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A Regularity Result for Polyharmonic Variational Inequalities with Thin Obstacles (*).

BERNHARD SCHILD

0. - Introduction.

In this work, we consider the polyharmonic variational inequality for $m = 2, 3, 4, \ldots$

Find $u \in \mathbf{K}$ *such that*

$$\text{(VI)} \qquad \langle(-\Delta)^m u - f,\ u - v\rangle \leqslant 0 \quad \textit{for all } v \in \mathbf{K}\,.$$

Here $\mathbf{K}$ stands for a closed convex subset of $g + H_0^{m,2}(\Omega)$, $g \in H^{m,2}(\Omega)$, $f \in L^p(\Omega)$, $(p > n/(2m-2))$, where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2,\ 2m \leqslant n+2)$ is a bounded domain, and $\langle\cdot,\cdot\rangle$ denotes the dual pairing of $H_0^{m,2}(\Omega)$ and $H^{-m,2}(\Omega)$. It is well-known that (VI) has a unique solution u provided that $\mathbf{K} \neq \emptyset$.

Now let Ω' be a k-dimensional C^2-surface $(k \geqslant n - 2m + 3)$ with $\Omega' \subset \Omega$, $\partial\Omega' \subset \partial\Omega$. Then for a thick obstacle $\Psi \in C^2(\bar{\Omega})$ and a thin obstacle $\Psi' \in C^2(\bar{\Omega}')$ we define

$$\mathbf{K}_\Psi := \{v \in g + H_0^{m,2}(\Omega)\colon\ v \geqslant \Psi \ \textit{ on } \Omega \ \textit{ in } H^{m,2}(\Omega)\}\,,$$

$$\mathbf{K}_{\Psi'} := \{v \in g + H_0^{m,2}(\Omega)\colon\ v \geqslant \Psi' \ \textit{ on } \Omega' \ \textit{ in } H^{m,2}(\Omega)\}\,.$$

Let u be the solution of (VI) for $\mathbf{K} = \mathbf{K}_\Psi$ or $\mathbf{K} = \mathbf{K}_{\Psi'}$, $\mathbf{K} \neq \emptyset$. In § 1, § 2 we prove that

$$u \in H^{2,\infty}_{\text{loc}}(\Omega) \cap H^{m+1,2}_{\text{loc}}(\Omega)\,,$$

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moreover,

$$u \in C^2(\Omega) \quad \text{for } 2m = n + 2 .$$

In the biharmonic case our results for the thick obstacle have already been proved by Frehse [2], [3], [5], and Caffarelli and Friedman [1]. Concerning the thin obstacle case, no regularity properties are known to us except for $H^{m,2}_{\mathrm{loc}}(\Omega)$-estimates of the tangential derivatives (Frehse [4]).

Our proof of the $H^{2,\infty}_{\mathrm{loc}}$-regularity rests on an inequality of type

$$-\Delta E \leqslant -c\cdot\Delta_k E , \quad \Delta_k := \sum_{i=1}^{k} \partial_{ii} , \quad (c > 0) .$$

This is valid for the fundamental solution E of the polyharmonic operator $(-\Delta)^m$, $(-\Delta)^m E = \delta$ (δ Dirac measure). Note that our regularity results are no longer valid for $k < n - 2m + 3$. This follows immediately from the theory of Riesz potentials resp. logarithmic potentials and the representations in our proof.

Finally, in § 3 we give a counterexample for higher regularity in the biharmonic case in two dimensions. We construct a thick obstacle problem with a quadratic polynomial Ψ and boundary data $g \in C^\infty(\bar{\Omega})$, where the solution u of (VI) with $\mathbb{K} = \mathbb{K}_\Psi$ satisfies $u \notin H^{3,\infty}_{\mathrm{loc}}(\Omega)$ and the set $[u = \Psi]$ is a line segment compact in Ω. In this connection we also mention a counterexample of Caffarelli and Friedman [1] showing that no a priori estimate of the modulus of continuity of the second derivatives in terms of the data holds. But this example is a one-dimensional problem generalized for higher dimensions, therefore the third derivatives are bounded.

Our example has a special meaning for a result of Kinderlehrer, Nirenberg and Spruck [6] on the regularity of the free boundary $[u = \Psi]$ for the biharmonic thick obstacle case in two dimensions. They need an assumption which includes the local boundedness of the third derivatives of the solution u in order to obtain that the set $[u = \Psi]$ is locally contained in a smooth curve. So our example shows that the free boundary $[u = \Psi]$ can be contained in an analytic curve in those cases where their assumptions are violated.

One should note that some additional regularity always holds for the solution u of a one-sided obstacle problem provided f is at least a measure on Ω. Using the fact, that $\mu := (-\Delta)^m u - f$ is a measure on Ω, Frehse proves $\nabla^{2m-1} u \in L^{n/(n-1)-\delta}_{\mathrm{loc}}(\Omega)$ for arbitrary $\delta \in (0, 1)$ in [4], [5].

The author wishes to express his gratitude to Prof. J. Frehse for suggesting the thin obstacle problem and for support, and Dr. P. Tolksdorf for a revision of § 2.

Notations.

By « C », we always mean a generic constant which may vary with the context possibly depending on the dimension n, the degree of polyharmonicity m, the bounded domain Ω and the surface Ω', and being independent of further quantities. E denotes the fundamental solution of $(-\Delta)^m$, for the explicit formula see the appendix. By $\Delta_k = \sum_{i=1}^{k} \partial_{ii}$, we denote the k-dimensional Laplacian.

The integration $\int$ runs over $\mathbb{R}^n$ if not otherwise indicated. Furthermore, let w be a lower semicontinuous function on $\mathbb{R}^n$, $\nu \geqslant 0$ a positive (Borel) measure on $\mathbb{R}^n$ with compact support supp (ν). Then one can always take the integral $\int w d\nu$ in the extended sense, that means with possible value ∞. Therefore $w * \nu := \int w(\cdot - y)\, d\nu(y)$ is a well defined lower semicontinuous function on $\mathbb{R}^n$. We use integration in the extended sense only in those cases, where a possible infinite value is indicated.

The mollifier $\omega_\varepsilon := \varepsilon^{-n} \cdot \omega(\cdot/\varepsilon)$, $\omega := c \cdot \exp\big(1/(|\cdot|^2 - 1)\big)$ for $|x| < 1$, $:= 0$ for $|x| \geqslant 1$, where $c > 0$ is a properly chosen constant, has well-known properties. Here we only state that *$w * \omega_\varepsilon \to w$ pointwise on $\mathbb{R}^n$* $(\varepsilon \to 0)$, if w is superharmonic on $\mathbb{R}^n$. For a proof see [1].

As usual, $\|w\|_p := \big(\int |w|^p\, dx\big)^{1/p}$, $1 \leqslant p < \infty$, $\|w\|_\infty := \operatorname{ess\,sup}\{|w(x)| : x \in \Omega\}$ denote the norms of the Lebesgue spaces, where we use the notation $\| \;\|_{\cdot, A}$, if the norm is taken over a set A other than Ω.

$H^{m,p}(\Omega)$ denotes the Sobolev space of functions with distributional derivatives in $L^p(\Omega)$ up to order m and provided with the norm $\|w\|_{m,p} := \sum_{i=0}^{m} \|\nabla^i w\|_p$. If the norm is taken over an open set A other than Ω, we use the notation $\| \;\|_{\cdot,\cdot,A}$. The space of functions w with $w|_{\Omega_0} \in H^{m,p}(\Omega_0)$ for any open set $\Omega_0 \subset\subset \Omega$ is denoted by $H^{m,p}_{\mathrm{loc}}(\Omega)$. Finally, $H^{m,p}_0(\Omega) \subset H^{m,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{m,p}$. For elements of Sobolev spaces it is possible to define inequalities on lower dimensional surfaces; for the definition see [9].

1. – Regularity results for the thick obstacle problem.

In this paragraph we prove the regularity results for the thick obstacle problem, which are already known in the biharmonic case. Our main intention is to make the reader familiar with the methods of proof and the utilization of some auxiliary statements. A suitable adaption replacing resp. changing some of these arguments will lead to similar results in case of a thin obstacle in § 2.

Now let u be *the solution of* (VI) *with* $\mathbb{K} = \mathbb{K}_\Psi \neq \emptyset$,

$$\mathbb{K}_\Psi := \{v \in g + H_0^{m,2}(\Omega)\colon\ v \geq \Psi \ on\ \Omega\}\,,$$

where $\Psi \in C^2(\bar{\Omega})$, $g \in H^{m,2}(\Omega)$, $f \in L^p(\Omega)$, $p > n/(2m-2)$.

We prove that $u \in H^{2,\infty}_{\text{loc}}(\Omega) \cap H^{m+1,2}_{\text{loc}}(\Omega)$ and besides $u \in C^2(\Omega)$ when $2m = n + 2$.

First, we want to study the local behaviour of u. Therefore let $x^0 \in \Omega$ be arbitrary and choose $\varrho = \varrho(x^0)$, $0 < \varrho < \frac{1}{2}$, so that $B_{2\varrho} := B_{2\varrho}(x^0) \subset \Omega$. In our notation the center of a ball will always be x^0, if not otherwise indicated. Furthermore, $\tau_\varrho \in C_0^\infty(B_{2\varrho})$ is a local cut-off function associated with x^0 with the properties

$$0 \leq \tau_\varrho \leq 1\,, \qquad \tau_\varrho \equiv 1 \ on\ B_{3\varrho/2}\,,$$
$$|\nabla^i \tau_\varrho| \leq C \cdot \varrho^i\,, \qquad i = 1, \ldots, 2m\,.$$

The following remark, which is a well-known fact in the theory of variational inequalities with one-sided obstacles, is crucial for our further considerations.

REMARK. $\mu := (-\Delta)^m u - f$ *is a measure on* Ω, $\mu \geq 0$.

PROOF. For $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, we have $u + \varphi \in \mathbb{K}_\Psi$. Inserting $u + \varphi$ in (VI) yields $-\langle \mu, \varphi \rangle \leq 0$, that means $\mu \in H^{-m,2}(\Omega) \subset \mathfrak{D}'(\Omega)$ is a positive distribution and hence a measure. ■

Now we set

$$\mu_\varrho := \tau_\varrho \cdot \mu \geq 0\,,$$

where μ_ϱ *is a positive measure on* $\mathbb{R}^n$. In the following lemma we split u, $-\Delta u$ into a potential and a regular function on B_ϱ. Using this representation it is possible to apply well-known principles of potential theory.

LEMMA 1.1. i) *For* $x \in \Omega$ *the following functions are well defined*

$$\bar{u}(x) := \lim_{\varepsilon \to 0} u * \omega_\varepsilon(x) \in \mathbb{R} \cup \{\infty\}\,,$$

$$\overline{\Delta u}(x) := \lim_{\varepsilon \to 0} \Delta u * \omega_\varepsilon(x) \in \mathbb{R} \cup \{-\infty\}\,.$$

ii) $\bar{u}$ *and* $\overline{\Delta u}$ *admit the splitting*

$$\bar{u} = E * \mu_\varrho + l_\varrho \qquad on\ B_\varrho , \tag{1.1}$$

$$\overline{\Delta u} = \Delta E * \mu_\varrho + \Delta l_\varrho \quad on\ B_\varrho , \tag{1.2}$$

with $l_\varrho \in C^2(\overline{B_\varrho}) \cap H^{m+1,2}(B_\varrho) \cap H^{2m-1,1}(B_\varrho)$,

$$\|l_\varrho\|_{2,\infty,B_\varrho} + \|l_\varrho\|_{m+1,2,B_\varrho} \leqslant C_\varrho \cdot \left(\|u\|_{m,2} + \|f\|_p\right) , \tag{1.3}$$

iii) $\bar{u}$, $-\overline{\Delta u}$ *are lower semicontinuous on* Ω.

Because of $u * \omega_\varepsilon \to u$ *in* $\mathcal{D}'(B_\varrho)$ $(\varepsilon \to 0)$, the following statement holds

COROLLARY 1. *Let* α *be a multiindex with* $|\alpha| \leqslant 2m - 1$. *Then*

$$\partial_\alpha u = \partial_\alpha E * \mu_\varrho + \partial_\alpha l_\varrho \quad on\ B_\varrho . \tag{1.4}$$

PROOF. Suppose that m is even, the case of odd m can be treated in the same way. The assumption $f \in L^p(\Omega)$ implies that $(-\Delta)^m$ is a *measure on* Ω, $\tau_\varrho \cdot (-\Delta)^m u$ a *measure on* $\mathbf{R}^n$, so we have at least $E * \left(\tau_\varrho \cdot (-\Delta)^m u\right) \in H^{2m-1,1}_{\text{loc}}(\mathbf{R}^n)$. This fact holds because of $|\cdot|^{1-n} * |\tau_\varrho \Delta^m u| \in L^1_{\text{loc}}(\mathbf{R}^n)$ and the estimate $|\nabla^i E| \leqslant C_{m,n} \cdot |\cdot|^{1-n}$ *on* $B_{\frac{1}{2}}(0)$, $i = 0, \ldots, 2m-1$. After integration by parts and application of Leibniz' rule, we obtain on account of $\nabla^i \tau_\varrho \equiv 0$ *on* $B_{3\varrho/2}$, $i \geqslant 1$,

$$\begin{aligned} E * \left(\tau_\varrho \cdot (-\Delta)^m u\right) &= (-\Delta)^{m/2} E * \left(\tau_p \cdot (-\Delta)^{m/2} u\right) + R_1(u) , \\ &= (-\Delta)^{m/2} E * (-\Delta)^{m/2} (\tau_\varrho \cdot u) + R_2(u) + R_1(u) ; \\ &= \tau_\varrho \cdot u + R_2(u) + R_1(u) \qquad in\ H^{m,1}(B_{3\varrho/2}) , \end{aligned}$$

where

$$R_1(u) := \sum_{|\alpha+\beta| \leqslant m, |\beta| > 0} a^1_{\alpha,\beta} \cdot \partial_\alpha E * \left(\partial_\beta \tau_\varrho \cdot (-\Delta)^{m/2} u\right) ,$$

$$R_2(u) := \sum_{|\alpha+\beta| \leqslant m, |\beta| > 0} a^2_{\alpha,\beta} \cdot (-\Delta)^{m/2} E * (\partial_\beta \tau_\varrho \cdot \partial_\alpha u) ,$$

and for $i = 1.2$ holds $a^i_{\alpha,\beta} \in \mathbf{Z}$, $R_i(u) \in C^\infty(B_{3\varrho/2})$,

$$\|R_i(u)\|_{2m,\infty,B_{4\varrho/3}} \leqslant C \cdot \varrho^{-C} \cdot \|u\|_{m,2} .$$

So we can conclude that

$$E * \left(\tau_\varrho \cdot (-\Delta)^m u\right) \in H^{m,2}(B_{4\varrho/3}) .$$

Using Hölder's inequality one shows that

$$E * (\tau_\varrho \cdot f) \in C^2(\mathbb{R}^n) \cap H^{m+1,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap H^{2m-1,1}_{\mathrm{loc}}(\mathbb{R}^n)\,,$$
$$\|E * (\tau_\varrho \cdot f)\|_{2,\infty} + \|E * (\tau_\varrho \cdot f)\|_{m+1,2} \leqslant C \cdot \|f\|_p\,.$$

For this well-known fact from the theory of linear integral operators we refer to [10, p. 40, p. 127]. Now we set

$$l_\varrho := -R_1(u) - R_2(u) + E * (\tau_\varrho \cdot f)\,,$$

and our construction yields

$$u = E * (\tau_\varrho \cdot (-\Delta)^m u) - E * (\tau_\varrho \cdot f) + l_\varrho\,,$$
$$= E * \mu_\varrho + l_\varrho \quad \textit{in } H^{m,2}(B_\varrho),\ H^{2m-1,1}(B_\varrho)\,,$$

and l_ϱ has all required properties.

Because $E * \mu_\varrho$, $-\Delta E * \mu_\varrho$ is continuous or superharmonic, assertion i) follows for $x \in B_\varrho$, so we have proved assertion ii). Further, $x^0 \in \Omega$ is arbitrary, and assertion i) holds on Ω. Finally, assertion iii) follows from the representation in ii). ■

COROLLARY 2. *In case of $\Delta u \in L^\infty_{\mathrm{loc}}(\Omega)$ for $x \in \Omega$ the following functions are well defined*

$$\overline{\partial_{ij} u}(x) := \lim_{\varepsilon \to 0} \partial_{ij} u * \omega_\varepsilon(x) \in \mathbb{R}\,, \qquad i, j = 1, \ldots, n\,,$$

and admit the splitting

$$\overline{\partial_{ij} u} = \partial_{ij} E * \mu_\varrho + \partial_{ij} l_\varrho \quad \textit{on } B_\varrho\,.$$

For the proof we need the following remark.

REMARK. *$\Delta u \in L^\infty_{\mathrm{loc}}(\Omega)$ implies $\mu(\{x\}) = 0$, $x \in \Omega$.*

PROOF OF THE REMARK. Without loss of generality we assume $2m < n + 2$, $x \in B_\varrho$. For $\zeta > 0$ we have, using (1.2),

$$C \cdot \mu_\varrho(B_\zeta(x)) \cdot \zeta^{2m-2-n} \leqslant -\Delta E * \mu_\varrho|_{B_\zeta(x)}(x) \leqslant -\Delta E * \mu_\varrho(x) < \infty\,,$$

that means $\mu_\varrho(B_\zeta(x)) \leqslant C \cdot \Delta E * \mu_\varrho(x) \cdot \zeta^{n+2-2m}$. Letting $\zeta \to 0$ we obtain $\mu(\{x\}) = \mu_\varrho(\{x\}) = 0$. ■

PROOF OF COROLLARY 2. For $x \in B_{2\varrho}$ we have the estimate

$$|\partial_{ij} E(x - \cdot)| \leqslant - C \cdot \Delta E(x - \cdot) \qquad on\ B_{2\varrho}(x)\,,$$

and further $\partial_{ij} E * \omega_\varepsilon(x - \cdot) \to \partial_{ij} E(x - \cdot)$ *pointwise on* $\mathbb{R}^n - \{x\}$ $(\varepsilon \to 0)$, where the estimate and the convergence is μ_ϱ—*a.e.* because of the remark. Now Lebesgue's theorem on dominated convergence yields $\partial_{ij} E * \mu_\varrho \omega_\varepsilon \to$ $\to \partial_{ij} E * \mu_\varrho$ *pointwise on* $B_{2\varrho}$ $(\varepsilon \to 0)$. Note that because of $\partial_{ij} E \in C^\infty(\mathbb{R}^n - \{0\})$ and the remark $\partial_{ij} E * u_\varrho$ is well defined. From (1.4) we infer the assertion. ■

CONVENTION. *For simplicity we shall assume* $f \equiv 0$ *in the sequel. Using the fact* $E * f \in H^{m,2}_{\text{loc}}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ *we can introduce* $(\widetilde{\text{VI}})$ *with* $\tilde f \equiv 0$ *and* $\widetilde{\mathbb{K}}$ with $\tilde g := g - E * f$. *Then one easily shows that* $\tilde u = u - E * f$ *is the solution of* $(\widetilde{\text{VI}})$ *with* $\widetilde{\mathbb{K}} = \mathbb{K}_{\tilde\Psi}$, *where* $\tilde\Psi := \Psi - E * f$.

In the following lemma we study the behaviour of $\bar u$ on supp (μ).

LEMMA 1.2.

$$\operatorname{supp}(\mu) \subset [\bar u = \Psi]\,.$$

PROOF. Let $x \in \Omega - [\bar u = \Psi]$, that means $d := (\bar u - \Psi)(x) > 0$

Because of lower semicontinuity for a $\sigma > 0$

$$(\bar u - \Psi) > d/2 \qquad on\ B_\sigma(x)\,.$$

Choose $\varphi \in C_0^\infty(B_\sigma(x))$, $\varphi \geqslant 0$, where $\|\varphi\|_\infty < d/2$, $\varphi > 0$ *on* $B_{\sigma/2}(x)$. After inserting the function $u - \varphi \in \mathbb{K}_\Psi$ in (VI) we obtain $\langle \mu, u - (u - \varphi)\rangle$ $= \langle \mu, \varphi \rangle \leqslant 0$, that means $\mu(B_{\sigma/2}(x)) = 0$, $x \notin \operatorname{supp}(\mu)$. ■

Now we state an adaption of the well-known continuity principle of potential theory.

LEMMA 1.3. i) $\bar u$ *continuous on* supp (μ) *implies* $\bar u \in C^0(\Omega)$.

ii) $\overline{\Delta u}$ *continuous on* supp (μ) *implies* $\overline{\Delta u} \in C^0(\Omega)$.

PROOF. i) Because $\bar u$ is always continuous for $2m > n$ we assume $2m \leqslant n$. Using (1.1) of lemma 1.1 twice we have

$$\begin{aligned} \bar u &= E * \mu_\varrho + l_\varrho \qquad on\ B_\varrho \text{ with } l_\varrho \in C^0(B_\varrho)\,,\\ &= E * \mu_{\varrho/2} + l_{\varrho/2} \quad on\ B_{\varrho/2} \text{ with } l_{\varrho/2} \in C^0(B_{\varrho/2})\,. \end{aligned}$$

Therefore the assumption $\overline{u}$ *continuous on* supp (μ) implies $E * u_\varrho$ *continuous on* supp $(\mu_{\varrho/2}) \subset B_\varrho$. In order to proceed we need the following principle.

HEREDITY PRINCIPLE [10, p. 229]. *Let* $0 \leqslant \nu_1 \leqslant \nu_2$ *be positive Borel measures on* $\mathbb{R}^n$ *with compact support,* $x \in \mathbb{R}^n$ *arbitrary. Then the continuity of* $E * \nu_2$ *in* x *implies the continuity of* $E * \nu_1$ *in* x.

From $0 \leqslant \mu_{\varrho/2} \leqslant \mu_\varrho$ we infer that $E * \mu_{\varrho/2}$ is continuous on supp $(\mu_{\varrho/2})$ and we can apply the continuity principle.

CONTINUITY PRINCIPLE [7, p. 365]. *Let* $\nu \geqslant 0$ *be a positive Borel measure on* $\mathbb{R}^n$ *with compact support. Then* $E * \nu$ *continuous on* supp (ν) *implies* $E * \nu \in C^0(\mathbb{R}^n)$.

So we obtain $E * \mu_{\varrho/2} \in C^0(\mathbb{R}^n)$ and therefore $\overline{u} \in C^0(B_{\varrho/2})$. Because $x^0 \in \Omega$ is arbitrary the assertion follows.

ii) can be proved like i) after replacing E by $-\Delta E$. ■

The following is an immediate consequence of lemma 1.2, 1.3 on account of $\Psi \in C^0(\Omega)$

LEMMA 1.4.

$$\overline{u} \in C^0(\Omega) .$$

Now we study the behaviour of $\overline{\Delta u}$ on supp(μ) and show the estimate

LEMMA 1.5.

$$-\overline{\Delta u} \leqslant -\Delta \Psi \leqslant \|\Delta \Psi\|_\infty \quad on \ \operatorname{supp}(\mu) .$$

PROOF. Let $x^c \in [\overline{u} = \Psi] \supset$ supp (μ), and for simplicity of notation assume $x^c = x^0$. We apply Green's formula for B_R, $0 < R < \varrho$, on $(u - \Psi) * \omega_\varepsilon$, $\varepsilon > 0$, where $G_R(\cdot, \cdot) \geqslant 0$ denotes Green's function of B_R, dS_R the $(n-1)$-dimensional surface element of $S_R = \partial B_R$. For small $\varepsilon > 0$ we have

$$(u - \Psi) * \omega_\varepsilon(x^0) = \int_{B_R} -\Delta(u - \Psi) * \omega_\varepsilon \cdot G_R(x^0, \cdot)\, dx + \int_{S_R} (u - \Psi) * \omega_\varepsilon \, dS_R .$$

Lower semicontinuity of $-\overline{\Delta u} + \Delta \Psi$ gives $\inf_{B_\varrho} -\overline{\Delta u} + \Delta \Psi = a > -\infty$. Letting $\varepsilon \to 0$ and observing $-\Delta(u - \Psi) * \omega_\varepsilon \cdot G_R(x^0, \cdot) \geqslant a \cdot G_R(x^0, \cdot)$ *on* B_R we infer from Fatou's lemma on account of $(u - \Psi) * \omega_\varepsilon \geqslant 0$

$$0 = \overline{u}(x^0) - \Psi(x^0) \geqslant \int_{B_R} (-\overline{\Delta u} + \Delta \Psi) \cdot G_R(x^0, \cdot)\, dx .$$

Because of $G_R(\cdot, \cdot) \geqslant 0$ for $B > 0$ we find $x^R \in B_R$ with $(-\Delta u + \Delta \Psi)(x^R) \leqslant 0$. Choosing a sequence $\{x^\varkappa\}_{\varkappa=1}^{\infty}$ of such $x^\varkappa := x^{R(\varkappa)}$, $R(\varkappa) \to 0$ $(\varkappa \to \infty)$, the lower semicontinuity of $-\overline{\Delta u}$ leads to

$$-\overline{\Delta u}(x^0) \leqslant \varliminf_{\varkappa \to \infty} -\overline{\Delta u}(x^\varkappa) \leqslant \lim_{\varkappa \to \infty} -\Delta \Psi(x^\varkappa) = -\Delta \Psi(x^0) .$$

Since $x^c = x^0 \in [\overline{u} = \Psi]$ is arbitrary, the assertion follows. ■

In order to employ lemma 1.5 we state an adaptation of the maximum principle of potential theory by the local estimate

LEMMA 1.6.

$$\|\Delta u\|_{\infty, B_{\varrho/2}} \leqslant C \cdot \sup_{\operatorname{supp}(\mu|B_\varrho)} -\overline{\Delta u} + C_\varrho \cdot \|u\|_{m,2} .$$

PROOF. We can assume $\sup_{\operatorname{supp}(\mu|B_\varrho)} -\overline{\Delta u} < \infty$ and obtain from (1.2) of lemma 1.1 that

$$\begin{aligned} -\overline{\Delta u} &= -\Delta E * \mu_\varrho - \Delta l_\varrho \qquad \text{on } B_\varrho, \ \|\Delta l_\varrho\|_{\infty, B_\varrho} \leqslant C_\varrho \cdot \|u\|_{m,2} , \\ &= -\Delta E * \mu_{\varrho/2} - \Delta l_{\varrho/2} \quad \text{on } B_{\varrho/2}, \ \|\Delta l_{\varrho/2}\|_{\infty, B_{\varrho/2}} \leqslant C_\varrho \cdot \|u\|_{m,2} . \end{aligned}$$

Now we need the following maximum principle.

MAXIMUM PRINCIPLE [7, p. 365]. *Let $\nu \geqslant 0$ be a positive Borel measure on $\mathbb{R}^n$ with compact support, $A \in \mathbb{R}$. Then $-\Delta E * \nu \leqslant A$ on* supp (μ) *implies $-\Delta E * \nu \leqslant C_n \cdot A$ on $\mathbb{R}^n$, where $C_n > 0$ is a constant depending only on the dimension n.*

Therefore because of $0 \leqslant \mu_{\varrho/2} \leqslant \mu_\varrho$ we get the estimate

$$\begin{aligned} 0 \leqslant -\Delta E * \mu_{\varrho/2} \leqslant -\Delta E * \mu_\varrho &\leqslant \sup_{\operatorname{supp}(\mu|B_\varrho)} -\overline{\Delta u} + C_\varrho \|u\|_{m,2} =: C_{1.1} \qquad \text{on supp } (\mu) , \\ &\leqslant C_n \cdot C_{1.1} \qquad \text{on } B_\varrho , \end{aligned}$$

and finally

$$|\overline{\Delta u}| \leqslant |\Delta E * \mu_{\varrho/2}| + |\Delta l_{\varrho/2}| \leqslant C_n \cdot C_{1.1} + C_\varrho \cdot \|u\|_{m,2} \qquad \text{on } B_{\varrho/2} .$$ ■

Inserting lemma 1.5 in lemma 1.6 yields

LEMMA 1.7.

$$\|\Delta u\|_{\infty, B_{\varrho/2}} \leqslant C_\varrho \cdot (\|\Delta \Psi\|_\infty + \|u\|_{m,2}) .$$

COROLLARY.

$$\Delta u \in L^\infty_{\mathrm{loc}}(\Omega) \,.$$

In the following lemma we show that a local bound for Δu gives a local bound for all second order derivatives.

LEMMA 1.8. *$\Delta u \in L^\infty(B_\varrho)$ implies $u \in H^{2,\infty}(B_\varrho)$, where*

$$\overline{|\nabla^2 u|} \leqslant -C \cdot \overline{\Delta u} + C_\varrho \cdot \|u\|_{m2} \qquad on\ B_\varrho \,.$$

PROOF. On account of (1.2) the assumption $\Delta u \in L^\infty(B_\varrho)$ implies $\Delta E * \mu_\varrho \in L^\infty(B_\varrho)$. Let α be a multiindex with $|\alpha| \leqslant 2$, the estimate $|\partial_\alpha E| \leqslant -C \cdot \Delta E$ *on* $B_{\frac{1}{2}}(0)$ gives, using (1.4)

$$\begin{aligned} |\partial_\alpha u| \leqslant |\partial_\alpha E * \mu_\varrho| + |\partial_\alpha l_\varrho| &\leqslant -C \cdot \Delta E * \mu_\varrho + |\partial_\alpha l_\varrho| \,, \\ &\leqslant -C \cdot \overline{\Delta u} + C \cdot \Delta l_\varrho + |\partial_\alpha l_\varrho| \,, \\ &\leqslant -C \cdot \overline{\Delta u} + C_\varrho \cdot \|u\|_{m,2} \qquad on\ B_\varrho \,. \end{aligned}$$

■

COROLLARY. *$\Delta u \in L^\infty_{\mathrm{loc}}(\Omega)$ implies $u \in H^{2,\infty}_{\mathrm{loc}}(\Omega)$.*

Again using tools from potential theory we can deduce local square integrability of the $m+1$-order derivatives from locally bounded Δu.

LEMMA 1.9. *$\Delta u \in L^\infty(B_\varrho)$ implies $u \in H^{m+1,2}(B_{\varrho/2})$.*

PROOF. The following notation is useful for our further considerations. We set

$$\begin{aligned} q_\beta &:= |\cdot|^{\beta-n} \,, \qquad 0 < \beta < n \,, \\ &:= -\log|\cdot| \,, \quad \beta = n \,. \end{aligned}$$

From potential theory one knows the following formula giving the connection of potential and energy of a measure for $0 < \beta \leqslant n/2$ [7, p. 80],

$$\|q_\beta * \mu_{\varrho/2}\|^2_{2,\mathbf{R}^n} = c_{\beta,n} \cdot \int q_{2\beta} * \mu_{\varrho/2} \, d\mu_{\varrho/2} \,.$$

On account of (1.2) the assumption $\Delta u \in L^\infty(B_\varrho)$ implies $\Delta E * \mu_\varrho \in L^\infty(B_\varrho)$, and we notice that $-\Delta E = c_{m,n} \cdot q_{2m-2}$. Let α be a multiindex with

$|\alpha| \leqslant m+1$, the estimate $|\partial_\alpha E| \leqslant c_{m,n}\cdot q_{m-1}$ *on* $B_{\frac{1}{2}}(0)$ gives

$$\begin{aligned}\|\partial_\alpha E * \mu_{\varrho/2}\|^2_{2,B_{\varrho/2}} &\leqslant \|q_{m-1} * \mu_{\varrho/2}\|^2_{2,\mathbb{R}^n}\,,\\ &= c_{m,n}\cdot\int q_{2(m-1)} * \mu_{\varrho/2}\, d\mu_{\varrho/2}\,,\\ &= -\,c_{m,n}\cdot\int \Delta E * \mu_{\varrho/2}\, d\mu_{\varrho/2}\,,\\ &\leqslant c_{m,n}\cdot \mu_\varrho(\mathbb{R}^n)\cdot \|\Delta E * \mu_{\varrho/2}\|_{\infty,B_\varrho}\,,\\ &\leqslant C_\varrho\cdot\|u\|_{m,2}\cdot\|\Delta E * \mu_\varrho\|_{\infty,B_\varrho} < \infty\,.\end{aligned}$$

From (1.4) of lemma 1.1 we obtain the assertion. ■

COROLLARY. *$\Delta u \in L^\infty_{\text{loc}}(\Omega)$ implies $u \in H^{m+1,2}_{\text{loc}}(\Omega)$.*

Combining the corollaries of lemma 1.7, 1.8, 1.9 yields

THEOREM 1.1. *Let u be the solution of* (VI) *with $\mathbf{K} = \mathbf{K}_\Psi$. Then*

$$u \in H^{2,\infty}_{\text{loc}}(\Omega) \cap H^{m+1,2}_{\text{loc}}(\Omega)\,.$$

In the following we want to study the case of $2m = n+2$ in order to get the continuity of the second derivatives. The essential steps are contained in two lemmata.

LEMMA 1.10. *Let $2m = n+2$, $i, j = 1, \ldots, n$, $i \neq j$.*

i) $\overline{\partial_{ij} u} \in C^0(\Omega)$

ii) $\overline{(\partial_{ii} - 1/n\Delta)\, u} \in C^0(\Omega)$

iii) $-\,\overline{\partial_{ii} u}$ *is lower semicontinuous on* Ω,

$$-\,\overline{\partial_{ii} u} = \overline{(\partial_{ii} - 1/n\Delta)\, u} + 1/n\cdot\overline{\Delta u} \tag{1.5}$$

iv) $\overline{\Delta u} \in C^0(\Omega)$ *implies* $u \in C^2(\Omega)$.

PROOF. i) From the remark after lemma 1.1 we know that $u \in H^{2,\infty}_{\text{loc}}(\Omega)$ implies $\mu_\varrho(\{x\}) = 0$, $x \in \mathbb{R}^n$, and moreover because of $\partial_{ij} E \in C^0(\mathbb{R}^n - \{0\}) \cap L^\infty(\mathbb{R}^n)$ that $\partial_{ij} E * \mu_\varrho \in C^0(\mathbb{R}^n)$. The 2. corollary of lemma 1.1 gives the assertion.

ii) can be proved like i). iii) follows from i), ii). Combining i), ii), iii) yields iv). ■

The following lemma shows the continuity of $\overline{\Delta u}$ on supp (μ).
The recursive technique is due to Caffarelli-Friedman [1].

LEMMA 1.11. *Let $2m = n + 2$, $x^0 \in \operatorname{supp}(\mu)$. Then for a sequence $\{x^\varkappa\}_{\varkappa=1}^\infty$ with $x^\varkappa \in \operatorname{supp}(\mu)$, $x^\varkappa \to x^0$ $(\varkappa \to \infty)$ holds*

$$\lim_{\varkappa\to\infty} \overline{\Delta u}(x^\varkappa) = \overline{\Delta u}(x^0).$$

PROOF. Using the invariance of $(-\Delta)^m$ under rotations we can assume after changing our coordinate system that $x^0 = 0$, and for a subsequence $\{x^{\varkappa'}\}_{\varkappa'=1}^\infty$, $|x^{\varkappa'}|^{-1} \cdot x^{\varkappa'} \to e_1$ $(\varkappa' \to \infty)$ holds, where e_1 denotes the first unit coordinate vector. This subsequence will be used later in the proof. Thanks to lemma 1.10 ii), iii) it suffices to show that

$$\lim_{\varkappa\to\infty} \overline{\partial_{11} u}(x^\varkappa) = \overline{\partial_{11} u}(x^0) = \overline{\partial_{11} u}(0).$$

Now we set $z := u - \Psi$, $z_\varepsilon := z * \omega_\varepsilon$, and observe from lemma 1.10 that $-\partial_{ij} z := -\overline{\partial_{ij} u} + \partial_{ij}\Psi$ is lower semicontinuous for $i = j$, continuous at 0 for $i \neq j$, $i, j = 1, \ldots, n$. On account of $\Psi \in C^2(\Omega)$ the assertion of lemma 1.11 follows from the validity of

$$\lim_{\varkappa\to\infty} \partial_{11} z(x^\varkappa) = \partial_{11} z(x^0) = \partial_{11} z(0) = 0,$$

which we state in two parts.

PART I. *We show*

$$-\partial_{11} z(0) \leqslant \varliminf_{\varkappa\to\infty} -\partial_{11} z(x^\varkappa) \leqslant \varlimsup_{\varkappa\to\infty} -\partial_{11} z(x^\varkappa) \leqslant 0.$$

From

$$z_\varepsilon(h\cdot e_1) - z_\varepsilon(0) = \int_0^h \int_0^t \partial_{11} z_\varepsilon(s\cdot e_1)\,ds + \partial_1 z_\varepsilon(0)\,dt,$$

$h > 0$, $\varepsilon > 0$, we infer from Lebesgue's theorem on dominated convergence for $\varepsilon \to 0$ on account of $z(0) = \partial_1 z(0) = 0$ that

$$0 \leqslant z(h\cdot e_1) = \int_0^h \int_0^t \partial_{11} z(s\cdot e_1)\,ds\,dt.$$

On this account for $h > 0$ there exists $x^h \in B_h(0)$ with $\partial_{11} z(x^h) \geqslant 0$.

Choosing a sequence of such $x^{h(\nu)}$ for $h(\nu) \to 0$ $(\nu \to \infty)$ we obtain, by lower semicontinuity

$$-\partial_{11} z(0) \leqslant \varliminf_{\nu\to\infty} -\partial_{11} z(x^{h(\nu)}) \leqslant 0.$$

In the same way we get $-\partial_{11}z(x^{\varkappa})\leqslant 0$. Finally the assertion of part I follows by the lower semicontinuity of $-\partial_{11}z$.

PART II. *We show* $-\partial_{11}z(0)\geqslant 0$.

Let us consider the subsequence $\{x^{\varkappa'}\}_{\varkappa'=1}^{\infty}$ again and let $\varkappa:=\varkappa'$. We may assume that $x_1^{\varkappa}>0$, $\varkappa\in\mathbb{N}$. Furthermore, we set

$$R_{\varkappa}:=R_{x_1^{\varkappa}}:=[0,x_1^{\varkappa}]\times[-x_1^{\varkappa}/2,x_1^{\varkappa}/2]^{n-1}=[0,x_1^{\varkappa}]\times R'_{\varkappa}\subset\mathbb{R}\times\mathbb{R}^{n-1}.$$

At first we show

$$\sigma(x_1^{\varkappa}):=\min\Big(0,\fint_{R_{\varkappa}}-\partial_{11}z\,dx\Big)\to 0\qquad(\varkappa\to\infty).$$

Integration by parts yields the following identity.

$$\int_{R_{\varkappa}}\partial_{11}z_{\varepsilon}\,dx=\int_{R'_{\varkappa}}\int_0^{x_1^{\varkappa}}\partial_{11}z_{\varepsilon}\,dx_1\,dx'=\int_{R'_{\varkappa}}\partial_1 z_{\varepsilon}(x_1^{\varkappa},x')-\partial_1 z_{\varepsilon}(0,x')\,dx'$$

$$=\int_{R'_{\varkappa}}\Big(\partial_1 z_{\varepsilon}(x_1^{\varkappa},x_2^{\varkappa},x_3,\dots,x_n)+\int_{x_2^{\varkappa}}^{x_2}\partial_{12}z_{\varepsilon}(x_1^{\varkappa},\xi_2,x_3,\dots,x_n)\,d\xi_2$$
$$-\partial_1 z_{\varepsilon}(0,0,x_3,\dots,x_n)-\int_0^{x_2}\partial_{12}z_{\varepsilon}(0,\xi_2,x_3,\dots,x_n)\,d\xi_2\Big)dx'$$

$$=\int_{R'_{\varkappa}}\Big(\sum_{i=2}^{n}\Big(\int_{x_i^{\varkappa}}^{x_i}\partial_{1i}z_{\varepsilon}(x_1^{\varkappa},\dots,x_{i-1}^{\varkappa},\xi_i,x_{i+1},\dots,x_n)\,d\xi_i$$
$$-\int_0^{x_i}\partial_{1i}z_{\varepsilon}(0,\dots,0,\xi_i,x_{i+1},\dots,x_n)\,d\xi_i\Big)\Big)dx'+\int_{R'_{\varkappa}}\partial_1 z_{\varepsilon}(x^{\varkappa})-\partial_1 z_{\varepsilon}(0)\,dx'$$

$$=\int_{R'_{\varkappa}}\Big(\sum_{i=2}^{n}\Big(\int_0^{x_i}\partial_{1i}z_{\varepsilon}(x_1^{\varkappa},\dots,x_{i-1}^{\varkappa},\xi_i,x_{i+1},\dots,x_n)$$
$$-\partial_{1i}z_{\varepsilon}(0,\dots,0,\xi_i,x_{i+1},\dots,x_n)\,d\xi_i\Big)\Big)dx'$$
$$-\int_{R'_{\varkappa}}\Big(\sum_{i=2}^{n}\int_0^{x_i^{\varkappa}}\partial_{1i}z_{\varepsilon}(x_1^{\varkappa},\dots,x_{i-1}^{\varkappa},\xi_i,x_{i+1},\dots,x_n)\,d\xi_i\Big)dx'+\int_{R'_{\varkappa}}\partial_1 z_{\varepsilon}(x^{\varkappa})-\partial_1 z_{\varepsilon}(0)\,dx'.$$

Letting $\varepsilon \to 0$ we can apply Lebesgue's theorem on dominated convergence and replacing $\partial_1 z_\varepsilon$, $\partial_{1i} z_\varepsilon$ by $\partial_1 z$, $\partial_{1i} z$, $i = 1, \dots, n$, we arrive at the following estimate

$$\begin{aligned}
\sigma(x_1^\varkappa) \geqslant & -(x_1^\varkappa)^{-n} \cdot \sum_{i=2}^{n} \int_{R'_\varkappa} \int_0^{|x_i|} d\xi_i \, dx' \cdot \max_{2 \leqslant i \leqslant n} \operatorname*{osc}_{R_\varkappa} \partial_{1i} z \\
& -(x_1^\varkappa)^{-n} \cdot \sum_{i=2}^{n} \int_{R'_\varkappa} \int_0^{|x_i^\varkappa|} d\xi_i \, dx' \cdot \|\nabla \partial_1 z\|_{\infty, R_\varkappa} \\
& + (x_1^\varkappa)^{-n} \cdot (x_1^\varkappa)^{n-1} \big(\partial_1 z(0) - \partial_1 z(x^\varkappa)\big) \\
\geqslant & -n \cdot \max_{2 \leqslant i \leqslant n} \operatorname*{osc}_{R_\varkappa} \partial_{1i} z - \max_{2 \leqslant i \leqslant n} |x_i^\varkappa / x_1^\varkappa| \cdot n \cdot \|\nabla \partial_1 z\|_{\infty, R_\varkappa} \\
& + \big(\partial_1 z(0) - \partial_1 z(x^\varkappa)\big) / x_1^\varkappa \, .
\end{aligned}$$

Inspecting the terms on the right hand in the above inequality we observe that continuity at 0 implies $\max_{2 \leqslant i \leqslant n} \operatorname{osc}_{R_\varkappa} \partial_{1i} z \to 0$ $(\varkappa \to \infty)$. Furthermore, the fact $|x^\varkappa|^{-1} \cdot x^\varkappa \to e_1$ $(\varkappa \to \infty)$ implies $\max_{2 \leqslant i \leqslant n} |x_i^\varkappa / (x_1^\varkappa)| \to 0$ $\sigma(\varkappa \to \infty)$. The last term of the sum vanishes because $\partial_1 z(0) = \partial_1 z(x^\varkappa) = 0$. Therefore we can conclude that $\sigma(x_1^\varkappa) \to 0$ $(\varkappa \to \infty)$.

Using (1.2), (1.5) we decompose $-\partial_{11} z$ on $B_\varrho(0)$ in a suitable way,

$$\begin{aligned}
-\partial_{11} z &= -\overline{\partial_{11} u} + \partial_{11} \Psi \\
&= -1/n \cdot \overline{\Delta u} - \overline{(\partial_{11} - 1/n\Delta)\, u} + \partial_{11} \Psi \\
&= -1/n \cdot \Delta E * \mu_\varrho - 1/n \cdot \Delta l_\varrho - \overline{(\partial_{11} - 1/n\Delta)\, u} + \partial_{11} \Psi \\
&= w_1 + w_2 \, .
\end{aligned}$$

Here w_1 is superharmonic on $B_\varrho(0)$, and w_2 is continuous at 0. Furthermore, we have $w_1 + w_2 > -A > -\infty$ *on* $B_\varrho(0)$, $A > 0$.

Choosing $\varkappa$ large enough and setting $s := x_1^\varkappa$ we obtain $(2n+1) \cdot s < \varrho$ and for $y \in B_s(0)$

$$R_s \subset B_{2ns}(y) \subset B_{(2n+1)s}(0) \subset B_\varrho(0).$$

Therefore

$$\alpha := \operatorname{vol}\big(B_{2ns}(y) - R_s\big) / \operatorname{vol}\big(B_{2ns}(y)\big)$$

is a constant, $0 < \alpha < 1$, independent of s and $y \in B_s(0)$. Now we estimate the integral mean of $w_1 + w_2$ over balls from below for $y \in B_s(0)$,

$$\fint_{B_{2ns}(y)} w_1 + w_2\,dx \geqslant \alpha \cdot \fint_{B_{2ns}(y) - R_s} - A\,dx + (1-\alpha)\cdot \fint_{R_s} w_1 + w_2\,dx$$
$$\geqslant - \alpha \cdot A + (1-\alpha)\cdot \sigma(s)\,.$$

Because w_1 is superharmonic on B_ϱ we infer from the above estimate for the mean values a pointwise estimate

$$w_1 + w_2 \geqslant - \alpha\cdot A + (1-\alpha)\cdot\sigma(s) - \eta(s) \qquad on\ B_s(0)\,,$$

where for the modulus of continuity $\eta(t) = \operatorname*{osc}_{B_{(2n+1)t}}, w_2$ tends to zero for $t \to 0+$ on account of the continuity of w_2 at 0. Let α', $\alpha < \alpha' < 1$, be a fixed constant and assume $\varkappa$ sufficiently large that $|\sigma(s)| + \eta(s) \leqslant (\alpha' - \alpha)\cdot A$, then

$$|\sigma(s)| + \eta(s) \leqslant (\alpha' - \alpha)\cdot A\,,$$

and the above pointwise estimate yields

$$w_1 + w_2 \geqslant - \alpha'\cdot A \qquad on\ B_s(0)\,.$$

Furthermore, we can assume for our chosen subsequence that for the above $\varkappa$ holds $\varkappa = 1$, and the sequence $\{x^\varkappa\}_{\varkappa=1}^{\infty}$ satisfies

$$|x^{\varkappa+1}| \leqslant (2n+1)^{-1}\cdot |x^\varkappa|\,,$$
$$|\sigma(x_1^{\varkappa+1})| + \eta(x_1^{\varkappa+1}) \leqslant (\alpha' - \alpha)\cdot(\alpha')^\varkappa\cdot A\,.$$

We are now in the position to improve the pointwise estimate for every smaller ball around 0 and establish that

$$w_1 + w_2 > - (\alpha')^\varkappa\cdot A \qquad on\ B_{x_1^\varkappa}(0),\ \varkappa \in \mathbb{N}\,.$$

The proof works by induction. For $\varkappa = 1$ the assertion has already been shown and we have to treat the step from $\varkappa$ to $\varkappa + 1$.

Therefore, let us assume that the assertion is true for a $\varkappa \in \mathbb{N}$ and set $s := x_1^{\varkappa+1}$. Because of the inclusions

$$R_s \subset B_{2ns}(y) \subset B_{(2n+1)s}(0) \subset B_{x_1^\varkappa}(0)\,, \qquad y \in B_s(0)\,,$$

the assumption for $\varkappa$ gives for the mean values

$$\fint_{B_{2ns}(y)} w_1 + w_2\, dx \geqslant -\alpha\cdot(\alpha')^{\varkappa}\cdot A + (1-\alpha)\cdot\sigma(s)\,, \qquad y\in B_s(0)\,,$$

and the pointwise estimate

$$\begin{aligned} w_1 + w_2 &\geqslant -\alpha\cdot(\alpha')^{\varkappa}\cdot A + (1-\alpha)\cdot\sigma(s) - \eta(s)\,,\\ &\geqslant -(\alpha')^{\varkappa+1}\cdot A \qquad on\ B_s(0) = B_{\alpha_1^{\varkappa+1}}(0)\,. \end{aligned}$$

Thus the improved pointwise estimate is valid and letting $\varkappa\to\infty$ we obtain

$$-\partial_{11}z(0) = w_1(0) + w_2(0) \geqslant 0\,,$$

and the assertion of Part II is proved. ■

Therewith our arguments are complete to show

THEOREM 1.2. *Let* $2m = n+2$ *and* u *be the solution of* (VI) *with* $\mathbf{K} = \mathbf{K}_{\psi}$. *Then* $u\in C^2(\Omega)$.

PROOF. We know from lemma 1.11 that $\overline{\Delta u}$ is continuous on $\operatorname{supp}(\mu)$. The application of lemma 1.2 ii) gives $\overline{\Delta u}\in C^0(\Omega)$. From lemma 1.10 iv), the assertion follows. ■

2. – Regularity results for the thin obstacle problem.

In this paragraph we present the regularity results for the thin obstacle problem. We obtain the same regularity of the solution as in the case of a thick obstacle provided the obstacle is given on a surface of sufficiently high dimension. Approximating with solutions of suitable thick obstacle problems we establish uniform local bounds for their Laplacian, using the methods of § 1 except the way of estimating the Laplacian on the coincidence set. During the approximation, uniform local bounds can only be derived for the sum of pure second derivatives in tangential directions with respect to the obstacle, for the so-called tangential Laplacian. The crucial fact in our proof is the possibility of giving a local estimate for the Laplacian of the approximate solution by its tangential Laplacian and the $H^{m,2}$-norm. We deduce this property from a corresponding inequality of the derivatives of the fundamental solution E.

Now let u be *the solution of* (VI) *with* $\mathbb{K} = \mathbb{K}_{\Psi'} \neq \emptyset$,

$$\mathbb{K}_{\Psi'} := \{v \in g + H_0^{m,2}(\Omega) \colon \ v \geqslant \Psi' \ \text{on} \ \Omega'\}\,,$$

where Ω' is a k-dimensional C^2-parametrized surface with $\partial\Omega' \subset \partial\Omega$, $k \geqslant n - 2m + 3$, $\Psi' \in C^2(\overline{\Omega'})$, $g \in H^{m,2}(\Omega)$, $f \in L^p(\Omega)$, $p > n/(2m-2)$.

In order to avoid unnecessary confusion in the proof we make the following additional assumptions. We require the existence of a *C^2-diffeomorphism $\Phi \in [C^2(\overline{V})]^n$, $\Phi\colon V \xrightarrow{\sim} \Omega$, where $V \subset \mathbb{R}^n$ is compact convex and symmetric with respect to $\mathbb{R}^k \times \{0\}^{n-k}$, furthermore $\Phi' := \Phi|_{\mathbb{R}^k \times \{0\}^{n-k}}$ is a parametrization of Ω': $V \cap (\mathbb{R}^k \times \{0\}^{n-k}) \to \Omega'$.*

By $\tilde{x} = (\tilde{x}', \tilde{x}'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ we denote the preimage of Φ, by $P_{\tilde{x}'}(\tilde{x}) := \tilde{x}'$, $P_{\tilde{x}''}(\tilde{x}) := \tilde{x}''$ being the projections. Our additional assumptions are always satisfied locally on Ω' and can be dropped, if one states the auxiliary steps in a suitable manner.

Now we want to construct local bounds for the second derivatives of u, in particular for the Laplacian, and approximate u by *the solution u_δ* of (VI) *with* $\mathbb{K} = \mathbb{K}_{\Psi_\delta} \neq \emptyset$. Here $\{\Psi_\delta\}_{\delta>0}$ are suitable thick obstacles such that $u_{\delta'} \to u$ *in* $H^{m,2}(\Omega)$, $\{\delta'\}$ is a sequence converging to 0. For the approximating solution u_δ all the results of § 1 are valid and using the notations from § 1 we write in particular $\mu_\delta := (-\Delta)^m u_\delta - f$, $u_\delta = E * \mu_{\delta,\varrho} + l_{\delta,\varrho}$ *on* B_ϱ.

Let $x^0 \in \Omega'$ be arbitrary as in § 1. In order to obtain an uniform bound for Δu_δ on $B_{\varrho/2}$ it suffices, by lemma 1.6, to give an uniform upper bound for $-\overline{\Delta u_\delta}$ on $\operatorname{supp}(\mu_\delta|B_\varrho)$. But during the approximation we can only give an upper bound for an expression not involving all the pure second derivatives.

In the case of Ω' being a k-dimensional hyperplane we can give an upper bound for $-\overline{\Delta_k u_\delta}$ on $\operatorname{supp}(\mu_\delta|B_\varrho)$. By $\Delta_k := \sum_{i=1}^{k} \partial_{ii}$ we denote the k-dimensional Laplacian. The crucial step is now a local estimate of the Laplacian of the approximate solution by its k-dimensional Laplacian which we derive from the corresponding inequality of the derivatives of the fundamental solution E; for the elementary calculations see the appendix.

PROPOSITION 2.1.

$$-\Delta E \leqslant -(2m-2)\cdot \Delta_k E + C_n\,.$$

REMARK. *Of course we assume $2m \leqslant n+2$ in the above proposition, and we have $C_n = 0$ unless $2m = n+2$.*

At this point we can give a short description how to proceed if Ω' is a k-dimensional hyperplane, that means $\Omega' = \Omega \cap (\mathbb{R}^k \times \{0\}^{n-k})$, $\Phi \equiv \mathrm{id}_\Omega$.

Then we set $\Psi_\delta(x) = \Psi'(x') - |x''|^2/\delta$, $x = (x', x'') \in \Omega$ and notice that $\|\Delta_k \Psi_\delta\|_\infty = \|\Delta_k \Psi'\|_\infty$. After proving the lower semicontinuity of $-\overline{\Delta_k u_\delta}$, Green's formula in $\mathbb{R}^k$ gives

$$(2.1) \qquad -\overline{\Delta_k u_\delta} \leqslant -\Delta_k \Psi_\delta \leqslant \|\Delta_k \Psi'\|_\infty \quad \text{on } \operatorname{supp}(\mu_\delta).$$

Using proposition 2.1 we show that

$$(2.2) \qquad -\overline{\Delta u_\delta} \leqslant -(2m-2)\cdot \overline{\Delta_k u_\delta} + C_\varrho \cdot \|u_\delta\|_{m,2} \quad \text{on } B_\varrho .$$

After inserting (2.1), (2.2) in lemma 1.6 we finally obtain an uniform bound for Δu_δ on $B_{\varrho/2}$ and establish $\Delta u \in L^\infty(B_{\varrho/2})$. Then we can proceed as in § 1.

In the more general case we have to replace the k-dimensional or tangential Laplacian in (2.1) by another differential operator which is a perturbation of the tangential Laplacian.

Therefore we need the following auxiliary result which states the existence of a suitable parameter transformation.

PROPOSITION 2.2. *For every $\xi \in \Omega'$ there exists a regular linear affine transformation $\theta_\xi\colon \theta_\xi^{-1}(V) \xrightarrow{\sim} V$ such that*

i) $\theta_\xi(0) = \Phi^{-1}(\xi)$,

ii) θ_ξ *leaves the $\tilde{x}''$-variables fixed.*

iii) *The vectors $Q_{\xi,i} := \partial_i(\Phi\circ\theta_\xi)(0) \in \mathbb{R}^n$, $i = 1, \dots, k$, form an orthonormal system of tangential vectors with respect to Ω' at ξ.*

iv) $Q_{\cdot,i} \in C^0(\Omega', \mathbb{R}^n)$, $i = 1, \dots, k$.

PROOF. Let $\Phi(\tilde{\xi}) = \xi \in \Omega'$. The successive application of the Gram-Schmidt method on $\{\partial_i \Phi(\tilde{\xi})\}_{i=1}^k$ yields an orthonormal system $\{Q_i\}_{i=1}^k$ of tangential vectors with respect to Ω' at ξ. Now we have for the Jacobian matrix $J_\Phi(\tilde{\xi}) = \big(\partial_1 \Phi(\tilde{\xi}), \dots, \partial_n \Phi(\tilde{\xi})\big)$ the decomposition

$$J_\Phi(\tilde{\xi}) = \big(Q_{\xi,1}, \dots, Q_{\xi,k}, \partial_{k+1}\Phi(\tilde{\xi}), \dots, \partial_n \Phi(\tilde{\xi})\big)\cdot A_\xi ,$$

where the $n\times n$ matrix A_ξ is regular and consists of two nonvanishing blocks, the first being a regular $k\times k$ matrix the second being the $(n-k)$-dimensional identity matrix. Hence, the inverse A_ξ^{-1} is of the same structure.

Identifying matrices with the associated linear mappings we observe, that A, A_ξ^{-1} do not change the $\tilde{x}''$-variables. If we set $\tilde{\tilde{\xi}} := A_\xi(\tilde{\xi}) = A_\xi(\Phi^{-1}(\xi))$, we have $\tilde{\tilde{\xi}}'' = \tilde{\xi}'' = 0$ recalling that $\tilde{\xi} \in \mathbb{R}^k \times \{0\}^{n-k}$. So the translation $T_{+\tilde{\tilde{\xi}}}$, $T_{+\tilde{\tilde{\xi}}}(\tilde{x}) := \tilde{x} + \tilde{\tilde{\xi}}$, $\tilde{x} \in \mathbb{R}^n$, is only a translation with respect to the $\tilde{x}'$-variables.

Now we arrive at the identity

$$\Phi = \Phi \circ A_\xi^{-1} \circ T_{+\tilde{\xi}} \circ T_{+\tilde{\xi}}^{-1} \circ A_\xi = \Phi \circ \theta_\xi \circ \theta_\xi^{-1} \colon V \xrightarrow{\sim} \Omega ,$$

where by construction $\theta_\xi := A_\xi^{-1} \circ T_{+\tilde{\xi}} = A_\xi^{-1} \circ T_{+A_\xi \circ \Phi^{-1}(\xi)}$ has all the required properties. ■

Completing the orthonormal system $\{Q_{\xi,i}\}_{i=1}^k$ to an orthonormal base $\{Q_{\xi,i}\}_{i=1}^n$ of the $\mathbb{R}^n$ we get the matrix

$$Q_\xi := (Q_{\xi,1}, \dots, Q_{\xi,n}) \in SO(n) .$$

Identifying Q_ξ with the associated linear mapping we obtain

COROLLARY.

$$\partial_i (Q_\xi^{-1} \circ \Phi \circ \theta_\xi)(0) = e_1 , \qquad i = 1, \dots, k .$$

Furthermore, we notice that $\Phi \circ \theta_\xi'$ is a *C^2-equivalent parametrization of Ω' with respect to Φ'*, where $\theta_\xi' := \theta_\xi|_{\mathbb{R}^k \times \{0\}^{n-k}}$.

Using Q_ξ we can express tangential derivatives with respect to Ω', for example the *tangential Laplacian Δ_t on Ω'* by

$$\Delta_{t,\xi} \varphi(\xi) := \Delta_k (\varphi \circ Q_\xi) \circ Q_\xi^{-1}(\xi) , \qquad \varphi \in C^2(\Omega), \ \xi \in \Omega' .$$

Unfortunately we can neither give an uniform upper bound of $-\overline{\Delta_t u_\delta}$ on $\operatorname{supp}(\mu_\delta | \Omega')$, nor a suitable extension of Δ_t on Ω.

Therefore for each $\xi \in \Omega'$ we introduce a linear differential operator $L_{\xi,\bullet}$ on $C^2(\Omega)$ using the transformations θ_ξ of proposition 2.2, whose principal part $L_{\xi,\bullet}^0$ satisfies $L_{\xi,\xi}^0 = -\Delta_{t,\xi}$. We set

$$L_{\xi,\bullet} \Phi := -\Delta_k (\varphi \circ \Phi \circ \theta_\xi) \circ (\Phi \circ \theta_\xi)^{-1} , \qquad \varphi \in C^2(\Omega), \ \xi \in \Omega',$$

and compute for $x \in \Omega$,

$$L_{\xi,x} = L_{\xi,x}^0 + \sum_{i=1}^n b_{i,\xi}(x) \cdot \partial_i ,$$
$$L_{\xi,x}^0 = -\Delta_{t,\xi} + \sum_{i,j=1}^n a_{ij,\xi}(x) \cdot \partial_{ij} .$$

The coefficients $a_{ij,\xi}$, $b_{i,\xi} \in C^0(\bar{\Omega})$ are linear combinations or products of the components of $\nabla\Phi$, $\nabla^2\Phi$, $\nabla\theta_\xi$, $\nabla^2\theta_\xi$, where

$$\|a_{ij,\xi}\|_\infty + \|b_{i,\xi}\|_\infty \leqslant C_\Phi < \infty ,$$

$$a_{ij,\xi}(\xi) = 0 , \quad |a_{ij,\xi}(x)| \leqslant C_\Phi \cdot |x - \xi| , \quad x \in \Omega , \tag{2.3}$$

$i, j = 1, \ldots, n$. In case of a k-dimensional hyperplane Ω' the above operators are reduced to

$$-\Delta_k \equiv -\Delta_t \equiv L^0_{\bullet,\bullet} \equiv L_{\bullet,\bullet} .$$

Now we infer from proposition 2.1 that $-\Delta_{t,\bullet} E$ has the same dominating property as $-\Delta_k E$.

PROPOSITION 2.3.

$$-\Delta E \leqslant -(2m-2)\cdot\Delta_{t,\xi} E + C_n , \quad \xi \in \Omega' .$$

PROOF.

$$\begin{aligned} -\Delta E = -\Delta E \circ Q_\xi^{-1} &\leqslant -(2m-2)\cdot\Delta_k E\circ Q_\xi^{-1} + C_n , \\ &\leqslant -(2m-2)\cdot\Delta_k(E\circ Q_\xi)\circ Q_\xi^{-1} + C_n , \\ &= -(2m-2)\cdot\Delta_{t,\xi} E + C_n . \end{aligned}$$ ■

Because we choose $\xi^0 = x^0 \in \Omega'$ fixed for our further considerations as in § 1, we suppress the dependence on ξ^0 and simplify the notation as follows

$$L := L_{\xi^0,\bullet} , \quad L^0 := L^0_{\xi^0,\bullet} , \quad \Delta_t := \Delta_{t,\xi^0} , \quad \theta := \theta_{\xi^0} ,$$

especially

$$L^0 E(x-y) := \big(L^0_{\xi^0,x} E(\cdot - y)\big)(x) , \quad x, y \in \Omega .$$

Now we are in the position to give local estimates for Δu.

Especially we have uniform bounds for Δu_δ on $B_{\varrho/2}$, where we have to choose $\varrho > 0$ sufficiently small. Therefore we construct suitable thick obstacles $\{\Psi_\delta\}_{\delta>0}$ and set

$$\Psi_\delta \circ \Phi \circ \theta := \Psi' \circ \Phi \circ \theta' \circ P_{\bar{x}'} - |P_{\bar{x}''}|^2/\delta \quad on\ \theta^{-1}(V),\ \delta > 0 .$$

Note that we use here the assumption that V is a convex set, symmetrical with respect to $\mathbb{R}^k \times \{0\}^{n-k}$, and the same holds for $\theta^{-1}(V)$ because θ^{-1} is linear and does not change the $\tilde{x}''$-variables. From this definition we derive

$$L\Psi_\delta = -\Delta_k(\Psi_\delta \circ \Phi \circ \theta) \circ (\Phi \circ \theta)^{-1} ,$$
$$= -\Delta_k(\Psi' \circ \Phi \circ \theta' \circ P_{\tilde{x}'} - |P_{\tilde{x}''}|^2/\delta) \circ (\Phi \circ \theta)^{-1} ,$$
$$= -\Delta_k(\Psi' \circ \Phi \circ \theta' \circ P_{\tilde{x}'}) \circ (\Phi \circ \theta)^{-1} \leqslant C \cdot \|\Psi'\|_{2,\infty} ,$$

and further, because of $\Psi_{\tilde{\delta}} \leqslant \Psi_\delta$, $0 < \tilde{\delta} < \delta$,

$$\mathbf{K}_{\Psi_\delta} \subset \mathbf{K}_{\Psi_{\tilde{\delta}}} \subset \mathbf{K}_{\Psi'} .$$

Recalling the definition of inequalities in Sobolev spaces we know that the set $cl_{H^{m,2}(\Omega)}\left(\bigcup_{\delta>0} \mathbf{K}_{\Psi_\delta}\right)$ consists of all limits $v \in g + H_0^{m,2}(\Omega)$ of sequences $\{\varphi_\varkappa\}_{\varkappa=1}^\infty$, $\varphi_\varkappa \in C^m(\bar{\Omega})$, $\varphi_\varkappa \geqslant \Psi'$ *on* Ω', $\varphi_\varkappa \to v$ *in* $H^{m,2}(\Omega)$ $(\varkappa \to \infty)$, and therefore we have

$$cl_{H^{m,2}(\Omega)}\left(\bigcup_{\delta>0} \mathbf{K}_{\Psi_\delta}\right) = \mathbf{K}_{\Psi'} .$$

As a consequence $\mathbf{K}_{\Psi_\delta} \neq \emptyset$ follows for sufficiently small $\delta > 0$ because of $\mathbf{K}_{\Psi'} \neq \emptyset$. Let u_δ *denote the solution of* (VI) *with* $\mathbf{K} = \mathbf{K}_{\Psi_\delta} \neq \emptyset$. Common arguments about minimal sequences well-known in the direct methods in the calculus of variations show that

$$u_{\delta'} \to u \quad in \ H^{m,2}(\Omega)(\delta' \to 0) ,$$

$\{\delta'\}$ being a sequence converging to 0. Theorem 1.1 gives $u_\delta \in H^{2,\infty}_{\mathrm{loc}}(\Omega)$, thus employing the 2.corollary of lemma 1.1 we can define $\overline{Lu_\delta}$ by

$$\overline{Lu_\delta}(x) := \lim_{\varepsilon \to 0} L(u_\delta * \omega_\varepsilon)(x) , \quad x \in \Omega ,$$

where we have term by term

$$\overline{Lu_\delta} = -\overline{\Delta_t u_\delta} + \sum_{i,j=1}^n a_{ij} \cdot \overline{\partial_{ij} u_\delta} + \sum^n b_i \cdot \partial_i u_\delta .$$

The following lemma extends lemma 1.1 iii).

LEMMA 2.1. *$\overline{Lu_\delta}$ is lower semicontinuous on B_ϱ for $\varrho > 0$ sufficiently small.*

PROOF. Lemma 1.1 gives $\overline{Lu_\delta} = LE * \mu_{\delta,\varrho} + Ll_{\delta,\varrho}$ *on* B_ϱ. We have to show, for sufficiently small $\varrho > 0$, that $L^0 E(x-y)$ is lower semicontinuous on $B_{2\varrho} \times B_{2\varrho}$ for $x \to y$, $x, y \in B_{2\varrho}$, and this is true if $L^0 E(x-y) \to \infty \, (x \to y)$.

But using (2.3) we have

$$|-\Delta_t E(x-y) - L^0 E(x-y)| = \Big|\sum_{i,j=1}^{n} a_{ij}(x)\cdot\partial_{ij}E(x-y)\Big| ,$$

$$\leqslant - C\cdot|x-\xi^0|\cdot\Delta E(x-y) , \qquad x, y\in\Omega .$$

Therefore the assertion follows from proposition 2.3. ■

Employing the lower semicontinuity, we are now able to give an uniform upper bound for $\overline{Lu_\delta}$ *on* supp $(\mu_\delta|B_\varrho)$,

LEMMA 2.2.

$$\overline{Lu_\delta}\leqslant L\Psi_\delta\leqslant C\cdot\|\Psi'\|_{2,\infty} \quad \textit{on } \operatorname{supp}(\mu_\delta|B_\varrho) .$$

PROOF. Let $x^c\in[u_\delta=\Psi_\delta]\cap B_\varrho\supset \operatorname{supp}(\mu_\delta|B_\varrho)$, $\Phi\circ\theta(\tilde{x}^c) = x^c$.
When $2m < n+2$ we apply Green's formula to

$$(u_\delta * \omega_\varepsilon)\circ\Phi\circ\theta - (\Psi_\delta * \omega_\varepsilon)\circ\Phi\circ\theta \qquad \textit{in } \mathbb{R}^k\times\{\tilde{x}^{c''}\} ,$$

where $\tilde{x}^c = (\tilde{x}^{c'}, \tilde{x}^{c''})$. By definition we have

$$\overline{Lu_\delta}(x) = \lim_{\varepsilon\to 0} -\Delta_k\big((u_\delta*\omega_\varepsilon)\circ\Phi\circ\theta\big)\circ\big(\Phi\circ\theta\big)^{-1}(x) , \quad x\in B_\varrho ,$$

and in virtue of lower semicontinuity we obtain

$$\overline{Lu_\delta}(x^c)\leqslant L\Psi_\delta(x^c)$$

as in the proof of lemma 1.5. When $2m = n+2$ we observe that $(u_\delta - \Psi_\delta)\circ\Phi\circ\theta'\circ P_{\tilde{x}'}$ which is twice continuously differentiable by theorem 1.2, has a minimum at $\tilde{x}^{c'}$ and the same result follows. ■

In the next lemma we utilize proposition 2.3 and get

LEMMA 2.3. $-\overline{\Delta u_\delta}\leqslant -(2m-2)\cdot\overline{\Delta_t u_\delta} + C_\varrho\cdot\|u_\delta\|_{m,2}$ *on* B_ϱ.

PROOF. With $\mu_{\delta,\varrho}(\mathbb{R}^n) = (\Delta^{m/2}u_\delta, \Delta^{m/2}\tau_\varrho)\leqslant C\cdot\|u_\delta\|_{m,2}$ we infer from proposition 2.3 using lemma 1.1 that

$$\begin{aligned}
-\overline{\Delta u_\delta} &= -\Delta E * \mu_{\delta,\varrho} - \Delta l_{\delta,\varrho}\\
&\leqslant -(2m-2)\cdot\Delta_t E * \mu_{\delta,\varrho} + C\cdot\mu_{\delta,\varrho}(\mathbb{R}^n) - \Delta l_{\delta,\varrho}\\
&\leqslant -(2m-2)\cdot\overline{\Delta_t u_\delta} + (2m-2)\cdot\Delta_t l_{\delta,\varrho} + C\cdot\mu_{\delta,\varrho}(\mathbb{R}^n) - \Delta l_{\delta,\varrho}\\
&\leqslant -(2m-2)\cdot\overline{\Delta_t u_\delta} + C_\varrho\cdot\|u_\delta\|_{m,2} \quad \textit{on } B_\varrho .
\end{aligned}$$

■

The following lemma gives a local bound of $|\nabla u_\delta|$ by $-\overline{\Delta u_\delta}$,

LEMMA 2.4. $|\nabla u_\delta| \leqslant - C\cdot\varrho\cdot\overline{\Delta u_\delta} + C_\varrho\cdot\|u_\delta\|_{m,2}$ *on* B_ϱ *for* ϱ, $0<\varrho<c_{m,n}$ *sufficiently small.*

PROOF. Because of

$$|\nabla E| = 0(|x|^{2m-1-n})\,, \qquad -\Delta E = 0(|x|^{2m-2-n})\,(|x|\to 0)\,, \qquad 2m<n+2\,,$$
$$|\nabla E| = 0(|x|\cdot\log|x|^{-1})\,, \qquad -\Delta E = 0(\log|x|^{-1})\,(|x|\to 0)\,, \qquad 2m=n+2\,,$$

we have, for $\varrho>0$ small enough,

$$|\nabla E| \leqslant - C\cdot\varrho\cdot\Delta E \qquad on\ B_\varrho(0)\,.$$

From lemma 1.1 then follows

$$\begin{aligned}|\nabla u_\delta| &\leqslant |\nabla E * \mu_{\delta,\varrho}| + |\nabla l_{\delta,\varrho}|\,,\\ &\leqslant - C\cdot\varrho\cdot\Delta E * \mu_{\delta,\varrho} + |\nabla l_{\delta,\varrho}|\,,\\ &\leqslant - C\cdot\varrho\cdot\overline{\Delta u_\delta} + C\cdot\varrho\cdot\Delta l_{\delta,\varrho} + |\nabla l_{\delta,\varrho}|\,,\\ &\leqslant - C\cdot\varrho\cdot\overline{\Delta u_\delta} + C_\varrho\cdot\|u_\delta\|_{m,2} \qquad on\ B_\varrho\,. \qquad \blacksquare\end{aligned}$$

Now we can give the following upper bound for $-\overline{\Delta_t u_\delta}$ on supp $(\mu_\delta|B_\varrho)$,

LEMMA 2.5.

$$-\overline{\Delta_t u_\delta} \leqslant - C\cdot\varrho\cdot\overline{\Delta u_\delta} + C_\varrho\cdot(\|\Psi'\|_{2,\infty} + \|u_\delta\|_{m,2}) \qquad on\ \mathrm{supp}\,(\mu_\delta|B_\varrho)$$

for $\varrho>0$ *sufficiently small.*

PROOF. Using successively lemma 2.2, 2.4, 1.8 and (2.3) we obtain

$$\begin{aligned}-\Delta_t u_\delta &= \overline{Lu_\delta} + (L^0 - L)\,u_\delta + (-\overline{\Delta_t u_\delta} - \overline{L^0 u_\delta})\\ &\leqslant C\cdot\|\Psi'\|_{2,\infty} + C\cdot\|b_i\|_\infty\cdot|\nabla u_\delta| + C\cdot\|a_{ij}\|_{\infty,B_\varrho}\cdot\overline{|\nabla^2 u_\delta|}\\ &\leqslant C\cdot\|\Psi'\|_{2,\infty} - C\cdot\|b_i\|_\infty\cdot\varrho\cdot\overline{\Delta u_\delta} - C\cdot\varrho\cdot\overline{\Delta u_\delta} + C_\varrho\cdot\|u_\delta\|_{m,2}\\ &\leqslant C\cdot\|\Psi'\|_{2,\infty} - C\cdot\varrho\cdot\overline{\Delta u_\delta} + C_\varrho\cdot\|u_\delta\|_{m,2} \qquad on\ \mathrm{supp}(\mu_\delta|B_\varrho)\,. \qquad \blacksquare\end{aligned}$$

Combination of the last result and lemma 2.3 yields

LEMMA 2.6.

$$-\overline{\Delta u_\delta} \leqslant C_\varrho\cdot(\|\Psi'\|_{2,\infty} + \|u_\delta\|_{m,2}) \qquad on\ \mathrm{supp}\,(\mu_\delta|B_\varrho)$$

for $\varrho>0$ *sufficiently small.*

By inserting lemma 2.6 in lemma 1.6 we obtain

LEMMA 2.7.

$$\|\Delta u_\delta\|_{\infty,B_{\varrho/2}} \leqslant C_\varrho \cdot (\|\Psi'\|_{2,\infty} + \|u_\delta\|_{m,2})$$

for $\varrho > 0$ sufficiently small.

COROLLARY.

$$\|\Delta u\|_{\infty,B_{\varrho/2}} \leqslant C_\varrho \cdot (\|\Psi'\|_{2,\infty} + \|u\|_{m,2}),$$

and further $\Delta u \in L^\infty_{\mathrm{loc}}(\Omega)$.

From the corollaries of lemma 2.7, 1.8, 1.9 we infer

THEOREM 2.1. *Let u be the solution of* (VI) *with $\mathbb{K} = \mathbb{K}_{\Psi'}$. Then*

$$u \in H^{2,\infty}_{\mathrm{loc}}(\Omega) \cap H^{m+1,2}_{\mathrm{loc}}(\Omega).$$

In case of $2m = n + 2$ we show again the continuity of the second derivatives of u and need the following modification of lemma 1.11.

LEMMA 2.8. *Let $2m = n + 2$, $\xi^0 \in \operatorname{supp}(\mu)$. Then for a sequence $\{\xi^\varkappa\}_{\varkappa=1}^\infty$ with $\xi^\varkappa \in \operatorname{supp}(\mu)$, $\xi^\varkappa \to \xi^0$ $(\varkappa \to \infty)$ holds*

$$\lim_{\varkappa\to\infty} \overline{\Delta u}(\xi^\varkappa) = \overline{\Delta u}(\xi^0).$$

PROOF. We adapt the proof of lemma 1.11 to the new situation and assume $\xi^0 = 0$. Using the fact $u \in C^0(\Omega)$, the assertion of lemma 1.2 holds and therefore $\xi^0, \xi^\varkappa \in \operatorname{supp}(\mu) \subset [u = \Psi'] \subset \Omega'$. Thus we have for a subsequence $\{\xi^{\varkappa'}\}_{\varkappa'=1}^\infty$

$$|\xi^{\varkappa'}|^{-1} \cdot \xi^{\varkappa'} \to e_t \qquad (\varkappa' \to \infty),$$

where e_t is a tangential unit vector with respect to Ω' at ξ^0. After rotating the coordinate system we can assume that $Q_i = e_i$, $i = 1, \dots, k$ and moreover $e_t = e_1$. Because of $\partial_i(\Phi \circ \theta)(0) = Q_i = e_i$, $i = 1, \dots, k$, the corresponding sequence $\{\tilde{x}^{\varkappa'}\}_{\varkappa'=1}^\infty$, $\tilde{x}^\varkappa := (\Phi \circ \theta)^{-1}(\xi^\varkappa) \in \mathbb{R}^k \times \{0\}^{n-k}$ also satisfies $|\tilde{x}^{\varkappa'}|^{-1} \cdot \tilde{x}^{\varkappa'} \to e_1$ $(\varkappa' \to \infty)$.

Setting $\tilde{\Phi} := \Phi \circ \theta' \circ P_{\tilde{x}'} + \sum_{i=k+1}^{n} P_{\tilde{x}_i} \cdot e_i$ we construct a C^2-diffeomorphism $\tilde{\Phi}\colon B_r(0) \xrightarrow{\sim} U(0) = U(\xi^0)$ of an open neighborhood $U(0)$ for $r > 0$ sufficiently small on account of $\partial_i \tilde{\Phi}(0) = e_i$, $i = 1, \dots, n$ and write $\Phi = \tilde{\Phi}$.

Recalling that $u \in H^{2,\infty}_{\mathrm{loc}}(\Omega)$, from the 2. corollary of lemma 1.1 we deduce the existence of

$$\overline{\partial_{1i}(u\circ\Phi)}(\tilde{x}) := \lim_{\varepsilon\to 0} \partial_{1i}\big((u * \omega_\varepsilon)\circ\Phi\big)(\tilde{x}) , \qquad \tilde{x}\in B_r(0),\ i = 1, \ldots, n ,$$

where differentiating term by term yields

$$\begin{aligned}\overline{\partial_{1i}(u\circ\Phi)} &= \sum_{j,l=1}^{n} \overline{\partial_{jl} u}\circ\Phi\cdot\partial_1\Phi_j\cdot\partial_i\Phi_l + \sum_{j=1}^{n}\partial_j u\circ\Phi\cdot\partial_{1i}\Phi_j\\ &=: \tilde{v}_i + v_i\\ &= \overline{\partial_{1i} u}\circ\Phi + (\tilde{v}_i - \overline{\partial_{1i} u}\circ\Phi) + v_i\\ &= \overline{\partial_{1i} u}\circ\Phi + \zeta_i + v_i .\end{aligned}$$

Here $v_i\in C^0(B_r(0))$ and further $\|\zeta_i\|_{\infty,B_\sigma(0)}\to 0$ $(\sigma\to 0)$, because

$$\|\partial_i\Phi - e_i\|_{\infty,B_\sigma(0)}\to 0 \qquad (\sigma\to 0) ,$$

$i = 1, \ldots, n$. Now we look at the proof of lemma 1.11 and setting

$$z := u\circ\Phi - \Psi'\circ\Phi'\circ P_{\tilde{x}'} ,$$

$$\partial_{1i} z := \overline{\partial_{1i}(u\circ\Phi)} - \partial_{1i}(\Psi'\circ\Phi'\circ P_{\tilde{x}'}) , \qquad i = 1, \ldots, n ,$$

we observe that

$$\lim_{\varkappa\to\infty}\partial_{11} z(\tilde{x}^\varkappa) = \partial_{11} z(0) ,$$

implies

$$\lim_{\varkappa\to\infty}\overline{\partial_{11} u}(\xi^\varkappa) = \overline{\partial_{11} u}(\xi^0) , \qquad \text{resp.}\lim_{\varkappa\to\infty}\overline{\Delta u}(\xi^\varkappa) = \overline{\Delta u}(\xi^0) .$$

As in part. I, II of the proof of lemma 1.11 one shows

$$\lim_{\varkappa\to\infty}\partial_{11} z(\tilde{x}^\varkappa) = \partial_{11} z(0) = 0 .$$

For this we set $z_\varepsilon := z\circ\Phi^{-1} * \omega_\varepsilon\circ\Phi$, where by definition follows

$$\partial_{1i} z_\varepsilon \to \partial_{1i} z \quad \textit{pointwise on } B_r(0)\ (\varepsilon\to 0),\ i = 1, \ldots, n .$$

Furthermore, we notice that $-\partial_{11} z$ consists of a sum of a lower semicontinuous function and functions continuous at 0, $\partial_{1i} z$ consists of a sum of functions continuous at 0, $i = 2, \ldots, n$.

In part II one has to give a suitable decomposition of $-\partial_{11}z$ in a sum of a superharmonic function w_1 and a function w_2 continuous at 0. Considering the term $\overline{\partial_{11}u}\circ\Phi$ *on* $\Phi^{-1}(B_\varrho(\xi^0))$, (1.5) of lemma 1.10 and (1.2) of lemma 1.1 yield

$$\begin{aligned}-\overline{\partial_{11}u}\circ\Phi &= -1/n\cdot\overline{\Delta u}\circ\Phi-\overline{(\partial_{11}-1/n\Delta)u}\circ\Phi\\ &= -1/n\cdot\Delta E*\mu_\varrho\circ\Phi-1/n\cdot\Delta l_\varrho\circ\Phi-\overline{(\partial_{11}-1/n\Delta)u}\circ\Phi\end{aligned}$$

and further, for $\tilde{x}\in\Phi^{-1}(B_{2\varrho}(\xi^0))\subset B_r(0)$, $\varrho>0$ sufficiently small,

$$\begin{aligned}-\Delta E*\mu_\varrho\circ\Phi(\tilde{x}) &= -c_n\cdot\int\log|\Phi(\tilde{x})-y|\,d\mu_\varrho(y)\,,\\ &= -c_n\cdot\int\log|\Phi(\tilde{x})-\Phi(\tilde{y})|\,d\tilde{\mu}_\varrho(\tilde{y})\,,\\ &= -c_n\cdot\int\log|\tilde{x}-\tilde{y}|\,d\tilde{\mu}_\varrho(\tilde{y})-c_n\cdot\int\log\big(|\Phi(\tilde{x})-\Phi(\tilde{y})|/|\tilde{x}-\tilde{y}|\big)\,d\tilde{\mu}_\varrho(\tilde{y})\,.\end{aligned}$$

Here $\tilde{\mu}_\varrho$ denotes the image of μ_ϱ with respect to Φ^{-1}.

Now we have $w_1:=-c_n\cdot\int\log|\tilde{x}-\tilde{y}|\,d\tilde{\mu}_\varrho(\tilde{y})$ superharmonically on $B_r(0)$ and $-\overline{\partial_{11}u}\circ\Phi-w_1$ continuously at 0. Setting

$$w_2:=-\overline{\partial_{11}u}\circ\Phi-w_1-\zeta_1-v_1+\partial_{11}(\Psi'\circ\Phi'\circ P_{\tilde{x}'})$$

which is continuous at 0 we can conclude as in lemma 1.11 and obtain the assertion of lemma 2.8. ■

Replacing lemma 1.11 by lemma 2.8 we repeat the proof of theorem 1.2 and obtain

THEOREM 2.2. *Let* $2m=n+2$ *and* u *be the solution of* (VI) *with* $\mathbb{K}=\mathbb{K}_{\Psi'}$. *Then* $u\in C^2(\Omega)$.

3. – A counterexample for higher regularity of the solution of the obstacle problem.

In this paragraph we construct an obstacle problem for the biharmonic case in the plane, $m=n=2$, where the obstacle Ψ is a quadratic polynomial, $f\equiv 0$, and the boundary data $g\in C^\infty(\bar{\Omega})$. We show that the solution u of (VI) with $\mathbb{K}=\mathbb{K}_\Psi$ satisfies $u\notin H^{3,\infty}_{\mathrm{loc}}(\Omega)$ and notice that the coincidence set $[u=\Psi]$ consists of one line segment compact in Ω. The same construction works for all cases $2m=n+2$, and one can cover the cases

$2m < n + 2$ by introducing the so-called dummy-variables, where of course the set $[u = \Psi]$ is no longer compact in Ω. Our construction relies heavily on a property of the logarithmic capacitary potential of a line segment stated in lemma 3.1 separately. The crucial observation leading to the assertion of the lemma is taken from an auxiliary result of Lewy's paper [8].

Now we set $\Omega_0' := [0, 1] \times \{0\}$ and inspect the following potential carefully at the points $(0, 0)$, $(1, 0)$.

LEMMA 3.1. *Let* $-\log * \nu$ *be the logarithmic capacitary potential with respect to* Ω_0' *Then one has* $\log * \nu \in C^0(\mathbb{R}^2)$, *but* $\log * \nu \notin H^{1,\infty}_{\text{loc}}(\mathbb{R}^2)$.

PROOF. For the existence of the capacitary potential $-\log * \nu$ with $-\log * \nu \equiv 1$ *on* Ω_0', where $\nu \geqslant 0$ *is a positive measure*, $\operatorname{supp}(\nu) \subset \Omega_0'$, see [7, chap. II, § 4]. Note that what we call capacitary potential is there called equilibrium potential. (1)

The continuity of $-\log * \nu$ *on* Ω_0' follows from the fact that each point of Ω_0' is regular in the sense of the Dirichlet problem for the Laplace equation. This is true in particular for the points $(0, 0)$, $(1, 0)$. But we show that the gradient $|\nabla \log * \nu|$ is unbounded in a neighborhood of $(0, 0)$, and because of symmetry the same is true for $(1, 0)$. The proof works by contradiction.

Let us assume that $\log * \nu \in H^{1,\infty}_{\text{loc}}(\mathbb{R}^2)$. Then because of

$$\partial_1 \log * \nu \in C^0(\mathbb{R}^2 - \Omega_0')$$

the restriction $\partial_1 \log * \nu|_{\mathbb{R} \times \{0\}}$ must be bounded on a neighborhood of 0 in $\mathbb{R}^-$. But we show that our assumption leads to

$$-\partial_1 \log * \nu|_{\mathbb{R} \times \{0\}}(x_1) \to \infty \qquad (x_1 \to 0-) ,$$

and therefore to a contradiction.

Setting $w' := -\log * \nu|_{\mathbb{R} \times \{0\}}$, the assumption gives $w' \in H^{1,\infty}_{\text{loc}}(\mathbb{R})$.

On account of $\operatorname{supp}(\nu) \subset [0, 1] \times \{0\}$ we have $w'|_{\mathbb{R}^-} \in C^\infty(-\infty, 0)$ and *on* $(-\infty, 0)$ writing $x = x_1$

$$\begin{aligned} d/dx\, w' &= d/dx - \int \log|x - y|\, d\nu(y) , \\ &= \int (y - x)^{-1}\, d\nu(y) > 0 , \\ d^2/dx^2\, w' &= \int (y - x)^{-2}\, d\nu(y) > 0 . \end{aligned}$$

Thus w' is strictly convex on $(-\infty, 0)$ and further $d/dx\, w' > 0$ strictly

(1) Note that $\log * \nu \in L^\infty_{\text{loc}}(\mathbb{R}^2)$ because of the maximum principle.

increasing on $(-\infty, 0)$. From the mean value theorem it follows on account of $w' \in H^{1,\infty}_{\text{loc}}(\mathbb{R})$ the existence of the derivative from the left at 0,

$$0 < \int y^{-1}\, d\nu(y) = d/dx^{-}\, w'(0) =: a_{-} < \infty\,,$$

and hence the integrability of y^{-1} with respect to ν which is crucial later in the proof. Since $w' \equiv 1$ *on* Ω_0', w' has a derivative from the right at 0,

$$0 = d/dx^{+}\, w'(0) =: a_{+}\,.$$

Now we show by Lewy's conclusions [8] that

$$|a_{-}|,\quad |a_{+}| < \infty\,,\qquad a_{-} \neq a_{+} \ \textit{is not possible}\,.$$

For our considerations we need the following auxiliary function $\lambda \in C^0(\mathbb{R})$,

$$\lambda(t) := \int_0^1 -\log|t - x| / \sqrt{x(1-x)}\, dx\,,\qquad t \in \mathbb{R}\,,$$

and we observe the properties

$$\lambda(t+1) = \lambda(-t)\,,\quad t \in \mathbb{R}\,,\qquad \lambda(t) - \lambda(t+1) \geq 0\,,\quad t \in [0, \infty)\,.$$

By averaging the difference quotients we obtain the following representations of the one-sided derivatives, recalling that

$$t^{-1} \cdot \int_0^t x/\sqrt{x(t-x)}\, dx = \int_0^1 \xi/\sqrt{\xi(1-\xi)}\, d\xi = \pi/2\,,$$

$$\pi/2 \cdot a_{+} = \lim_{t \downarrow 0}\ t^{-1} \cdot \int_0^t 1/\sqrt{x(t-x)}\, dx \int \log|y| - \log|y - x|\, d\nu(y)\,,$$

$$\begin{aligned} -\pi/2 \cdot a_{-} &= \lim_{t \uparrow 0} -t^{-1} \cdot \int_0^{-t} 1/\sqrt{x(-t-x)}\, dx \int \log|y| - \log|y - x|\, d\nu(y) \\ &= \lim_{t \downarrow 0}\ t^{-1} \cdot \int_0^t 1/\sqrt{x(t-x)}\, dx \int \log|y| - \log|y + x|\, d\nu(y)\,. \end{aligned}$$

Subtraction yields

$$\pi/2\cdot(a_+ + a_-) = \lim_{t\downarrow 0} t^{-1}\cdot\int_0^t 1/\sqrt{x(t-x)}\,dx\int -\log|y-x| + \log|y+x|\,d\nu(y)\,,$$

$$= \lim_{t\downarrow 0} t^{-1}\cdot\int_0^1 1/\sqrt{\xi(1-\xi)}\,d\xi\int -\log|y-t\cdot\xi| + \log|y+t\cdot\xi|\,d\nu(y)\,,$$

$$= \lim_{t\downarrow 0} t^{-1}\cdot\int \lambda(y/t) - \lambda(-y/t)\,d\nu(y)\,,$$

$$= \lim_{t\downarrow 0} t^{-1}\cdot\int \lambda(y/t) - \lambda(1+y/t)\,d\nu(y)\,,$$

$$= \lim_{t\downarrow 0} \int (y/t)\cdot\big(\lambda(y/t) - \lambda(1+y/t)\big)\cdot y^{-1}\,d\nu(y)\,,$$

$$= \lim_{t\downarrow 0} \int F(y/t)\cdot y^{-1}\,d\nu(y)\,,$$

recalling that $\operatorname{supp}(\nu)\subset[0,\infty)$ and $\nu(\{0\}) = 0$ on account of $-\log * \nu(0,0) = 1 < \infty$. Considering the function $F\in C^0(\mathbf{R})$,

$$F(s) := s\cdot\big(\lambda(s) - \lambda(1+s)\big)\,,\qquad s\in\mathbf{R}\,,$$

we notice that $F|_{\mathbf{R}^+}\geqslant 0$, moreover $F|_{\mathbf{R}^+}\in L^\infty(0,\infty)$ because of

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty}\int_0^1 s\cdot\log|(1+s-x)/(s-x)|/\sqrt{x(1-x)}\,dx\,,$$

$$= \int_0^1 1/\sqrt{x(1-x)}\,dx = \pi\,.$$

Setting $s := y/t$ for fixed $y > 0$ we obtain

$$\lim_{t\downarrow 0} F(y/t) = \lim_{s\to\infty} F(s) = \pi\,,$$

and the theorem on dominated convergence yields

$$\pi/2\cdot(a_- + a_+) = \lim_{r\downarrow 0}\int F(y/t)\cdot y^{-1}\,d\nu(y) = \pi\cdot\int y^{-1}\,d\nu(y) = \pi\cdot a_-\,.$$

So the assumption $w' \in H^{1;\infty}_{\mathrm{loc}}(\mathbb{R})$ implies $a_- = a_+$, but we have also $a_+ = 0 < a_- < \infty$, and this contradiction leads to $a_- = \infty$. Thus lemma 3.1 is proved. ∎

Now we give a sufficient condition that a function $w \in H^{m,2}(\Omega)$ is the solution of an obstacle problem.

LEMMA 3.2. *Let $w \in g + H^{m,2}_0(\Omega) \cap C^0(\bar{\Omega})$ such that $\mu_w := (-\Delta)^m w - f$ is a positive measure,* $\mathrm{supp}(\mu_w) \subset\subset \Omega$. *If for some $\Psi \in C^0(\bar{\Omega})$ we have $\Psi \leqslant w$ and $\Psi = w$ on* $\mathrm{supp}\,(\mu_w)$, *then w is the solution of* (VI) *for* $\mathbb{K} = \mathbb{K}_\Psi$.

PROOF. Let $v \in \mathbb{K}_\Psi$ be arbitrary. By definition of inequalities in Sobolev spaces there exists a sequence $\{\varphi_\varkappa\}_{i=1}^\infty$ such that $\varphi_\varkappa \in C^m(\bar{\Omega})$, $\varphi_\varkappa \geqslant \Psi$ *on* Ω, $\varphi_\varkappa \to v$ *in* $H^{m,2}(\Omega)$ $(\varkappa \to \infty)$. Choosing $\chi \in C^\infty_0(\Omega)$, $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ *on a neighborhood U of* $\mathrm{supp}\,(\mu_w)$, and setting $v_\varkappa := (1-\chi)\cdot v + \chi \cdot \varphi_\varkappa \in \mathbb{K}_\Psi$ we have $v_\varkappa \to v$ *in* $H^{m,2}(\Omega)$ $(\varkappa \to \infty)$, and because of $v_\varkappa \in C^0(U)$ that

$$\langle (-\Delta)^m w - f,\, w - v_\varkappa \rangle = \langle \mu_w,\, w - v_\varkappa \rangle = \int w - v_\varkappa \, d\mu_w = \int \Psi - v_\varkappa \, d\mu_w \leqslant 0\,.$$

This yields

$$\begin{aligned}\langle (-\Delta)^m w - f,\, w - v \rangle = \langle (-\Delta)^m w - f,\, w - v_\varkappa \rangle \\ + \big(\Delta^{m/2} w,\, \Delta^{m/2}(v_\varkappa - v)\big) + (f,\, v_\varkappa - v)\,, \\ \leqslant \big(\|w\|_{m,2} + \|f\|_p\big)\cdot \|v_\varkappa - v\|_{m,2}\,,\end{aligned}$$

where $(\cdot\,,\,\cdot)$ denotes the scalar product in $L^2(\Omega)$. Letting $\varkappa \to \infty$ the right-hand side of the inequality becomes arbitrarily small.

Hence the assumption of lemma 3.2 follows because $w \in \mathbb{K}_\Psi$ is trivial. ∎

EXAMPLE 3.1. *Let $n = m = 2$, $\Omega := B_2(0)$, and $f \equiv 0$. There exists a quadratic polynomial Ψ such that the solution u of* (VI) *with $\mathbb{K} = \mathbb{K}_\Psi$ satisfies $u \notin H^{3,\infty}_{\mathrm{loc}}(\Omega)$ and $[u = \Psi] \subset\subset \Omega$ is a line segment.*

CONSTRUCTION. We consider $E * \nu$, *ν the capacity measure* from lemma 3.1 and have $-\Delta E * \nu = -c \cdot \log * \nu \in C^0(\mathbb{R}^2)$. Furthermore we are in the situation that all lemmata of § 1 are available, which are independent of the obstacle, and obtain by lemma 1.10 iv) that

$$E * \nu \in C^2(\mathbb{R}^2)\,.$$

Now we have to construct a suitable obstacle Ψ. Therefore we inspect the restriction $b' := E * \nu|_{\mathbb{R}\times\{0\}}$ and infer from the identity

$$\partial_{11} E(x) = c_0 \cdot \big(\log|x| + x_1^2/|x|^2 - \tfrac{1}{2}\big)\,, \qquad x \in \mathbb{R}^2\,,$$

for the second derivative

$$d^2/dx_1^2\, b' = c_0 \cdot (\log|\cdot| + \tfrac{1}{2}) * \nu|_{\mathbb{R}\times\{0\}},$$
$$= c_0 \cdot \log|\cdot| * \nu + c_1, \quad c_0, c_1 > 0 .$$

Because $-\log * \nu$ is the logarithmic capacitary potential with respect to Ω_0' we have

$$d^2/dx_1^2\, b' \equiv -c_0 \cdot 1 + c_1 =: c_2 \quad on \ \Omega_0',$$

and moreover by the maximum principle on account of $c_0 > 0$

$$(3.1) \qquad -d^2/dx_1^2\, b' = -c_0 \cdot \log|\cdot| * \nu|_{\mathbb{R}\times\{0\}} - c_1 \leqslant -c_2 \quad on \ \mathbb{R} .$$

Thus b' coincides with a quadratic polynomial P' with respect to the x_1-variable on Ω_0', where $d^2/dx_1^2\, P' \equiv c_2$. In order to identify the linear term of P' we notice that b' is symmetric with respect to $\frac{1}{2}$, hence

$$b' \equiv P' := b'(\tfrac{1}{2}) + c_2/2 \cdot (x_1 - \tfrac{1}{2})^2 \quad on \ \Omega_0' .$$

On account of (3.1) we have $d^2/dx_1^2\, b' \geqslant c_2 = d^2/dx_1^2\, P'$ *on* $\mathbb{R}$, and infer from the mean value theorem

$$b' \geqslant P' \quad on \ \mathbb{R} .$$

Defining the quadratic polynomial Ψ,

$$\Psi(x_1, x_2) := P'(x_1) - \|E * \nu\|_{2,\infty,\Omega} \cdot x_2^2 ,$$

we have $\partial_{22} E * \nu \geqslant \partial_{22} \Psi$ *on* Ω. We notice that $\partial_2 E * \nu = \partial_2 \Psi = 0$ *on* $\mathbb{R}\times\{0\}$ because of symmetry. Thus, again the mean value theorem gives

$$E * \nu \geqslant \Psi \quad on \ \Omega .$$

Choosing $\chi \in C_0^\infty(\Omega)$, $\chi \equiv 1$ *on* $U(\Omega_0')$, we set

$$g := (1-\chi) \cdot E * \nu \in C^\infty(\bar{\Omega}) ,$$

and notice that $E * \nu - g \in H_0^{2,2}(\Omega)$. By construction one has

$$E * \nu \geqslant \Psi \quad on \ \Omega , \qquad E * \nu = \Psi \quad on \ \Omega_0' ,$$

where $\mu := \Delta^2 E * \nu = \nu \geqslant 0$ is a positive measure with $\mathrm{supp}\,(\nu) \subset \Omega_0' \subset\subset \Omega$.

Therefore, we know from lemma 3.2 that $E * \nu$ solves (VI) with $\mathbf{K} = \mathbf{K}_{\Psi}$, that means $E * \nu = u$ in our terminology. The construction is complete. [2] ∎

4. – Addendum.

1) The regularity of the solutions of polyharmonic quasi-variational inequalities has been studied in [11], where we have stated our regularity results for the first time giving a proof different in some details.

2) Obviously we can obtain theorem 1.1, 2.1 under the assumption that $\Psi \in C^0(\bar{\Omega}) \cap H^{2,\infty}_{\mathrm{loc}}(\Omega)$ resp. $\Psi' \in C^0(\overline{\Omega'}) \cap H^{2,\infty}_{\mathrm{loc}}(\Omega')$ by a suitable approximation. Furthermore, the assumption $f \in L^p_{\mathrm{loc}}(\Omega)$ is sufficient to obtain all our results.

3) On account of the lower semicontinuity of the solution which is stated in lemma 1.1 independent of the obstacle, all results remain true under the assumption that $\Psi \in H^{2,\infty}_{\mathrm{loc}}(\Omega_0)$ resp. $\Psi' \in H^{2,\infty}_{\mathrm{loc}}(\Omega'_0)$, where $\Omega_0 \subset \Omega$ is an open set, $\Omega'_0 := \Omega' \cap \Omega_0$, and

$$\Psi(x) \to -\infty \;\; (\Omega_0 \ni x \to \partial\Omega_0) \qquad \text{resp. } \Psi'(x) \to -\infty \;\; (\Omega'_0 \ni x \to \partial\Omega'_0)\,.$$

Therefore, Ψ resp. Ψ' admits the natural extension

$$\Psi := -\infty \quad on\ \Omega - \Omega_0 \qquad \text{resp.} \qquad \Psi' := -\infty \quad on\ \Omega' - \Omega'_0\,.$$

4) Another application of our methods yields $H^{2,\infty}_{\mathrm{loc}}$-regularity of the solution in case of a thick obstacle $\Psi \in C^0(\bar{\Omega})$ under the assumption that $-\Delta_k \Psi \leqslant C_{\Psi} < \infty$ in the sense of distributions, $k \geqslant n - 2m + 3$. Corresponding results can be proved in case of thin obstacles.

5) We state theorem 2.1, 2.2 for Ω' being a k-dimensional C^2-surface with $\partial\Omega' \subset \partial\Omega$, $k \geqslant n - 2m + 3$. Of course this assumption includes compact k-dimensional C^2-surfaces $\Omega' \subset\subset \Omega$ with $\partial\Omega' = \emptyset$. Using our methods of proof one can also show the regularity results for $k+1$-dimensional compact C^2-surfaces $\Omega' \subset\subset \Omega$ where the boundary $\partial\Omega'$ is a k-dimensional C^2-surface.

6) Finally, we arrive at the case of a two-sided thin obstacle. Recalling that the maximum principle of potential theory is only valid for potentials of positive measures, it is natural to treat two-sided problems only for biharmonic variational inequalities where we can use the maximum principle for harmonic functions. Now we consider the two-sided thin obstacle

(2) Finally, lemma 3.1 shows that $\Delta u = \log * \nu \notin H^{1,\infty}_{\mathrm{loc}}(\Omega)$, that means $u \notin H^{3,\infty}_{\mathrm{loc}}(\Omega)$.

problem in two dimensions. We assume that Ω' is a C^4-curve with $\partial\Omega' \subset \partial\Omega$ where the case of $\Omega' \subset\subset \Omega$ being a closed curve is included. Now let u *be the solution of* (VI) *with* $\mathbb{K} = \mathbb{K}_{\Psi', \Upsilon'} \neq \emptyset$,

$$\mathbb{K}_{\Psi', \Upsilon'} := \{v \in g + H_0^{2,2}(\Omega) \colon \Psi' \leqslant v \leqslant \Upsilon' \text{ on } \Omega' \text{ in } H^{2,2}(\Omega)\} ,$$

where Ψ', $\Upsilon' \in C^2(\overline{\Omega'})$, $[\Psi' = \Upsilon'] \subset\subset \Omega$. Elaborating our methods of proof we show in [12] that

$$u \in H^{2,\infty}_{\mathrm{loc}}(\Omega) .$$

Immediately the conclusions leading to theorem 2.2 yield

$$u \in C^2(\Omega - [\Psi' = \Upsilon']) .$$

In order to obtain C^2-regularity even on a neighborhood of $[\Psi' = \Upsilon'] \subset\subset \Omega$, we have to assume the following regularity condition

(D) *All points of* $[\Psi' = \Upsilon']$ *are regular relative to* $\Omega - [\Psi' = \Upsilon']$ *with respect to the Dirichlet problem for the Laplace equation.*

Under the assumption (D) we show in [12] by techniques which are different from the proof of theorem 2.2 that

$$u \in C^2(\Omega) .$$

5. – Appendix: The formula of the fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^n$.

Let $E := E_{m,n}$ be the fundamental solution of $(-\Delta)^m$ *in* $\mathbb{R}^n$, $(-\Delta)^m E = \delta$, δ Dirac measure. We give the explicit formula for $E_{m,n}$ and the derivatives up to the second order. Considering $E_{m,n}$ only for n, $m \geqslant 2$, $2m \leqslant n + 2$, we have four different cases. By $c_{m,n} > 0$, we denote constants depending only on n, m.

1) *case*: $2m = n + 2$

$$E_{m,n}(x) = c_{m,n} \cdot |x|^2 \cdot \big(\log|x| - (2 + n)/(2n)\big) ,$$

$$\partial_i E_{m,n}(x) = c_{m,n} \cdot 2 \cdot \big(x_i \cdot \log|x| - 1/n \cdot x_i\big) , \quad i = 1, \ldots, n ,$$

$$\partial_{ij} E_{m,n}(x) = c_{m,n} \cdot 2 \cdot x_i \cdot x_j \cdot |x|^{-2} , \quad i \neq j,\ i, j = 1, \ldots, n ,$$

$$\partial_{ii} E_{m,n}(x) = c_{m,n} \cdot 2 \cdot \big(\log|x| + x_i^2/|x|^2 - 1/n\big) , \quad i = 1, \ldots, n ,$$

$$- \Delta_k E_{m,n}(x) = c_{m,n} \cdot 2 \cdot \Big(k \cdot (-\log|x|) - \sum_{i=1}^{k} x_i^2 \Big/ \sum_{i=1}^{n} x_i^2 + k/n\Big) ,$$

$$\begin{aligned}
- \Delta E_{m \cdot n}(x) &= c_{m,n} \cdot 2 \cdot n \cdot (-\log|x|) , \\
&= c_{m,n} \cdot 2 \cdot (2m-2) \cdot (-\log|x|) , \\
&= c_{m-1,n} \cdot (-\log|x|) , \\
&= E_{m-1,n}(x) .
\end{aligned}$$

2) *case:* $2m = n + 1$

$$\begin{aligned}
E_{m,n}(x) &= c_{m,n} \cdot (-|x|) , \\
\partial_i E_{m,n}(x) &= c_{m,n} \cdot (-1) \cdot x_i \cdot |x|^{-1} , \qquad i = 1, \ldots, n , \\
\partial_{ij} E_{m,n}(x) &= c_{m,n} \cdot x_i \cdot x_j \cdot |x|^{-3} , \qquad i \neq j, \ i, j = 1, \ldots, n , \\
\partial_{ii} E_{m,n}(x) &= c_{m,n} \cdot (-1) \cdot \big(1 - x_i^2/|x|^2\big) \cdot |x|^{-1} , \qquad i = 1, \ldots, n , \\
- \Delta_k E_{m,n}(x) &= c_{m,n} \cdot \Big(k - \sum_{i=1}^{k} x_i^2 \Big/ \sum_{i=1}^{n} x_i^2\Big) \cdot |x|^{-1} , \\
- \Delta E_{m,n}(x) &= c_{m,n} \cdot (n-1) \cdot |x|^{-1} , \\
&= c_{m,n} \cdot (2m-2) \cdot |x|^{-1} , \\
&= c_{m-1,n} \cdot |x|^{-1} , \\
&= E_{m-1,n}(x) .
\end{aligned}$$

3) *case:* $2m = n$

$$\begin{aligned}
E_{m,n}(x) &= c_{m,n} \cdot (-\log|x|) , \\
\partial_i E_{m,n}(x) &= c_{m,n} \cdot (-1) \cdot x_i \cdot |x|^{-2} , \qquad i = 1, \ldots, n , \\
\partial_{ij} E_{m,n}(x) &= c_{m,n} \cdot 2 \cdot x_i \cdot x_j \cdot |x|^{-4} , \qquad i \neq j, \ i, j = 1, \ldots, n , \\
\partial_{ii} E_{m,n}(x) &= c_{m,n} \cdot (-1) \cdot \big(1 - 2 \cdot x_i^2/|x|^2\big) \cdot |x|^{-2} , \qquad i = 1, \ldots, n , \\
- \Delta_k E_{m,n}(x) &= c_{m,n} \cdot \Big(k - 2 \cdot \sum_{i=1}^{k} x_i^2 \Big/ \sum_{i=1}^{n} x_i^2\Big) \cdot |x|^{-2} , \\
- \Delta E_{m,n}(x) &= c_{m,n} \cdot (n-2) \cdot |x|^{-2} , \\
&= c_{m,n} \cdot (2m-2) \cdot |x|^{-2} , \\
&= c_{m-1,n} \cdot |x|^{-2} , \\
&= E_{m-1,n}(x) .
\end{aligned}$$

4) *case:* $2m < n$

$$E_{m,n}(x) = c_{m,n} \cdot |x|^{2m-n} ,$$

$$\partial_i E_{m,n}(x) = c_{m,n} \cdot (2m-n) \cdot x_i \cdot |x|^{2m-2-n} , \quad i = 1, \ldots, n ,$$

$$\partial_{ij} E_{m,n}(x) = c_{m,n} \cdot (2m-n) \cdot (2m-2-n) \cdot x_i \cdot x_j \cdot |x|^{2m-4-n} ,$$

$$i \neq j, \; i, j = 1, \ldots, n ,$$

$$\partial_{ii} E_{m,n}(x) = c_{m,n} \cdot (2m-n) \cdot \big(1 + (2m-2-n) \cdot x_i^2/|x|^2\big) \cdot |x|^{2m-2-n} ,$$

$$i = 1, \ldots, n ,$$

$$-\Delta_k E_{m,n}(x) = c_{m,n} \cdot (n-2m) \cdot \Big(k + (2m-2-n) \cdot \sum_{i=1}^{k} x_i^2 \Big/ \sum_{i=1}^{n} x_i^2\Big) \cdot |x|^{2m-2-n} ,$$

$$-\Delta E_{m,n}(x) = c_{m,n} \cdot (n-2m) \cdot (2m-2) \cdot |x|^{2m-2-n} ,$$

$$= c_{m-1,n} \cdot |x|^{2m-2-n} ,$$

$$= E_{m-1,n}(x) .$$

Note that for $m \geqslant 1$, $n \geqslant 2$ $E_{m,n}$ is superharmonic in case of $2m \leqslant n$, continuous in case of $2m > n$.

Added in proof.

1) To 6) of the Addendum: In case of a two-sided thin obstacle let us now replaec the assumption (D) by the stronger one

$$(F) \qquad [\Psi' = \Upsilon'] = \bigcup_{i=1}^{i_0} \Omega_i'$$

where $\Omega_i' \subset \Omega'$ is a nondegenerated subcurve with endpoints $\zeta^{i,1}, \zeta^{i,2} \in \Omega'$, $i = 1, \ldots, i_0$. Further we assume $\Psi', \Upsilon' \in C^4(\bar{\Omega}')$ and the monotonicity condition (M) that for $i = 1, \ldots, i_0$, $j = 1, 2$,

(M) *d^4/ds^4 $\Upsilon' - \Psi'$ is (not necessarily strictly) increasing resp. decreasing on a neighborhood of $\zeta^{i,j}$*

where s denotes the arc length of Ω'. Then under the assumptions (F) and (M) we show in [12] carefully analysing the behaviour of Δu on a neighborhood of $[\Psi' = \Upsilon'] \subset\subset \Omega$ the stronger result

$$u \in C^2(\Omega) \cap H^{3,2}_{\mathrm{loc}}(\Omega) .$$

The work [12] will appear in the series « Bonner Mathematische Schriften », Bonn (1984).

2) For a general discussion of thin obstacle problems see the recent book of A. Friedman, *Variational principles and free-boundary problems*, New York, Wiley (1982).

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