Mathematik, **60**, Springer-Verlag, Berlin (1971).
[10] J.-P. Vigué, *Le groupe des automorphismes analytiques dun domaine borné d'un espace de Banach complexe. Application aux domaines bornés symétriques,* Ann. Sci. École Norm. Sup. (4), **9** (1976), pp. 203-282.
[11] J.P. Vigué, *Frontière des domaines bornés cerclés homogènes,* C. R. Acad. Sci. Sci. Paris, **288** A (1979), pp. 657-660.
[12] J.-P. Vigué - J.-M. Isidro, *Sur la topologie du groupe des automorphismes analytiques d'un domaine cerclé borné,* Bull. Sci. Math. (2), **106** (1982), pp. 417-426.

Departamento de Teoría de Funciones
Facultad de Matemáticas
Santiago de Compostela, Spain

Université de París VI
Tour 45-46; 5^e étage
4 Place Jussieu
Paris, France