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The Singular Set of the Minima of Certain Quadratic Functionals (*).

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Let Ω be an open set in $\mathbb{R}^n$ and let $A_{ij}^{\alpha\beta}(x, u)$ be continuous functions in $\Omega \times \mathbb{R}^n$ satisfying

$$|A_{ij}^{\alpha\beta}(x, u)| \leqslant M$$

$$\sum_{\alpha,\beta=1}^{n} \sum_{ij=1}^{N} A_{ij}^{\alpha\beta}(x, u)\,\xi_\alpha^i \xi_\beta^j \geqslant |\xi|^2 \qquad \forall \xi \in \mathbb{R}^{nN}.$$

Denote by F the functional

$$F(u; E) = \int_E A_{ij}^{\alpha\beta}(x, u)\, D_\alpha u^i D_\beta u^i\, dx . \tag{1}$$

A function $u\colon \Omega \to \mathbb{R}^N$ is a local minimum of F in Ω if for every φ with compact support in Ω we have

$$F(u; \operatorname{spt}\varphi) \leqslant F(u + \varphi; \operatorname{spt}\varphi) .$$

We have proved in [2] that every local minimum $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is Hölder-continuous in an open subset $\Omega_0 \subset \Omega$. Moreover, the singular set $\Sigma = \Omega - \Omega_0$, which is in general non empty (see e.g. [4]), has Hausdorff dimension strictly less than $n - 2$.

The question can be raised whether the dimension of Σ is $n - 3$ or less.

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In this paper we give a partial answer to this question. More precisely, in the case in which the coefficients are of the form

$$A_{ij}^{\alpha\beta}(x, u) = g_{ij}(x, u)\, G^{\alpha\beta}(x); \qquad G^{\alpha\beta} = G^{\beta\alpha}, \tag{2}$$

we prove that the singular set Σ of a bounded local minimum u has Hausdorff dimension not greater than $n-3$ (theorem 2), and that in dimension $n=3$ it consists at most of isolated points (theorem 1).

We note that, although of particular type, quadratic functionals with coefficients given by (2) are of interest in the theory of harmonic mappings of Riemannian manifolds.

The methods used in the proof follow closely those developed in the theory of minimal surfaces, see for example [1] [3].

Let us start with the following lemma.

LEMMA 1. *Let $A^{\nu}(x, z) = A_{ij}^{\alpha\beta(\nu)}(x, z)$ be a sequence of continuous functions in $B \times \mathbb{R}^N$ (B is the unit ball in $\mathbb{R}^n$) converging uniformly to $A(x, z)$ and satisfying the inequalities*

$$|A^{(\nu)}(x, z)| \leqslant M \tag{3}$$

$$A^{(\nu)}\xi\cdot\xi = A_{ij}^{\alpha\beta(\nu)}(x, z)\,\xi_\alpha^i \xi_\beta^j \geqslant |\xi|^2 \qquad \forall \xi \in \mathbb{R}^{nN} \tag{4}$$

$$|A^{(\nu)}(x, z) - A^{(\nu)}(x', z')| \leqslant \omega\big(|x-x'|^2 + |z-z'|^2\big) \tag{5}$$

where $\omega(t)$ is a bounded continuous concave function with $\omega(0)=0$. For each $\nu = 1, 2, \ldots$ let $u^{(\nu)}$ be a local minimum in B for the functional

$$F^{(\nu)}(u^{(\nu)}) = \int A^{(\nu)}(x, u^{(\nu)})\, Du^{(\nu)} Du^{(\nu)}\, dx$$

and suppose that $u^{(\nu)} \to v$ weakly in $L^2(B; \mathbb{R}^N)$.

Then v is a local minimum in B for the functional

$$F(v) = \int A(x, v)\, Dv\, Dv\, dx\,.$$

Moreover, if x_ν is a singular point for $u^{(\nu)}$, and $x_\nu \to x_0$, then x_0 is a singular point for v.

PROOF. We begin by recalling some results from [2]. For each ball $B_r = B_r(x_0) \subset B$ we have

$$\int_{B_{r/2}} |Du^{(\nu)}|^2\, dx \leqslant k_1 r^{-2} \int_{B_r} |u^{(\nu)} - u_r^{(\nu)}|^2\, dx \tag{6}$$

where

$$u_r^{(\nu)} = \fint_{B_r} u^{(\nu)}\,dx = \frac{1}{|B_r|}\int_{B_r} u^{(\nu)}\,dx\,.$$

Moreover there exists a $q > 2$, independent of ν, such that

$$\left(\fint_{B_{r/2}} |Du^{(\nu)}|^q\,dx\right)^{1/q} \leqslant \gamma_2\left(\fint_{B_r} |Du^{(\nu)}|^2\,dx\right)^{1/2}. \tag{7}$$

It follows from (6) and (7) that $Du^{(\nu)} \in L^q_{\mathrm{loc}}(B)$ and that for every $R < 1$

$$\int_{B_R(0)} |Du^{(\nu)}|^q\,dx \leqslant c(R) \tag{8}$$

where $c(R)$ is independent of ν.

The above inequality and the weak L^2 convergence of $u^{(\nu)}$ imply that for every $R < 1$ we have

$$\begin{cases} u^{(\nu)} \to v & \text{strongly in } L^2(B_R) \\ Du^{(\nu)} \to Dv & \text{weakly in } L^q(B_R)\,. \end{cases} \tag{9}$$

Passing possibly to a subsequence we may suppose that $u^{(\nu)} \to v$ a.e. in B. We first show that

$$F(v;\, B_R) \leqslant \liminf_{\nu\to\infty} F^{(\nu)}(u^{(\nu)};\, B_R)\,. \tag{10}$$

We have actually

$$F^{(\nu)}(u^{(\nu)};\, B_R) = \int_{B_R} A(x, v)\, Du^{(\nu)} Du^{(\nu)}\,dx + \int_{B_R} [A^{(\nu)}(x, u^{(\nu)}) - A(x, v)]\, Du^{(\nu)} Du^{(\nu)}\,dx$$

and hence from (8)

$$F^{(\nu)}(u^{(\nu)};\, B_R) \geqslant \int_{B_R} A(x, v) Du^{(\nu)} Du^{(\nu)}\,dx - c(R)^{2/q}\left(\int_{B_R} |A^{(\nu)}(x, u^{(\nu)}) - A(x, v)|^{q/(q-2)}\,dx\right)^{1-2/q}.$$

When $\nu \to \infty$, the last term on the right-hand side tends to zero, whereas the first is lower semi-continuous. This proves (10).

Let now w be an arbitrary function coinciding with v outside B_R, and let $\eta(x)$ be a C^1 function in B, with $0\leqslant\eta\leqslant1$ $\eta=0$ in B_ϱ $(\varrho<R)$ and $\eta=1$ outside B_R. The function

$$v^{(\nu)}=w+\eta(u^{(\nu)}-v)$$

coincides with $u^{(\nu)}$ outside B_R, and therefore

$$F^{(\nu)}(u^{(\nu)};B_R)\leqslant F^{(\nu)}(v^{(\nu)};B_R)\,. \tag{11}$$

Taking (3) (8) into account we get

$$F^{(\nu)}(v^{(\nu)};B_R)\leqslant\int\limits_{B_R}A^{(\nu)}(x,v^{(\nu)})DwDw\,dx+\gamma_3(R)\|\eta\|_{q/(q-2),R} \\ +\gamma_4(R,\eta)\|u^{(\nu)}-v\|_{2,R}\big(1+\|u^{(\nu)}-v\|_{2,R}\big)\,.$$

Letting $\nu\to\infty$, we get from (9) (10) and (11)

$$F(v;B_R)\leqslant\liminf_{\nu\to\infty}F^{(\nu)}(v^{(\nu)};B_R)\leqslant F(w;B_R)+\gamma_3\|\eta\|_{q/(q-2),R}\,.$$

Taking ϱ close to R the last term can be made arbitrarily small, thus proving the first assertion of the lemma.

In order to conclude the proof, we recall that a point $\bar{x}$ is singular if and only if

$$\liminf_{\varrho\to0}\varrho^{2-n}\int\limits_{B_\varrho(\bar{x})}|Du|^2dx\geqslant\varepsilon_0$$

(see [2], theorem 5.1), where ε_0 depends only on ω and therefore is independent of ν. It follows from (6) that $\bar{x}$ is a singular point for u if and only if

$$\liminf_{\varrho\to0}\varrho^{-n}\int\limits_{B_\varrho(\bar{x})}|u-u_R|^2\,dx\geqslant\varepsilon_1\,.$$

Suppose now that x_0 is a regular point for v, and let $x_\nu\to x_0$. Let $\varrho>0$ be such that $B_\varrho(x_0)\subset B_R\subset B$ and

$$\varrho^{-n}\int\limits_{B_\varrho(x_0)}|v-v_\varrho|^2\,dx<\varepsilon_1\,.$$

We have from (9)

$$\lim_{\nu\to0}\varrho^{-n}\int\limits_{B_\varrho(x^{(\nu)})}|u^{(\nu)}-u_\varrho^{(\nu)}|^2dx=\varrho^{-n}\int\limits_{B_\varrho(x_0)}|v-v_\varrho|^2\,dx<\varepsilon_1$$

and therefore $x^{(\nu)}$ is regular for $u_{(\nu)}$, provided ν is sufficiently large. This concludes the proof of the lemma. q.e.d.

The next step is the proof of a monotonicity result. The reader will recognize here a strict similarity with the theory of minimal surfaces.

We need for this lemma to restrict ourselves to the special form of the coefficients in (2), namely:

$$A_{ij}^{\alpha\beta}(x, u) = g_{ij}(x, u)\, G^{\alpha\beta}(x); \qquad G^{\alpha\beta} = G^{\beta\alpha}. \tag{12}$$

It is easily seen that we may assume without loss of generality that

$$G^{\alpha\beta}(0) = \delta_{\alpha\beta}\,. \tag{13}$$

We shall suppose of course that the coefficients A satisfy (3), (4) and (5). Moreover we assume that

$$\int_0^1 \frac{\omega(t^2)}{t}\, dt < +\infty\,. \tag{14}$$

LEMMA 2. *Let u be a local minimum of F in B, with coefficients A given by* (12) *and satisfying* (3), (4), (5), (13), (14). *Then for every ϱ, R with $0 < \varrho < R < 1$ we have*

$$\int_{\partial B} |u(Rx) - u(\varrho x)|^2\, dH_{n-1}(x) \leqslant \gamma_6 \log \frac{R}{\varrho}\, [\Phi(R) - \Phi(\varrho)] \tag{15}$$

where

$$\Phi(t) = t^{2-n} \exp\left(\gamma_5 \int_0^t \frac{\omega(s^2)}{s}\, ds\right) \int_{B_t} A(x, u)\, Du\, Du\, dx\,. \tag{16}$$

REMARK. This lemma is the only place in which we need the special form (12) of the coefficients. Any extension of the lemma to a more general class of coefficients will therefore imply an immediate extension of the results of this paper.

PROOF. For $t < 1$ let $x_t = t(x/|x|)$ and $u_t(x) = u(x_t)$. We have

$$F(u_t; B_t) = \tag{17}$$
$$= \int_{B_t} A_{ij}^{\alpha\beta}(x, u(x_t))\, \frac{t^2}{|x|^2} \left(\delta_{\alpha h} - \frac{x_\alpha x_h}{|x|^2}\right) \left(\delta_{\beta k} - \frac{x_\beta x_k}{|x|^2}\right) D_h u^i(x_t)\, D_k u^j(x_t)\, dx\,.$$

We write now

$$A(x, u(x_t)) = A(0, u(x_t)) + [A(x, u(x_t)) - A(0, u(x_t))] \tag{18}$$

so that the integral on the right-hand side of (17) splits naturally into two parts: $F = F_1 + F_2$.

The first integral can be easily transformed by observing that for every f we have

$$\int_{B_t} |x|^{-2} f(x_t)\, dx = \frac{t^{-1}}{n-2} \int_{\partial B_t} f(x)\, dH_{n-1}(x)\,.$$

We get therefore

$$F_1 = \frac{t}{n-2}\left\{ \int_{\partial B_t} A(0, u)\, Du\, Du\, dH_{n-1}(x) - \int_{\partial B_t} A_{ij}^{\alpha\beta} \frac{x_\alpha x_h}{|x|^2}\left[2\delta_{\beta k} - \frac{x_\beta x_k}{|x|^2}\right] D_h u^i D_k u^j\, dH_{n-1}(x)\right\}.$$

Taking into account the special form (12) of the coefficients and inequality (4), we conclude that

$$F_1 \leqslant \frac{t}{n-2}\left\{ \int_{\partial B_t} A(0, u)\, Du\, Du\, dH_{n-1}(x) - \int_{\partial B_t} \frac{|\langle x, Du\rangle|^2}{|x|^2}\, dH_{n-1}(x)\right\}$$

where $\langle x, Du\rangle = x_\alpha D_\alpha u$, and in conclusion

$$F_1 \leqslant \frac{t}{n-2}\left\{ \int_{\partial B_t} A(x, u)\, Du\, Du\, dH_{n-1}(x) - \int_{\partial B_t} \frac{|\langle x, Du\rangle|^2}{|x|^2}\, dH_{n-1}(x) + \gamma_7 \omega(t^2) \int_{\partial B_t} A\, Du\, Du\, dH_{n-1}(x)\right\}.$$

In a similar way we estimate the second term

$$F_2 \leqslant \gamma_8\, t\omega(t^2) \int_{\partial B_t} A\, Du\, Du\, dH_{n-1}(x)\,.$$

From the minimality of u we have $F(u, B_t) \leqslant F(u_t, B_t)$ and therefore

$$\int_{B_t} A\, Du\, Du\, dx \leqslant \frac{t}{n-2}\left\{ (1 + \gamma_9 \omega(t^2)) \int_{\partial B_t} A\, Du\, Du\, dH_{n-1}(x) - \int_{\partial B_t} \frac{|\langle x, Du\rangle|^2}{|x|^2}\, dH_{n-1}(x)\right\}. \tag{19}$$

If we set

$$\varphi(t) = t^{2-n}\int_{B_t} A(x, u)\, Du\, Du\, dx$$

we have

$$t^{n-2}\int_{\partial B_t} A(x, u)\, Du\, Du\, dH_{n-1}(x) = \varphi'(t) + (n-2)\,\frac{\varphi(t)}{t}\,.$$

From (19) we get

$$\varphi'(t) + \gamma_5\,\frac{\omega(t^2)}{t}\,\varphi(t) \geqslant \gamma_{10}\, t^{2-n}\int_{\partial B_t} \frac{|\langle x, Du\rangle|^2}{|x|^2}\, dH_{n-1}(x)$$

and hence

$$\Phi'(t) \geqslant \gamma_{10}\, t^{2-n}\int_{\partial B_t} \frac{|\langle x, Du\rangle|^2}{|x|^2}\, dH_{n-1}(x)\,. \tag{20}$$

Integrating (20) we easily obtain

$$\Phi(R) - \Phi(\varrho) \geqslant \gamma_{10}\int_{\varrho}^{R} t^{2-n}\, dt \int_{\partial B_t} \frac{|\langle x, Du\rangle|^2}{|x|^2}\, dx\,. \tag{21}$$

On the other hand

$$|u(Rx) - u(\varrho x)|^2 \leqslant \left(\int_{\varrho}^{R} |\langle x, Du(tx)\rangle|\, dt\right)^2 \leqslant \log\frac{R}{\varrho}\int_{\varrho}^{R} t|\langle x, Du(tx)\rangle|^2\, dt$$

from which the conclusion follows at once integrating on ∂B and taking (21) into account. q.e.d.

We are now ready to prove our first result, dealing with the three-dimensional case.

THEOREM 1. *Let u be a bounded local minimum of the functional F in B and let the conclusion of lemma 2 hold. If $n = 3$, u may have at most isolated singular points.*

PROOF. We first observe that the function $\Phi(t)$ defined by (16) is increasing, and therefore tends to a finite limit when $t \to 0$. Suppose now that u has a sequence of singular points, x_ν, converging to $x_0 = 0$, and let $R_\nu = 2|x_\nu| < 1$. The function

$$u^{(\nu)}(x) = u(R_\nu x)$$

is a local minimum in B for the functional

$$F^{(\nu)}(u^{(\nu)}; B) = \int_B A^{(\nu)}(x, u^{(\nu)}) Du^{(\nu)} Du^{(\nu)} dx$$

with

$$A^{(\nu)}(x, z) = A(R_\nu x, z) .$$

Moreover, each $u^{(\nu)}$ has a singular point y_ν with $|y_\nu| = \frac{1}{2}$. Since the $u^{(\nu)}$ are uniformly bounded, we can suppose (passing to a subsequence) that $u^{(\nu)}$ converge weakly in $L^2(B)$ to some function v and that $y_\nu \to y_0$. The coefficients $A^{(\nu)}(x, u)$ are bounded and uniformly continuous in $\bar{B} \times B_L$ (L being a bound for $|u|$) and hence we may apply lemma 1 and conclude that v is a local minimum for

$$F_0(v; B) = \int_B A(0, v) Dv\, Dv\, dx .$$

Also from lemma 1 it follows that v has a singular point at y_0. Let now $0 < \lambda < \mu < 1$, and let us apply inequality (15) to $\varrho = \lambda R_\nu$ and $R = \mu R_\nu$. We have

$$\int_{\partial B} |u^{(\nu)}(\lambda x) - u^{(\nu)}(\mu x)|^2 dH_{n-1} \leqslant \gamma_6 \log \frac{\mu}{\lambda} [\Phi(\mu R_\nu) - \Phi(\lambda R_\nu)] .$$

If we let $\nu \to \infty$ the right-hand side converges to zero and hence for almost every value of λ and μ we have

$$\int_{\partial B} |v(\lambda x) - v(\mu x)|^2 dH_{n-1} = 0$$

so that v is homogeneous of degree zero.

We may therefore conclude that the whole segment joining 0 with y_0 is made of singular points for v. This contradicts theorem 5.1 of [2] and in particular the conclusion that the set of singular points has dimension strictly less than $3 - 2 = 1$. q.e.d.

We pass now to the general case of arbitrary dimension. The techniques involved follow very closely those introduced for minimal surfaces, see in particular [1].

If $A \subset \mathbb{R}^n$, $0 \leqslant k < \infty$ and $0 < \delta \leqslant +\infty$ we define

$$H_k^\delta(A) = \omega_k 2^{-k} \inf \left\{ \sum_{i=1}^\infty (\operatorname{diam} s_i)^k;\ A \subset \bigcup_{i=1}^\infty s_i;\ \operatorname{diam} s_i < \delta \right\}$$

where $\omega_k = \Gamma(\frac{1}{2})^k/\Gamma(k/2+1)$.

The quantity

$$H_k(A) = \lim_{\delta\to 0} H_k^\delta(A) = \sup_\delta H_k^\delta(A)$$

is the k-dimensional Hausdorff measure of A.

Although H_k and H_k^∞ may be extremely different, it is easily seen that $H_k(A)=0$ if and only if $H_k^\infty(A)=0$. This fact and the results below make H_k^∞ more convenient for our purposes. A first property of H_k^∞ is that for every set $\Sigma \subset \mathbf{R}^n$ we have

$$\limsup_{r\to 0} r^{-k} H_k^\infty(\Sigma \cap B_r(x)) \geqslant \omega_k \cdot 2^{-k} \tag{22}$$

for H_k-almost all $x \in \Sigma$, see [1].

Moreover, if Q, Q_ν $(\nu = 1, 2, \ldots)$ are compact sets and if every open set $A \supset Q$ contains Q_ν for ν sufficiently large, then

$$H_k^\infty(Q) \geqslant \limsup_{\nu\to\infty} H_k^\infty(Q_\nu) \tag{23}$$

see [1] again.

The next result is the analogous of theorem 1 when $n > 3$.

THEOREM 2. *Let u be a bounded local minimum for F in B and suppose that the conclusion of lemma 2 holds. Then the dimension of the singular set Σ of u cannot exceed $n-3$.*

PROOF. Suppose that for some $k > 0$ we have $H_k(\Sigma) > 0$. Then $H_k^\infty(\Sigma) > 0$ and there exists a point x_0, which we may take as the origin, such that (22) holds. Let R_ν be an infinitesimal sequence such that

$$H_k^\infty(\Sigma \cap B_{R_\nu}(0)) \geqslant 2^{-1-k} \omega_k R_\nu^k \tag{24}$$

and let $u^{(\nu)}(x) = u(2R_\nu x)$. Arguing as in the proof of theorem 1, we conclude that a subsequence of $u^{(\nu)}$ converges to a function v homogeneous of degree zero and minimizing locally the functional

$$F_0(v; E) = \int_E A(0, v)\, Dv\, Dv\, dx\,. \tag{25}$$

We note that the coefficients are now independent of x.

If we call $\Sigma^{(\nu)}$ the singular set of $u^{(\nu)}$ we have from (24)

$$H_k^\infty(\Sigma^{(\nu)} \cap B_{1/2}) \geqslant 2^{-1-2k}\omega_k \,. \tag{26}$$

By (23) and lemma 1 the same inequality holds for the singular set Σ of v. Since $k > 0$, there exists a point $x_0 \neq 0$ for which (22) holds, Σ now denoting the singular set of v. We may suppose that $x_0 = (0, 0, \ldots, a)$. We blow up now near x_0 by taking

$$v^{(\nu)}(x) = v(x_0 + R_\nu x) \,.$$

Arguing as above and recalling that the coefficients in F_0 do not depend on x, we arrive to a function w independent of x_n, minimizing F_0 locally in $\mathbb{R}^n$, and whose singular set has positive k-dimensional measure.

The restriction of w to the plane $x_0 = 0$, which we denote again by w, minimizes F_0 locally in $\mathbb{R}^{n-1}$; moreover its singular set Σ satisfies $H_{k-1}(\Sigma) > 0$.

By repeating the procedure we construct for each $s < k$ a local minimum of F_0 in $\mathbb{R}^{n-s}$ whose singularities have positive $(k-s)$-dimensional measure.

Suppose now that $k > n-3$. Taking $s = n-3$ we obtain a local minimum in $\mathbb{R}^3$ whose singular set has positive H_{k-n+3}-measure. This contradicts theorem 1 and therefore proves the assertion. q.e.d.

REMARK. The results of theorem 1 and 2 apply to harmonic mappings of Riemannian manifolds $u\colon M^n \to M^N$, provided that every point of M^n has a neighborhood which is mapped into a bounded co-ordinate chart of M^N. In fact, in that case, if u minimize the energy

$$\mathcal{E}(u) = \int |du|^2$$

a representative of u in local coordinates minimizes locally the functional

$$\mathcal{E}(u; A) = \int_A g_{ij}(u)\, G^{\alpha\beta}(x)\, D_\alpha u^i D_\beta u^i \sqrt{G}\, dx$$

where g_{ij} and $G_{\alpha\beta}$ are the metric tensors of M^N and M^n respectively. We note that no assumption on M^N, involving for example its sectional curvature, is needed.

Added, June 1982. The regularity result for harmonic mappings has been proved independently by R. Schoen and K. Uhlenbeck [5], without the assumption that the image of u is locally contained in a bounded coordinate chart.

REFERENCES

[1] H. Federer, *The singular set of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc., **76** (1970), pp. 767-771.

[2] M. Giaquinta - E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math., **148** (1982), pp. 31-46.

[3] E. Giusti, *Minimal surfaces and functions of bounded variation*, Note on Pure Math., **10**, Canberra (1977).

[4] E. Giusti - M. Miranda, *Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni*, Boll. Un. Mat. Ital., **2** (1968), pp. 1-8.

[5] R. Schoen - K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geometry, **17** (1982), pp. 307-335.

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