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Microlocal Analysis for Spatially Inhomogeneous Pseudo Differential Operators.

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Introduction.

A spatially inhomogeneous pseudo differential operator A in the class $\mathfrak{L}^{\lambda}_{\Phi,\varphi}(\Omega)$, $\Omega\subset\mathbb{R}^n$, is a properly supported operator of the form

$$Af(x) = a(x, D)f(x) = (2\pi)^{-n}\int \exp[ix\xi]\, a(x, \xi)\hat{f}(\xi)\, d\xi\,, \tag{0.1}$$

where $a(x, \xi)\in S^{\lambda}_{\Phi,\varphi}(\Omega)$, i.e. for every $K\subset\subset\Omega$

$$|D^{\alpha}_x D^{\beta}_{\xi} a(x, \xi)| \leqslant c_{\alpha\beta K} \exp[\lambda(x, \xi)]\, \varphi(x, \xi)^{-\alpha}\, \Phi(x, \xi)^{-\beta}, \quad x\in K\,,\ \xi\in\mathbb{R}^n, \tag{0.2}$$

with standard vectorial notations; $\Phi(x, \xi) = \big(\Phi_1(x, \xi), \dots, \Phi_n(x, \xi)\big)$, $\varphi(x, \xi) = \big(\varphi_1(x, \xi), \dots, \varphi_n(x, \xi)\big)$ is a pair of local weight vectors in Ω and $\lambda(x, \xi)\in O(\Phi, \varphi)$, according to the definitions in Beals [2], for example (see also the recent works [9], where Hörmander has given for similar classes of operators an equivalent definition which is invariant under linear symplectic transformations, and Nagel-Stein [11], [12], Beals [4], where the L^p-boundedness of A is proved under suitable conditions on Φ, φ).

In this paper we set up a new approach to the study of the action of $A\in\mathfrak{L}^{\lambda}_{\Phi,\varphi}(\Omega)$ on the singularities of a distribution $f\in\mathcal{E}'(\Omega)$ and we outline some applications; the analysis will be microlocal, that is we shall not only be concerned with the location of the singularities in Ω, but also with their local harmonic analysis in $T^*\Omega$, following a well known idea of Hörmander [7], [8]. Actually, the action of $A\in\mathfrak{L}^{\lambda}_{\Phi,\varphi}(\Omega)$ on the wave front set (WF) of Hörmander was already studied in Parenti-Rodino [13], [14] where it was observed that $A\in\mathfrak{L}^{\lambda}_{\Phi,\varphi}(\Omega)$ is always pseudolocal but it may dispace

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the WF inside the fibers of $T^*\Omega$, if the components of the vector φ do not satisfy suitable estimates.

The negative results of [13], [14] suggest here a microlocal analysis of a more general type. Let $\Psi(\xi) = (\Psi_1(\xi), \ldots, \Psi_n(\xi))$ be a *basic weight vector*, i.e. let $\Phi(x, \xi) = \Psi(\xi)$, $\varphi(x, \xi) = 1$ be a pair of local weight vectors, and write simply $S^\mu_\Psi(\Omega)$ for the related class of symbols $S^\lambda_{\Psi,1}(\Omega)$, $\lambda = \mu \log \Psi$ (the properties of the operators in $\mathfrak{L}^\mu_\Psi(\Omega)$ are reviewed in the next Section 1). We shall say that $f \in \mathcal{E}'(\Omega)$ is *Ψ-smooth* in $\Lambda \subset T^*\Omega$ if there exists an elliptic symbol in Λ, $a(x, \xi) \in S^0_\Psi(\Omega)$, such that $a(x, D)f \in C^\infty(\Omega)$. In the following Section 2 the singularities of f are then located by means of the collection of subsets of $T^*\Omega$

$$\mathcal{F}_\Psi(f) = \{\Gamma \subset T^*\Omega,\ f \text{ is } \Psi\text{-smooth in } \Lambda = T^*\Omega \setminus \Gamma\}, \tag{0.3}$$

which we shall call the *Ψ-filter of the singularities* of f; localizing (0.3) at a point $x_0 \in \Omega$ we shall consider in particular

$$\mathcal{F}_\Psi(f; x_0) = \Big\{ Y \subset \mathbb{R}^n,\ f \text{ is } \Psi\text{-smooth in } \{x_0\} \times (\mathbb{R}^n \setminus Y) \Big\}. \tag{0.3}'$$

Choosing $\Psi_j(\xi) = 1 + |\xi|$, $j = 1, \ldots, n$, one recovers essentially in (0.3) the filter of the neighborhoods of the WF of Hörmander. Another important example is given by the anisotropic wave front set in Lascar [10], Grushin-Sananin [6], Parenti-Rodino [16], Parenti-Segala [17], corresponding to the pseudo differential operators of quasi-homogeneous type.

In Section 3 we study the action of an arbitrary $A \in \mathfrak{L}^\lambda_{\Phi,\varphi}(\Omega)$ on the Ψ-filter; it is proved that if the condition

$$\Psi_j(\xi)\varphi_j(x, \xi) \geq c|\xi|^c, \qquad j = 1, \ldots, n, \tag{0.4}$$

is satisfied for some $c > 0$, then Af is Ψ-smooth in every subset $\Lambda \subset T^*\Omega$ where f is Ψ-smooth. In particular, if

$$\varphi_j(x, \xi) \geq c(1 + |\xi|)^{c-1}, \qquad j = 1, \ldots, n, \tag{0.5}$$

then we have $\mathrm{WF}\, Af \subset \mathrm{WF}\, f$ for every $A \in \mathfrak{L}^\lambda_{\Phi,\varphi}(\Omega)$.

At the end of Section 3 we present some applications to the microlocal analysis of linear partial differential operators. We shall say that a differential operator P is *Ψ-hypoelliptic* in Ω if the Ψ-filters of the singularities of f and Pf coincide for every $f \in \mathcal{E}'(\Omega)$. Suppose P has a left parametrix $E \in \mathfrak{L}^\lambda_{\Phi,\varphi}(\Omega)$, with suitable Φ, φ, λ; then P, which is certainly hypoelliptic, is Ψ-hypoelliptic for all Ψ which satisfy the condition (0.4). In particular, if (0.5) is valid, taking $\Psi_j(\xi) = 1 + |\xi|$, $j = 1, \ldots, n$, one obtains that $\mathrm{WF}\, Pf = \mathrm{WF}\, f$ for every $f \in \mathcal{E}'(\Omega)$. We shall also introduce a related notion of *Ψ-solvability*; this will allow us to solve, in a suitable microlocal sense, certain operators which are not locally solvable in standard sense, as for example $D_{x_1} + ix_1 D_{x_2}$ in a neighborhood of the origin in $\mathbf{R}^2$. Micro-solvability and micro-hypoellipticity with respect to anisotropic wave front sets were already discussed in Parenti-Rodino [16] for a class of degenerate quasi-elliptic equations.

Section 4 is devoted to the study of the Fourier integral operators

$$Ff(x) = (2\pi)^{-n}\int \exp\,[i\omega(x,\eta)]\, b(x,\eta)\hat{f}(\eta)\, d\eta\,, \tag{0.6}$$

where the amplitude $b(x,\eta)$ is in $S^{\mu}_{\Psi}(\mathbf{R}^n)$ and the phase $\omega(x,\eta)$ satisfies in a neighborhood of supp b conditions of the type

$$\partial\omega(x,\eta)/\partial\eta_j \in S^0_{\Psi}\,, \qquad j = 1, \dots, n\,, \tag{0.7}$$

$$\Psi_j(\eta)^{-1}\,\partial\omega(x,\eta)/\partial x_j \in S^0_{\Psi}\,, \qquad j = 1, \dots, n\,. \tag{0.8}$$

We shall give a formula for the composition of F with a pseudo differential operator and we shall study the action of F on the Ψ-filter; it will follow in particular that the Ψ-filter is invariant under the changes of variables which satisfy suitable conditions of compatibility with the basic weight vector Ψ.

Fourier integral operators of the form (0.6) are the natural tool for the analysis of the singularities, when one deals with spatially inhomogeneous evolution equations. Here we shall limit ourselves to a simple application, concerning a translation invariant model. General results on propagation and reflection of anisotropic wave front sets can be found in Parenti-Segala [17], where a calculus is developed for operators F with phase $\omega(x,\eta)$ of quasi-homogeneous type.

Further applications of our machinery to the theory of the linear partial differential operators will be discussed in future papers.

1. – Classes $S^{\mu}_{\Psi}(\Omega)$.

We say that the n-tuple of positive continuous functions $\Psi(\xi) = (\Psi_1(\xi), \dots, \Psi_n(\xi))$ in $\mathbf{R}^n$ is a *basic weight vector* if there are positive constants c, C such that

$$c(1+|\xi|)^c \leqslant \Psi_j(\xi) \leqslant C(1+|\xi|)^C\,, \qquad j = 1, \dots, n\,; \tag{1.1}$$

$$c \leqslant \Psi_j(\xi+\vartheta)\,\Psi_j(\xi)^{-1} \leqslant C\,, \quad j = 1, \dots, n\,, \quad \text{if } \sum_{k=1}^{n} |\vartheta_k|\Psi_k(\xi)^{-1} \leqslant c. \tag{1.2}$$

Let Ω be an open subset of $\mathbb{R}^n$. For $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ we define $S^\mu_\Psi(\Omega)$ to be the set of all $a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ which satisfy in every $K \subset\subset \Omega$ the estimates

$$(1.3) \qquad |D^\alpha_x D^\beta_\xi a(x, \xi)| \leqslant c_{\alpha\beta K} \Psi(\xi)^{\mu-\beta}, \qquad x \in K, \ \xi \in \mathbb{R}^n,$$

with standard vectorial notations. Let

$$(1.4) \qquad Af(x) = a(x, D) f(x) = (2\pi)^{-n} \int \exp[ix\xi]\, a(x, \xi) \hat{f}(\xi)\, d\xi, \qquad f \in C^\infty_0(\Omega),$$

with $a(x, \xi) \in S^\mu_\Psi(\Omega)$. The rules of the calculus of the pseudo differential operators hold for operators of the form (1.4); in fact, if $\Psi(\xi)$ is a basic weight vector, the pair $\Phi(x, \xi) = \Psi(\xi)$, $\varphi(x, \xi) = 1$ is a pair of local weight vectors in the sense of [2], for example. Let us review shortly the properties which we shall use in the following.

Recall first that for every given basic weight vector $\Psi^0(\xi)$ we may find a smooth basic weight vector $\Psi(\xi)$, which is equivalent to $\Psi^0(\xi)$ (i.e. $\Psi_j(\xi)\Psi^0_j(\xi)^{-1}$ and $\Psi_j(\xi)^{-1}\Psi^0_j(\xi)$ are bounded in $\mathbb{R}^n$), such that

$$(1.5) \qquad |D^\beta_\xi \Psi_j(\xi)| \leqslant c_\beta \Psi_j(\xi) \Psi(\xi)^{-\beta}, \qquad \xi \in \mathbb{R}^n, \ j = 1, \ldots, n.$$

Equivalent basic weight vectors define the same class of symbols; therefore we may assume in the following that $\Psi_1(\xi), \ldots, \Psi_n(\xi)$ satisfy (1.5).

The operator A in (1.4) maps continuously $C^\infty_0(\Omega)$ into $C^\infty(\Omega)$ and it extends to a linear continuous operator from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$. We shall denote by $\mathcal{L}^\mu_\Psi(\Omega)$ the set of all properly supported maps $A = a(x, D)$, with $a(x, \xi) \in S^\mu_\Psi(\Omega)$; for every given $a(x, \xi) \in S^\mu_\Psi(\Omega)$ one can find $a'(x, \xi) \in S^\mu_\Psi(\Omega)$ such that $a'(x, D)$ is properly supported and $a(x, \xi) \sim a'(x, \xi)$, i.e. $a(x, \xi) - a'(x, \xi) \in \bigcap_\mu S^\mu_\Psi(\Omega)$.

Write H^μ_Ψ for the Hilbert space of the distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ which satisfy

$$(1.6) \qquad \|f\|^2_{\mathrm{H}^\mu_\Psi} = \int \Psi(\xi)^{2\mu} |\hat{f}(\xi)|^2\, d\xi < \infty.$$

The properties of the spaces H^μ_Ψ are similar to those of the usual Sobolev spaces; we have in particular $\bigcup_\mu \mathrm{H}^\mu_\Psi = \mathcal{S}'(\mathbb{R}^n)$, $\bigcap_\mu \mathrm{H}^\mu_\Psi = \mathcal{S}(\mathbb{R}^n)$, also in the topological sense. Let $\mathrm{H}^\mu_{\Psi,\mathrm{comp}}(\Omega)$, $\mathrm{H}^\mu_{\Psi,\mathrm{loc}}(\Omega)$ be defined in the standard way; if A is in $\mathcal{L}^\mu_\Psi(\Omega)$, then for every $\nu \in \mathbb{R}^n$:

$$(1.7) \qquad A: \quad \mathrm{H}^{\mu+\nu}_{\Psi,\mathrm{comp}}(\Omega) \to \mathrm{H}^\nu_{\Psi,\mathrm{comp}}(\Omega) \ \textit{continuously},$$

$$(1.8) \qquad A: \quad \mathrm{H}^{\mu+\nu}_{\Psi,\mathrm{loc}}(\Omega) \to \mathrm{H}^\nu_{\Psi,\mathrm{loc}}(\Omega) \ \textit{continuously}.$$

A map $A\colon C_0^\infty(\Omega)\to\mathcal{D}'(\Omega)$ is said to be smoothing if it has a continuous extension mapping $\mathcal{E}'(\Omega)$ into $C^\infty(\Omega)$; for given operators $A_1, A_2\colon \mathcal{E}'(\Omega)\to\mathcal{D}'(\Omega)$ we shall write $A_1\sim A_2$ if the difference A_1-A_2 is smoothing. If $a(x,\xi)$ is in $\bigcap_\mu S^\mu_\Psi(\Omega)$, then $a(x,D)$ is smoothing.

THEOREM 1.1. *Let $a_1(x,D)$ be in $\mathcal{L}^{\mu^1}_\Psi(\Omega)$, let $a_2(x,D)$ be in $\mathcal{L}^{\mu^2}_\Psi(\Omega)$. Then the product $a_1(x,D)\,a_2(x,D)$ is in $\mathcal{L}^{\mu^1+\mu^2}_\Psi(\Omega)$ with symbol*

$$a(x,\xi)\sim\sum_\alpha(\alpha!)^{-1}\,\partial^\alpha_\xi a_1(x,\xi)\,D^\alpha_x a_2(x,\xi)\,, \tag{1.9}$$

in the sense that for any finite set J of multi-indices

$$a-\sum_{\alpha\in J}(\alpha!)^{-1}\partial^\alpha_\xi a_1 D^\alpha_x a_2\in\bigcup_{\alpha\notin J}S^{\mu^1+\mu^2-\alpha}_\Psi(\Omega)\,. \tag{1.10}$$

THEOREM 1.2. *If $a(x,D)\in\mathcal{L}^\mu_\Psi(\Omega)$, then the restriction to $C_0^\infty(\Omega)$ of the L^2-adjoint, $a(x,D)^*$, is also in $\mathcal{L}^\mu_\Psi(\Omega)$, with symbol*

$$a^\#(x,\xi)\sim\sum_\alpha(\alpha!)^{-1}\,\partial^\alpha_\xi D^\alpha_x\bar a(x,\xi)\,. \tag{1.11}$$

If $\Psi(\xi)$ is a basic weight vector, then ${}^t\Psi(\xi)=\Psi(-\xi)$ is still a basic weight vector; we have the following:

THEOREM 1.3. *If $a(x,D)\in\mathcal{L}^\mu_\Psi(\Omega)$, then the formal adjoint ${}^ta(x,D)$ is in $\mathcal{L}^\mu_{{}^t\Psi}(\Omega)$ with symbol*

$$a^\sim(x,\xi)\sim\sum_\alpha(\alpha!)^{-1}\,\partial^\alpha_\xi D^\alpha_x a(x,-\xi)\,. \tag{1.12}$$

We shall say that $a_0(x,\xi)\in S^\mu_\Psi(\Omega)$ is a principal symbol for $a(x,D)\in\mathcal{L}^\mu_\Psi(\Omega)$ if $a(x,\xi)-a_0(x,\xi)\in S^{\mu-\nu}_\Psi(\Omega)$, for some $\nu=(\nu_1,\dots,\nu_n)\in\mathbb{R}^n$, $\nu_j\geq 0$ for all j and $|\nu|\neq 0$. Observe that $a(x,\xi)$ in (1.9), $a^\#(x,\xi)$ in (1.11), $a^\sim(x,\xi)$ in (1.12) have principal part $a_1(x,\xi)\,a_2(x,\xi)$, $\bar a(x,\xi)$, $a(x,-\xi)$, respectively.

DEFINITION 1.4. *The symbol $a(x,\xi)\in S^\mu_\Psi(\Omega)$ is said to be elliptic (with respect to Ψ) in Ω if for every $K\subset\subset\Omega$ there are positive constants c_K, C_K such that*

$$|a(x,\xi)|\geq c_K\Psi(\xi)^\mu\qquad \text{for } x\in K\,,\ |\xi|\geq C_K\,. \tag{1.13}$$

If $a(x,\xi)$ has principal part $a_0(x,\xi)$, $a(x,\xi)$ is elliptic if and only if $a_0(x,\xi)$ is elliptic; it is then clear that products and adjoints of operators with elliptic symbols have elliptic symbols.

THEOREM 1.5. *Let $a(x, D)$ be in $\mathfrak{L}^{\mu}_{\Psi}(\Omega)$ with elliptic symbol in Ω; there exists $b(x, D) \in \mathfrak{L}^{-\mu}_{\Psi}(\Omega)$ such that*

$$b(x, D)\, a(x, D) = \textit{identity} + c(x, D) \tag{1.14}$$

where $c(x, D) \in \bigcap_{\mu} \mathfrak{L}^{\mu}_{\Psi}(\Omega)$.

We want to study now operators in $\mathfrak{L}^{\mu}_{\Psi}(\Omega)$ which are « microlocally » elliptic in a subset of $\Omega \times \mathbf{R}^n$. First of all, let us introduce some new notations. Take $\Lambda \subset \Omega \times \mathbf{R}^n$ and assume that its projection

$$\pi(\Lambda) = \{x \in \Omega,\ (x, \xi) \in \Lambda \textit{ for some } \xi \in \mathbf{R}^n\} \tag{1.15}$$

has compact closure in Ω. We define for $\varepsilon > 0$

$$\Lambda^{\varepsilon\Psi} = \bigcup_{(x_0, \xi^0) \in \Lambda} \{(x, \xi) \in \Omega \times \mathbf{R}^n,\ |x - x_0| < \varepsilon \textit{ and } |\xi_j - \xi^0_j| < \varepsilon \Psi_j(\xi^0) \textit{ for } j = 1, \ldots, n\}; \tag{1.16}$$

$\Lambda^{\varepsilon\Psi}$ will be called the *Ψ-neighborhood of ray ε of Λ in $\Omega \times \mathbf{R}^n$*. In (1.16) we understand that the ray ε is so small that $\pi(\Lambda^{\varepsilon\Psi})$ too has compact closure in Ω.

Then let X be a subset of $\mathbf{R}^n$; we define for $\varepsilon > 0$ (cf. (1.13) in [11]):

$$X_{\varepsilon\Psi} = \bigcup_{\xi^0 \in X} \{\xi \in \mathbf{R}^n,\ |\xi_j - \xi^0_j| < \varepsilon \Psi_j(\xi^0) \textit{ for } j = 1, \ldots, n\}\,. \tag{1.17}$$

Setting

$$\Lambda_{x_0} = \pi^{-1}(x_0) \cap \Lambda = \{\xi \in \mathbf{R}^n,\ (x_0, \xi) \in \Lambda\} \subset \mathbf{R}^n\,, \tag{1.18}$$

we have $\Lambda = \bigcup_{x_0 \in \pi(\Lambda)} \Lambda_{x_0}$ and

$$\Lambda^{\varepsilon\Psi} = \bigcup_{x_0 \in \pi(\Lambda)} \{x \in \Omega,\ |x - x_0| < \varepsilon\} \times (\Lambda_{x_0})_{\varepsilon\Psi}\,. \tag{1.19}$$

DEFINITION 1.6. *Let Λ be a subset of $\Omega \times \mathbf{R}^n$, and assume $\pi(\Lambda)$ has compact closure in Ω; let $a(x, \xi)$ be in $S^{\mu}_{\Psi}(\Omega)$. We say that $a(x, \xi)$ is elliptic (with respect to Ψ) in Λ if there are positive constants c, C such that*

$$|a(x, \xi)| \geqslant c\Psi(\xi)^{\mu} \qquad \textit{for } (x, \xi) \in \Lambda\,,\ |\xi| \geqslant C\,. \tag{1.20}$$

If $a(x, \xi)$ is elliptic in Λ then it is elliptic in $\Lambda^{\varepsilon\Psi}$, for a sufficiently small $\varepsilon > 0$; this follows easily from the estimates (1.3) for $|\alpha + \beta| = 1$.

DEFINITION 1.7. *We say that $a(x, \xi) \in S^{\mu}_{\Psi}(\Omega)$ is rapidly decreasing in $\Theta \subset \Omega \times \mathbf{R}^n$ if there exists $a_0(x, \xi) \in S^{\mu}_{\Psi}(\Omega)$ such that $a(x, \xi) \sim a_0(x, \xi)$ and* supp $a_0 \subset (\Omega \times \mathbf{R}^n) \setminus \Theta$.

THEOREM 1.8. *Let $a(x, D)$ be in $\mathfrak{L}^{\mu}_{\Psi}(\Omega)$ with elliptic symbol in $\Lambda \subset \Omega \times \mathbf{R}^n$ (as in Definition 1.4 $\pi(\Lambda)$ has compact closure in Ω). There exists $b(x, D) \in \mathfrak{L}^{-\mu}_{\Psi}(\Omega)$ such that*

$$b(x, D)\, a(x, D) = \textit{identity} + c(x, D)\,, \tag{1.21}$$

where $c(x, \xi) \in S^0_{\Psi}(\Omega)$ is rapidly decreasing in $\Lambda^{r\Psi}$, for some $r > 0$.

This microlocal version of Theorem 1.5 is new, with respect to the references; we shall prove it by means of the following lemmas.

LEMMA 1.9. *Let Ξ be a subset of $\Omega \times \mathbf{R}^n$, and assume $\pi(\Xi)$ has compact closure in Ω. Let $a(x, D)$ be in $\mathfrak{L}^0_{\Psi}(\Omega)$ with elliptic symbol in Ξ; let $\tau(x, D)$ be in $\mathfrak{L}^0_{\Psi}(\Omega)$ with rapidly decreasing symbol in $(\Omega \times \mathbf{R}^n) \setminus \Xi$. Then there exists $b(x, D) \in \mathfrak{L}^0_{\Psi}(\Omega)$ such that $b(x, D)\, a(x, D) \sim \tau(x, D)$.*

PROOF. We may take $\tau_0(x, \xi) \in S^0_{\Psi}(\Omega)$ such that $\tau(x, \xi) \sim \tau_0(x, \xi)$ and supp $\tau_0 \subset \Xi$. Set

$$b_0(x, \xi) = \begin{cases} \tau_0(x, \xi)/a(x, \xi) & \text{for } (x, \xi) \in \Xi\,, \\ 0 & \text{for } (x, \xi) \notin \Xi\,. \end{cases} \tag{1.22}$$

Since $a(x, \xi)$ is elliptic in Ξ, $b_0(x, \xi)$ is well defined in $S^0_{\Psi}(\Omega)$ for large $|\xi|$. Then, arguing by recurrence, we introduce

$$b_{-j}(x, \xi) = \begin{cases} -\sum\limits_{0<|\alpha|\leqslant j} (\alpha!)^{-1} \partial^{\alpha}_{\xi} b_{j-|\alpha|}(x, \xi) D^{\alpha}_x a(x, \xi)/a(x, \xi) & \text{for } (x, \xi) \in \Xi\,, \\ 0 & \text{for } (x, \xi) \notin \Xi\,, \end{cases} \tag{1.23}$$

$j = 1, 2, \ldots.$

For large $|\xi|$, $b_{-j}(x, \xi)$ is well defined in $\bigcup\limits_{|\alpha|\geqslant j} S^{-\alpha}_{\Psi}(\Omega)$, with supp $b_{-j} \subset \Xi$. Using a standard argument we can construct $b(x, \xi) \in S^0_{\Psi}(\Omega)$ such that $b(x, \xi) \sim \sum\limits_{j=0}^{\infty} b_{-j}(x, \xi)$; we may also assume that $b(x, D)$ is properly supported. Noting by $c(x, \xi)$ the symbol of the product $b(x, D)\, a(x, D) \in \mathfrak{L}^0_{\Psi}(\Omega)$, we have easily from Theorem 1.1 that $c(x, \xi) \sim \tau_0(x, \xi)$. Lemma 1.9 is therefore proved.

LEMMA 1.10. *Let* $\varepsilon > 0$ *and* $X \subset \mathbb{R}^n$ *be fixed. There exists* $\sigma(\xi) \in S^0_\Psi(\mathbb{R}^n)$ *such that* $\mathrm{supp}_\xi\, \sigma \subset X_{\varepsilon\Psi}$ *and* $\sigma(\xi) = 1$ *if* $\xi \in X_{\varepsilon'\Psi}$, *for a suitable* ε', $0 < \varepsilon' < \varepsilon$, *which depends only on* ε *and* Ψ.

In the proof we shall use the following result, which is a straight consequence of the property (1.2):

LEMMA 1.11. *For every fixed* $\varepsilon > 0$, *we can find* ε^*, $0 < \varepsilon^* < \varepsilon$, *such that*

$$(X_{\varepsilon^*\Psi})_{\varepsilon^*\Psi} \subset X_{\varepsilon\Psi}\,, \tag{1.24}$$

$$(\mathbb{R}^n \setminus X_{\varepsilon\Psi})_{\varepsilon^*\Psi} \subset \mathbb{R}^n \setminus X_{\varepsilon^*\Psi}\,, \tag{1.25}$$

for every $X \subset \mathbb{R}^n$.

PROOF OF LEMMA 1.10. Applying (1.25) in Lemma 1.11, we begin by taking $\varepsilon_0 > 0$ such that

$$(\mathbb{R}^n \setminus X_{(\varepsilon/2)\Psi})_{\varepsilon_0\Psi} \subset \mathbb{R}^n \setminus X_{\varepsilon_0\Psi}; \tag{1.26}$$

the constant ε' in the statement of Lemma 1.10 will be chosen in such a way that

$$(X_{\varepsilon'\Psi})_{\varepsilon'\Psi} \subset X_{\varepsilon_0\Psi}\,, \tag{1.27}$$

according to (1.24) in Lemma 1.11. From (1.26), (1.27) it follows

$$(\mathbb{R}^n \setminus X_{(\varepsilon/2)\Psi})_{\varepsilon'\Psi} \cap (X_{\varepsilon'\Psi})_{\varepsilon'\Psi} = \emptyset\,, \tag{1.28}$$

for every $X \subset \mathbb{R}^n$. Let u denote the characteristic function of the subset $(X_{\varepsilon'\Psi})_{\varepsilon'\Psi} \subset \mathbb{R}^n$. Take $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\int \varphi(t)\, dt = 1$, $\varphi(t) \geqslant 0$ and $\varphi(t) = 0$ for $|t| \geqslant \frac{1}{2}$. We define:

$$\sigma(\xi) = \varepsilon'^{-n}\, \Psi_1(\xi)^{-1} \dots \Psi_n(\xi)^{-1} \int u(\eta)\, \varphi\left(\frac{\xi_1 - \eta_1}{\varepsilon' \Psi_1(\xi)}, \dots, \frac{\xi_n - \eta_n}{\varepsilon' \Psi_n(\xi)}\right) d\eta\,. \tag{1.29}$$

It is easy to check that $\sigma(\xi) \in S^0_\Psi(\mathbb{R}^n)$. Fixing $\xi \in \mathbb{R}^n$, then we consider the function of η

$$g_\xi(\eta) = \varphi\left(\frac{\xi_1 - \eta_1}{\varepsilon' \Psi_1(\xi)}, \dots, \frac{\xi_n - \eta_n}{\varepsilon' \Psi_n(\xi)}\right) \tag{1.30}$$

and we observe that

$$\mathrm{supp}\, g_\xi \subset \{\eta \in \mathbb{R}^n,\ |\eta_j - \xi_j| < \varepsilon' \Psi_j(\xi)\ \textit{for}\ j = 1, \dots, n\}\,. \tag{1.31}$$

If $\xi \in X_{\varepsilon'\Psi}$, then $\operatorname{supp} g_\xi \subset (X_{\varepsilon'\Psi})_{\varepsilon'\Psi}$ and thus

$$\sigma(\xi) = \varepsilon'^{-n}\Psi_1(\xi)^{-1} \dots \Psi_n(\xi)^{-1}\int g_\xi(\eta)\, d\eta = \int \varphi(t)\, dt = 1 \,. \tag{1.32}$$

On the other hand, if $\xi \notin X_{(\varepsilon/2)\Psi}$ then $\operatorname{supp} g_\xi \subset (\mathbb{R}^n \setminus X_{(\varepsilon/2)\Psi})_{\varepsilon'\Psi}$ and it follows from (1.28) that $\operatorname{supp} g_\xi \cap \operatorname{supp} u = \emptyset$. Therefore we have $\sigma(\xi) = 0$ for $\xi \notin X_{(\varepsilon/2)\Psi}$ and $\operatorname{supp}_\xi \sigma \subset X_{\varepsilon\Psi}$.

LEMMA 1.12. *Let $\varepsilon > 0$ and $\Lambda \subset \Omega \times \mathbb{R}^n$ be fixed; assume $\pi(\Lambda)$ has compact closure in Ω. There exists $\tau_0(x, \xi) \in S^0_\Psi(\Omega)$ such that* $\operatorname{supp} \tau_0 \subset \Lambda^{\varepsilon\Psi}$ *and $\tau_0(x, \xi) = 1$ if $(x, \xi) \in \Lambda^{\varepsilon'\Psi}$, for some ε', $0 < \varepsilon' < \varepsilon$.*

PROOF. The constant ε' will be chosen in such a way that the conclusions of Lemma 1.10 hold, for the same ε, ε', and moreover $8\varepsilon' < \varepsilon$ and $\pi(\Lambda^{8\varepsilon'\Psi})$ has compact closure in Ω. Take note of the obvious inclusions: $\pi(\Lambda) \subset \pi(\Lambda^{\varepsilon'\Psi}) \subset \pi(\Lambda^{\varepsilon\Psi})$.

Let $S_1, \dots, S_H$ be balls in Ω with centres $x_0^1, \dots, x_0^H \in \pi(\Lambda)$ and radius $2\varepsilon'$, such that

$$\pi(\Lambda^{\varepsilon'\Psi}) \subset \bigcup_{h=1}^{H} S_h \,. \tag{1.33}$$

Since $8\varepsilon' < \varepsilon$, we have

$$\bigcup_{h=1}^{H} S_h \subset \pi(\Lambda^{\varepsilon\Psi}) \,. \tag{1.34}$$

Denote by S'_h the ball with centre x_0^h and radius $4\varepsilon'$; observe that (1.33), (1.34) are still satisfied if we replace S_h with S'_h. Keeping in mind the definition of Λ_{x_0} in (1.18), we set for $h = 1, \dots, H$:

$$X_h = \bigcup_{x_0 \in S' \cap \pi(\Lambda)} \Lambda_{x_0} \tag{1.35}$$

and moreover

$$\Lambda_{(h)} = S_h \times (X_h)_{\varepsilon'\Psi} \,, \qquad \Lambda_* = \bigcup_{h=1}^{H} \Lambda_{(h)} \,. \tag{1.36}$$

Observe that $(X_h)_{\varepsilon'\Psi} = \bigcup_{x_0 \in S' \cap \pi(\Lambda)} (\Lambda_{x_0})_{\varepsilon'\Psi}$. We define also:

$$\Lambda'_{(h)} = S'_h \times (X_h)_{\varepsilon\Psi} \,, \quad \Lambda'_* = \bigcup_{h=1}^{H} \Lambda'_{(h)} \,. \tag{1.37}$$

We shall prove:

$$\Lambda^{\varepsilon'\Psi} \subset \Lambda_* \subset \Lambda'_* \subset \Lambda^{\varepsilon\Psi} \,. \tag{1.38}$$

The central inclusion is obvious, since $S_h \subset S'_h$ and $(X_h)_{\varepsilon'\Psi} \subset (X_h)_{\varepsilon\Psi}$. Assume $(x, \xi) \in \Lambda^{\varepsilon'\Psi}$; this means that there exists $(x_0, \xi^0) \in \Lambda$ with $|x - x_0| < \varepsilon'$ and $|\xi_j - \xi_j^0| < \varepsilon' \Psi_j(\xi^0)$, $j = 1, \dots, n$. Since $x \in \pi(\Lambda^{\varepsilon'\Psi})$, we have $x \in S_{h_0}$, for some index h_0; therefore it is $x_0 \in S'_{h_0}$. Writing $\xi_0 \in \Lambda_{h_0}$, we have from (1.35) $\xi^0 \in X_{h_0}$ and thus $\xi \in (X_{h_0})_{\varepsilon'\Psi}$. Summing up we obtain $(x, \xi) \in \Lambda_{(h_0)}$ and the first inclusion in (1.37) is so proved, in view of (1.36).

Assume now $(x, \xi) \in \Lambda'_*$; this means that, for some index h_0, $x \in S'_{h_0}$ and $\xi \in (X_{h_0})_{\varepsilon\Psi}$, i.e. $\xi \in (\Lambda_{\bar{x}})_{\varepsilon\Psi}$ for some $\bar{x} \in S'_{h_0} \cap \pi(\Lambda)$. We have $|x - \bar{x}| < 8\varepsilon' < \varepsilon$ and moreover $|\xi_j - \bar{\xi}_j| < \varepsilon \Psi_j(\bar{\xi})$, $j = 1, \dots, n$, for some $\bar{\xi} \in \Lambda_{\bar{x}}$; this implies $(x, \xi) \in \Lambda^{\varepsilon\Psi}$ and completes the proof of (1.38). Let $\sigma_h(\xi) \in S^0_\Psi(\Omega)$ be defined as in Lemma 1.10, with $\operatorname{supp}_\xi \sigma_h \subset (X_h)_{\varepsilon\Psi}$, $\sigma_h(\xi) = 1$ for $\xi \in (X_h)_{\varepsilon'\Psi}$. Then take $\chi_h \in C_0^\infty(\Omega)$ such that $\operatorname{supp} \chi_h \subset S'_h$ and $\chi_h(x) = 1$ for $x \in S_h$. The product $\tau_h(x, \xi) = \chi_h(x)\sigma_h(\xi)$ is in $S^0_\Psi(\Omega)$, with $\operatorname{supp} \tau_h \subset \Lambda'_{(h)}$ and $\tau_h(x, \xi) = 1$ for $(x, \xi) \in \Lambda_{(h)}$. Define

$$\tau_0(x, \xi) = 1 - \prod_{h=1}^{H} \bigl(1 - \tau_h(x, \xi)\bigr) \in S^0_\Psi(\Omega)\,. \tag{1.39}$$

We have $\tau_0(x, \xi) = 0$ if $(x, \xi) \notin \Lambda'_*$ and $\tau_0(x, \xi) = 1$ if $(x, \xi) \in \Lambda_*$. Therefore Lemma 1.12 is proved, in view of (1.38).

Proof of Theorem 1.8. Multiplying $a(x, D)$ by an operator $a'(x, D) \in \mathfrak{L}^{-\mu}_\Psi(\Omega)$ with elliptic symbol in Ω (take for example $a'(x, \xi) \sim \Psi(\xi)^{-\mu}$), we are reduced to prove the theorem in the case $\mu = 0$. Then let $a(x, D)$ be in $\mathfrak{L}^0_\Psi(\Omega)$, and choose $\varepsilon > 0$ such that $a(x, \xi)$ is still elliptic in $\Lambda^{\varepsilon\Psi}$. Take $\tau_0(x, \xi)$ as in Lemma 1.12, with $\operatorname{supp} \tau_0 \subset \Lambda^{\varepsilon\Psi}$ and $\tau_0(x, \xi) = 1$ if $(x, \xi) \in \Lambda^{\varepsilon'\Psi}$, for some ε', $0 < \varepsilon' < \varepsilon$. Let $\tau(x, D)$ be in $\mathfrak{L}^0_\Psi(\Omega)$, with $\tau(x, \xi) \sim \tau_0(x, \xi)$; $\tau(x, \xi)$ is rapidly decreasing in $(\Omega \times \mathbb{R}^n) \setminus \Lambda^{\varepsilon\Psi}$ and we may apply Lemma 1.9 with $\Xi = \Lambda^{\varepsilon\Psi}$. So we get $b(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$, such that $b(x, D)a(x, D) \sim \tau(x, D)$. The symbol of the operator $c(x, D) = b(x, D)a(x, D) -$ identity is rapidly decreasing in $\Lambda^{r\Psi}$ for $0 < r < \varepsilon'$ and Theorem 1.8 is therefore proved.

Finally let us discuss some important examples of basic weight vectors.

Example 1.13. We shall say that the n-tuple $\Psi(\xi) = \bigl(\Psi_1(\xi), \dots, \Psi_n(\xi)\bigr)$ is a *basic weight vector of rational type* if it satisfies (1.1), (1.2) and moreover

(1.40) *there exists a polynomial* $Q(\xi) \in S^{\mu^0}_\Psi(\mathbb{R}^n)$, $\mu^0 = (\mu^0_1, \dots, \mu^0_n) \in \mathbb{R}^n$, $\mu^0_j \geq 0$ *for all* j *and* $|\mu^0| \neq 0$, *such that*

$$|Q(\xi)| \geq c\Psi(\xi)^{\mu_0} \qquad \textit{for } |\xi| \geq C\,.$$

That is, $Q(\xi)$ is elliptic with respect to Ψ in $\mathbf{R}^n$; it follows in particular that the operator with constant coefficients $Q(D)$ is hypoelliptic.

For example, let $M = (M_1, \ldots, M_n)$ be a n-tuple of positive integers, define $[\xi]_M = 1 + \sum_{j=1}^{n} |\xi_j|^{1/M_j}$ and consider the basic weight vector (see [6], [10], [16], [17]):

$$\Psi(\xi) = [\xi]_M^M = \left([\xi]_M^{M_1}, \ldots, [\xi]_M^{M_n}\right). \tag{1.41}$$

$\Psi(\xi)$ in (1.41) is of rational type, since (1.40) is satisfied by the quasi-elliptic polynomial $Q(\xi) = 1 + \sum_{j=1}^{n} \xi_j^{2\tau/M_j}$ where τ is the least common multiple of the M_j's.

EXAMPLE 1.14. Let $\Psi(\xi) = \left(\Psi_1(\xi), \ldots, \Psi_n(\xi)\right)$ be a basic weight vector whose components coincide; we shall identify $\Psi(\xi)$ with the function $\Psi_1(\xi) = \ldots = \Psi_n(\xi)$ and we shall call it a ***basic weight function*** (cf. [1], [3], [5]). A positive continuous function $\Psi(\xi)$ in $\mathbf{R}^n$ is a basic weight function if and only if there are positive constants c, C such that (see (1.18) in [2]):

$$c(1 + |\xi|)^c \leqslant \Psi(\xi) \leqslant C(1 + |\xi|), \tag{1.42}$$

$$c \leqslant \Psi(\xi + \vartheta)\Psi(\xi)^{-1} \leqslant C \qquad \textit{if } |\vartheta| \leqslant c\Psi(\xi). \tag{1.43}$$

For example, $\Psi(\xi) = (1 + |\xi|)^\varrho$ is a basic weight function of rational type for $0 < \varrho \leqslant 1$.

2. – Ψ-filter of the singularities of a distribution.

Here as in the preceding section Ψ is a fixed basic weight vector in $\mathbf{R}^n$ and Ω an open subset of $\mathbf{R}^n$.

DEFINITION 2.1. *Let f be in $\mathcal{D}'(\Omega)$; let Λ be a subset of $\Omega \times \mathbf{R}^n$ and assume $\pi(\Lambda)$ has compact closure in Ω. We shall say that f is Ψ-smooth in Λ if there exists $a(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ with elliptic symbol in Λ such that $a(x, D) f \in C^\infty(\Omega)$. We shall call Ψ-filter of the singularities of f the collection of subsets of $\Omega \times \mathbf{R}^n$*

(2.1) $\mathcal{F}_\Psi(f) = \{\Gamma \subset \Omega \times \mathbf{R}^n$; *the projection* $\pi(\Lambda)$, $\Lambda = (\Omega \times \mathbf{R}^n) \setminus \Gamma$, *has compact closure in* Ω *and* f *is* Ψ-*smooth in* $\Lambda\}$.

Observe that if f is Ψ-smooth in Λ, then it is Ψ-smooth in $\Lambda^{\varepsilon\Psi}$, for a sufficiently small $\varepsilon > 0$.

Let us check that $\mathcal{F}_\Psi(f)$ is actually a filter. Arguing on the collection of the complements, we have to prove:

(i) *if f is Ψ-smooth in Λ, and $\Lambda' \subset \Lambda$, then f is Ψ-smooth in Λ';*

(ii) *if f is Ψ-smooth in $\Lambda_1, \dots, \Lambda_H$, then f is Ψ-smooth in $\bigcup_{h=1}^{H} \Lambda_h$.*

The first point is obvious. As for (ii), we assume there exist $a_h(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$, $h = 1, \dots, H$, with elliptic symbol in Λ_h, such that $a_h(x, D) f \in C^\infty(\Omega)$. Consider the operator $a(x, D) = \sum_{h=1}^{H} a_h(x, D)^* a_h(x, D)$. We have $a(x, D) f \in C^\infty(\Omega)$ and moreover it follows from Theorems 1.1, 1.2 that $a(x, D)$ has principal symbol $\sum_{h=1}^{H} |a_h(x, \xi)|^2$, which is elliptic in $\bigcup_{h=1}^{H} \Lambda_h$. Therefore f is Ψ-smooth in $\bigcup_{h=1}^{H} \Lambda_h$ and it is proved that $\mathcal{F}_\Psi(f)$ is a filter.

THEOREM 2.2. *Let $A = a(x, D)$ be in $\mathfrak{L}^\mu_\Psi(\Omega)$. Then*

$$\mathcal{F}_\Psi(f) \subset \mathcal{F}_\Psi(Af)\,, \qquad \textit{for every } f \in \mathcal{D}'(\Omega)\,. \tag{2.2}$$

LEMMA 2.3. *Let Λ be a subset of $\Omega \times \mathbb{R}^n$ and assume $\pi(\Lambda)$ has compact closure in Ω. If $f \in \mathcal{D}'(\Omega)$ is Ψ-smooth in Λ, then there exists $c(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ with rapidly decreasing symbol in $\Lambda^{\varepsilon\Psi}$, for a suitable $\varepsilon > 0$, such that $f - c(x, D) f \in C^\infty(\Omega)$.*

PROOF. We assume the existence of $a(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ with elliptic symbol in Λ such that $a(x, D) f \in C^\infty(\Omega)$. Let us apply Theorem 1.8 to the operator $a(x, D)$: there exists $b(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ such that $b(x, D) a(x, D) = \text{identity} + c(x, D)$, where $c(x, \xi)$ is rapidly decreasing in $\Lambda^{\varepsilon\Psi}$ for some $\varepsilon > 0$. To obtain the lemma then it will be sufficient to observe that $f - c(x, D) f = b(x, D)\, a(x, D)$ is in $C^\infty(\Omega)$.

PROOF OF THEOREM 2.2. Arguing on the collection of the complements, we have to prove that if f is Ψ-smooth in Λ then Af is Ψ-smooth in Λ. Let $c(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ satisfy the conclusions of Lemma 2.3, i.e. assume $f - c(x, D) f \in C^\infty(\Omega)$ and $c(x, \xi)$ is rapidly decreasing in $\Lambda^{\varepsilon\Psi}$, $\varepsilon > 0$. Then take $\tau_0(x, \xi) \in S^0_\Psi(\Omega)$ as in Lemma 1.12, with $\operatorname{supp} \tau_0 \subset \Lambda^{\varepsilon\Psi}$ and $\tau_0(x, \xi) = 1$ for $(x, \xi) \in \Lambda^{\varepsilon'\Psi}$, $0 < \varepsilon' < \varepsilon$. Let $\tau(x, D)$ be in $\mathfrak{L}^0_\Psi(\Omega)$, $\tau(x, \xi) \sim \tau_0(x, \xi)$. To prove that Af is Ψ-smooth in Λ we shall check that $\tau(x, D) Af \in C^\infty(\Omega)$. Write

$$\tau(x, D) Af = \tau(x, D) A c(x, D) f + \tau(x, D) A\big(f - c(x, D) f\big)\,. \tag{2.3}$$

The second term in the right-hand side is in $C^\infty(\Omega)$. On the other hand it is $c(x, \xi) \sim c_0(x, \xi)$, with $\operatorname{supp} c_0 \cap \operatorname{supp} \tau_0 = \emptyset$; then it follows from Theorem 1.1 that $\tau(x, D)Ac(x, D) \sim 0$, and therefore we have also $\tau(x, D) Ac(x, D) f \in C^\infty(\Omega)$. Theorem 2.2 is proved.

Let Ξ be a subset of $\Omega \times \mathbb{R}^n$ and assume $\pi(\Xi)$ has compact closure in Ω; we define the filter

$$\mathcal{F}_\Psi(f)|_\Xi = \{\Gamma' \subset \Xi;\ \Gamma' = \Xi \cap \Gamma,\ \textit{for some}\ \Gamma \in \mathcal{F}_\Psi(f)\}\ . \tag{2.4}$$

From Theorem 2.2 we have for all $A \in \mathfrak{L}^\mu_\Psi(\Omega)$:

$$\mathcal{F}_\Psi(f)|_\Xi \subset \mathcal{F}_\Psi(Af)|_\Xi\,, \qquad \textit{for every}\ f \in \mathcal{D}'(\Omega)\ . \tag{2.5}$$

PROPOSITION 2.4. *Let $A = a(x, D)$ be in $\mathfrak{L}^\mu_\Psi(\Omega)$ with elliptic symbol in Ξ; then*

$$\mathcal{F}_\Psi(f)|_\Xi = \mathcal{F}_\Psi(Af)|_\Xi\,, \qquad \textit{for every}\ f \in \mathcal{D}'(\Omega)\ . \tag{2.6}$$

PROOF. It remains to prove the inclusion $\mathcal{F}_\Psi(f)|_\Xi \supset \mathcal{F}_\Psi(Af)|_\Xi$. Since we may also define $\mathcal{F}_\Psi(f)|_\Xi = \{\Gamma' \subset \Xi,\ f$ is Ψ-smooth in $\Lambda' = \Xi \setminus \Gamma'\}$, we are reduced to prove that if Af is Ψ-smooth in $\Lambda' \subset \Xi$ then also f is Ψ-smooth in Λ'. Let $a_1(x, D)$ be in $\mathfrak{L}^0_\Psi(\Omega)$ with elliptic symbol in Λ', such that $a_1(x, D)Af \in C^\infty(\Omega)$. If we take $a_2(x, D) \in \mathfrak{L}^{-\mu}_\Psi(\Omega)$ with elliptic symbol in Ω, we have

$$a_2(x, D)\, a_1(x, D) Af \in C^\infty(\Omega)\ ;$$

on the other hand the operator $a_2(x, D)\, a_1(x, D) A$ has principal symbol $a_2(x, \xi)\, a_1(x, \xi)\, a(x, \xi)$, which is elliptic in Λ'. It is therefore proved that f is Ψ-smooth in Λ' .

PROPOSITION 2.5. *Assume $\Gamma \in \mathcal{F}_\Psi(f)$ and let $A = a(x, D)$ be in $\mathfrak{L}^\mu_\Psi(\Omega)$ with rapidly decreasing symbol in Γ. Then $Af \in C^\infty(\Omega)$.*

PROOF. Note $\Lambda = (\Omega \times \mathbb{R}^n) \setminus \Gamma$. Applying Lemma 2.3 we find $c(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ with rapidly decreasing symbol in $\Lambda^{\varepsilon\Psi}$, $\varepsilon > 0$, such that $f - c(x, D) f \in C^\infty(\Omega)$. Write

$$Af = Ac(x, D) f + A\big(f - c(x, D)\big) f\ . \tag{2.7}$$

The second term in the right-hand side is in $C^\infty(\Omega)$. On the other hand the operator $Ac(x, D)$ is smoothing, as it follows from Theorem 1.1; therefore we have also $Ac(x, D) f \in C^\infty(\Omega)$ and Proposition 2.5 is proved.

Now we want to study in detail the filter $\mathcal{F}_\Psi(f)|_\Xi$ in the case

$$\Xi = \{(x, \xi) \in \Omega \times \mathbb{R}^n,\ x = x_0\}\ .$$

DEFINITION 2.6. *We shall note $\mathcal{F}_\Psi(f, x_0)$ and we shall call Ψ-filter of the singularities of $f \in \mathcal{D}'(\Omega)$ at $x_0 \in \Omega$ the collection of all the subsets $Y \subset \mathbb{R}^n$ such that f is Ψ-smooth in $\{x_0\} \times (\mathbb{R}^n \setminus Y)$.*

The following theorem gives for $\mathcal{F}_\Psi(f, x_0)$ an equivalent definition where pseudo differential operators do not appear explicitly.

THEOREM 2.7. *$Y \in \mathcal{F}_\Psi(f, x_0)$ if and only if there exists $\chi \in C_0^\infty(\Omega)$, $\chi(x) = 1$ in a neighborhood of x_0, such that for every integer $N \geqslant 0$*

$$|(\chi f)\hat{\ }(\xi)| \leqslant C_N |\xi|^{-N} \qquad \textit{for } \xi \in (\mathbb{R}^n \setminus Y)_{\varepsilon\Psi}\,, \tag{2.8}$$

where ε is a suitable positive constant independent of N.

PROOF. Let (2.8) be satisfied for suitable χ and ε. Note $X = \mathbb{R}^n \setminus Y$. Applying Lemma 1.10, we find $\sigma(\xi) \in S^0_\Psi(\mathbb{R}^n)$ with $\operatorname{supp}_\xi \sigma \subset X_{\varepsilon\Psi}$ and $\sigma(\xi) = 1$ for $\xi \in X_{\varepsilon'\Psi}$, $0 < \varepsilon' < \varepsilon$. The function $\sigma(\xi)(\chi f)\hat{\ }(\xi)$ is rapidly decreasing in $\mathbb{R}^n$. Regarding $\sigma(\xi)$ as a symbol in $S^0_\Psi(\Omega)$, with $\sigma(D)$: $\mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$, we have $\sigma(D)(\chi f) \in C^\infty(\Omega)$. Let $\sigma'(x, D)$ be in $\mathfrak{L}^0_\Psi(\Omega)$ with $\sigma'(x, \xi) \sim \sigma(\xi)$ in Ω. Define $\tau(x, D) f = \sigma'(x, D)(\chi f)$; the operator $\tau(x, D)$ is in $\mathfrak{L}^0_\Psi(\Omega)$ and its principal symbol $\chi(x)\sigma(\xi)$ is elliptic in $\{x_0\} \times X$. On the other hand we have $\tau(x, D) f \in C^\infty(\Omega)$ and it is therefore proved that $Y \in \mathcal{F}_\Psi(f, x_0)$.

In the opposite direction, let f be Ψ-smooth in $\{x_0\} \times X$, i.e. assume the existence of $a(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ with elliptic symbol in $\{x_0\} \times X$ such that $a(x, D) f \in C^\infty(\Omega)$. The symbol $a(x, \xi)$ is still elliptic in the Ψ-neighborhood

$$(\{x_0\} \times Y)^{\varepsilon\Psi} = \{x \in \Omega,\ |x - x_0| < \varepsilon\} \times X_{\varepsilon\Psi}\,, \tag{2.9}$$

for a sufficiently small $\varepsilon > 0$. Take $\chi \in C_0^\infty(\Omega)$ with $\chi(x) = 1$ in a neighborhood of x_0 and $\operatorname{supp} \chi \subset \{x \in \Omega,\ |x - x_0| < \varepsilon\}$. Let $\sigma(\xi) \in S^0_\Psi(\Omega)$ be fixed again according to Lemma 1.10 and define $\tau(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ as in the first part of the proof. From Theorem 1.1 we have

$$\tau(x, \xi) \sim \sum_\alpha (\alpha!)^{-1}\, \partial_\xi^\alpha \sigma(\xi)\, D_x^\alpha \chi(x)\,; \tag{2.10}$$

hence we can construct $\tau_0(x, \xi) \in S^0_\Psi(\Omega)$ such that $\tau(x, \xi) \sim \tau_0(x, \xi)$ and $\operatorname{supp} \tau_0 \subset \{x \in \Omega,\ |x - x_0| < \varepsilon\} \times X_{\varepsilon\Psi}$. Then, applying Lemma 1.9, we find $b(x, D) \in \mathfrak{L}^0_\Psi(\Omega)$ such that $b(x, D)\, a(x, D) \sim \tau(x, D)$. We obtain $\tau(x, D) f \in$

$\in C^\infty(\Omega)$ and therefore $\sigma(D)(\chi f) \in C^\infty(\Omega)$. On the other hand, regarding $\sigma(D)$ as pseudolocal map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, we have also $\sigma(D)(\chi f) \in C^\infty(\mathbb{R}^n \setminus \setminus \operatorname{supp} \chi)$ and, summing up, $\sigma(D)(\chi f) \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$. Thus $\sigma(\xi)(\chi f)^\wedge(\xi)$ is rapidly decreasing in $\mathbb{R}^n$ and the estimates (2.8) are satisfied in the subset $X_{\varepsilon'\Psi}$ where it is $\sigma(\xi) = 1$. Theorem 2.7 is proved.

PROPOSITION 2.8. *The following conditions are equivalent*:

(I) $\emptyset \in \mathcal{F}_\Psi(f, x_0)$,

(II) $x_0 \notin \operatorname{sing\,supp} f$,

(III) *there exist* $X_1, \ldots, X_H \subset \mathbb{R}^n$, *with* $\bigcup_{h=1}^{H} X_h = \mathbb{R}^n$, *such that* f *is* Ψ-*smooth in* $\{x_0\} \times X_h$, *for* $h = 1, \ldots, H$.

In fact, it follows easily from Theorem 2.7 that (I) $\leftrightarrow$ (II), and (III) is equivalent to (I) since $\mathcal{F}_\Psi(f, x_0)$ is a filter.

3. – Action of $A \in \mathcal{L}^{\lambda}_{\Phi,\varphi}(\Omega)$ on the Ψ-filter.

DEFINITION 3.1. *A linear map* A: $C_0^\infty(\Omega) \to C^\infty(\Omega)$, $\mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$ *is said to be* Ψ-*local in* Ω *if for all* $x_0 \in \Omega$

$$\mathcal{F}_\Psi(f, x_0) \subset \mathcal{F}_\Psi(Af, x_0)\,, \quad \text{for every } f \in \mathcal{E}'(\Omega)\,. \tag{3.1}$$

From Proposition 2.8 it follows immediately:

PROPOSITION 3.2. *If* A *is* Ψ-*local in* Ω, *then it is pseudolocal in* Ω, *i.e.* $\operatorname{sing\,supp} Af \subset \operatorname{sing\,supp} f$ *for every* $f \in \mathcal{E}'(\Omega)$.

From (2.5) we have that every $A \in \mathcal{L}^{\mu}_{\Psi}(\Omega)$ is Ψ-local in Ω. We want now to discuss the action on the Ψ-filter of an operator A in the classes $\mathcal{L}^{\lambda}_{\Phi,\varphi}(\Omega)$ of the Introduction. For the sake of definiteness we shall suppose that Φ, φ are a pair of local weight vectors and $\lambda \in O(\Phi, \varphi)$ in the sense of Beals (see the definitions of Section 8 in [2]); however, as it will be clear from the proof of the subsequent Theorem 3.3, we could argue on generic Φ, φ, λ assuming only the validity of (0.1), (0.2), an algebraic upper bound for λ and the pseudolocal property for A (which actually follows from the hypotheses on Φ, φ in [2]).

THEOREM 3.3. *Let* $\Psi(\xi)$ *be a fixed basic weight vector. Let* $\Phi(x, \xi)$, $\varphi(x, \xi)$ *be a pair of local weight vectors in* Ω *and let* $\lambda(x, \xi)$ *be in* $O(\Phi, \varphi)$. *Suppose* *for each $K \subset\subset \Omega$ there exists a positive constant c_K such that*

$$\Psi_j(\xi)\varphi_j(x,\xi) \geqslant c_K|\xi|^{c_K}, \quad x \in K, \ \xi \in \mathbb{R}^n, \ j = 1, \dots, n. \tag{3.2}$$

Then every $A \in \mathfrak{L}^{\lambda}_{\Phi,\varphi}(\Omega)$ is Ψ-local in Ω.

EXAMPLE 3.4. Every $A \in \mathfrak{L}^{\lambda}_{\Phi,\varphi}(\Omega)$ is $(1+|\xi|)^{\varrho}$-local, $0<\varrho\leqslant 1$, if for all $K \subset\subset \Omega$ there exists $c_K > 0$ such that

$$\varphi_j(x,\xi) \geqslant c_K(1+|\xi|)^{c_K-\varrho}, \qquad x \in K, \ \xi \in \mathbb{R}^n, \ j = 1, \dots, n. \tag{3.3}$$

In particular if (3.3) is satisfied with $\varrho = 1$ then

$$\mathrm{WF}\, Af \subset \mathrm{WF}\, f, \qquad \textit{for every } f \in \mathcal{E}'(\Omega); \tag{3.4}$$

in fact $(x_0, \xi^0) \notin \mathrm{WF}\, f$, $|\xi^0| \neq 0$, means that f is $(1+|\xi|)$-smooth in $\{x_0\} \times \times \{\xi;\ \xi = t\xi^0,\ t \in \mathbb{R}_+\}$ and therefore (3.4) follows from the inclusion

$$\mathcal{F}_{(1+|\xi|)}(f, x_0) \subset \mathcal{F}_{(1+|\xi|)}(Af, x_0).$$

If $\varphi(x,\xi) = 1$ the conditions (3.3) is satisfied for every ϱ, $0<\varrho\leqslant 1$; actually, in this case (3.2) holds for any basic weight vector Ψ, in view of the assumption (1.1).

EXAMPLE 3.5. Let the vector $\Psi(\xi) = [\xi]^M_M$ be defined as in (1.41). The anisotropic wave front set WF_M of a distribution is defined in the following way: $(x_0, \xi^0) \notin \mathrm{WF}_M f$, $|\xi_0| \neq 0$, if and only if f is $[\xi]^M_M$-smooth in $\{x_0\} \times \times \{\xi;\ \xi_j = t^{M_j}\xi^0_j,\ t \in \mathbb{R}_+\}$. Therefore if A is $[\xi]^M_M$-local then in particular

$$\mathrm{WF}_M\, Af \subset \mathrm{WF}_M\, f, \qquad \textit{for every } f \in \mathcal{E}'(\Omega). \tag{3.5}$$

For example, every operator $A \in L^m_{M,\varrho,\delta}(\Omega) = \mathfrak{L}^{m \log [\xi]_M}_{[\xi]^{\varrho M}_M,\, [\xi]^{-\delta M}_M}(\Omega)$, $0 \leqslant \delta \leqslant \varrho \leqslant 1$, $\varrho \neq 0$, $\delta \neq 1$, $m \in \mathbb{R}$, is $[\xi]^M_M$-local and satisfies (3.5).

EXAMPLE 3.6. Let Φ, φ be a fixed pair of local weight vectors; assume Φ, φ are independent of x in Ω. We may always choose the basic weight vector $\Psi(\xi)$ in such a way that (3.2) is valid. Define for example

$$\Psi_j(\xi) = [\Phi_j(\xi)\varphi_j(\xi)^{-1}]^{\frac{1}{2}+\varepsilon}, \qquad j = 1, \dots, n. \tag{3.6}$$

If $\varepsilon > 0$ is sufficiently small, $\Psi(\xi)$ is a basic weight vector, as it follows from (2.5) in [2]. Moreover from (2.8), (2.13) in [2] we have

$$\Psi_j(\xi)\varphi_j(\xi) \geqslant c_1[\Phi_j(\xi)\varphi_j(\xi)^{-1}]^{\varepsilon} \geqslant c_2\Phi_j(\xi)^{\varepsilon} \geqslant c_3|\xi|^{c_3}, \tag{3.7}$$

for suitable positive constants c_1, c_2, c_3, and therefore (3.2) is satisfied.

PROOF OF THEOREM 3.3. For arbitrary $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, $f \in \mathcal{E}'(\Omega)$ we want to prove that if f is Ψ-smooth in $\{x_0\} \times X$ then also Af is Ψ-smooth in $\{x_0\} \times X$. In view of Theorem 2.7, if f is Ψ-smooth in $\{x_0\} \times X$ we can find $\chi \in C_0^\infty(\Omega)$, $\chi(x) = 1$ in a neighborhood of x_0, such that $(\chi f)^\wedge(\xi)$ is rapidly decreasing in $X_{\varepsilon\Psi}$ for some $\varepsilon > 0$. Take $\sigma(\xi) \in S_\Psi^0(\mathbb{R}^n)$ as in Lemma 1.10, with $\operatorname{supp}_\xi \sigma \subset X_{\varepsilon\Psi}$, $\sigma(\xi) = 1$ if $\xi \in X_{\varepsilon'\Psi}$, $0 < \varepsilon' < \varepsilon$; clearly we have $\sigma(D)(\chi f) \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$.

Then, applying again Lemma 1.10, we construct $\sigma_0(\xi) \in S_\Psi^0(\mathbb{R}^n)$ with $\operatorname{supp}_\xi \sigma_0 \subset X_{r\Psi}$, $\sigma(\xi) = 1$ for $\xi \in X_{r'\Psi}$, $0 < r' < r$, where we may choose r in such a way that

$$(\mathbb{R}^n \setminus X_{\varepsilon'\Psi})_{r\Psi} \cap (X_{r\Psi})_{r\Psi} = \emptyset, \tag{3.8}$$

arguing as in the first part of the proof of Lemma 1.10. Finally we take $u \in C_0^\infty(\Omega)$ with $\operatorname{supp} u \subset \{x \in \Omega, \chi(x) = 1\}$, $u(x) = 1$ in a neighborhood of x_0. We want to prove that

$$\sigma_0(D)(uAf) \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n). \tag{3.9}$$

In fact, if (3.9) is satisfied then $\sigma_0(\xi)(uAf)^\wedge(\xi)$ is rapidly decreasing in $\mathbb{R}^n$ and $(uAf)^\wedge(\xi)$ is rapidly decreasing in $X_{r'\Psi}$; in view of Theorem 2.7 this will imply the Ψ-smoothness of Af in $\{x_0\} \times X$. We begin by splitting

$$\sigma_0(D)(uAf) = \sigma_0(D)[uA(\chi f)] + \sigma_0(D)[uA((1-\chi)f)]. \tag{3.10}$$

Since every $A \in \mathcal{L}^\lambda_{\Phi,\varphi}(\Omega)$ is pseudolocal in Ω we have $uA((1-\chi)f) \in C_0^\infty(\Omega)$; then to prove (3.9) it will be sufficient to prove that $\sigma_0(D)[uA(\chi f)]$ is in $C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$, or else that its restriction to Ω is smooth. Split again

$$\sigma_0(D)[uA(\chi f)] = \sigma_0(D)[uA\sigma(D)(\chi f)] + \sigma_0(D)[uA(1-\sigma)(D)(\chi f)], \tag{3.11}$$

where now $\sigma_0(D)$, $\sigma(D)$, $(1-\sigma)(D)$ are regarded as maps from $\mathcal{E}'(\Omega)$ into $\mathcal{D}'(\Omega)$. The first term in the right-hand side of (3.11) is in $C^\infty(\Omega)$ and we may limit ourselves to check that

$$B = \sigma_0(D)uA(1-\sigma)(D)\colon\ \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega) \tag{3.12}$$

is smoothing in Ω. The operator B can be easily written in the pseudo differential form, $B = b(x, D)$, with

$$b(x, \xi) = \iint \exp[i(x-y)(\eta-\xi)]\big(1-\sigma(\xi)\big)\sigma_0(\eta)\,u(y)\,a(y, \xi)\,dy\,d\eta\,, \tag{3.13}$$

where a is the symbol of A. From the hypothesis (3.2) and from condition (1.1) we deduce that for suitable positive constants τ, c

$$\Psi_j(\xi)^{1-\tau}\varphi_j(x, \xi) \geqslant c|\xi|^c\,, \qquad x \in \operatorname{supp} u\,,\ \xi \in \mathbf{R}^n\,,\ j = 1, \ldots, n\,. \tag{3.14}$$

Set

$$\mathfrak{L} = \mathfrak{L}(\xi, D_y) = \sum_{j=1}^{n} \Psi_j(\xi)^{-2(1-\tau)} D_{y_j}^2\,, \tag{3.15}$$

where τ is the constant in (3.14), and take note of the identity

$$\begin{aligned}\exp[i(x-y)(\eta-\xi)] &= \\ &= \Big[\sum_{j=1}^{n} \Psi_j(\xi)^{-2(1-\tau)}(\eta_j-\xi_j)^2\Big]^{-1} \mathfrak{L}\big(\exp[i(x-y)(\eta-\xi)]\big)\,.\end{aligned} \tag{3.16}$$

Using (3.16) and integrating by parts repeatedly in (3.13) we obtain

$$D_x^\alpha b(x, \xi) = \iint \exp[i(x-y)(\eta-\xi)]\,H_{\alpha N}(y, \xi, \eta)\,dy\,d\eta\,, \tag{3.17}$$

with

$$\begin{aligned}H_{\alpha N}(y, \xi, \eta) &= \\ &= i^{|\alpha|}(\eta-\xi)^\alpha \Big[\sum_{j=1}^{n} \Psi_j(\xi)^{-2(1-\tau)}(\eta_j-\xi_j)^2\Big]^{-N} \big(1-\sigma(\xi)\big)\sigma_0(\eta)\,\mathfrak{L}^N\big(u(y)\,a(y, \xi)\big)\,,\end{aligned} \tag{3.18}$$

where N may be taken arbitrarily large; observe that $\eta - \xi \neq 0$ for

$$(\xi, \eta) \in \operatorname{supp}\big(1-\sigma(\xi)\big)\sigma_0(\eta) \subset \mathbf{R}^n_\xi \times \mathbf{R}^n_\eta\,.$$

From (3.14) we get

$$|\mathfrak{L}^N\big(u(y)\,a(y, \xi)\big)| \leqslant C_N(1+|\xi|)^{-2cN}\,|u(y)\,\lambda(y, \xi)|\,, \tag{3.19}$$

where we can estimate

$$|u(y)\,\lambda(y, \xi)| \leqslant C(1+|\xi|)^T \tag{3.20}$$

for a suitable T, in view of (3.15) in [2]. On the other hand, keeping in mind (3.8), we have in supp $(1-\sigma(\xi))\sigma_0(\eta)$

$$|\eta_{j_0}-\xi_{j_0}|\geqslant r\Psi_{j_0}(\xi)\,, \qquad |\eta_{j_0}-\xi_{j_0}|\geqslant r\Psi_{j_0}(\eta)\,, \tag{3.21}$$

for some index j_0, and therefore from (1.1)

$$|\eta_{j_0}-\xi_{j_0}|\geqslant c'(1+|\xi|)^{c'}\,, \qquad |\eta_{j_0}-\xi_{j_0}|\geqslant c'(1+|\eta|)^{c'} \tag{3.22}$$

with $c'>0$; from (3.21), (3.22) it follows

$$\Psi_{j_0}(\xi)^{-2(1-\tau)}(\eta_{j_0}-\xi_{j_0})^2\geqslant r^{2(1-\tau)}c'^{2\tau}(1+|\eta|)^{2\tau c'}\,. \tag{3.23}$$

Thus we can estimate in the right-hand side of (3.18)

$$\left[\sum_{j=1}^{n}\Psi_j(\xi)^{-2}(\eta_j-\xi_j)^2\right]^{-N}\leqslant C'_N(1+|\eta|^{-2\tau c'N}). \tag{3.24}$$

Summing up, from (3.18), (3.19), (3.20), (3.24) we have for large N

$$|H_{\alpha N}(y,\xi,\eta)|\leqslant C_{\alpha N}(1+|\eta|)^{|\alpha|-2\tau c'N}(1+|\xi|)^{|\alpha|+T-2cN}\,. \tag{3.25}$$

Since $H_{\alpha N}$ has compact support with respect to the variable y, from (3.17), (3.25) it follows

$$|D_x^\alpha b(x,\xi)|\leqslant C'_{\alpha M}(1+|\xi|)^{-M}\,, \qquad x\in\Omega\,,\ \xi\in\mathbf{R}^n\,, \tag{3.26}$$

for every integer $M\geqslant 0$. Using (3.26) we check readily that $b(x,D)$ has smooth kernel in Ω; therefore B in (3.12) is smoothing and Theorem 3.3 is proved.

Adding a technical hypothesis in Theorem 3.3, we may obtain easily the more general inclusion

$$\mathcal{F}_\Psi(f)\subset\mathcal{F}_\Psi(Af)\,, \qquad \textit{for every } f\in\mathcal{D}'(\Omega)\,, \tag{3.27}$$

which we have already proved in Theorem 2.2 for an operator $A\in\mathfrak{L}^\mu_\Psi(\Omega)$.

THEOREM 3.7. *Under the hypotheses of Theorem* 3.3, *set*

$$\Phi_j^*(x,\xi)=\min\big(\Phi_j(x,\xi),\Psi_j(\xi)\big)\,, \qquad j=1,\dots,n\,,$$

and assume

(3.28) $\Phi^* = (\Phi_1^*, \dots, \Phi_n^*)$, $\varphi = (\varphi_1, \dots, \varphi_n)$ *is a pair of local weight vectors in* Ω; $\lambda, \log\Psi_1, \dots, \log\Psi_n \in O(\Phi^*, \varphi)$.

Then every $A \in \mathfrak{L}^{\lambda}_{\Phi,\varphi}(\Omega)$ *satisfies* (3.27).

The proof proceeds as in Theorem 2.2, with the only difference that' to prove $\tau(x, D)Ac(x, D) \sim 0$ in (2.3), we shall use the following

LEMMA 3.8. *Let the hypotheses of Theorem* 3.7 *be satisfied. Assume* $A \in \mathfrak{L}^{\lambda}_{\Phi,\varphi}(\Omega)$. *Let* $\tau(x, D)$, $c(x, D)$ *be in* $S^0_{\Psi}(\Omega)$ *with rapidly decreasing symbol in* Γ, Λ, *respectively*; *suppose* $\Gamma \cup \Lambda = \Omega \times \mathbb{R}^n$. *Then* $\tau(x, D)Ac(x, D)f \in C^{\infty}(\Omega)$ *for every* $f \in \mathcal{D}'(\Omega)$.

Essentially: regarding $\tau(x, D)$, $c(x, D)$ and A as operators in $\mathfrak{L}^0_{\Phi^*,\varphi}(\Omega)$ and $\mathfrak{L}^{\lambda}_{\Phi^*,\varphi}(\Omega)$, one can apply Theorem 1.1 to their product and conclude that it is smoothing. For a detailed proof of Lemma 3.8 see [14], Lemma 2.4.

Translating into the language of the Ψ-filters the second part of Proposition 1.8 in [14] we may also obtain a partial converse of Theorem 3.3.

THEOREM 3.9. *Let* $\Phi(\xi) = (\Phi_1(\xi), \dots, \Phi_n(\xi))$, $\varphi(\xi) = (\varphi_1(\xi), \dots, \varphi_n(\xi))$ *be a pair of x-independent local weight vectors, and assume*

(3.29) $$\Psi_j(\xi)\varphi_j(\xi) \leqslant C \qquad \textit{for some } j \in \{1, \dots, n\} \textit{ and } C > 0 .$$

Then there exists $A \in \mathfrak{L}^0_{\Phi,\varphi}(\mathbb{R}^n)$ *which is not* Ψ*-local in* $\mathbb{R}^n$.

PROOF. It will be sufficient actually to assume

(3.30) $$\Psi_n(\xi_1, 0, \dots, 0)\varphi_n(\xi_1, 0, \dots, 0) \leqslant C , \qquad \xi_1 \in \mathbb{R}_+$$

(in the following we shall understand $n \geqslant 2$; the case $n = 1$ could be discussed separately in a lightly different way). Making use of the hypotheses of Beals on Φ, φ and recalling in particular (2.3), (2.8), (2.14) in [2], we may suppose for the same constant C of (3.30) and for suitable $c > 0$, $\varepsilon > 0$:

(3.31) $\Phi_j(\eta)^{-1} \leqslant C\Phi_j(\xi)^{-1}$, $\varphi_j(\eta)^{-1} \leqslant C\varphi_j(\xi)^{-1}$, $j = 1, \dots, n$, *if for each* $k = 1, \dots, n$, $|\xi_k - \eta_k| < \Phi_k(\eta)$;

(3.32) $$\Phi_1(\xi_1, 0, \dots, 0) \leqslant \xi_1 \qquad \textit{for } \xi_1 \geqslant 1 ;$$

(3.33) $$\Phi_j(\xi_1, 0, \dots, 0) \geqslant c\xi_1^{\varepsilon} \qquad \textit{for } \xi_1 \in \mathbb{R}_+ , \ j = 1, \dots, n .$$

In view of (1.1), (1.2), taking a multiple of Ψ as a new basic weight vector, we may also assume without loss of generality:

$$4\Psi_1(\xi_1, 0, \ldots, 0) \leqslant \xi_1 \quad \textit{for } \xi_1 \geqslant 1 ; \tag{3.34}$$

$$\Psi_j(\xi_1, 0, \ldots, 0) \geqslant c\xi_1^\varepsilon \quad \textit{for } \xi_1 \in \mathbb{R}_+ , \; j = 1, \ldots, n , \tag{3.35}$$

for the same constants c, ε of (3.33).

The operator A is constructed in the following way. First choose $\vartheta \in C_0^\infty(\mathbb{R})$ such that $\vartheta(t) = 0$ for $|t| \geqslant \frac{1}{4}$, $\vartheta(t) = 1$ for $|t| \leqslant \frac{1}{8}$. Write $\omega_s = (2^s, 0, \ldots, 0)$ and define for $s = 1, 2, \ldots$:

$$\zeta_s(\xi) = \vartheta\big(\Phi_1(\omega_s)^{-1}(\xi_1 - 2^s)\big) \prod_{j=2}^{n} \vartheta\big(\Phi_j(\omega_s)^{-1}\xi_j\big) ; \tag{3.36}$$

$\zeta_s(\xi)$ is in $C_0^\infty(\mathbb{R}^n)$ and

$$\operatorname{supp} \zeta_s \subset \{\xi \in \mathbb{R}^n, \; |\xi_1 - 2^s| \leqslant \Phi_1(\omega_s)/4, \; |\xi_j| \leqslant \Phi_j(\omega_s)/4 \; \textit{for } j = 2, \ldots, n\} . \tag{3.37}$$

Since $\Phi_1(\omega_s) \leqslant 2^s$ in view of (3.32), all $\operatorname{supp} \zeta_s$ are disjoint; moreover $\Phi_j(\omega_s)^{-1} \leqslant C\Phi_j(\xi)^{-1}$ for $\xi \in \operatorname{supp} \zeta_s$, $j = 1, \ldots, n$, in virtue of (3.31), and therefore

$$|D_\xi^\beta \zeta_s(\xi)| \leqslant c_\beta \Phi(\xi)^{-\beta} , \qquad \xi \in \mathbb{R}^n , \tag{3.38}$$

with constants which do not depend on s. Observe also that in view of (3.33) we have $\zeta_s(\xi) = 1$ for every ξ in the subset I_s,

$$I_s = \{\xi \in \mathbb{R}^n, \; |\xi_1 - 2^s| \leqslant c2^{\varepsilon s}/8, \; |\xi_j| \leqslant c2^{\varepsilon s}/8 \; \textit{for } j = 2, \ldots, n\} . \tag{3.39}$$

Let M_s be the least integer such that

$$M_s \geqslant 3C\varphi_n(\omega_s)^{-1} ; \tag{3.40}$$

in view of (2.1) in [2] we have for a suitable $\tau > 0$ independent of s:

$$M_s \leqslant \tau\varphi_n(\omega_s)^{-1} . \tag{3.41}$$

Fix $x_0 \in \mathbb{R}^n$ and let u be in $C_0^\infty(\mathbb{R}^n)$, $u(x) = 1$ in a neighborhood of x_0; we define

$$Af(x) = (2\pi)^{-n} \int \exp[ix\xi]\, u(x)\, a(x, \xi)\, (uf)\hat{\ }(\xi)\, d\xi \tag{3.42}$$

with

$$a(x, \xi) = \sum_{s=1}^{\infty} \exp[iM_s x_n]\, \zeta_s(\xi) . \tag{3.43}$$

It follows easily from (3.31), (3.37), (3.38), (3.41) that $a(x, \xi) \in S^0_{\Phi,\varphi}(\mathbf{R}^n)$ and hence $A \in \mathfrak{L}^0_{\Phi,\varphi}(\mathbf{R}^n)$. We claim that A is not Ψ-local. This will be tested on

$$f(x) = \chi(x) \sum_{s=1}^{\infty} 2^{-s} \exp[i2^s x_1] \in L^2_{\mathrm{comp}}(\mathbf{R}^n) , \tag{3.44}$$

where $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi(x) = 1$ in a neighborhood of x_0 and $\operatorname{supp} \chi \subset \{x \in \mathbf{R}^n, u(x) = 1\}$. Note Z_s the ball with centre ω_s and radius $2^{\varepsilon s/2}$, and set $Z = \bigcup_{s=1}^{\infty} Z_s$; observing that the series in the right-hand side of (3.44) is lacunary one gets the estimates (cf. [14], Lemma 2.2 and subsequent proof):

$$|\hat{f}(\xi)| \leq c_N |\xi|^{-N} \quad \textit{for } \xi \in \mathbf{R}^n \setminus Z . \tag{3.45}$$

Write $E = \{\omega_1, \omega_2, \dots\}$ and observe that in view of (3.35) we have for every $r > 0$ $Z \subset E_{r\Psi} \cap F_r$, where F_r is a suitable bounded set depending on r; it follows easily from (3.45), (1.25) and Theorem 2.7 that all the Ψ-neighborhoods $E_{r\Psi}$, $r > 0$, are in $\mathcal{F}_\Psi(f, x_0)$.

Let us prove that the assumption $E_\Psi \in \mathcal{F}_\Psi(Af, x_0)$ leads to a contradiction. First introduce

$$g(x) = \chi(x) \sum_{s=1}^{\infty} 2^{-s} \exp[i(2^s x_1 + M_s x_n)] , \tag{3.46}$$

with χ as before. Write $\omega'_s = (2^s, 0, \dots, 0, M_s)$, let T_s be the ball with centre ω'_s and radius $2^{\varepsilon s/2}$, and set $T = \bigcup_{s=1}^{\infty} T_s$; for any $\chi_1 \in C_0^\infty(\mathbf{R}^n)$ we have in this case estimates of the type:

$$|(\chi_1 g)\hat{}\,(\xi)| \leq c'_N |\xi|^{-N} \quad \textit{for } \xi \in \mathbf{R}^n \setminus T . \tag{3.47}$$

On the other hand, inserting (3.44) in (3.42) and using (3.39), (3.43), one obtains that $Af - g$ is C^∞ in a neighborhood of the origin (the argument is similar to those in [13], [14] and we omit the details). Hence $E_\Psi \in \mathcal{F}_\Psi(Af, x_0)$ implies $E_\Psi \in \mathcal{F}_\Psi(g, x_0)$ and for some $\chi_1 \in C_0^\infty(\mathbf{R}^n)$, $\chi_1(x) = 1$ in a neighborhood of x_0, we have

$$|(\chi_1 g)\hat{}\,(\xi)| \leq c''_N |\xi|^{-N} \quad \textit{for } \xi \in \mathbf{R}^n \setminus E_\Psi . \tag{3.48}$$

Writing $E_\Psi = \bigcup_{s=1}^{\infty} \{\omega_s\}_\Psi$, observing that all the sets $\{\omega_s\}_\Psi$ are disjoint in view of (3.34) and applying (3.30), (3.40), (3.35), one obtains that $T \cap E_\Psi$ is a bounded set. Therefore $(\chi_1 g)\hat{}\,(\xi)$ is of rapid decrease in all $\mathbb{R}^n$ and $x_0 \notin \operatorname{sing\,supp} g$. So we get the contradiction, since the series in (3.46) is lacunary and $\operatorname{sing\,supp} g = \operatorname{supp} g$. Theorem 3.9 is proved.

Let now P be a linear partial differential operator with smooth coefficients in Ω; obviously P is Ψ-local in Ω for every choice of Ψ.

DEFINITION 3.10. *P is said to be Ψ-hypoelliptic in Ω if for all $x_0 \in \Omega$*

$$\mathcal{F}_\Psi(f, x_0) = \mathcal{F}_\Psi(Pf, x_0)\,, \quad \textit{for every } f \in \mathcal{E}'(\Omega)\,. \tag{3.49}$$

From Proposition 2.8 it follows immediately:

PROPOSITION 3.11. *If P is Ψ-hypoelliptic in Ω, then it is hypoelliptic in Ω, i.e.* $\operatorname{sing\,supp} f = \operatorname{sing\,supp} Pf$ *for every $f \in \mathcal{E}'(\Omega)$.*

Let $\Psi'(\xi)$, $\Psi''(\xi)$ be two basic weight vectors such that for some $C > 0$

$$\Psi'_j(\xi) \leqslant C\Psi''_j(\xi)\,, \quad \xi \in \mathbb{R}^n\,,\ j = 1, \ldots, n\,. \tag{3.50}$$

Since $X_{C^{-1}\Psi''} \subset X_{\Psi''}$ for every $X \subset \mathbb{R}^n$, it follows from Theorem 2.7 that $\mathcal{F}_{\Psi''}(f, x_0) \subset \mathcal{F}_{\Psi'}(f, x_0)$ for every $f \in \mathcal{E}'(\Omega)$.

PROPOSITION 3.12. *Under the assumption* (3.50), *if P is Ψ'-hypoelliptic in Ω then it is also Ψ''-hypoelliptic in Ω.*

PROOF. We have to prove $\mathcal{F}_{\Psi''}(Pf, x_0) \subset \mathcal{F}_{\Psi''}(f, x_0)$ for every $x_0 \in \Omega$ and all $f \in \mathcal{E}'(\Omega)$. Assume Pf is Ψ''-smooth in $\{x_0\} \times X$, $X \subset \mathbb{R}^n$; then Pf is also Ψ''-smooth in $\{x_0\} \times X_{\varepsilon\Psi''}$, for some $\varepsilon > 0$. Since $\mathcal{F}_{\Psi''}(Pf, x_0) \subset \mathcal{F}_{\Psi'}(Pf, x_0)$, Pf is Ψ'-smooth in the same set $\{x_0\} \times X_{\varepsilon\Psi''}$. Therefore, if P is Ψ'-hypoelliptic, f is Ψ'-smooth in $\{x_0\} \times X_{\varepsilon\Psi''}$ and then it is clear from Theorem 2.7 that f is Ψ''-smooth in $\{x_0\} \times X$.

PROPOSITION 3.13. *Assume there exists a left parametrix of P, $E \in \mathcal{L}^{\lambda}_{\Phi,\varphi}(\Omega)$, for certain local weight vectors Φ, φ in Ω and $\lambda \in O(\Phi, \varphi)$; then P is Ψ-ipoelliptic for all the basic weight vectors Ψ which satisfy* (3.2).

In fact from Theorem 3.3 we have $\mathcal{F}_\Psi(Pf, x_0) \subset \mathcal{F}_\Psi(EPf, x_0)$ and on the other hand $\mathcal{F}_\Psi(EPf, x_0) = \mathcal{F}_\Psi(f, x_0)$ since the difference $EPf - f$ is smooth.

DEFINITION 3.14. *P is said to be Ψ-solvable at $\{x_0\} \times X$, $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, if for every $g \in \mathcal{E}'(\Omega)$ there exists $f \in \mathcal{E}'(\Omega)$ such that $Pf - g$ is Ψ-smooth in $\{x_0\} \times X$.*

The following obvious statement can be regarded as dual of Propositions 3.11, 3.12.

PROPOSITION 3.15. *Assume P is solvable in some neighborhood $V \subset \Omega$ of $x_0 \in \Omega$ in the following sense: for every $g \in \mathcal{E}'(\Omega)$ there exists $f \in \mathcal{E}'(\Omega)$ such that $Pf - g$ is C^∞ in V; then P is Ψ-solvable at $\{x_0\} \times X$ for every choice of the basic weight vector Ψ and all $X \subset \mathbb{R}^n$.*

PROPOSITION 3.16. *Under the assumption* (3.50), *if P is Ψ''-solvable at $\{x_0\} \times X$, $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, then it is also Ψ'-solvable at $\{x_0\} \times X$.*

REMARK 3.17. Note that the $(1 + |\xi|)$-solvability of P at $\{x_0\} \times \{\xi \in \mathbb{R}^n, \xi = t\xi^0, t \in \mathbb{R}_+\}$, $x_0 \in \Omega$, $\xi^0 \neq 0$, is equivalent to the solvability of P at (x_0, ξ^0) in the sense of Hörmander (see [8], Definition 3.3.3), whereas the $(1 + |\xi|)$-hypoellipticity of P in Ω implies the identity

$$\mathrm{WF}\, f = \mathrm{WF}\, Pf\,, \qquad \textit{for every } f \in \mathcal{E}'(\Omega) \tag{3.51}$$

(strict hypoellipticity, according to the terminology of [8]).

EXAMPLE 3.18. Let us apply the preceding arguments to the model in $\mathbb{R}^2$

$$P_{h,k} = D_{x_1} + i x_1^h D_{x_2}^k\,. \tag{3.52}$$

From the results in [15], [16], for example, we have:

(3.53) *$P_{h,k}$ is hypoelliptic in $\mathbb{R}^2$ if and only if one at least of the positive integers h, k is even.*

In the hypoelliptic case using the methods of [2] one can construct for $P_{h,k}$ a left parametrix $E_{h,k} \in \mathcal{L}^{\lambda}_{\Phi,\varphi}(\mathbb{R}^2)$, where noting

$$\sigma_k(\xi) = 1 + |\xi_1| + |\xi_2|^k\,, \tag{3.54}$$

$$g_{h,k}(x, \xi) = 1 + |\xi_1| + |x_1|^h\,|\xi_2|^k + \sigma_k(\xi)^{1/(h+1)}\,, \tag{3.55}$$

we have $\lambda = \lambda_{h,k} = \log g_{h,k}$ and $\Phi = (\Phi_1, \Phi_2)$, $\varphi = (\varphi_1, \varphi_2)$ with

$$\begin{cases} \Phi_1 = \Phi_{1,h,k} = g_{h,k}^{1/h} \sigma_k^{(1-1/h)/(h+1)}\,, \\ \Phi_2 = \Phi_{2,h,k} = \sigma_k^{1/k}\,; \end{cases} \tag{3.56}$$

$$\varphi_1 = \varphi_{1,h,k} = (g_{h,k}/\sigma_k)^{1/h}\,, \qquad \varphi_2 = \varphi_{2,h,k} = 1\,. \tag{3.57}$$

Observing that $g_{h,k}(x,\xi) \geqslant \sigma_k(\xi)^{1/(h+1)}$ and $\sigma_k(\xi) \leqslant C(1+|\xi|)^k$ for C independent of ξ, we get

$$\Phi_{1,h,k} \geqslant \sigma_k^{1/(h+1)}, \tag{3.58}$$

$$\varphi_{1,h,k} \geqslant \sigma_k^{-1/(h+1)} \geqslant C^{-1/(h+1)}(1+|\xi|)^{-k/(h+1)}. \tag{3.59}$$

First let us define the basic weight vector $\Psi(\xi)$ as in (1.41), fixing $M=(k,1)$ in $\mathbb{R}^2$. In view of (3.56), (3.57), (3.58) and of the first inequality in (3.59), $E_{h,k}$ can be regarded as an element of the class $L^k_{(k,1),1/(h+1),1/(h+1)}(\mathbb{R}^2)$ in the Example 3.5; hence, applying Proposition 3.12 and referring to the anisotropic wave front set in Example 3.5 we may conclude (cf. [16]):

(3.60) *if* $P_{h,k}$ *is hypoelliptic then* $\mathrm{WF}_{(k,1)}f = \mathrm{WF}_{(k,1)}P_{h,k}f$ *for every* $f \in \mathcal{E}'(\mathbb{R}^2)$.

We shall now investigate the $(1+|\xi|)^\varrho$-hypoellipticity, $0<\varrho\leqslant 1$, of $P_{h,k}$, we claim:

(3.61) $P_{h,k}$ *is* $(1+|\xi|)^\varrho$*-hypoelliptic in* $\mathbb{R}^2$ *if and only if it is hypoelliptic and* $k/(h+1)<\varrho$.

In fact, if one at least of the integers h, k is even and $k/(h+1)<\varrho$, we may apply Proposition 3.13 using the second inequality in (3.59). On the other hand if $P_{h,k}$ is not hypoelliptic certainly it is not $(1+|\xi|)^\varrho$-hypoelliptic either, in view of Proposition 3.11; moreover in the case $\varrho\leqslant k/(h+1)$ one can construct $f\in \mathcal{S}'(\mathbb{R}^2)$ which is not $(1+|\xi|)^\varrho$-smooth in the ray

$$\Gamma = \{(0,0)\}\times\{(\xi_1,\xi_2)\in\mathbb{R}^2,\ \xi_1=0,\ \xi_2>0\} \tag{3.62}$$

and such that $P_{h,k}f$ is $(1+|\xi|)^\varrho$-smooth there. To define f consider a solution $u(t)$ of

$$(i)^{h+1}u^{(h)}+tu=0\,,\qquad u(0)=1\,. \tag{3.63}$$

Then take $\nu\in C_0^\infty(\mathbb{R})$, $\nu(t)=1$ for $|t|\leqslant L$, $\nu(t)=0$ for $|t|\geqslant 2L$, write $u_1=\nu u$ and set

$$H(\xi)=\begin{cases} u_1(\xi_2^{-k/(h+1)}\xi_1) & \textit{for } \xi_2>0\,,\\ 0 & \textit{for } \xi_2\leqslant 0\,.\end{cases} \tag{3.64}$$

Extend H to $\tilde H\in\mathcal{S}'(\mathbb{R}^2)$ quasi-homogeneous with respect to the weight $(k,h+1)$, take $\tau\in C^\infty(\mathbb{R}^2)$, $\tau(\xi)=0$ for $|\xi|\leqslant\frac12$, $\tau(\xi)=1$ for $|\xi|\geqslant 1$, and define

$$f = \mathcal{F}^{-1}(\tau\tilde{H}), \qquad F = \mathcal{F}^{-1}(\tilde{H}), \tag{3.65}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in $\mathbb{R}^2$. The distribution $F \in \mathcal{S}'(\mathbb{R}^2)$ is quasi-homogeneous with respect to the same weight $(k, h+1)$ and $\operatorname{sing\,supp} F = \operatorname{sing\,supp} f = \{(0, 0)\}$. Fix an arbitrary $\chi \in C_0^\infty(\mathbb{R}^2)$ with $\chi(x) = 1$ in a neighborhood of the origin; it is easy to check that

$$\lim_{\xi_2 \to +\infty} (\chi f)^\wedge(0, \xi_2) = u(0) = 1 \tag{3.66}$$

and, in view of Theorem 2.7, this implies that f is not Ψ-smooth in Γ in (3.62) for any basic weight vector Ψ. On the other hand, if $\varrho \leqslant k/(h+1)$, in the region

$$\{(\xi_1, \xi_2) \in \mathbb{R}^2,\ \xi_1^2 + \xi_2^2 > 1,\ \xi_2 > 0,\ |\xi_1| < L\xi_2^\varrho\} \tag{3.67}$$

we have

$$(P_{h,k} f)^\wedge(\xi) = [(i)^{h+1}\xi_2^k \partial_{\xi_1}^h + \xi_1] H(\xi) = 0 \tag{3.68}$$

and using Theorem 2.7 we conclude that $P_{h,k}f$ is $(1 + |\xi|)^\varrho$-smooth in Γ. In particular we may discuss the validity for $P_{h,k}$ of (3.51); looking again at f in (3.65) and applying Remark 3.17 we obtain:

(3.69) $\mathrm{WF}\, f = \mathrm{WF}\, P_{h,k} f$ *for every* $f \in \mathcal{E}'(\mathbb{R}^2)$ *if and only if* $P_{h,k}$ *is hypoelliptic and* $k/(h+1) < 1$.

Incidentally, note that (3.61), (3.69) give a new proof of Theorem 3.9 for Φ, φ in (3.56), (3.57) and $\Psi(\xi) = (1 + |\xi|)^\varrho$. Let us refer to [15] for results on the propagation of $\mathrm{WF}\, f$ when $k/(h+1) \geqslant 1$. Fixing in (3.52) $k = 1$ we get

$$Q_h = D_{x_1} + i x_1^h D_{x_2}, \tag{3.70}$$

which is hypoelliptic if and only if h is even; for h even Q_h satisfies always (3.51) whereas it is $(1 + |\xi|)^\varrho$-hypoelliptic if and only if $1/(h+1) < \varrho$. As for the solvability, say in a neighborhood of the origin, we recall that Q_h is solvable if h is even; if h is odd Q_h is $(1 + |\xi|)$-solvable at every ray $\{(0, 0)\} \times \{(\xi_1, \xi_2) \in \mathbb{R}^2;\ \xi_1 = \lambda\xi_1^0,\ \xi_2 = \lambda\xi_2^0,\ \lambda \in \mathbb{R}_+,\ \xi_1^0 \neq 0 \text{ or } \xi_2^0 < 0\}$, but it is not $(1 + |\xi|)$-solvable at the ray Γ in (3.62) (see for example [8]). However:

(3.71) Q_h *is* $(1 + |\xi|)^\varrho$*-solvable at* Γ, *if* $\varrho \leqslant 1/(h+1)$.

In view of Proposition 3.16 it will be sufficient to prove $(1+|\xi|)^{\varrho}$-solvability for $\varrho = 1/(h+1)$. Trying to solve $Q_h f = g \in \mathcal{E}'(\mathbb{R}^2)$ by means of $f \in \mathcal{S}'(\mathbb{R}^2)$ we are reduced to consider

$$[(i)^{h+1}\xi_2\,\partial^h_{\xi_1} + \xi_1]\,H(\xi) = \hat{g}(\xi)\,. \tag{3.72}$$

By regarding (3.72) as an ordinary differential equation depending on the parameter ξ_2, one proves easily that there exists $H \in \mathcal{S}'(\mathbb{R}^2)$ solution in the region (3.67) for an arbitrary fixed L, and the difference $Q_h f - g$, $f = \mathcal{F}^{-1}(H)$, is indeed $(1+|\xi|)^{\varrho}$-smooth in Γ.

4. – A class of Fourier integral operators.

Let Λ be a fixed subset of $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ and assume the projection $\pi(\Lambda)$ is bounded in $\mathbb{R}^n_x$. We define $S^\mu_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})$, $\varepsilon > 0$, to be the class of all $a(x,\xi) \in C^\infty(\Lambda^{\varepsilon\Psi})$, $\Lambda^{\varepsilon\Psi}$ as in (1.16), which satisfy for every $\varepsilon' < \varepsilon$

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \leqslant c_{\alpha\beta\varepsilon'}\Psi(\xi)^{\mu-\beta} \qquad \textit{for } (x,\xi) \in \Lambda^{\varepsilon'\Psi}\,, \tag{4.1}$$

whereas we write $S^\mu_{\Psi,\mathrm{comp}}(\Lambda)$ for the class of all $a(x,\xi) \in S^\mu_\Psi(\mathbb{R}^n)$ such that $\operatorname{supp} a \subset \Lambda$.

In this section we shall refer to basic weight vectors Ψ of rational type, according to the definition in Example 1-13; we shall also assume that for suitable positive constants c, C

$$c \leqslant \Psi_j(\xi + t\vartheta)\,\Psi_j(\xi)^{-1} \leqslant C\,, \quad j = 1, \dots, n,\ t \in [-1,1]\,, \quad \text{if} \quad \min_k \Psi_k(\vartheta)\,\Psi_k(\xi)^{-1} \leqslant c\,. \tag{4.2}$$

Note that this new property holds for $\Psi(\xi) = [\xi]^M_M$ in (1.41), since $[\xi + t\vartheta]_M \leqslant [\xi]_M + |t|\,[\vartheta]_M$ if $t \in [-1,1]$. For a basic weight function condition (4.2) reads

$$c \leqslant \Psi(\xi + t\vartheta)\,\Psi(\xi)^{-1} \leqslant C\,, \quad t \in [-1,1]\,, \quad \textit{if } \Psi(\vartheta) \leqslant c\Psi(\xi)\,, \tag{4.3}$$

and joined with (1.42) implies (1.43); for example (4.3) is valid for $\Psi(\xi) = (1+|\xi|)^\varrho$, $0 < \varrho \leqslant 1$.

We want to study the Fourier integral operator

$$Ff(x) = (2\pi)^{-n}\int \exp\,[i\omega(x,\eta)]\,b(x,\eta)\hat{f}(\eta)\,d\eta\,, \qquad f \in C^\infty_0(\mathbb{R}^n)\,, \tag{4.4}$$

where the amplitude b and the phase ω satisfy the following properties.

Fix two basic weight vectors of rational type:

$$\Psi(\eta) = \big(\Psi_1(\eta), \ldots, \Psi_n(\eta)\big)\,, \qquad \Psi^*(\xi) = \big(\Psi_1^*(\xi), \ldots, \Psi_n^*(\xi)\big)\,;$$

let (4.2) be valid at least for $\Psi^*(\xi)$. As for the amplitude, we suppose $b(x, \eta) \in S^{\mu}_{\Psi,\mathrm{comp}}(\Lambda)$, where $\Lambda \subset \mathbb{R}^n_x \times \mathbb{R}^n_\eta$ has bounded projection $\pi(\Lambda) \subset \mathbb{R}^n_x$ (to guarantee the existence of non-trivial amplitude-functions, it will be convenient, though unnecessary in the following, to assume $\Lambda = \Gamma^{\delta\Psi}$, for some $\delta > 0$ and $\Gamma \subset \mathbb{R}^n_x \times \mathbb{R}^n_\eta$). The phase $\omega(x, \eta)$ is a real-valued C^∞ function in $\Lambda^{\varepsilon\Psi}$, for some $\varepsilon > 0$, and it satisfies the three conditions:

$$(4.5)\quad \begin{cases} \partial^2\omega(x, \eta)/\partial x_j\, \partial\eta_k \in S^0_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})\,, \\ \Psi_k(\eta)\, \partial^2\omega(x, \eta)/\partial\eta_j\, \partial\eta_k \in S^0_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})\,, \qquad \textit{for } j, k = 1, \ldots, n; \end{cases}$$

$$(4.6)\quad \begin{cases} \Psi_j(\eta)^{-1}\Psi_k(\eta)\, \partial^2\omega(x, \eta)/\partial x_j \partial\eta_k \in S^0_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})\,, \\ \Psi_j(\eta)^{-1}\, \partial^2\omega(x, \eta)/\partial x_j\, \partial x_k \in S^0_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})\,, \qquad \textit{for } j, k = 1, \ldots, n; \end{cases}$$

$$(4.7)\qquad \Psi_j^*\big(d_x\omega(x, \eta)\big) \sim \Psi_j(\eta) \qquad \textit{for } (x, \eta) \in \Lambda^{\varepsilon\Psi}\,,\ j = 1, \ldots, n\,.$$

In (4.7) we mean: the quotients of the functions $\Psi_j^*\big(d_x\omega(x, \eta)\big)$ and $\Psi_j(\eta)$ are bounded in $\Lambda^{\varepsilon'\Psi}$, for every $\varepsilon' < \varepsilon$. The product $\exp[i\omega(x, \eta)]\, b(x, \eta)$ in (4.4) (as well as similar products later on) is understood to be defined $= 0$ for $(x, \eta) \notin \Lambda^{\varepsilon\Psi}$. It will be useful to observe that the conditions in the Introduction

$$(4.8)\qquad \partial\omega(x, \eta)/\partial\eta_j \in S^0_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})\,, \qquad j = 1, \ldots, n\,,$$

$$(4.9)\qquad \Psi_j(\eta)^{-1}\, \partial\omega(x, \eta)/\partial x_j \in S^0_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})\,, \quad j = 1, \ldots, n;$$

imply trivially the weaker conditions (4.5), (4.6). Condition (4.7) is the spatially inhomogeneous version of the standard ellipticity condition on $|d_x\omega|$.

THEOREM 4.1. *Under the preceding assumptions we have*

(I) F: $C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)$ *continuously.*

(II) *Let* $A = a(x, D)$ *be in* $\mathcal{L}^{\nu}_{\Psi^*}(\mathbb{R}^n)$. *Then the product* AF: $\mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)$ *is the sum of a smoothing operator and an operator of the form* (4.4), *with the same phase* $\omega(x, \eta)$ *and with amplitude* $h(x, \eta) \in S^{\mu+\nu}_{\Psi,\mathrm{comp}}(\Lambda)$, $h(x, \eta) \sim$

$\sim \sum_\alpha h_\alpha(x, \eta)$, *where* $h_\alpha(x, \eta) \in S^{\mu+\nu-\alpha/2}_{\Psi,\mathrm{comp}}(\Lambda)$ *is given by:*

$$h_\alpha(x, \eta) = (\alpha!)^{-1}\, \partial_\xi^\alpha a\big(x, d_x\omega(x, \eta)\big)\, D_z^\alpha\big(\exp[i\bar\omega(z, x, \eta)]\, b(z, \eta)\big)\big|_{z=x}, \tag{4.10}$$

with

$$\bar\omega(z, x, \eta) = \omega(z, \eta) - \omega(x, \eta) - \langle d_x\omega(x, \eta), z - x\rangle . \tag{4.11}$$

LEMMA 4.2. *If* $a(x, \xi) \in S^\nu_{\Psi*}(\mathbb{R}^n)$, *then* $a\big(x, d_x\omega(x, \eta)\big) \in S^\nu_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})$.

PROOF. Noting $a^\#(x, \eta) = a\big(x, d_x\omega(x, \eta)\big)$, we want to prove that for every α, β we can write

$$D_x^\alpha D_\eta^\beta a^\#(x, \eta) = \sum_{(\alpha', \beta')\in I} (D_x^{\alpha'} D_\xi^{\beta'} a)\big(x, d_x\omega(x, \eta)\big)\, g_{\alpha'\beta'}(x, \eta), \tag{4.12}$$

where I is a suitable finite set of multi-indices and $g_{\alpha'\beta'}(x, \eta)$ are suitable symbols in $S^{\beta'-\beta}_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})$. Since we have from (4.7)

$$|(D_x^{\alpha'} D_\xi^{\beta'} a)\big(x, d_x\omega(x, \eta)\big)| \leqslant c_{\alpha'\beta'\varepsilon'}\Psi(\eta)^{\nu-\beta'}, \qquad (x, \eta) \in \Lambda^{\varepsilon'\Psi}, \tag{4.13}$$

for every $\varepsilon' < \varepsilon$, it is clear that (4.12) will imply $a^\#(x, \eta) \in S^\nu_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})$. Observe that (4.12) is trivial for $|\alpha + \beta| = 0$. Then assume formula (4.12) has been proved for every α, β with $|\alpha + \beta| = k$; differentiation with respect to x_j gives

$$\begin{aligned} D_{x_j} D_x^\alpha D_\eta^\beta a^\#(x, \eta) = \sum_{(\alpha', \beta')\in I} \Big\{ & (D_x^{\alpha'} D_\xi^{\beta'} a)\big(x, d_x\omega(x, \eta)\big)\, D_{x_j} g_{\alpha'\beta'}(x, \eta) + \\ & + \Big[\sum_{k=1}^n (D_{\xi_k} D_x^{\alpha'} D_\xi^{\beta'} a)\big(x, d_x\omega(x, \eta)\big)\, \partial^2\omega(x, \eta)/\partial x_j\, \partial x_k + \\ & + (D_{x_j} D_x^{\alpha'} D_\xi^{\beta'} a)\big(x, d_x\omega(x, \eta)\big)\Big]\, g_{\alpha'\beta'}(x, \eta)\Big\}. \end{aligned} \tag{4.14}$$

Applying the second assumption in (4.6) we recognize that (4.14) is still an expression of the type (4.12). Developing in the same way $D_{\xi_j} D_x^\alpha D_\eta^\beta a^\#(x, \eta)$, $|\alpha + \beta| = k$, and applying the first assumption in (4.6) we easily conclude that formula (4.12) is valid for $|\alpha + \beta| = k + 1$ and therefore, by induction, for every α, β. Lemma 4.2 is proved.

LEMMA 4.3. *Let* $a(x, D)$ *be a linear partial differential operator in the class* $\mathfrak{L}^\nu_{\Psi*}(\mathbb{R}^n)$. *Then*

$$h(x, \eta) = \exp[-i\omega(x, \eta)]\, a(x, D_x)\big(\exp[i\omega(x, \eta)]\, b(x, \eta)\big) \tag{4.15}$$

is a symbol in the class $S^{\mu+\nu}_{\Psi,\mathrm{comp}}(\Lambda)$ *with principal part* $h_0(x,\eta) = a(x, d_x\omega(x,\eta))\, b(x,\eta)$.

Letting $a(x, D)$ act under the integral sign in (4.4), Lemma 4.3 essentially proves the second part of Theorem 4.1 for a partial differential operator in $\mathfrak{L}^{\nu}_{\Psi*}(\mathbb{R}^n)$.

PROOF. A standard computation shows that $h(x,\eta)$ in (4.15) is given by

$$h(x,\eta) = \sum_{\alpha} h_\alpha(x,\eta)\,, \tag{4.16}$$

where $h_\alpha(x,\eta)$ is exactly as in (4.10), (4.11), but the sum in α is obviously finite. Hence, to obtain Lemma 4.3 it will be sufficient to prove that $h_\alpha(x,\eta)$ is actually a symbol in $S^{\mu+\nu-\alpha/2}_{\Psi,\mathrm{comp}}(\Lambda)$.

Observe first that the term $\partial^\alpha_\xi a(x, d_x\omega(x,\eta))$ in the right-hand side of (4.10) is in $S^{\nu-\alpha}_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})$, in view of Lemma 4.2. On the other hand we may develop

$$D^\alpha_z\big(\exp[i\bar\omega(z,x,\eta)]\,b(z,\eta)\big)\big|_{z=x} = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta} D^\beta_z\big(\exp[i\bar\omega(z,x,\eta)]\big)\big|_{z=x} D^{\alpha-\beta}_x b(x,\eta)\,, \tag{4.17}$$

where $D^{\alpha-\beta}_x b(x,\eta)\in S^{\mu}_{\Psi,\mathrm{comp}}(\Lambda)$. From (4.6) and from the fact that $\bar\omega(z,x,\eta)$ vanishes of order two at $z=x$ we deduce easily

$$D^\beta_z\big(\exp[i\bar\omega(z,x,\eta)]\big)\big|_{z=x}\in S^{\beta/2}_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon\Psi})\,.$$

Therefore the function in (4.17) is a symbol in $S^{\mu+\alpha/2}_{\Psi,\mathrm{comp}}(\Lambda)$ and it is proved that $h_\alpha(x,\eta)\in S^{\mu+\nu-\alpha/2}_{\Psi,\mathrm{comp}}(\Lambda)$.

LEMMA 4.4. *We can find*

(i) *a linear partial differential operator with constant coefficients* $P(D)\in\mathfrak{L}^{\nu^0}_{\Psi*}(\mathbb{R}^n)$, $\nu^0=(\nu^0_1,\dots,\nu^0_n)$, $\nu^0_j\geqslant 0$ *for all* j *and* $|\nu^0|\neq 0$;

(ii) *a symbol* $q(x,\eta)\in S^{-\nu^0}_{\Psi,\mathrm{comp}}(\Lambda^{\varepsilon'\Psi})$, *for some* ε', $0<\varepsilon'<\varepsilon$;

(iii) *a symbol* $r(x,\eta)\in\bigcap_\mu S^{\mu}_{\Psi,\mathrm{comp}}(\Lambda^{\varepsilon'\Psi})$,

such that the identity

$$\exp[i\omega(x,\eta)] = \big(q(x,\eta)P(D_x) + r(x,\eta)\big)\exp[i\omega(x,\eta)] \tag{4.18}$$

is valid for $(x,\eta)\in\Lambda$.

PROOF. Since Ψ^* is of rational type, we can find in $S^{\nu^0}_{\Psi^*}(\mathbb{R}^n)$ a polynomial $P(\xi)$ which is elliptic with respect to Ψ^*. Applying Lemma 4.3 we obtain easily that

$$p(x, \eta) = \exp[-i\omega(x, \eta)]P(D_x)\exp[i\omega(x, \eta)] \tag{4.19}$$

is a symbol in $S^{\nu^0}_{\Psi,\mathrm{loc}}(\Lambda^{\varepsilon'\Psi})$, $0 < \varepsilon' < \varepsilon$, whose principal part $P(d_x\omega(x, \eta))$ satisfies for suitable positive constants c, C:

$$|P(d_x\omega(x, \eta))| \geqslant c\Psi(\eta)^{\nu^0}, \qquad (x, \eta) \in \Lambda^{\varepsilon'\Psi}, \qquad |\eta| \geqslant C, \tag{4.20}$$

in view of (4.7). Let $\tau_0(x, \eta)$ be defined as in Lemma 1.12, with supp $\tau_0 \subset \Lambda^{\varepsilon'\Psi}$ and $\tau_0(x, \eta) = 1$ in a Ψ-neighborhood of Λ; set

$$q(x, \eta) = \begin{cases} \tau_0(x, \eta)/p(x, \eta) & \textit{for } (x, \eta) \in \Lambda^{\varepsilon'\Psi}, \\ 0 & \textit{for } (x, \eta) \notin \Lambda^{\varepsilon'\Psi}. \end{cases} \tag{4.21}$$

It follows from (4.20) that $q(x, \eta)$ is well defined in $S^{-\nu^0}_{\Psi,\mathrm{comp}}(\Lambda^{\varepsilon'\Psi})$ for large $|\eta|$. Therefore, if we fix a smooth extension of $q(x, \eta)$ and define

$$r(x, \eta) = \tau_0(x, \eta) - q(x, \eta)p(x, \eta) \tag{4.22}$$

in $\bigcap_\mu S^{\mu}_{\Psi,\mathrm{comp}}(\Lambda^{\varepsilon'\Psi})$, the identity (4.18) is trivially satisfied for $(x, \eta) \in \Lambda$.

PROOF OF THEOREM 4.1 (I). Using (4.7) and applying the condition (1.1) to Ψ^*, we obtain in every $\Lambda^{\varepsilon'\Psi}$, $\varepsilon' < \varepsilon$, the estimates

$$c(1 + |d_x\omega(x, \eta)|)^c \leqslant \Psi_j(\eta) \leqslant C(1 + |d_x\omega(x, \eta)|)^C, \tag{4.23}$$

where the positive constants c, C depend only on ε'; applying (1.1) to Ψ, from (4.23) we have for some other constants c', C'

$$c'(1 + |\eta|)^{c'} \leqslant 1 + |d_x\omega(x, \eta)| \leqslant C'(1 + |\eta|)^{C'}. \tag{4.24}$$

Differentiating under the integral sign in (4.1) and using the second inequality in (4.24) we obtain easily that $F\colon C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$. To prove $F\colon \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)$ we shall argue on the adjoint ${}^tF$:

$${}^tFf(x) = (2\pi)^{-n}\iint \exp[i[\omega(y, \eta) - x\eta]]\, b(y, \eta) f(y)\, dy\, d\eta, \tag{4.25}$$

where for the moment the integral is purely formal. We want to show that ${}^tF\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ continuously. We begin by assuming in (4.25) $b(y, \eta) \in$ $\in S^{\mu}_{\Psi,\mathrm{comp}}(\Lambda)$ with

$$\Psi(\eta)^\mu \leqslant C(1 + |\eta|)^{-n-1}\,, \qquad \eta \in \mathbb{R}^n\,, \tag{4.26}$$

for some positive constant C. In this case the integral with respect to η in (4.25) is absolutely convergent; moreover, if we fix an arbitrary $\chi \in C_0^\infty(\mathbb{R}^n)$ and if we take $\chi_0 \in C_0^\infty(\mathbb{R}^n)$, $\chi_0(y) = 1$ for $y \in \pi(\Lambda)$, we have

$$\|\chi\, {}^tFf\|_{L^2(\mathbb{R}^n)} \leqslant c_\chi \|\chi_0 f\|_{L^2(\mathbb{R}^n)}\,, \qquad f \in C^\infty(\mathbb{R}^n)\,, \tag{4.27}$$

with c_χ independent of f.

In the general case, we shall first apply Lemma 4.4 under the integral sign in (4.25). Actually, from (4.18) we have the identity

$$\exp[i\omega(y, \eta)]\, b(y, \eta) = b(y, \eta)\, q(y, \eta)\, P(D_y) \exp[i\omega(y, \eta)] + s(y, \eta) \tag{4.28}$$

where $P(D_y) \in \mathfrak{L}^{\nu^0}_{\Psi^*}(\mathbb{R}^n)$, $q(y, \eta) \in S^{-\nu^0}_{\Psi,\mathrm{comp}}(\Lambda^{\varepsilon'\Psi})$, $0 < \varepsilon' < \varepsilon$, $s(y, \eta) \in \bigcap_\mu S^{\mu}_{\Psi,\mathrm{comp}}(\Lambda)$. Inserting (4.28) in (4.25) and integrating by parts we obtain easily that

$${}^tFf(x) = Rf(x) + \iint \exp\big[i[\omega(y, \eta) - x\eta]\big]\, {}^tP(D_y)\big(q(y, \eta)\, b(y, \eta)\, f(y)\big)\, dy\, d\eta \tag{4.29}$$

where R is a smoothing operator. Now, applying the generalized Leibnitz formula, we have

$${}^tP(D_y)\big(q(y, \eta)\, b(y, \eta)\, f(y)\big) = \sum_\alpha (\alpha!)^{-1} D_y^\alpha\big(q(y, \eta)\, b(y, \eta)\big)\, L_\alpha(D_y) f(y)\,, \tag{4.30}$$

with

$$L_\alpha(\xi) = i^{|\alpha|} D^\alpha[{}^tP(\xi)] \in S^{\nu^0-\alpha}_{t(\Psi^*)}(\mathbb{R}^n)\,. \tag{4.31}$$

Then, inserting (4.30) in (4.29) we may write the integral as a sum of integrals of the type (4.25) where the amplitude-functions are in $S^{\mu-\nu^0}_{\Psi,\mathrm{comp}}(\Lambda)$ and f is replaced by the functions $L_\alpha(D) f$, where $L_\alpha(D) \in \mathfrak{L}^{\nu^0}_{t(\Psi^*)}(\mathbb{R}^n)$ for every α. Iterating the preceding argument, say M times, the condition (4.26) will be finally satisfied, so that in view of (4.27)

$${}^tF\colon H^{M\nu^0}_{t(\Psi^*),\mathrm{loc}}(\mathbb{R}^n) \to L^2_{\mathrm{loc}}(\mathbb{R}^n) \text{ continuously}\,. \tag{4.32}$$

Since Ψ is also of rational type, there exists a polynomial $Q(\eta) \in S^{\mu^0}_{\Psi}(\mathbb{R}^n)$, $\mu^0 = (\mu_1^0, \ldots, \mu_n^0)$, $\mu_j^0 \geqslant 0$ for all j, $|\mu^0| \neq 0$, which is elliptic with respect to Ψ; consider the function

$$(4.33)\qquad [{}^tQ(D)]^N\,{}^tFf(x) = (2\pi)^{-n}\iint \exp\,[i[\omega(y,\eta) - x\eta]]\,Q(\eta)^N b(y,\eta) f(y)\,dy\,d\eta\,.$$

Observing that $Q(\eta)^N b(y,\eta) \in S^{N\mu^0+\mu}_{\Psi,\mathrm{comp}}(\Lambda)$ we may argue as before in (4.33), and reach the following conclusion: for every $\mu^1 \in \mathbb{R}^n$ there exists $\mu^2 \in \mathbb{R}^n$ such that ${}^tF\colon H^{\mu^2}_{t(\Psi^*),\mathrm{loc}}(\mathbb{R}^n) \to H^{\mu^1}_{t\Psi,\mathrm{loc}}(\mathbb{R}^n)$ continuously. This is sufficient to obtain ${}^tF\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, and the first part of Theorem 4.1 is therefore proved.

PROOF OF THEOREM 4.1 (II). Letting $A = a(x, D)$ act under the integral sign in (4.4), we are easily reduced to prove

$$(4.34)\qquad a(x, D_x)\big(\exp\,[i\omega(x,\eta)]\,b(x,\eta)\big) = \exp\,[i\omega(x,\eta)]\,h(x,\eta) + w(x,\eta)\,,$$

where $w(x,\eta) \in \bigcap_\mu S^\mu_\Psi(\mathbb{R}^n)$ and $h(x,\eta) \sim \sum_\alpha h_\alpha(x,\eta)$, with $h_\alpha(x,\eta)$ as in (4.10), (4.11). Note that we have already seen in the proof of Lemma 4.3 that $h_\alpha(x,\eta)$ is actually in $S^{\mu+\nu-\alpha/2}_{\Psi,\mathrm{comp}}(\Lambda)$; in fact, the arguments there rely only on the assumption $a(x, D) \in \mathfrak{L}^\nu_{\Psi^*}(\mathbb{R}^n)$.

We can write:

$$(4.35)\qquad a(x, D_x)\big(\exp\,[i\omega(x,\eta)]\,b(x,\eta)\big) = \\ = (2\pi)^{-n}\iint \exp\,[i(x-z)\vartheta + i\omega(z,\eta)]\,b(z,\eta)\,a(x,\vartheta)\,dz\,d\vartheta\,.$$

Take $u \in C_0^\infty(\mathbb{R}^n)$, $u(z) = 1$ for $|z| \leqslant \sigma/2$, $u(t) = 0$ for $|z| \geqslant \sigma$, where the constant $\sigma > 0$ will be chosen later. Define

$$(4.36)\qquad b'(z,x,\eta) = u(z-x)\,b(z,\eta)\,;\qquad b''(z,x,\eta) = \big(1 - u(z-x)\big)\,b(z,\eta)$$

and denote by $g'(x,\eta)$, $g''(x,\eta)$ the functions which we get from the integral in the right-hand side of (4.35) by replacing there $b(z,\eta)$ with $b'(z,x,\eta)$, $b''(z,x,\eta)$, respectively. Using the obvious identity

$$(4.37)\qquad \exp\,[i(x-z)\vartheta] = |x-z|^{-2N}(-\Delta_\vartheta)^N \exp\,[i(x-z)\vartheta]$$

and applying repeatedly formula (4.18) of Lemma 4.4, we obtain by means of integrations by parts

$$(4.38)\qquad g''(x,\eta) = (2\pi)^{-n}\iint \exp\,[i\omega(z,\eta)]\,|x-z|^{-2N}\cdot \\ \cdot(-\Delta_\vartheta)^N a(x,\vartheta)\,\mathcal{M}^M_\eta\big(\exp\,[i(x-z)\vartheta]\,b''(z,x,\eta)\big)\,dz\,d\vartheta\,,$$

where the operator $\mathcal{M}_\eta = \mathcal{M}_\eta(z, D_z)$ is defined by

$$\mathcal{M}_\eta \psi(z) = {}^tP(D_z)\big(q(z,\eta)\psi(z)\big) + r(z,\eta)\psi(z) \tag{4.39}$$

with $q(x,\eta) \in S^{-\nu^0}_{\Psi,\mathrm{comp}}(\Lambda^{\varepsilon'\Psi})$, $r(x,\eta) \in \bigcap_\mu S^\mu_{\Psi,\mathrm{comp}}(\Lambda^{\varepsilon'\Psi})$, $0 < \varepsilon' < \varepsilon$, as in Lemma 4.4. Taking large N and M, we deduce easily from (4.38), (4.39) that $g''(x,\eta) \in \bigcap_\mu S^\mu_\Psi(\mathbb{R}^n)$. Thus we are reduced to consider the remainder

$$R(x,\eta) = \exp[-i\omega(x,\eta)]\, g'(x,\eta) - \sum_{|\alpha|<k} h_\alpha(x,\eta)\,. \tag{4.40}$$

A standard computation gives the following expression for R (see [18], for example):

$$R(x,\eta) = \sum_{|\alpha|=k} k(\alpha!)^{-1} R_\alpha(x,\eta)\,, \tag{4.41}$$

where R_α is defined by the oscillatory integral

$$R_\alpha(x,\eta) = (2\pi)^{-n}\iint \exp[-i(x-z)\vartheta]\, r_\alpha(x,\eta,\vartheta)\, b_\alpha(z,x,\eta)\, dz\, d\vartheta\,, \tag{4.42}$$

with

$$r_\alpha(x,\eta,\vartheta) = \int_0^1 (1-t)^{k-1}(\partial_\xi^\alpha a)\big(x, d_x\omega(x,\eta) - t\vartheta\big)\, dt\,, \tag{4.43}$$

$$b_\alpha(z,x,\eta) = D_z^\alpha\big(\exp[i\tilde\omega(z,x,\eta)]\, b'(z,x,\eta)\big)\,. \tag{4.44}$$

Note that $b_\alpha(z,x,\eta) = 0$ for $(z,\eta) \notin \Lambda$, or $(x,\eta) \notin \Lambda^{\sigma\Psi}$, where σ is the constant in the definition of the function u in (4.36); suppose $\sigma < \varepsilon/2$, for example. It will be sufficient to prove that $R_\alpha(x,\eta) \in S_\Psi^{\bar\mu-\alpha/2}(\mathbb{R}^n)$, for a fixed $\bar\mu \in \mathbb{R}^n$. As a matter of fact, we shall content ourselves with proving

$$|R_\alpha(x,\eta)| \leqslant c\Psi(\eta)^{\bar\mu-\alpha/2}\,, \qquad (x,\eta) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{4.45}$$

since the argument for the estimates of $D_x^\gamma D_\eta^\delta R_\alpha(x,\eta)$ is almost exactly the same.

Define for $\tau > 0$

$$W^1_{\tau,\eta} = \{\vartheta \in \boldsymbol{R}^n,\ \min_k \big(\Psi_k^*(\vartheta)\Psi_k(\eta)^{-1}\big) < \tau\} \subset \boldsymbol{R}^n_\vartheta \tag{4.46}$$

and consider the complement

$$W^2_{\tau,\eta} = \mathbb{R}^n_\vartheta \setminus W^1_{\tau,\eta} = \{\vartheta \in \mathbb{R}^n;\ \Psi_k^*(\vartheta) \geqslant \tau\Psi_k(\eta),\ k = 1, \ldots, n\}\,. \tag{4.47}$$

Arguing as in the proof of Lemma 1.10, we can easily construct a function $\lambda(\vartheta, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, with $\lambda(\vartheta, \eta) = 1$ if $\vartheta \in W^2_{\tau,\eta}$, $\lambda(\vartheta, \eta) = 0$ if $\vartheta \in W^1_{\tau',\eta}$, for some τ', $0 < \tau' < \tau$, and such that

$$(4.48) \qquad |D^\gamma_\vartheta \lambda(\vartheta, \eta)| \leqslant c_\gamma \Psi(\eta)^{-\gamma},$$

where the constants c_γ do not depend on ϑ. Set

$$(4.49) \quad r'_\alpha(x, \eta, \vartheta) = \big(1 - \lambda(\vartheta, \eta)\big)\, r_\alpha(x, \eta, \vartheta)\,; \quad r''_\alpha(x, \eta, \vartheta) = \lambda(\vartheta, \eta)\, r_\alpha(x, \eta, \vartheta)$$

and denote $R_{\alpha,1}(x, \eta)$, $R_{\alpha,2}(x, \eta)$ the functions which we get from the integral in the right-hand side of (4.42) by replacing there $r_\alpha(x, \eta, \vartheta)$ with $r'_\alpha(x, \eta, \vartheta)$, $r''_\alpha(x, \eta, \vartheta)$, respectively.

First let us estimate $R_{\alpha,1}(x, \eta)$. Applying the identity

$$(4.50) \quad \exp[-i(x-z)\vartheta] = \\ = \Big(1 + \sum_{j=1}^n \Psi_j(\eta)^2 |x_j - z_j|^2\Big)^{-N} \Big(1 + \sum_{j=1}^n \Psi_j(\eta)^2 D^2_{\vartheta_j}\Big)^N \exp[-i(x-z)\vartheta]$$

and integrating by parts, we may write

$$(4.51) \quad R_{\alpha,1}(x, \eta) = (2\pi)^{-n} \iint \exp[-i(x-z)\vartheta] \cdot \\ \cdot \Big(1 + \sum_{j=1}^n \Psi_j(\eta)^2 |x_j - z_j|^2\Big)^{-N} r_{\alpha,N}(x, \eta, \vartheta)\, b_\alpha(z, x, \eta)\, dz\, d\vartheta,$$

with

$$(4.52) \quad r_{\alpha,N}(x, \eta, \vartheta) = \Big(1 + \sum_{j=1}^n \Psi_j(\eta)^2 D^2_{\vartheta_j}\Big)^N r'_\alpha(x, \eta, \vartheta)\,.$$

If the constant τ in (4.46) is chosen sufficiently small, for $\vartheta \in W^1_{\tau,\eta}$ and $(x, \eta) \in \Lambda^{\varepsilon\Psi/2}$ we have

$$(4.53) \quad |(\partial^\alpha_\xi a)\big(x, d_x\omega(x, \eta) - t\vartheta\big)| \leqslant C_\alpha \Psi^*\big(d_x\omega(x, \eta)\big)^{\nu - \alpha},$$

since $\min_k \Psi^*_k(\vartheta) \Psi^*_k\big(d_x\omega(x, \eta)\big)^{-1} \leqslant C \min_k \Psi^*_k(\vartheta) \Psi_k(\eta)^{-1}$ in view of (4.7), and one may apply (4.2) to Ψ^* (here and in the following C, C_α, ... are suitable positive constants). Using (4.48), (4.53) and (4.7) from (4.43), (4.49), (4.52) we get

$$(4.54) \quad |r_{\alpha,N}(x, \eta, \vartheta)| \leqslant c_{\alpha,N} \Psi(\eta)^{\nu - \alpha} \quad \text{for } \vartheta \in \mathbb{R}^n,\ (x, \eta) \in \Lambda^{\varepsilon\Psi/2}\,.$$

On the other hand we have

$$(4.55)\qquad |b_\alpha(z,x,\eta)|\leqslant C'_\alpha \Psi(\eta)^{\mu+\alpha/2}\left(1+\sum_{j=1}^{n}\Psi_j(\eta)^2|x_j-z_j|^2\right)^{L_\alpha},$$

for suitable integers L_α, as one obtains easily from the arguments in the proof of Lemma 4.3 and from the estimates

$$(4.56)\qquad |\partial\omega(x,\eta)/\partial x_k-\partial\omega(z,\eta)/\partial z_k|\leqslant c_k\sum_{j=1}^{n}\Psi_j(\eta)|x_j-z_j|,$$

$$(x,\eta)\in\Lambda^{\varepsilon\Psi/2},\ (z,\eta)\in\Lambda,\ k=1,\dots,n,$$

which are consequences of (4.6). Applying (4.54), (4.55) in the integral (4.51) we obtain

$$(4.57)\qquad |R_{\alpha,1}(x,\eta)|\leqslant C_{\alpha,N}\Psi(\eta)^{\mu+\nu-\alpha/2}V(\eta)I(\eta)$$

where $V(\eta)$ is the volume of $W^1_{\tau,\eta}$ in (4.46) and

$$(4.58)\qquad I(\eta)=\int\left(1+\sum_{j=1}^{n}\Psi_j(\eta)^2v_j^2\right)^{L_\alpha-N}dv\,.$$

If N is fixed sufficiently large, we have

$$(4.59)\qquad I(\eta)\leqslant c_1\Psi_1(\eta)^{-1}\dots\Psi_n(\eta)^{-1}\,.$$

On the other hand, it follows from (1.1) that $W^1_{\tau,\eta}$ is included in the ball with centre the origin and radius $H(1+|\eta|)^H$, for a suitable $H>0$, and therefore

$$(4.60)\qquad V(\eta)\leqslant(2H)^n(1+|\eta|)^{nH}\leqslant c_2\Psi(\eta)^{\tilde\mu},$$

for a suitable $\tilde\mu\in\mathbb{R}^n$. Applying (4.59), (4.60) in (4.57), we conclude that an estimate of the type (4.45) is valid for $R_{\alpha,1}(x,\eta)$.

To handle $R_{\alpha,2}(x,\eta)$ it will be convenient to develop (4.44):

$$(4.61)\qquad b_\alpha(z,x,\eta)=\exp[i\tilde\omega(z,x,\eta)]\sum_\beta b_{\alpha\beta}(z,x,\eta),$$

where the sum in β is finite and the functions $b_{\alpha\beta}$ satisfy

$$(4.62)\qquad |D_z^\gamma b_{\alpha\beta}(z,x,\eta)|\leqslant c_{\alpha\beta\gamma}\Psi(\eta)^{\mu_{\alpha\beta}},\qquad (z,x,\eta)\in\mathbb{R}^{3n},$$

with suitable $\mu_{\alpha\beta} \in \mathbb{R}^n$; moreover:

$$b_{\alpha\beta}(z, x, \eta) = 0 \qquad \text{for } |z - x| \geqslant \sigma . \tag{4.63}$$

We are reduced to consider

$$T_{\alpha\beta}(x, \eta) = (2\pi)^{-n} \int \exp[-ix\vartheta]\, r''_\alpha(x, \eta, \vartheta) F_{\alpha\beta}(x, \eta, \vartheta)\, d\vartheta , \tag{4.64}$$

where $r''_\alpha(x, \eta, \vartheta)$ is defined in (4.49) and

$$F_{\alpha\beta}(x, \eta, \vartheta) = \int \exp[i\varrho(z, x, \eta, \vartheta)]\, b_{\alpha\beta}(z, x, \eta)\, dz , \tag{4.65}$$

with

$$\varrho(z, x, \eta, \vartheta) = z\vartheta + \bar{\omega}(z, x, \eta) . \tag{4.66}$$

Observe that $r''_\alpha(x, \eta, \vartheta) = 0$ for $\vartheta \notin W^2_{\tau',\eta}$, where $\tau' > 0$ is the constant in the definition of the function λ in (4.49); therefore we may limit ourselves to estimate $F_{\alpha\beta}(x, \eta, \vartheta)$ in $W^2_{\tau',\eta}$, where $\Psi^*_k(\vartheta) \geqslant \tau' \Psi_k(\eta)$, $k = 1, \ldots, n$, in view of (4.47). Applying (1.2) to Ψ^*, we have that in this subregion

$$\bar{c} \leqslant \Psi^*_j(\vartheta + \zeta)\Psi^*_j(\vartheta) \leqslant \bar{C} ,\ j = 1, \ldots, n , \qquad \text{if } \sum_{k=1}^{n} |\zeta_k| \Psi_k(\eta)^{-1} < \bar{c} , \tag{4.67}$$

where $\bar{c}$, $\bar{C}$ are suitable positive constants which do not depend on ϑ, ζ, η. Consider now

$$d_z \varrho(z, x, \eta, \vartheta) = \vartheta + d_z\omega(z, \eta) - d_x\omega(x, \eta) . \tag{4.68}$$

If σ is sufficiently small, for $|z - x| < \sigma$ we have

$$\sum_{k=1}^{n} |\partial\omega(z, \eta)/\partial z_k - \partial\omega(x, \eta)/\partial x_k| \Psi_k(\eta)^{-1} < \bar{c} , \tag{4.69}$$

as we obtain from (4.6); then from (4.67) it follows that

$$\Psi^*(d_z \varrho(z, x, \eta, \vartheta)) \sim \Psi^*(\vartheta) \tag{4.70}$$

for $(z, x, \eta) \in \operatorname{supp} b_{\alpha\beta}$ and $\vartheta \in W^2_{\tau',\eta}$. Let $P(D) \in \mathfrak{L}^{\nu^0}_{\Psi^*}(\mathbb{R}^n)$ be as in Lemma 4.4 and consider

$$p(z, x, \eta, \vartheta) = \exp[-i\varrho(z, x, \eta, \vartheta)]\, P(D_z) \exp[i\varrho(z, x, \eta, \vartheta)] . \tag{4.71}$$

Observing that $D_z^\alpha \varrho(z, x, \eta, \vartheta) = D_z^\alpha \omega(z, \eta)$ for $|\alpha| \geqslant 2$, arguing as in the proofs of Lemmas 4.3, 4.4, using (4.70) and keeping in mind that $\Psi_k^*(\vartheta) \geqslant \tau' \Psi_k(\eta)$, we see easily that

$$q(z, x, \eta, \vartheta) = p(z, x, \eta, \vartheta)^{-1} \tag{4.72}$$

is well defined for $(z, x, \eta) \in \operatorname{supp} b_{\alpha\beta}$, $\vartheta \in W^2_{\tau',\eta}$, $|\eta|$ large, and it satisfies there

$$|D_z^\gamma q(z, x, \eta, \vartheta)| \leqslant c_\gamma^1 \Psi^*(\vartheta)^{-\nu^0} \leqslant c_\gamma^2 \Psi(\eta)^{-\nu^0} \tag{4.73}$$

for constants c_γ^1, c_γ^2 which do not depend on z, x, η, ϑ. Applying in (4.65) the identity $\exp[i\varrho] = qP(D_z)\exp[i\varrho]$ and integrating by parts repeatedly, we obtain from (4.62), (4.73) that in $\operatorname{supp} r''_\alpha(x, \eta, \vartheta)$

$$F_{\alpha\beta}(x, \eta, \vartheta) \leqslant C_{\alpha\beta\mu^1\mu^2} \Psi(\eta)^{-\mu^1} \Psi^*(\vartheta)^{-\mu^2}, \tag{4.74}$$

for arbitrary $\mu^1, \mu^2 \in \mathbb{R}^n$. Hence, observing that $r''_\alpha(x, \eta, \vartheta) \leqslant C_\alpha \Psi^*(\vartheta)^{\bar\nu}$ for some $\bar\nu \in \mathbb{R}^n$ and fixing a suitable μ^2 in (4.74), we conclude:

$$|T_{\alpha\beta}(x, \eta)| \leqslant C'_{\alpha\beta\mu^1} \Psi(\eta)^{-\mu^1}, \qquad (x, \eta) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.75}$$

This ends the proof of the second part of Theorem 4.1.

We can now study the action of F in (4.4) on the Ψ-filter of a distribution. Let us associate to the phase $\omega(x, \eta)$ the transformations

$$\chi\colon \Lambda^{\varepsilon\Psi} \to \mathbb{R}^n_y \times \mathbb{R}^n_\eta, \qquad \chi((x, \eta)) = (y = d_\eta \omega(x, \eta), \eta), \tag{4.76}$$

$$\psi\colon \Lambda^{\varepsilon\Psi} \to \mathbb{R}^n_x \times \mathbb{R}^n_\xi, \qquad \psi((x, \eta)) = (x, \xi = d_x \omega(x, \eta)), \tag{4.77}$$

and assume ψ is a diffeomorphism with inverse

$$\psi^{-1}\colon \psi(\Lambda^{\varepsilon\Psi}) \to \Lambda^{\varepsilon\Psi}; \qquad \psi^{-1}((x, \xi)) = (x, \eta = \eta(x, \xi)); \tag{4.78}$$

suppose moreover $\psi(\Lambda)^{\delta\Psi^*} \subset \psi(\Lambda^{\varepsilon\Psi})$ for a sufficiently small $\delta > 0$ and

$$\begin{cases} \Psi_j^*(\xi)^{-1} \Psi_k^*(\xi)\, \partial\eta_j(x, \xi)/\partial\xi_k \in S^0_{\Psi^*,\mathrm{loc}}(\psi(\Lambda)^{\delta\Psi^*}), \\ \Psi_j^*(\xi)^{-1}\, \partial\eta_j(x, \xi)/\partial x_k \in S^0_{\Psi^*,\mathrm{loc}}(\psi(\Lambda)^{\delta\Psi^*}), \qquad j, k = 1, \ldots, n. \end{cases} \tag{4.79}$$

THEOREM 4.5. *Under the hypotheses of Theorem* 4.1 *and the additional assumptions* (4.79), *let* $f \in \mathcal{E}'(\mathbb{R}^n)$ *be* Ψ-*smooth in* $\Theta \subset \chi(\Lambda)$; *then* Ff *is* Ψ^*-*smooth in* $\Xi = \psi(\chi^{-1}(\Theta))$.

PROOF. In view of Lemma 2.3 there exists $c(x, D) \in \mathfrak{L}^0_\Psi(\mathbb{R}^n)$ with rapidly decreasing symbol in $\Theta^{\delta_1\Psi}$, for a suitable $\delta_1 > 0$, such that $f - c(x, D)f \in C_0^\infty(\mathbb{R}^n)$. Recalling (1.19) and Lemma 1.11, we can find $\delta_2 > 0$ such that

$$(4.80) \qquad (\Theta^{\delta_2\Psi})^{\delta_2\Psi} \subset \Theta^{\delta_1\Psi}.$$

Moreover, using (4.5) and (4.79), we see easily that there exist $\delta_3 > 0$, $\delta_4 > 0$ such that

$$(4.81) \qquad \psi(\chi^{-1}(\Theta)^{\delta_3\Psi}) \subset \Theta^{\delta_2\Psi}; \qquad \psi^{-1}(\Xi^{\delta_4\Psi^*}) \subset \chi^{-1}(\Theta)^{\delta_3\Psi}.$$

Then, applying Lemma 1.12, we take $\tau_0(x, \xi) \in S^0_{\Psi^*}(\mathbb{R}^n)$ with $\operatorname{supp} \tau_0 \subset \Xi^{\delta_4\Psi^*}$ and $\tau_0(x, \xi) = 1$ if $(x, \xi) \in \Xi^{\delta_5\Psi^*}$, for a suitable constant $\delta_5 > 0$. Let $\tau(x, D)$ be in $\mathfrak{L}^0_{\Psi^*}(\mathbb{R}^n)$, with $\tau(x, \xi) \sim \tau_0(x, \xi)$; to get the conclusion in Theorem 4.5 it will be sufficient to prove $\tau(x, D)Ff \in C_0^\infty(\mathbb{R}^n)$. Split

$$(4.82) \qquad \tau(x, D)Ff = \tau(x, D)F(f - c(x, D)f) + \tau(x, D)Fc(x, D)f.$$

The first term in the right-hand side is in $C_0^\infty(\mathbb{R}^n)$; therefore we may limit ourselves to prove that the operator $\tau(x, D)Fc(x, D)$ is smoothing. We shall begin by applying Theorem 4.1 (II) to the product $F_1 = \tau(x, D)F$; it follows that, modulo the addition of a function in $C_0^\infty(\mathbb{R}^n)$,

$$(4.83) \qquad F_1 f(x) = (2\pi)^{-n}\int \exp[i\omega(x, \eta)]\, b_1(x, \eta)\hat{f}(\eta)\, d\eta$$

where $b_1(x, \eta) \in S^\mu_{\Psi,\text{comp}}(\Lambda^{\varepsilon\Psi})$ with

$$(4.84) \qquad b_1(x, \eta) \sim \sum_\alpha (\alpha!)^{-1}\, \partial_\xi^\alpha \tau_0(x, d_x\omega(x, \eta))\, D_z^\alpha(\exp[i\tilde{\omega}(z, x, \eta)]\, b(z, \eta))|_{z=x}.$$

In view of (4.84) and of the second inclusion in (4.81) we may assume in (4.83) $\operatorname{supp} b_1 \subset \chi^{-1}(\Theta)^{\delta_3\Psi}$. To prove that $F_1c(x, D)$ maps $\mathcal{E}'(\mathbb{R}^n)$ into $C_0^\infty(\mathbb{R}^n)$, we shall argue on the adjoint ${}^tc(x, D)\,{}^tF_1$. From Theorem 1.3 we have that ${}^tc(x, D)$ is in $\mathfrak{L}^0_{t\Psi}(\mathbb{R}^n)$ with symbol

$$(4.85) \qquad c^\sim(x, \xi) \sim \sum_\alpha (\alpha!)^{-1}\, \partial_\xi^\alpha D_x^\alpha c(x, -\xi).$$

Then, arguing on the oscillatory integral in (4.25), we may write

$$(4.86) \qquad {}^tc(x, D)\,{}^tF_1 f(x) = (2\pi)^{-n}\iint \exp[i[\omega(y, \eta) - x\eta]]\, c^\sim(x, -\eta)\, b_1(y, \eta) f(y)\, dy\, d\eta.$$

Observe that $c^{\sim}(x, -\eta)$ is in $S^0_\Psi(\mathbb{R}^n)$; in view of (4.85) it will be not restrictive to assume $\operatorname{supp} c^{\sim}(x, -\eta) \subset \mathbb{R}^n \setminus \Theta^{\delta_1\Psi}$, if (4.86) is understood to be valid modulo the addition of functions in $C^\infty(\mathbb{R}^n)$. In (4.86) we may apply the identity

$$\text{(4.87)} \qquad \exp[i[\omega(y, \eta) - x\eta]] = |d_\eta \omega(y, \eta) - x|^{-2N}(-\Delta_\eta)^N \exp[i[\omega(y, \eta) - x\eta]]$$

and we integrate by parts. From (4.80) and from the first inclusion in (4.81) we have

$$\text{(4.88)} \qquad |d_\eta \omega(y, \eta) - x| \geqslant \delta_1 \quad \textit{for } (x, y, \eta) \in \operatorname{supp} c^{\sim}(x, -\eta)\, b_1(y, \eta);$$

hence the function

$$\text{(4.89)} \qquad e_N(x, y, \eta) = |d_\eta \omega(y, \eta) - x|^{-2N} c^{\sim}(x, -\eta)\, b_1(y, \eta)$$

satisfies the estimates

$$\text{(4.90)} \qquad |D_x^\alpha D_y^\beta D_\eta^\gamma e_N(x, y, \eta)| \leqslant c_{\alpha\beta\gamma N} \Psi(\eta)^{\mu-\gamma}, \qquad (x, y, \eta) \in \mathbb{R}^{3n},$$

in view of (4.5). Since N may be taken arbitrarily large, it follows easily that ${}^t c(x, D)\, {}^t F_1$: $\mathcal{D}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ continuously and this concludes the proof of Theorem 4.5.

EXAMPLE 4.6 (*Changes of variables*). Let $y = \sigma(x)$ be a diffeomorphism from a neighborhood U of $x_0 \in \mathbb{R}^n$ into a neighborhood $V = \sigma(U)$ of $y_0 = \sigma(x_0) \in \mathbb{R}^n$ and let $x = \sigma^{-1}(y)$ be its inverse. Fix a small neighborhood U_1 of x_0, $\overline{U}_1 \subset\subset U$, and write $V_1 = \sigma(U_1)$; the map

$$\text{(4.91)} \qquad F\colon \mathcal{E}'(V_1) \to \mathcal{E}'(U_1), \qquad (Ff)(x) = f \circ \sigma,$$

can be expressed in the form (4.4), with $\omega(x, \eta) = \langle \sigma(x), \eta \rangle$ and $b \in C_0^\infty(U)$, $b(x) = 1$ for $x \in U_1$.

Fix $\Psi(\eta)$ of rational type; assume for simplicity $\Psi(\eta)$ is a basic weight function, and let (4.3) be valid. Conditions (4.5), (4.6) are satisfied for F in (4.91) if and only if

$$\text{(4.92)} \qquad \Psi(\eta)^{-1}\, \partial^2 \langle \sigma(x), \eta \rangle / \partial x_j\, \partial x_k \in S^0_\Psi(U), \quad j, k = 1, \ldots, n.$$

Consider the linear transformation

$$\text{(4.93)} \qquad \sigma_*\colon \mathbb{R}^n_\eta \to \mathbb{R}^n_\xi, \qquad \xi = \sigma_*(\eta) = d_x \langle \sigma(x), \eta \rangle|_{x=x_0}$$

and define

$$\Psi^*(\xi) = \Psi((\sigma_*)^{-1}(\xi)); \tag{4.94}$$

it is easy to check that $\Psi^*(\xi)$ is a basic weight function of rational type satisfying (4.3). Moreover, possibly after a shrinking of U, from (4.92) we have that the property (4.7) is valid for $\Psi^*(\xi)$:

$$\Psi^*(d_x\langle\sigma(x), \eta\rangle) \sim \Psi(\eta) \qquad \textit{for } (x, \eta) \in U \times \mathbb{R}^n . \tag{4.95}$$

Let us introduce the new condition

$$\Psi^*(\xi)^{-1}\, \partial^2\langle\sigma^{-1}(y), \xi\rangle/\partial y_j\, \partial y_k \in S^0_{\Psi^*}(V) , \quad j, k = 1, \ldots, n . \tag{4.96}$$

We shall say that a (germ of) diffeomorphism σ is *Ψ-consistent* at x_0 if (4.92), (4.96) are both satisfied. Observe that every diffeomorphism σ is $(1 + |\xi|)$-consistent; a linear σ is Ψ-consistent for all Ψ, whereas all Ψ-consistent σ are linear if $\Psi(\xi) \leqslant C(1 + |\xi|)^\varrho$ for some $C > 0$ and $\varrho < 1$.

For a given collection $\mathcal{G}$ of subsets of $\mathbb{R}^n$ let us write

$$\sigma^*(\mathcal{G}) = \{Y \subset \mathbb{R}^n, (\sigma_*)^{-1}(Y) \in \mathcal{G}\} , \tag{4.97}$$

with σ_* as in (4.93); we claim:

(4.98) *if σ is Ψ-consistent at x_0 then $\mathcal{F}_{\Psi^*}(f\circ\sigma, x_0) = \sigma^*(\mathcal{F}_\Psi(f, x_0))$ for every distribution f defined in a neighborhood of $y_0 = \sigma(x_0)$.*

In fact, if σ is Ψ-consistent we can apply Theorem 4.5 to both F in (4.91) and its inverse F^{-1}; we have in particular that f is Ψ-smooth in $\{x_0\} \times X$ if and only if $f\circ\sigma$ is Ψ^*-smooth in $\{y_0\} \times \sigma_*(X)$ and therefore $Y \in \mathcal{F}_{\Psi^*}(f\circ\sigma, x_0)$ if and only if $Y \in \sigma^*(\mathcal{F}_\Psi(f, x_0))$.

EXAMPLE 4.7 (*Propagation for evolution equations*). We shall refer to the following translation invariant model. Let $\lambda(\xi)$ be a real x-independent symbol in $S^\mu_\Psi(\mathbb{R}^n)$, where the basic weight vector Ψ is of rational type and satisfies (4.2); assume also that for some constant $C > 0$

$$\Psi(\xi)^\mu \leqslant C \min_j \Psi_j(\xi) , \qquad \xi \in \mathbb{R}^n. \tag{4.99}$$

The unique solution $u \in C^\infty(\mathbb{R}_t; \mathcal{S}'(\mathbb{R}^n_x))$ of the problem

$$\partial_t u - i\lambda(D_x) u = 0 , \qquad u(0, x) = f_0(x) \in \mathcal{E}'(\mathbb{R}^n) , \tag{4.100}$$

is expressed formally in $\mathbb{R}_t \times K$, $K \subset\subset \mathbb{R}^n_x$, by

$$u(t, x) = (2\pi)^{-n} \int \exp [ix\eta + it\lambda(\eta)]\, b(x) \hat{f}_0(\eta)\, d\eta\,, \tag{4.101}$$

where $b \in C_0^\infty(\mathbb{R}^n)$, $b(x) = 1$ in a neighborhood of K. For every fixed $t \in \mathbb{R}$ we recognize in (4.101) a Fourier integral operator of the form (4.4); in fact, for $\omega_t(x, \eta) = x\eta + t\lambda(\eta)$ the condition (4.7) is trivially satisfied with $\Psi_t^* = \Psi$ and (4.5), (4.6) follow from (4.99). Define for $\Theta_0 \subset \mathbb{R}^n_x \times \mathbb{R}^n_\xi$

$$\Theta_t = \{(x, \xi),\ x = x_0 - t d_\xi \lambda(\xi^0),\ \xi = \xi^0 \textit{ for some } (x_0, \xi^0) \in \Theta_0\}\,. \tag{4.102}$$

Applying Theorem 4.5 in (4.101) and keeping in mind the reversibility of the problem (4.100) we obtain for $f_t(x) = u(t, x)$, $t \in \mathbb{R}$:

(4.103) *$f_t \in S'(\mathbb{R}^n)$ is Ψ-smooth in Θ_t if and only if $f_0 \in S'(\mathbb{R}^n)$ is Ψ-smooth in Θ_0.*

For example, fix $\Psi(\xi) = [\xi]^M_M$ as in (1.41) and consider the quasi-elliptic polynomial $Q(\xi) = 1 + \sum_{j=1}^n \xi_j^{2\tau/M_j}$, where τ is the least common multiple of the M_j's; from the resolution into pseudo differential factors of the operator $D_t^{2\tau} - Q(D_x)$ one obtains the first order terms $\partial_t \pm iQ^{1/2\tau}(D_x)$, to which proposition (4.103) applies in an obvious way (cf. [10], [17] and Theorem 1.6.5 in [8]). A more exotic example is given by $\lambda(\xi) = |\xi|^\varrho \sin |\xi|^{1-\varrho}$, where $0 < \varrho < 1$ and we argue on large $|\xi|$; for the corresponding equation $\partial_t u - i\lambda(D_x) u = 0$ the preceding arguments apply with $\Psi(\xi) = (1 + |\xi|)^\varrho$.

REFERENCES

[1] R. Beals, *Spatially inhomogeneous pseudodifferential operators, II*, Comm. Pure Appl. Math., **27** (1974), pp. 161-205.

[2] R. Beals, *A general calculus of pseudodifferential operators*, Duke Math. J., **42** (1975), pp. 1-42.

[3] R. Beals, *Characterization of pseudodifferential operators and applications*, Duke Math. J., **44** (1977), pp. 45-57.

[4] R. Beals, *L^p and Hölder estimates for pseudodifferential operators: sufficient conditions*, Ann. Inst. Fourier (Grenoble), **29** (1979), pp. 239-260.

[5] R. Beals - C. Fefferman, *Spatially inhomogeneous pseudodifferential operators, I*, Comm. Pure Appl. Math., **27** (1974), pp. 1-24.

[6] V. V. Grushin - N. A. Sananin, *Some theorems on the singularities of solutions of differential equations with weighted principal symbol*, Mat. Sb., **103** (1977), pp. 37-51; Math. USSR Sb., **32** (1977), pp. 32-44.

[7] L. Hörmander, *Fourier integral operators, I*, Acta Math., **121** (1971), pp. 79-183.
[8] L. Hörmander, *On the existence and the regularity of solutions of linear pseudo-differential equations*, Enseignement Math., **17** (1971), pp. 99-163.
[9] L. Hörmander, *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math., **32** (1979), pp. 359-443.
[10] R. Lascar, *Propagation des singularités des solutions d'équations pseudo-différentielles quasi-homogènes*, Ann. Inst. Fourier (Grenoble), **27** (1977), pp. 79-123.
[11] A. Nagel - E. M. Stein, *A new class of pseudo-differential operators*, Proc. Nat. Acad. Sci. U.S.A., **75** (1978), pp. 582-585.
[12] A. Nagel - E. M. Stein, *Lectures on pseudo-differential operators*, Princeton University Press (1979), Princeton.
[13] C. Parenti - L. Rodino, *A pseudo differential operator which shifts the wave front set*, Proc. Amer. Math. Soc., **72** (1978), pp. 251-257.
[14] C. Parenti - L. Rodino, *On general pseudodifferential operators*, Comm. Partial Diff. Equat., **5** (1980), pp. 561-594.
[15] C. Parenti - L. Rodino, *Examples of hypoelliptic operators which are not micro-hypoelliptic*, Boll. Un. Mat. Ital., **17**-B (1980), pp. 390-409.
[16] C. Parenti - L. Rodino, *Parametrices for a class of pseudo differential operators, I, II*, Ann. Mat. Pura Appl., **125** (1980), pp. 221-278.
[17] C. Parenti - F. Segala, *Propagation and reflection of singularities for a class of evolution equations*, Comm. Partial Diff. Equat., **6** (1981).
[18] F. Trèves, *Introduction to pseudodifferential and Fourier integral operators*, voll. 1, 2, Plenum Publishing Corporation (1980), New York.

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