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Olle Stormark

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On the Theorem of Frobenius for Complex Vector Fields.

OLLE STORMARK

Generalizing some results of Lewy, Andreotti and Hill proved in [1] that any system of first order homogeneous linear partial differential equations with complex valued real analytic coefficients locally may be reduced to a real part, consisting of the de Rham system, and a complex part, which consists of the tangential Cauchy-Riemann equations $\bar{\partial}_M$ on a generic locally closed real analytic submanifold M of some $\mathbb{C}^n$. Then the question arises whether the $\bar{\partial}_M$-equations can be simplified in their turn, for instance in such a way that they are composed of equations of Cauchy-Riemann and Lewy type. Geometrically this would mean that M admits complex foliations and Lewy foliations. In this paper we present a method for finding such and related foliations, which is based on Vessiot's theory for vector field systems.

1. – Some theorems of Frobenius type.

Let

$$X_k = \sum_{i=1}^{n} a_k^i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}, \qquad k = 1, \ldots, m, \tag{1}$$

be m real linearly independent vector fields on an open set U in $\mathbb{R}^n$, and let $\mathcal{F}$ be the vector field system generated by these. If we are only interested in solutions $u(x_1, \ldots, x_n)$ of the system

$$X_k u = 0, \quad k = 1, \ldots, m \tag{2}$$

we may as well assume (by maybe having added same brackets and thereby increased m) that $\mathcal{F}$ is complete, *i.e.*, closed under Lie brackets.

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Then the classical theorem of Frobenius says that it is possible to find local coordinates $(x_1', \ldots, x_n')$ such that the vector fields

$$X_k' = \frac{\partial}{\partial x_k'}, \qquad k = 1, \ldots, m, \tag{3}$$

form a basis for $\mathcal{F}$.

With these coordinates, (2) goes over into

$$\frac{\partial u}{\partial x_k'} = 0, \qquad k = 1, \ldots, m,$$

i.e., $u(x')$ is a function of $(x_{m+1}', \ldots, x_n')$ alone.

Or, in other words, to (1) there is associated a local foliation with the leaves $x_{m+1}' = c_{m+1}, \ldots, x_n' = c_n$ (where the c_i, $i = m+1, \ldots, n$, are constants), such that the solutions of (2) are constant along the leaves, but may vary arbitrarily from one leaf to another. In such a situation we say that the variables $x_1', \ldots, x_m'$ are principal, while $x_{m+1}', \ldots, x_n'$ are parametric.

Another way of stating the Frobenius theorem (in a way so that it can be generalized to the complex case) is by saying that (1) locally admits n-m functionally independent invariants $\xi_{m+1}, \ldots, \xi_n$, *i.e.* functions $\xi_i(x)$ satisfying (2) and the condition $d\xi_{m+1} \wedge \ldots \wedge d\xi_n \neq 0$. Namely, if one introduces the ξ_i and m other functions t_j, such that $d_1 \wedge \ldots \wedge dt_m \wedge d\xi_{m+1} \wedge \ldots \wedge d\xi_n \neq 0$, as new coordinates, then the X_k are expressed by means of the derivations $\partial/\partial t_j$ only, and Gauss elimination yields (3). Thus the ξ_i, $i = m+1, \ldots, n$, appear as parametric variables.

In this paper we want to consider what happens when the vector fields in (1) are allowed to be complex valued. To see what difference this makes, we first look at the case of just one vector field, say $P(x) = P'(x) + iP''(x)$, where P' and P'' are real vector fields defined on $U \subset \mathbf{R}^n$. If P' and P'' are linearly independent in U, P attaches to each point $x \in U$ two different directions (instead of one, as in the real case), which together span a 2-dimensional plane in $T_x(U)$ (where $T_x(M)$ denotes the real tangent space of a (real or complex) manifold M at the point x). In the most favourable case these planes fit together to form a foliation of U with 2-dimensional leaves. By the real Frobenius theorem this occurs if and only if the vector field system $\mathcal{T}$ spanned by P' and P'' is complete. Moreover, it may happen that these leaves can be given the structure of complex 1-dimensional manifolds, so that P appears as the Cauchy-Riemann equations on each leaf. Then the solutions $u(x)$ of $Pu = 0$ are holomorphic along all leaves (instead of being constant, as in the real case), but are arbitrary in the parametric variables.

However, if $\mathfrak{T}$ is not complete, it is natural to consider the vector field system $\mathfrak{T}'$ generated by P', P'' and $[P', P'']$. If this is complete, one gets a foliation of U with 3-dimensional leaves, so that P acts only within each leaf. Then it may happen that each leaf can be imbedded in a complex 2-dimensional manifold in such a way that the restriction to any leaf of a solution $u(x)$ to $Pu = 0$ is the boundary value of a holomorphic function (in two complex variables).

And if $\mathfrak{T}'$ is not complete, one adds more Lie brackets until a complete system is obtained, whereupon the Frobenius theorem gives a foliation.

Let us now consider systems of complex valued C^∞ vector fields $P_k(x) = P'_k(x) + iP''_k(x)$, $k = 1, \dots, m$, where P'_k and P''_k are real. The simplest (nonreal) case that occurs is handled by Nirenberg's complex Frobenius theorem (see [10]):

THEOREM 1. *Let $\mathfrak{T}$ be the vector field system generated by $P_1, \dots, P_m$ on an open neighbourhood of 0 in $\mathbf{R}^n$, and let $\tilde{\mathfrak{T}}$ be the vector field system generated by $P_1, \dots, P_m, \overline{P}_1, \dots, \overline{P}_m$. Assume that $\mathfrak{T}$ has constant rank m and that $\tilde{\mathfrak{T}}$ has constant rank $m + r$ (where $r \leqslant \min(m, n-m)$). Then, if*

$$[\mathfrak{T}, \mathfrak{T}] \subseteq \mathfrak{T} \quad \textit{and} \quad [\tilde{\mathfrak{T}}, \tilde{\mathfrak{T}}] \subseteq \tilde{\mathfrak{T}},$$

it is possible to introduce new coordinates x_j, y_j $(j = 1, \dots, r)$, t_k $(k = 1, \dots, m-r)$ and s_l $(l = 1, \dots, n-m-r)$ for $\mathbf{R}^n$ such that $\mathfrak{T}$ is generated by the vectorfields

$$\frac{\partial}{\partial \bar{z}_j}, \quad j = 1, \dots, r, \qquad \textit{and} \qquad \frac{\partial}{\partial t_k}, \quad k = 1, \dots, m-r,$$

where $z_j = x_j + iy_j$.

Here z_j and t_k are principal variables, while the s_l are parametric. And the foliation associated to $\mathfrak{T}$ has leaves of the form $\mathbf{C}^r \times \mathbf{R}^{m-r}$. On the $\mathbf{R}^{m-r}$-factor $\mathfrak{T}$ acts as the de Rham system, and on the $\mathbf{C}^r$-factor $\mathfrak{T}$ forms the Cauchy-Riemann system.

In the general case it is natural to preserve the condition $[\mathfrak{T}, \mathfrak{T}] \subseteq \mathfrak{T}$, since this changes nothing as far as the solutions of $\mathfrak{T}u = 0$ are concerned. Under this assumption Andreotti and Hill have proved the following generalization of the real Frobenius theorem (see [1]):

THEOREM 2. *Suppose that the complete vector field system $\mathfrak{T}$ is real analytic. Then locally one can find $n - m$ functionally independent complex valued real analytic invariants $\zeta_1(x_1, \dots, x_n), \dots, \zeta_{n-m}(x_1, \dots, x_n)$ for $\mathfrak{T}$, where $m = $ rank $\mathfrak{T}$.*

We remark that Nirenberg has shown in [11] that this theorem is not always true in the C^∞ category. For this reason (and for others, that will emerge later on) we assume from now on that all vector fields considered, as well as the manifolds they live on, are real analytic.

However, we do not want to make such a restrictive assumption on the solutions $u(x)$ of the system

$$P_k u = 0\,, \quad k = 1, \ldots, m\,, \tag{4}$$

but u may be a distribution or a hyperfunction for instance. But since we are interested in foliations, it is natural to suppose that the solutions u may be restricted to all leaves under consideration, and this we do from now on.

Now, just as in the real case, it is natural to use the invariants of $\mathcal{T}$ as new coordinates in order to reduce $\mathcal{T}$ to canonical form. More precisely, assume that $\mathcal{T}$ is complete, that rank $\mathcal{T} = m$ and that rank $\{\mathcal{T}, \overline{\mathcal{T}}\} = m + r$ (where $r \leqslant \min(m, n - m)$) on an open set $U \subseteq \mathbf{R}^n$. Then the following has been proved by Andreotti-Hill in [1]:

THEOREM 3. *Consider the map* $\zeta\colon U \to \mathbf{C}^{n-m}$ *defined by* $x \mapsto (\zeta_1(x), \ldots, \ldots, \zeta_{n-m}(x))$, *and a* lifting $\psi\colon U \to \mathbf{C}^{n-m} \times \mathbf{R}^{m-r}$ *of* ζ, *such that the diagram*

$$\begin{tikzcd} & \mathbf{C}^{n-m} \times \mathbf{R}^{m-r} \arrow[d, "\mathrm{pr}_{\mathbf{C}^{n-m}}"] \\ U \arrow[ur, "\psi"] \arrow[r, "\zeta"] & \mathbf{C}^{n-m} \end{tikzcd}$$

commutes.

Then for each point $x \in U$ there is an open neighbourhood ω of x such that $M := \zeta(\omega)$ is a generic locally closed real analytic submanifold of $\mathbf{C}^{n-m}$ of real dimension equal to $n - m + r$, and such that $N := \psi(\omega)$ is diffeomorphic to ω, and is an open set in $M \times \mathbf{R}^{m-r}$.

The system (4) *on ω is then equivalent to the system*

$$\begin{cases} \bar{\partial}_M u = 0 & (r \textit{ equations})\,, \\ d_{\mathbf{R}^{m-r}} u = 0 & (m - r \textit{ equations}) \end{cases} \tag{5}$$

on $N \subset M \times \mathbf{R}^{m-r}$, where $\bar{\partial}_M$ is the induced Cauchy-Riemann system on M and $d_{\mathbf{R}^{m-r}}$ is the de Rham system on an open set in $\mathbf{R}^{m-r}$.

As usual, functions u on M satisfying $\bar{\partial}_M u = 0$ are called *CR* functions.

We see that when the system (4) is written in the form of (5), it is splitted into a real part, namely the de Rham system, and a complex part. Since the real part is more or less trivial (at least locally), we consider only the complex part $\bar{\partial}_M$ in the following.

The $\bar{\partial}_M$-equations are in general not as pleasant as the de Rham or Cauchy-Riemann equations. For instance, $\bar{\partial}_M$ gives rise to a complex on M in a natural way, corresponding to the de Rham and Dolbeault sequences on $\mathbf{R}^n$ and $\mathbf{C}^n$ respectively. But it turns out that the cohomology groups of this complex are in general not locally trivial, as one can see from [2] or [14]. And this fact gives an explanation of the famous counter example of Lewy (see [2], part 1, § 5).

However, in the situation of Theorem 1 the $\bar{\partial}_M$-equations are easy to understand, because then M is locally of the form $\mathbf{R}^{n-m-r}\times\mathbf{C}^r$, where $\mathbf{R}^{n-m-r}$ is a parameter space, and $\bar{\partial}_M$ equals the Cauchy-Riemann system on the factor $\mathbf{C}^r$. Thus $\bar{\partial}_M u = 0$ means in this case that u is holomorphic on each leaf of the form $\mathbf{C}^r$, and is arbitrary in the parametric variables.

In the general case it can happen that M has a foliation where the leaves are perhaps not complex manifolds, but where each leaf can be imbedded in a complex manifold with some (real) codimension k, such that the restriction of a *CR* function u to each leaf is a boundary value of a holomorphic function. In particular, if $k = 1$ (in which case the foliation is said to be a hypersurface foliation), we then have a lot of information concerning the restrictions of u to the different leaves thanks to the work in [2] for instance. The purpose of this paper is to indicate how foliations such as these may be found.

Since we shall have to deal a lot with vector field systems in the sequel, we introduce the following convenient notation: If $X_1, \ldots, X_N$ are real analytic vector fields defined on a real analytic manifold M, then $(X_1, \ldots, X_N)$ denotes the vector field system generated by $X_1, \ldots, X_N$ (as a module over the ring of real analytic functions on M).

2. – Examples.

Before proceeding further, we present some examples which show how one in an intrinsic manner can find foliations related to the tangential Cauchy-Riemann equations $\bar{\partial}_M$. Since the foliations are eventually given by the real Frobenius theorem, we will mainly work with real vector fields.

EXAMPLE 1. Define the real 4-dimensional submanifold M of $\mathbf{C}^3$ by

$$\begin{cases} |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\,, \\ z_1 = \bar{z}_1\,. \end{cases}$$

Then a holomorphic vector field $\sum_{i=1}^{3} a_i \cdot \partial/\partial z_i$ on $\mathbf{C}^3$ is tangent to M if

$a_1 = 0$ and $a_2\bar{z}_2 + a_3\bar{z}_3 = 0$. The choice $a_2 = \bar{z}_3$ and $a_3 = -\bar{z}_2$ gives the vector field

$$Z = \bar{z}_3 \cdot \frac{\partial}{\partial z_2} - \bar{z}_2 \cdot \frac{\partial}{\partial z_3} = \frac{1}{2}(X - iJX),$$

where $X = x_3 \cdot \partial/\partial x_2 - x_2 \cdot \partial/\partial x_3 - y_3 \cdot \partial/\partial y_2 + y_2 \cdot \partial/\partial y_3$ (with $z_k = x_k + iy_k$), and J is the complex structure of $\mathbf{C}^3$. The system $\bar{\partial}_M u = 0$ then consists of the single equation $\bar{Z}u = 0$.

Calculations show that

$$\begin{aligned}
&JX = y_3 \cdot \partial/\partial x_2 - y_2 \cdot \partial/\partial x_3 + x_3 \cdot \partial/\partial y_2 - x_2 \cdot \partial/\partial y_3, \\
&[JX, X] = -2(y_2 \cdot \partial/\partial x_2 + y_3 \cdot \partial/\partial x_3 - x_2 \cdot \partial/\partial y_2 - x_3 \cdot \partial/\partial y_3), \\
&J[JX, X] = -2(x_2 \cdot \partial/\partial x_2 + x_3 \cdot \partial/\partial x_3 + y_2 \cdot \partial/\partial y_2 + y_3 \cdot \partial/\partial y_3), \\
&[X, [JX, X]] = 4JX, \\
&[JX, [JX, X]] = -2X,
\end{aligned}$$

and that $J[JX, X]$ commutes with X, JX and $[JX, X]$.

Hence (X, JX) is not complete, so there is no complex foliation of M. However, $(X, JX, [JX, X])$ is complete, and it is easily seen that the corresponding foliation of M has the leaves

$$L_{x_1} := M \cap \{z \in \mathbf{C}^3 : z_1 = x_1\} \quad \text{for } -1 < x_1 < 1,$$

i.e., x_1 is a parametric variable.

But also $(X, JX, [JX, X], J[JX, X])$ is complete, and gives a foliation of $\mathbf{C}^3 \setminus \{z \in \mathbf{C}^3 : z_2 = z_3 = 0\}$. Since this vector field system is invariant under J, the leaves inherit the complex structure J from $\mathbf{C}^3$, and hence are complex manifolds. In fact, they are

$$\tilde{L}_{x_1} := \{z \in \mathbf{C}^3 : z_1 = x_1;\ z_2 \text{ or } z_3 \neq 0\} \quad \text{for } -1 < x_1 < 1.$$

Thus $\{L_{x_1}\}_{x_1 \in (-1,1)}$ forms a hypersurface foliation of M.

Now the Bochner theorem shows that if u is a CR function on M then for each $x_1 \in (-1, 1)$, $u_{x_1} := u|_{L_{x_1}}$ is the boundary value of a holomorphic function $\tilde{u}_{x_1}$ defined on $\{z \in \mathbf{C}^3 : z_1 = x_1, |z_2|^2 + |z_3|^2 < 1 - x_1^2\}$. (In case u_{x_1} is a hyperfunction we refer to [12] or [14] for the existence of $\tilde{u}_{x_1}$.)

The idea behind the considerations above is the following: If (X, JX) were a complete system, M would have a complex 1-dimensional foliation. Since this is not the case, we first add $[JX, X]$ so as to get the complete system $(X, JX, [JX, X])$, which gives the foliation $\{L_{x_1}\}$. However, $(X, JX, [JX, X])$ is not invariant under J, so we enlarge this system to $(X, JX, [JX, X], J[JX, X])$, which is both complete and invariant under J. Hence it gives rise to a foliation $\{\tilde{L}_{x_1}\}$ with complex leaves, which is an « analytic continuation » of $\{L_{x_1}\}$.

REMARK 1. *On* $J[JX, X]$. Let M be a real hypersurface defined in a neighbourhood of the origin in $\mathbf{C}^2$ by a real equation $\varphi(z, \bar{z}) = 0$ such that $\varphi(0) = 0$ and $\operatorname{grad}\varphi(0) \neq 0$. We assume that the coordinates are chosen such that M is tangent to the hyperplane $\{z \in \mathbf{C}^2 : \operatorname{Re} z_2 = 0\}$ at 0, *i.e.*, $\partial\varphi/\partial x_1(0) = \partial\varphi/\partial y_1(0) = \partial\varphi/\partial y_2(0) = 0$, and $\partial\varphi/\partial x_2(0) = \operatorname{grad}\varphi(0) \neq 0$, where $z_k = x_k + iy_k$, $k = 1, 2$.

Up to multiplication with a function, there is just one holomorphic vector field which is tangent to M, namely,

$$Z = \frac{\partial\varphi}{\partial z_2}\cdot\frac{\partial}{\partial z_1} - \frac{\partial\varphi}{\partial z_1}\cdot\frac{\partial}{\partial z_2} = \frac{1}{2}(X - iJX),$$

where

$$X = \frac{\partial\varphi}{\partial x_2}\cdot\frac{\partial}{\partial x_1} - \frac{\partial\varphi}{\partial x_1}\cdot\frac{\partial}{\partial x_2} - \frac{\partial\varphi}{\partial y_2}\cdot\frac{\partial}{\partial y_1} + \frac{\partial\varphi}{\partial y_1}\cdot\frac{\partial}{\partial y_2}.$$

Then $X|_M$ and $JX|_M$ generate the module of *CR* vector fields on M (*i.e.*, the vector fields Y on M, such that also JY is tangent to M).

Straightforward calculations now show that

$$\langle J[JX, X], d\varphi\rangle(0) = -4\cdot(\operatorname{grad}\varphi(0))^2\cdot\frac{\partial^2\varphi}{\partial z_1\partial\bar{z}_1}(0), \tag{6}$$

where $\partial^2\varphi/\partial z_1\partial\bar{z}_1(0)$ is the Levi form of M at 0.

Hence the following statements are equivalent at 0:

1) $J[JX, X]|_M$ has a nonvanishing normal component;

2) the Levi form of M is nonvanishing;

3) M is strictly pseudoconvex.

Furthermore, the local Bochner theorem (see [12] or [14] for the hyperfunction case) shows that in this situation each *CR* function on M has an analytic continuation to a one-sided neighbourhood of M in $\mathbf{C}^2$ at 0. And by (6), this neighbourhood lies on that side of M into which $J[JX, X]|_M$ points.

Let ω be a neighbourhood of 0 in M. Then (6) also implies that the following statements are equivalent:

1) the Levi form of M vanishes identically in ω;

2) the normal component of $J[JX, X]|_M$ vanishes identically in ω;

3) $[JX, X]|_\omega$ is a CR vector field;

4) ω has a complex foliation.

Here 3) implies 4), since 3) shows that $[JX, X]$ is a linear combination of X and JX on ω, so that $(X, JX)|_\omega$ is complete, and this vector field system is obviously closed under J. Conversely, if ω has a complex foliation, this is associated to $(X, JX)|_\omega$, which must therefore be complete. Thus $[JX, X]|_\omega$ is a linear combination of $X|_\omega$ and $JX|_\omega$, and hence is a CR vector field.

REMARK 1′. *One-sided continuation and the heat equation.* Let M be a strictly pseudoconvex hypersurface defined in an open set U in $\mathbf{C}^2$, where U is so small that the vector fields X, JX, $[JX, X]$ and $J[JX, X]$, given in the above remark, are defined and linearly independent in all of U.

Set $Y := J[JX, X]$ and $Z := X - iJX$, $W := Y - iJY$. Then Z and W form a basis for the holomorphic vector fields on U. Hence a function $F = F_1 + iF_2 \in C^\infty(U)$ is holomorphic in U if and only if

$$\bar{Z}F = \overline{W}F = 0 \tag{7}$$

in U, that is,

$$0 = (X + iJX)(F_1 + iF_2) = XF_1 - JXF_2 + i(JXF_1 + XF_2)\,,$$

and

$$0 = (Y + iJY)(F_1 + iF_2) = YF_1 - JYF_2 + i(JYF_1 + YF_2)\,.$$

Thus (7) is equivalent to the system of real equations

$$\begin{cases} XF_1 = JXF_2\,, & XF_2 = -JXF_1\,, \\ YF_1 = JYF_2\,, & YF_2 = -JYF_1\,. \end{cases} \tag{8}$$

Now let f be a C^∞ CR function on M, and consider the following Cauchy problem in U:

$$\begin{cases} \bar{Z}F = \overline{W}F = 0\,, \\ F|_M = f\,. \end{cases} \tag{9}$$

Of course, this problem is not well-posed, but following [6] we can formulate it in such a way that we at least obtain a formal solution. Namely, by (8),

$$\begin{aligned} YF_1 = JYF_2 = -[JX, X]F_2 = -JX(XF_2) + X(JXF_2) = \\ = -JX(-JXF_1) + X(XF_1) = (X\circ X + JX\circ JX)F_1 . \end{aligned}$$

This equation remains true with F_1 replaced by F_2, and hence

$$YF = (X\circ X + JX\circ JX)F . \tag{10}$$

Now $Y|_M$ has a nonvanishing normal component, while $X|_M$ and $JX|_M$ are vector fields which are tangent to M. Thus

$$(YF)|_M = (X\circ X + JX\circ JX)f ,$$

and an iteration of this equation shows that F is uniquely determined in a formal neighbourhood of M in U.

If $J[JX, X]$ commutes with the vector field system $(X, JX, [JX, X])$ and if the latter is complete (as was the case in example 1), we can make this argument more explicit in the following way (we give the calculations in some detail, since they indicate a general procedure for finding suitable local coordinates associated to complete vector field systems):

Let ξ_1, ξ_2 and ξ_3 be real functionally independent invariants of $J[JX, X]$, and let η be a real invariant of $(X, JX, [JX, X])$.

Then

$$\begin{cases} X = \sum_{i=1}^{3} a_i(\xi, \eta)\cdot\dfrac{\partial}{\partial\xi_i} , \\ JX = \sum_{i=1}^{3} b_i(\xi, \eta)\cdot\dfrac{\partial}{\partial\xi_i} , \\ [JX, X] = \sum_{i=1}^{3} c_i(\xi, \eta)\cdot\dfrac{\partial}{\partial\xi_i} , \\ J[JX, X] = h(\xi, \eta)\cdot\dfrac{\partial}{\partial\eta} . \end{cases}$$

The commutativity assumption shows that

$$0 = [J[JX, X], X] = h(\xi, \eta)\cdot\sum_{i=1}^{3}\frac{\partial a_i}{\partial\eta}\cdot\frac{\partial}{\partial\xi_i} - \left(\sum_{i=1}^{3} a_i\cdot\frac{\partial h}{\partial\xi_i}\right)\cdot\frac{\partial}{\partial\eta} ,$$

that is,

$$\frac{\partial a_i}{\partial \eta} = 0 \qquad \text{for } i = 1, 2, 3 \qquad \text{and} \qquad \sum_{i=1}^{3} a_i \cdot \frac{\partial h}{\partial \xi_i} = 0 \,.$$

Analogously,

$$\frac{\partial b_i}{\partial \eta} = \frac{\partial c_i}{\partial \eta} = 0 \qquad \text{for } i = 1, 2, 3$$

and

$$\sum_{i=1}^{3} b_i \cdot \frac{\partial h}{\partial \xi_i} = \sum_{i=1}^{3} c_i \cdot \frac{\partial h}{\partial \xi_i} = 0 \,.$$

Since X, JX and $[JX, X]$ are linearly independent, this shows that $\partial h/\partial \xi_i = 0$ for $i = 1, 2, 3$. Hence a_i, b_i and c_i are functions of ξ only, while h only depends on η. Since M is an integral manifold of $(X, JX, [JX, X])$, the functions $\xi_i|_M$, $i = 1, 2, 3$, form a system of local coordinates for M. Setting

$$t = \int \frac{d\eta}{h(\eta)}, \tag{11}$$

we get

$$\begin{cases} X = \sum_{i=1}^{3} a_i(\xi) \cdot \dfrac{\partial}{\partial \xi_i}, \\[2ex] JX = \sum_{i=1}^{3} b_i(\xi) \cdot \dfrac{\partial}{\partial \xi_i}, \\[2ex] J[JX, X] = \dfrac{\partial}{\partial t}. \end{cases}$$

Since t is independent of ξ, t is constant on M. Hence we can choose the integration constant in (11) so that $t = 0$ on M. With these new coordinates, (10) goes over into

$$\frac{\partial F}{\partial t} = \Delta_\xi F, \tag{12}$$

where $\Delta_\xi = X \circ X + JX \circ JX$ is a second order differential operator in the ξ-variables. Hence the Cauchy problem (9) has the formal solution

$$F(\xi, t) = f(\xi) + \sum_{n=1}^{\infty} \frac{\Delta_\xi^{(n)} f(\xi)}{n!} \cdot t^n \,. \tag{13}$$

As (12) resembles the heat equation, one suspects that (9) will have an actual solution when $t > 0$ (and small) and this is confirmed by the local

Bochner theorem. Perhaps this observation (besides the usual disc arguments) may give an intuitive explanation of the one-sidedness of analytic continuation from pseudoconvex hypersurfaces.

EXAMPLE 2. The real 4-dimensional manifold M in $\mathbf{C}^3$ is defined by

$$\begin{cases} |z_1|^2 - |z_2|^2 + i(z_3 - \bar{z}_3) = 0 \\ z_3 + \bar{z}_3 = 0 \,. \end{cases}$$

Then there is (up to multiplication with a function) only one holomorphic vector field which is tangent to M, namely

$$Z = \bar{z}_2 \cdot \frac{\partial}{\partial z_1} + \bar{z}_1 \cdot \frac{\partial}{\partial z_2} = \frac{1}{2}(X - iJX)\,,$$

where $X = x_2 \cdot \partial/\partial x_1 + x_1 \cdot \partial/\partial x_2 - y_2 \cdot \partial/y_1 - y_1 \cdot \partial/\partial y_2$.

Furthermore,

$$JX = y_2 \cdot \partial/\partial x_1 + y_1 \cdot \partial/\partial x_2 + x_2 \cdot \partial/\partial y_1 + x_1 \cdot \partial/\partial y_2\,,$$

$$[JX, X] = 2(y_1 \cdot \partial/\partial x_1 + y_2 \cdot \partial/\partial x_2 - x_1 \cdot \partial/\partial y_1 - x_2 \cdot \partial/\partial y_2)\,,$$

$$J[JX, X] = 2(x_1 \cdot \partial/\partial x_1 + x_2 \cdot \partial/\partial x_2 + y_1 \cdot \partial/\partial y_1 + y_2 \cdot \partial/\partial y_2) = \\ = \operatorname{grad}(x_1^2 + x_2^2 + y_1^2 + y_2^2)\,,$$

$$[X, [JX, X]] = -4JX\,,$$

$$[JX, [JX, X]] = 4X\,,$$

and $J[JX, X]$ commutes with X, JX and $[JX, X]$.

The vector fields X, JX and $[JX, X]$ are linearly independent on M save on the exceptional set

$$M_0 := M \cap \{y_3 = 0\} = \{z \in \mathbf{C}^3 : z_3 = 0,\ |z_1| = |z_2|\}\,.$$

Hence M_0 has a complex foliation, which one also sees directly:

$$M_0 = \bigcup_{s \in \mathbf{R}} \{z \in \mathbf{C}^2 : z_2 = e^{is} \cdot z_1\}\,.$$

On $M \setminus M_0$ the complete vector field system $(X, JX, [JX, X])$ defines a foliation with real 3-dimensional leaves

$$L_{y_3} := M \cap \{z \in \mathbf{C}^3 : \operatorname{Re} z_3 = 0,\ \operatorname{Im} z_3 = y_3\} \quad \text{for } y_3 \neq 0\,.$$

To get an «analytic continuation» of $\{L_{y_3}\}_{y_3\neq 0}$, we add on $J[JX, X]$ so as to get the complete J-invariant system $(X, JX, [JX, X], J[JX, X])$. The rank of this is equal to 4 when $y_3 \neq 0$, and the associated foliation has the leaves

$$\tilde{L}_{y_3} := \{z \in \mathbb{C}^3 \colon \operatorname{Re} z_3 = 0, \operatorname{Im} z_3 = y_3\} \quad \text{for } y_3 \neq 0 .$$

Since $J[JX, X]|_{M\setminus M_0}$ has a nonvanishing normal component, the local Bochner theorem gives an analytic continuation of CR functions on L_{y_3} into $\tilde{L}_{y_3}$ on that side of L_{y_3} into which $J[JX, X]|_{M\setminus M_0}$ points.

So if u is a CR function on M, u has the following properties:

i) u is arbitrary in the parametric variable y_3;

ii) for fixed $y_3 \neq 0$, $u|_{L_{y_3}}$ is the boundary value of a holomorphic function defined in a one-sided neighbourhood of L_{y_3} in $\tilde{L}_{y_3}$;

iii) $u|_{M_0}$ is holomorphic along the leaves in the complex foliation of M_0.

Moreover, the exceptional case $y_3 = 0$ is related to the fact that the analytic continuation of u from L_{y_3} into $\tilde{L}_{y_3}$ changes side at the parameter value $y_3 = 0$—cf. figure 1.

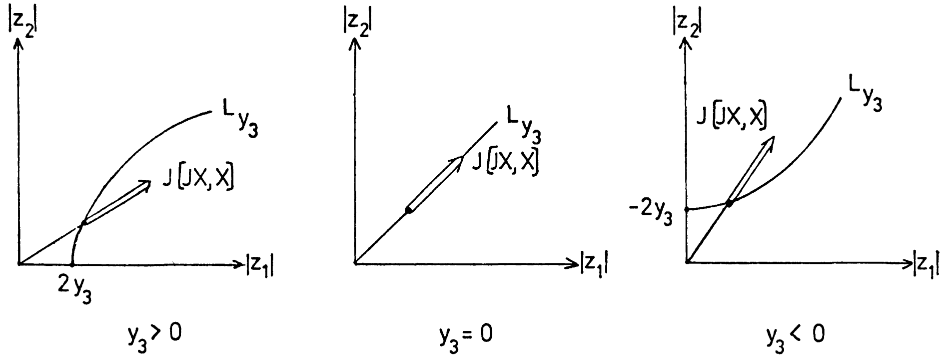

Figure 1

EXAMPLE 3. We now consider a simple example where there are two linearly independent holomorphic vector fields tangent to M.

Let $M := \{z \in \mathbb{C}^3 \colon |z_1|^2 + |z_2|^2 - |z_3|^2 = 1\}$. Then $\sum_{i=1}^{3} a_i \cdot \partial/\partial z_i$ is tangent to M if $a_1\bar{z}_1 + a_2\bar{z}_2 - a_3\bar{z}_3 = 0$. The choices $a_1 = \bar{z}_2$, $a_2 = -\bar{z}_1$, $a_3 = 0$ and $a_1 = \bar{z}_3$, $a_2 = 0$, $a_3 = \bar{z}_1$ respectively, give the holomorphic vector fields

$$Z = \bar{z}_2 \cdot \partial/\partial z_1 - \bar{z}_1 \cdot \partial/\partial z_2 = \tfrac{1}{2}(X - iJX)$$

and

$$W = \bar{z}_3 \cdot \partial/\partial z_1 + \bar{z}_1 \cdot \partial/\partial z_3 = \tfrac{1}{2}(Y - iJY)\,.$$

Here

$$X = x_2 \cdot \partial/\partial x_1 - x_1 \cdot \partial/\partial x_2 - y_2 \cdot \partial/\partial y_1 + y_1 \cdot \partial/\partial y_2$$

and

$$Y = x_3 \cdot \partial/\partial x_1 + x_1 \cdot \partial/\partial x_3 - y_3 \cdot \partial/\partial y_1 - y_1 \cdot \partial/\partial y_3\,.$$

Then $(X, JX, [JX, X])$ and $(X, JX, [JX, X], J[JX, X])$ are complete and give foliations $\{L^X_{z_3}\}_{z_3\in\mathbf{C}}$ and $\{\tilde{L}^X_{z_3}\}_{z_3\in\mathbf{C}}$ respectively, where

$$\tilde{L}^X_{z_3} := \{z \in \mathbf{C}^3 \colon z_3 = \text{constant}\} \quad \text{and} \quad L^X_z := M \cap \tilde{L}^X_{z_3}\,.$$

Similarly, $(Y, JY, [JY, Y])$ and $(Y, JY, [JY, Y], J[JY, Y])$ are complete. The associated foliations are $\{L^Y_{z_2}\}_{z_2\in\mathbf{C}}$ and $\{\tilde{L}^Y_{z_2}\}_{z_2\in\mathbf{C}}$ respectively, with

$$\tilde{L}^Y_{z_2} := \{z \in \mathbf{C}^3 \colon z_2 = \text{constant}\} \quad \text{and} \quad L^Y_{z_2} := M \cap \tilde{L}^Y_{z_2}\,.$$

Here $|z_2| = 1$ is an exceptional case in the same way as $y_3 = 0$ was in the preceding example.

REMARK 2. *Foliations of nondegenerate hypersurfaces.* The last example can be generalized to arbitrary nondegenerate hypersurfaces in $\mathbf{C}^n$ by using nonintrinsic methods (which, however, do not generally work when the codimension is bigger than one).

In fact, let M be such a hypersurface passing through the origin in $\mathbf{C}^n$. It follows from [4], sections 13, 14 and 18, that it is possible to introduce so called normal coordinates near 0, such that M is the zero set of a function

$$\varphi(z, \bar{z}) = \operatorname{Re} z_n + \sum_{j=1}^{n-1} \varepsilon_j \cdot |z_j|^2 + O\big(|z_n| \cdot |z'| + |z'|^3\big)\,, \tag{14}$$

where $z' = (z_1, \ldots, z_{n-1})$ and $\{\varepsilon_j\}_{j=1}^{n-1}$, with $\varepsilon_j = \pm 1$, is the signature of the Levi form of M at 0.

From (14) it follows immediately that the foliations $\{\tilde{L}^k_c\}_{c\in\mathbf{C}^{n-2}}$, $k = 1, \ldots, n-1$, where

$$\tilde{L}^k_c := \{z \in \mathbf{C}^n \colon z_j = c_j = \text{constant},\ j \neq k,\ n\}\,,$$

have the property that for each $c \in \mathbf{C}^{n-2}$, $L^k_c := M \cap \tilde{L}^k_c$ is a real 3-dimensional strictly pseudoconvex hypersurface in the complex 2-dimensional manifold $\tilde{L}^k_c$ near 0. Hence we get $n-1$ pairs of foliations $\{L^k_c, \tilde{L}^k_c\}_{c\in\mathbf{C}^{n-2}}$,

$k = 1, \dots, n-1$, where each pair is of the same kind as those considered in the examples above.

REMARK 3. *On misleading examples.* The examples discussed here are somewhat misleading in the sense that the considered vector fields have been defined on all of $\mathbb{C}^n$, and not only on the real submanifold $M \subset \mathbb{C}^n$. This depends on the fact that the holomorphic vector fields Z have been defined by requiring that $Z(\varphi_i) = 0$ for a set of defining equations $\{\varphi_i = 0\}$ for M, *i.e.*, Z is tangent to all manifolds $\{\varphi_i = \text{constant}\}$ as well. But then the definition of Z outside of M depends on the choice of the functions φ_i, and hence is rather arbitrary.

When considering $\bar{\partial}_M$-equations arising as in Theorem 3, there is in general no obvious privileged extension of the vector fields in question outside of M. As a consequence, the complex leaves $\tilde{L}_t$ will no longer present themselves, but require some construction.

3. – The theory of Vessiot.

In order to find complete vector field systems of the kind that we are interested in, we will use the theory of Vessiot given in [15]. For the convenience of the reader, and in order to establish terminology, we give a rough sketch of that theory in this section. We work in the real analytic category throughout, and all functions and vector fields are real valued.

The problems to which the Vessiot theory can be applied are of the following kind: For a given integer p and a given noncomplete vector field system $\mathcal{F}$ defined on an open set U in $\mathbb{R}^n$, one wants to find a foliation of U with p-dimensional leaves, such that all leaves are invariant under the action of p linearly independent vector fields from $\mathcal{F}$. By Frobenius this is equivalent to finding a complete sub vector field system $\mathcal{G}$ of $\mathcal{F}$ of rank p.

By Gauss elimination and by renumbering the indices if necessary, we may suppose that $\mathcal{F}$ has a basis of the form

$$X_k = \frac{\partial}{\partial x_k} + \sum_{\alpha=m+1}^{n} \xi_{k\alpha}(x_1, \dots, x_n) \cdot \frac{\partial}{\partial x_\alpha}, \qquad k = 1, \dots, m,$$

i.e., the basis is in resolved form (with respect to $\partial/\partial x_k$) in Vessiot's terminology. Assuming that $\mathcal{F}' := \mathcal{F} + [\mathcal{F}, \mathcal{F}]$ has the basis $\{X_1, \dots, X_m; Z_1, \dots, Z_s\}$, there are structure functions $c_{ij}^k(x_1, \dots, x_n)$ such that

$$[X_i, X_j] = \sum_{k=1}^{s} c_{ij}^k(x) \cdot Z_k\,; \quad i, j = 1, \dots, m. \tag{15}$$

By above we look for a sub vector field system $\mathcal{G}$ of $\mathcal{F}$ with a basis

$$Y_i = \sum_{\alpha=1}^{m} a_{i\alpha}(x_1, \dots, x_n) \cdot X_\alpha, \quad i = 1, \dots, p,$$

such that

$$[Y_i, Y_j] = 0 \quad \text{for } i, \quad j = 1, \dots, p. \tag{16}$$

(15) shows that

$$[Y_i, Y_j] = \sum_{\alpha=1}^{s} \Big(\sum_{\gamma=1}^{m} \sum_{\delta=1}^{m} c^{\alpha}_{\gamma\delta}(x) \cdot a_{i\gamma}(x) \cdot a_{j\delta}(x) \Big) \cdot Z_\alpha + \sum_{\beta=1}^{m} \left(Y_i a_{j\beta}(x) - Y_j a_{i\beta}(x) \right) \cdot X_\beta,$$

so that the problem of finding Y_i satisfying (16) can be divided into two subproblems (where $i, j = 1, \dots, p$):

(I) $\sum_{\gamma=1}^{m} \sum_{\delta=1}^{m} c^{\alpha}_{\gamma\delta}(x) \cdot a_{i\gamma}(x) \cdot a_{j\delta}(x) = 0$, $\alpha = 1, \dots, s$; *i.e.*, $[Y_i, Y_j] \equiv 0 \pmod{\mathcal{F}}$,

which is a problem in linear algebra, and

(II) $Y_i a_{j\beta}(x) = Y_j a_{i\beta}(x)$, $\beta = 1, \dots, m$,

which is a system of partial differential equations of a rather special type.

To solve (I), start with an arbitrary $Y_1 = \sum_{\alpha=1}^{m} a_{1\alpha}(x) \cdot X_\alpha$, and determine the coefficients in $Y_2 = \sum_{\alpha=1}^{m} a_{2\alpha}(x) \cdot X_\alpha$ from

(I_1) $\sum_{\gamma=1}^{m} \sum_{\delta=1}^{m} c^{\alpha}_{\gamma\delta}(x) \cdot a_{1\gamma}(x) \cdot a_{2\delta}(x) = 0$, $\alpha = 1, \dots, s$; *i.e.*, $[Y_1, Y_2] \equiv 0 \pmod{\mathcal{F}}$.

For fixed $a_{1\gamma}$, $\gamma = 1, \dots, m$, this is a system of linear equations in the $a_{2\delta}$. We suppose that the coefficients $a_{1\gamma}$ are chosen such that the rank of (I_1) is maximal, in which case the $a_{1\gamma}$ are said to be generic. If this rank equals q_1, then $m - q_1$ of the $a_{2\delta}$, $\delta = 1, \dots, m$, are left arbitrary.

Then we try to find a suitable $Y_3 = \sum_{\alpha=1}^{m} a_{3\alpha}(x) \cdot X_\alpha$ by solving

$$[Y_1, Y_3] \equiv [Y_2, Y_3] \equiv 0 \pmod{\mathcal{F}}, \tag{I_2}$$

with Y_2 the general solution of (I_1). Also here we choose $a_{1\gamma}$ and the $a_{2\delta}$ left arbitrary in a generic way, so that the rank q_2 of (I_2), as a system of linear equations for $a_{3\alpha}$, is maximal.

And then we continue in this way, until at last $Y_p = \sum_{\alpha=1}^{m} a_{p\alpha}(x) \cdot X_\alpha$ is determined from

$$(\mathrm{I}_{p-1}) \qquad [Y_1, Y_p] \equiv \dots \equiv [Y_{p-1}, Y_p] \equiv 0 \pmod{\mathcal{F}},$$

where again $a_{1\gamma}, \dots, a_{p-1,\gamma}$, $\gamma = 1, \dots, m$, are chosen generically, so that the rank q_{p-1} of (I_{p-1}) is maximal.

If all the systems (I_1)-(I_{p-1}) have nontrivial solutions, we say that $\mathcal{F}$ is involutive of order p, and the vector fields $Y_1, \dots, Y_p$ found in this way form an involution of p-th order. In this case we continue to solve (II).

Suppose that with a suitable choice of local coordinates one wants $x_1, \dots, x_p$ to be functionally independent on the sought—for integral manifolds. Then a basis for $\mathcal{G} = (Y_1, \dots, Y_p)$ can be written in the form

$$V_k = X_k + \sum_{\alpha=1}^{m-p} v_{k\alpha}(x_1, \dots, x_n) \cdot X_{p+\alpha}, \qquad k = 1, \dots, p,$$

where, according to what was done above, all $v_{1\alpha}$ are arbitrary (though generic), $m - p - q_1$ of the $v_{2\alpha}$ are arbitrary, and so on. Because of the resolved form of V_k, the partial differential equations for $v_{k\alpha}(x)$ corresponding to (II) can now be solved by a clever induction argument using the Cauchy-Kowalewski theorem repeatedly. This determines the $v_{k\alpha}(x)$ up to the following arbitrariness:

$r_1 := m - p - q_{p-1}$	of the	$v_{1\alpha}$	are	arbitrary	functions	of n	variables,
$r_2 := m - p - q_{p-2}$	»	$v_{2\alpha}$	»	»	»	» $n - 1$	»
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
$r_{p-1} := m - p - q_1$	»	$v_{p-1,\alpha}$	»	»	»	» $n - p + 2$	»
$r_p := m - p$	»	$v_{p\alpha}$	»	»	»	» $n - p + 1$	»

The whole theory is really invariant under coordinate changes, except for the last steps involving the Cauchy-Kowalewski theorem. And it is precisely these which give the curious-looking result above. However, it has been shown in [9], chapter I, that in such a result only the arbitrariness in the biggest number of variables is really invariant (of course this was known already to Cartan—see e.g. letter XXXVII in [3]). So if r_a is the first nonzero integer in the sequence $r_1, r_2, \dots, r_p$, we say that the solution depends on « r_a arbitrary functions of $n - a + 1$ variables » (see [9] for the precise meaning of this).

The solutions found in this way (with generic $v_{k\alpha}(x)$) are called regular. Then there may also exist singular solutions corresponding to nongeneric $v_{k\alpha}(x)$. However, each of these can under rather general circumstances—as is indicated in [15], and proved in [9], chapter III, for the dual case of exterior differential systems—be « regularized » by means of a finite number of prolongations, and then be obtained by the preceding method. We refer to [15], section 15, for this.

4. – Complex submanifolds of $\mathbb{C}^n$ and the Cartan-Kähler theory.

The examples in section 2 show that the following problem is reasonable:

Given a real submanifold M of $\mathbb{C}^n$ we want to find a family $\{\tilde{L}_t\}$ (where t belongs to some parameter space) of complex manifolds defined near M, so that $L_t := \tilde{L}_t \cap M$ is a nondegenerate hypersurface in $\tilde{L}_t$ for each t.

One way to find $\{\tilde{L}_t\}$ is by first constructing a suitable foliation $\{L_t\}$ of M by means of the Vessiot theory, and then extending each L_t to a complex manifold. It is the latter step that we shall describe in this section.

Suppose that $\mathcal{M}$ is a complex m-dimensional submanifold of $\mathbb{C}^n$. Then locally one can find complex coordinates $z_1, \ldots, z_m, z_{m+1}, \ldots, z_n$ for $\mathbb{C}^n$ so that $\mathcal{M}$ is defined by

$$z_k = f_k(z_1, \ldots, z_m), \qquad k = m+1, \ldots, n,$$

where the functions $f_k\ (z_1, \ldots, z_m)$ are holomorphic.

Hence

$$dz_k = \sum_{j=1}^{m} \frac{\partial f_k}{\partial z_j} dz_j, \qquad k = m+1, \ldots, n,$$

on $\mathcal{M}$, so that

$$dz_{j_1} \wedge \ldots \wedge dz_{j_{m+1}}|_{\mathcal{M}} = 0$$

for all $(m+1)$-tuples $(j_1, \ldots, j_{m+1})$. By complex conjugation, also

$$d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_{m+1}}|_{\mathcal{M}} = 0.$$

Now let $\mathfrak{J}^m$ be the differential ideal on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ generated by

$$\{dz_{j_1} \wedge \ldots \wedge dz_{j_{m+1}},\ d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_{m+1}},\ 1 \leqslant j_1, \ldots, j_{m+1} \leqslant n\},$$

or, in real form, by

$$\{\mathrm{Re}\,(dz_{j_1} \wedge \ldots \wedge dz_{j_{m+1}}),\ \mathrm{Im}\,(dz_{j_1} \wedge \ldots \wedge dz_{j_{m+1}}),\ 1 \leqslant j_1, \ldots, j_{m+1} \leqslant n\}.$$

Then the argument above shows that $\mathcal{M}$ is an integral manifold of $\mathfrak{J}^m$ with the real dimension $2m$.

Conversely, suppose that $\mathcal{N}$ is a real analytic submanifold of $\mathbf{C}^n \cong \mathbf{R}^{2n}$ of real dimension $2m$, such that $\mathcal{N}$ is an integral manifold of $\mathfrak{J}^m$. Then one can find local coordinates $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ in $\mathbf{C}^n \cong \mathbf{R}^{2n}$ such that

$$dz_{j_1} \wedge \ldots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \ldots \wedge d\bar{z}_{k_q}|_{\mathcal{M}} \neq 0 \,, \tag{17}$$

for certain $j_1, \ldots, j_p, k_1, \ldots, k_q$ with $p + q = 2m$.

In fact, since $\mathcal{N}$ is an integral manifold of $\mathfrak{J}^m$, necessarily $p = q = m$.

For $j \neq j_1, \ldots, j_m$, $z_j = g_j(z_{j_1}, \ldots, z_{j_m}, \bar{z}_{k_1}, \ldots, \bar{z}_{k_m})$ locally on $\mathcal{N}$, where g_j is a convergent power series in the indicated variables. Hence

$$dz_j = \sum_{l=1}^{m} \frac{\partial g_j}{\partial z_{j_l}} dz_{j_l} + \sum_{l=1}^{m} \frac{\partial g_j}{\partial \bar{z}_{k_l}} d\bar{z}_{k_l}$$

on $\mathcal{N}$. By hypothesis,

$$0 = dz_j \wedge dz_{j_1} \wedge \ldots \wedge dz_{j_m} = \sum_{l=1}^{m} \frac{\partial g_j}{\partial \bar{z}_{k_l}} d\bar{z}_{k_l} \wedge dz_{j_1} \wedge \ldots \wedge dz_{j_m} \qquad \text{on } \mathcal{N} \,,$$

which by (17) shows that $\partial g_j / \partial \bar{z}_{k_l} = 0$ for $l = 1, \ldots, m$.

Thus g_j is a holomorphic function in the variables $z_{j_1}, \ldots, z_{j_m}$. The following lemma is a consequence of all this:

LEMMA. *A real analytic submanifold of $\mathbf{C}^n$ of real dimension $2m$ is complex analytic (of complex dimension m) if and only if it is an integral manifold of the differential ideal $\mathfrak{J}^m$.*

The importance of this lemma is that it makes it possible to construct complex m-dimensional submanifolds of $\mathbf{C}^n$ starting from real analytic integral manifolds of $\mathfrak{J}^m$ of real dimension $< 2m$ by means of the Cartan-Kähler theory (see e.g. [8] or [9]).

To be able to discuss integral elements of $\mathfrak{J}^m$, we first introduce a convenient notation:

Let V be a real vector subspace of $T_x\mathbf{C}^n$, where $x \in \mathbf{C}^n$, and let $V^\perp$ be the orthogonal complement of V in $T_x\mathbf{C}^n$. Then the complex structure $J_x\colon T_x\mathbf{C}^n \to T_x\mathbf{C}^n$ and the projection $P\colon V \oplus V^\perp \to V^\perp$ together define a mapping $Q := P \circ J_x|_V\colon V \to V^\perp$. If a vector $X \in V$ belongs to $\ker Q$, then also $JX \in \ker Q$ since $J(JX) = -X \in V$. So $\dim_{\mathbf{R}} \ker Q$ is an even number, equal to $2p$ say. In this situation we say that V is of the form $\mathbf{C}^p \times \mathbf{R}^q$ (where $\dim_{\mathbf{R}} V = 2p + q$), or that the complex dimension of V is equal to p.

Then it is easy to see that the integral elements $V \subset T_x\mathbf{C}^n$ of the differential ideal $\mathfrak{J}^m$ are of the following form:

1) any V with $\dim_{\mathbb{R}} V \leqslant m$, and

2) those V with $\dim_{\mathbb{R}} V = m + a$ (where $1 \leqslant a \leqslant m$), for which the complex dimension of V is at least equal to a.

Furthermore, in the latter case V is a regular integral element of $\mathfrak{J}^m$ if the complex dimension of V is equal to a, *i.e.*, if V is of the form $\mathbb{C}^a \times \mathbb{R}^{m-a}$.

So if L is a real analytic integral manifold of $\mathfrak{J}^m$ of real dimension $m + a$, such that $T_x(L)$ is of the form $\mathbb{C}^a \times \mathbb{R}^{m-a}$ for some $x \in L$, then L can be locally extended near x to a complex submanifold $\tilde{L}$ of $\mathbb{C}^n$ of complex dimension m. And this extension is unique, as one can see for instance by calculating the characters of $\mathfrak{J}^m$ (cf. [8], pp. 60-61).

In particular, consider a foliation $\{L_t\}_{t \in T}$ of a real submanifold M in $\mathbb{C}^n$ associated to a complete vector field system $(X, JX, [JX, X])$, where X, JX and $[JX, X]$ are linearly independent vector fields on M, and J is the complex structure of $\mathbb{C}^n$. Then the tangent spaces of L_t are of the form $\mathbb{C} \times \mathbb{R}$, where the « $\mathbb{C}$ » is formed by X and JX, and the « $\mathbb{R}$ » corresponds to $[JX, X]$. Thus each L_t can locally be extended to a unique complex 2-dimensional manifold $\tilde{L}_t$, such that $J[JX, X]$ gives one of the two normal directions for L_t in $\tilde{L}_t$.

Moreover, the Cartan-Kähler theory is valid also in the presence of parameters, so that if $\{L_t\}$ depends real analytically on t, $\{\tilde{L}_t\}$ will do so too.

5. – The Levi module and its derivatives.

Now we want to consider the tangential Cauchy-Riemann equations on a submanifold M as in Theorem 3, *i.e.*, we assume that M is a generic locally closed real analytic submanifold of $\mathbb{C}^n$ of real dimension m, where $n < m < 2n$. That M is generic means that for each x in M the *CR* tangent space to M at x, defined by

$$H_x := T_x(M) \cap J_x T_x(M)\,,$$

is of minimal complex dimension, namely $\dim_{\mathbb{C}} H_x(M) = m - n$.

Then

$$H(M) := \bigcup_{x \in M} H_x(M)$$

has the structure of a real analytic subbundle of $T(M)$, and is called the *CR* bundle of M.

To avoid confusion about what is regarded as real and as complex, we will from now on try to consider everything from a real point of view.

In particular, $\mathbb{C}^n$ is identified with the pair $(\mathbb{R}^{2n}, J)$, where J is an integrable complex structure.

Let $\mathcal{A} := \Gamma(M, H(M))$, or equivalently,

$$\mathcal{A} = \{X \in \Gamma(M, T(M)) : JX \in \Gamma(M, T(M))\}.$$

Then $\mathcal{A}$ is a module of vector fields (the so called CR vector fields) over the ring of real analytic functions on M. It is called the Levi module.

The connection between $\mathcal{A}$ and the $\bar{\partial}_M$-equations is the following: any $\bar{\partial}_M$-operator $\bar{Z}$ is of the form $X + iJX$ with $X, JX \in \mathcal{A}$, and conversely, each $X \in \mathcal{A}$ can be written as $X = \operatorname{Re} \bar{Z}$, with $\bar{Z}$ a $\bar{\partial}_M$-operator.

By definition $\mathcal{A}$ is invariant under the action of J. However, in general $\mathcal{A}$ is not closed under Lie brackets. This makes it natural to consider the derivatives of $\mathcal{A}$ (cf. [15]):

$$\mathcal{A}' := \mathcal{A} + [\mathcal{A}, \mathcal{A}], \quad \text{and}$$

$$\mathcal{A}^{(k)} := \mathcal{A}^{(k-1)} + [\mathcal{A}^{(k-1)}, \mathcal{A}^{(k-1)}] \quad \text{for } k > 1.$$

Then $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A}'' \subseteq \ldots \subseteq \Gamma(M, T(M))$. We assume (by shrinking M if necessary) that all $\mathcal{A}^{(k)}$ (as well as all other vector field systems on M to be considered in the sequel) are of constant rank on M. Since $\operatorname{rank} \Gamma(M, T(M)) = \dim M = m$ is finite, there is an integer c such that all $\mathcal{A}^{(k)}$ are equal for $k \geqslant c$. Then $\mathcal{A}^{(c)}$ is a complete vector field system, which is called the Levi algebra of M (because it is a Lie algebra under the bracket operation). The number $e := \operatorname{rank} \mathcal{A}^{(c)}$-$\operatorname{rank} \mathcal{A}$ is sometimes called the excess dimension of M. Since $\operatorname{rank} \mathcal{A} = 2(m-n)$ and $\operatorname{rank} \mathcal{A}^{(c)} \leqslant \operatorname{rank} \Gamma(M, T(M)) = m$, we see that $e \leqslant 2n - m$.

The simplest case occurs if $c = 0$. Then $\mathcal{A}$ is complete, and thus defines a foliation of M. Moreover, since $\mathcal{A}$ is invariant under J, the restriction of J to the leaves of that foliation will provide them with an integrable complex structure. Thus $\mathcal{A}$ defines a complex foliation of M. (This is really just a special case of Theorem 1.) *CR* functions on M are then characterized by being holomorphic along the complex leaves, but arbitrary in the parametric variables.

Suppose now that $c > 0$. Then also $e > 0$, and $\operatorname{rank} \mathcal{A}^{(c)} = 2(m-n) + e \leqslant m$. If $e < 2n - m$, $\operatorname{rank} \mathcal{A}^{(c)} < m$, and the complete vector field system $\mathcal{A}^{(c)}$ defines a nontrivial foliation $\{B_t\}_{t \in T}$ of M, where the parameter space $T \cong \mathbb{R}^{2n-m-e}$. Since the invariants of $\mathcal{A}^{(c)}$ are also invariants of $\mathcal{A}$, *CR* functions may vary quite arbitrarily from one leaf to another, and only their behaviour along the leaves B_t is restricted.

Consider now a fixed leaf B_t, and let $x \in B_t$ be arbitrary. By definition of $\mathcal{A}$ and $\mathcal{A}^{(c)}$, $T_x(B_t)$ will be of the form $\mathbb{C}^{m-n} \times \mathbb{R}^e$. Hence B_t is a regular integral manifold of the differential ideal $\mathfrak{J}^{m-n+e}$ on $\mathbf{C}^n$. By the Cartan-Kähler theory we can thus extend B_t locally to a unique complex manifold $\tilde{B}_t$ of complex dimension $m-n+e$ (which is less than n if $e<2n-m$).

Since no derivatives with respect to the parametric variables occur in $\mathcal{A}$ and $\mathcal{A}^{(c)}$, we can restrict these to B_t, so that they form vector field systems $\mathcal{A}_t$ and $\mathcal{A}_t^{(c)}$ there. Then $\mathcal{A}_t$ is the Levi module of B_t considered as a real submanifold of $\tilde{B}_t$, and $\mathcal{A}_t^{(c)}=\Gamma(B_t, T(B_t))$. Clearly the excess dimension of B_t is equal to e.

Now from a deep result due to Hunt and Wells (see [7]) it follows that if f is a C^1 CR function on B_t, then f has a unique local extension to a CR function $\tilde{f}$ defined on a CR manifold $U_t\subset\tilde{B}_t$ with $\dim U_t=\dim B_t+e=2(m-n+e)$ and $\overline{U}_t\supseteq B_t$. Consequently $\dim U_t=\dim\tilde{B}_t$, so that U_t is in fact an open subset of $\tilde{B}_t$, and $\tilde{f}$ is holomorphic on U_t.

Hence we have obtained the following

THEOREM. *M has a foliation $\{B_t\}_{t\in T}$ with leaves B_t of real dimension $2(m-n)+e$ (where $e\leqslant 2n-m$; if $e=2n-m$, the foliation is trivial in the sense that there is only one leaf, M itself). Each B_t has a unique local extension to a complex $(m-n+e)$-dimensional manifold $\tilde{B}_t$. If u is a CR function on M, then u is arbitrary in the parametric variables t. If $u_t:=u|_{B_t}\in$ $\in C^1(B_t)$, then u_t has a unique local extension to a holomorphic function $\tilde{u}_t$ defined on an open set U_t in $\tilde{B}_t$.*

If e (which is equal to the real codimension of B_t in $\tilde{B}_t$) is big, u_t is not expected to inherit so many of the analytic properties of $\tilde{u}_t$. However, by making local extensions of u_t into $\tilde{B}_t$ and gluing these together by means of the unique continuation theorem for holomorphic functions on open subsets of $\tilde{B}_t$, we have at least:

COROLLARY. *C^1 CR functions on M have the unique continuation property along the leaves B_t.*

In favourable cases there may exist subfoliations of $\{B_t\}$ such that the corresponding e is smaller, and which give further information about the CR functions on M. The existence of such foliations will be investigated in the following sections.

6. – Complex foliations.

As remarked before, when $\mathcal{A}$ is complete the leaves of the corresponding foliation have a complex structure. But also when $\mathcal{A}$ is not complete there may exist complex foliations of M, and in this section we will try to find maximal such foliations by means of the Vessiot theory. For the sake of simplicity we only consider foliations which are regular in the sense of section 3. We remark that similar problems have been treated by Sommer ([13]), Freeman ([5]) and others.

A foliation of M with complex leaves of as big dimension as possible corresponds on the vector field level to a maximal sub vector field system $\mathcal{B}$ of $\mathcal{A}$ satisfying

i) $[\mathcal{B}, \mathcal{B}] \subseteq \mathcal{B}$, and

ii) $J\mathcal{B} \subseteq \mathcal{B}$.

Let $\mathcal{A}_1$ be the biggest regular (in the sense of Vessiot) submodule of $\mathcal{A}$ satisfying i). To find $\mathcal{A}_1$ we solve as many as possible of the systems of linear equations (I_1), (I_2), ... in section 3, with $\mathcal{F}$ replaced by $\mathcal{A}$, until we reach a system (I_p) with no nontrivial solutions. This gives an involution of order p, where p is maximal. Solving (II) for this involution then yields the vector field system $\mathcal{A}_1$. With notations as in section 3 we suppose that $\mathcal{A}$ has a basis $\{X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q}\}$ (where $p + q = 2(m - n)$), and that $\mathcal{A}_1$ has a basis

$$V_k = X_k + \sum_{=1}^{q} v_{k\alpha}(x_1, \ldots, x_m) \cdot X_{p+\alpha}, \qquad k = 1, \ldots, p,$$

where $x_1, x_2, \ldots, x_m$ are local coordinates for M.

As explained before, all the $v_{k\alpha}(x)$ can be considered as determined functions of $(x_1, \ldots, x_m)$ except $r_a = q - q_{p-a}$ of the $v_{a\alpha}(x)$, which depend in an arbitrary way on $m - a + 1$ variables. So really $\mathcal{A}_1$ is a family of vector field systems parameterized by the « arbitrary » functions $v_{a\alpha}(x)$.

Now $\mathcal{A}_1$ need not satisfy ii). So let $\mathcal{A}_2$ be the biggest submodule of $\mathcal{A}_1$ which is invariant under J, i.e., $\mathcal{A}_2 = \{V \in \mathcal{A}_1 : JV \in \mathcal{A}_1\}$.

Suppose that the complex structure $J: \mathcal{A} \to \mathcal{A}$ is defined by

$$JX_k = \sum_{j=1}^{p+q} J_k^j(x) \cdot X_j \qquad \text{for } k = 1, \ldots, p + q.$$

Then

$$JV_k = \sum_{j=1}^{p} \Big\{ J_k^j(x) + \sum_{\alpha=1}^{q} v_{k\alpha}(x) \cdot J_{p+\alpha}^j(x) \Big\} \cdot V_j + \sum_{\beta=1}^{q} A_k^{p+\beta}(x) \cdot X_{p+\beta},$$

where

$$A_k^{p+\beta}(x) = J_k^{p+\beta}(x) + \sum_{\alpha=1}^{q} v_{k\alpha}(x) \cdot J_{p+\alpha}^{p+\beta}(x) - \sum_{j=1}^{p} \Big(J_k^j(x) + \sum_{\alpha=1}^{q} v_{k\alpha}(x) \cdot J_{p+\alpha}^j(x) \Big) \cdot v_{j\beta}(x).$$

Hence $V=\sum_{k=1}^{p} a_k(x)\cdot V_k \in \mathcal{A}_2$ if and only if the $a_k(x)$ satisfy

$$\sum_{k=1}^{p} a_k(x)\cdot A_k^{p+\beta}(x) = 0 \qquad \text{for } \beta = 1, \dots, q\,. \tag{18}$$

The rank r of the system (18) may or may not depend on the « arbitrary » $v_{a\alpha}(x)$. In the latter case we suppose that r is constant on M (shrinking M if necessary), and then we get a vector field system $\mathcal{A}_2$ of the rank $p-r$. In the former case we can, by using linear algebra, for instance choose the $v_{a\alpha}(x)$ so that r is as small as possible (and constant on M). If this minimal value is zero, then we are done. Otherwise we obtain an $\mathcal{A}_2$ which satisfies ii), but not necessarily i), and which may depend on some arbitrary functions.

If $\mathcal{A}_2$ is not complete, let $\mathcal{A}_3$ be the biggest regular submodule of $\mathcal{A}_2$ satisfying i). The possible arbitrariness can here be exploited to get an involution of as big order as possible. Since the Cauchy-Kowalewski theorem is valid also in the presence of parametric functions (see e.g. [9], section 1). the differential equations corresponding to (II) can be solved as before.

And so on.

Since the rank drops all the time, after a finite number of steps one will find a regular vector field system $\mathcal{A}_{\mathbf{C}}$ satisfying both i) and ii). If $\mathcal{A}_{\mathbf{C}}$ is nonempty and depends on a set of arbitrary functions in a certain number of variables, then $\mathcal{A}_{\mathbf{C}}$ corresponds to a family of complex foliations which is parameterized by these functions.

If u is a CR function on M, then of course u is holomorphic along the leaves of such a foliation.

7. – Lewy foliations.

A complex foliation with complex 1-dimensional leaves is by Frobenius equivalent to a complete vector field system of the form (X, JX) with $X \in \mathcal{A}$. A somewhat more complicated foliation is that associated to a complete vector field system $(X, JX, [JX, X])$, where $X \in \mathcal{A}$, and X, JX and $[JX, X]$ are linearly independent. By the results in section 4, the leaves $\{L_t\}$ of such a foliation can locally be extended to complex 2-dimensional leaves $\{\tilde{L}_t\}$, such that L_t is a strictly pseudoconvex hypersurface in $\tilde{L}_t$ for each t in the parameter space. Since the tangential Cauchy-Riemann equations on L_t induced from the $\bar{\partial}$-equations on $\tilde{L}_t$ give rise to a counter example à la Lewy (cf. [2], part 1, § 5), such a foliation is called a Lewy foliation.

By adding more brackets one obtains a partition of $\mathcal{A}$ into a finite number of different classes:

$\mathcal{C}_0 := \{X \in \mathcal{A}: (X, JX) \text{ is complete}\}$,

$\mathcal{C}_1 := \{X \in \mathcal{A}\setminus\mathcal{C}_0: (X, JX, [JX, X]) \text{ is complete}\}$,

$\mathcal{C}_2 := \{X \in \mathcal{A}\setminus(\mathcal{C}_0 \cup \mathcal{C}_1): (X, JX, [JX, X], [X, [JX, X]]) \text{ is complete}\}$,

$\mathcal{C}_3 := \{X \in \mathcal{A}\setminus(\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2): (X, JX, [JX, X], [X, [JX, X]], [JX, [JX, X]])$ is complete$\}$,

. . . . .

and so on. Observe that the $\mathcal{C}_i$ are not vector field systems in general, but just sets.

In this section we want to indicate a method by which one can determine the sets $\mathcal{C}_i$. Since $\mathcal{C}_0$ is rather simple, and is related to what was done in the preceding section, we start with $\mathcal{C}_1$.

Suppose that $X \in \mathcal{C}_1$. Because $X \notin \mathcal{C}_0$, X, JX and $[JX, X]$ are linearly independent. By definition X and JX both belong to $\mathcal{A}$. If also $[JX, X] \in \mathcal{A}$, then $J[JX, X] \in \mathcal{A}$ too, and we have two cases (assuming the ranks to be constant, as always):

1) $(X, JX, [JX, X], J[JX, X])$ is complete. Then this vector field system defines a foliation of M with complex 2-dimensional leaves.

2) $(X, JX, [JX, X], J[JX, X])$ is not complete. Then the complex leaves $\tilde{L}_t$ obtained by Cartan-Kähler continuation of the foliation $\{L_t\}$ associated to $(X, JX, [JX, X])$ have a nontransversal intersection with M along L_t.

Since these cases seem to be rather nongeneric, we exclude them and consider only $\mathcal{C}_1^* := \{X \in \mathcal{C}_1: [JX, X] \notin \mathcal{A}\} = \{X \in \mathcal{A}: (X, JX, [JX, X])$ is complete and $[JX, X] \notin \mathcal{A}\}$. Then to each $X \in \mathcal{C}_1^*$ there corresponds a Lewy foliation $\{L_t^X\}$, such that the associated complex leaves $\tilde{L}_t^X$ cut M transversally along L_t^X.

Unfortunately the Vessiot theory cannot be applied directly in the search for $\mathcal{C}_1^*$ (since JX and $[JX, X]$ are completely determined by X, there are too few arbitrary coefficients to play around with), so we first make a suitable « prolongation », inspired by section 15 in [15].

Choose a resolved basis $X_1, \ldots, X_N$ (where $N = 2(m - n)$) for $\mathcal{A}$, so that each $X \in \mathcal{A}$ can be written in the form

$$X = \sum_{i=1}^{N} a_i(x_1, \ldots, x_m) \cdot X_i$$

with the functions $a_i(x)$ real analytic on M. If we regard such an X as a section of the CR bundle $H(M)$, then for each $x \in M$, $(a_1(x), \ldots, a_N(x))$ can be considered as coordinates for the N-dimensional fiber $H_x(M)$.

If the complex structure $J\colon \mathcal{A} \to \mathcal{A}$ is defined by

$$JX_k = \sum_{i=1}^{N} J_k^i(x) \cdot X_i\,, \qquad k = 1, \ldots, N\,, \tag{19}$$

we have

$$JX = \sum_{i,j=1}^{N} a_i(x) \cdot J_i^j(x) \cdot X_j$$

and

$$[JX, X] = \sum_{i,j,k=1}^{N} \{a_i(x) \cdot J_i^j(x) \cdot \big(X_j a_k(x_j)\big) \cdot X_k - \\ - a_k(x) \cdot \big(X_k\big(a_i(x) \cdot J_i^j(x)\big)\big) \cdot X_j + a_i(x) \cdot J_i^j(x) \cdot a_k(x) \cdot [X_j, X_k]\}\,.$$

With $\mathcal{F}_X := \big(X, JX, [JX, X]\big)$ we want to determine X (and hence the coefficients $a_i(x)$, $i = 1, \ldots, N$) so that $\mathcal{F}_X' = \mathcal{F}_X$ and $[JX, X] \notin \mathcal{A}$. As a first step we replace the condition $\mathcal{F}_X' = \mathcal{F}_X$ by the weaker $\mathcal{F}_X' \subseteq \mathcal{A}'$.

Suppose that $\{X_1, \ldots, X_N;\, Y_1, \ldots, Y_p\}$ and $\{X_1, \ldots, X_N;\; Y_1, \ldots, Y_p;\; Z_1, \ldots, Z_q\}$ are bases for $\mathcal{A}'$ and $\mathcal{A}''$ respectively. Define the structure functions $c_{ij}^l(x)$ and $c_{ijk}^l(x)$ by

$$[X_i, X_j] = \sum_{l=1}^{p} c_{ij}^l(x) \cdot Y_l,$$

and

$$[X_i, [X_j, X_k]] \equiv \sum_{l=1}^{q} c_{ijk}^l(x) \cdot Z_l \qquad (\mathrm{mod}\ \mathcal{A}')\,,$$

respectively. Then

$$[JX, X] = \sum_{i,j,k=1}^{N} \Big\{a_i(x) \cdot J_i^j(x) \cdot \big(X_j a_k(x)\big) \cdot X_k - a_k(x) \cdot \Big(X_k\big(a_i(x) \cdot J_i^j(x)\big)\Big) \cdot X_j\Big\} + \\ + \sum_{l=1}^{p} \Big\{ \sum_{i,j,k=1}^{p} c_{ij}^l(x) \cdot J_i^j(x) \cdot a_i(x)\, a_k(x)\Big\} \cdot Y_l, \tag{20}$$

$$[X, [JX, X]] \equiv \sum_{l=1}^{q} \Big\{ \sum_{i,j,k,\alpha=1}^{N} c_{k\alpha j}^l(x) \cdot J_i^\alpha(x) \cdot a_i(x)\, a_j(x)\, a_k(x)\Big\} \cdot Z_l\ (\mathrm{mod}\ \mathcal{A}')\,,$$

and

$$[JX, [JX, X]] \equiv \sum_{l=1}^{q} \Big\{ \sum_{i,j,k,\beta,\gamma=1}^{N} c_{\beta\gamma k}^l(x) \cdot J_i^\beta(x) \cdot J_j^\gamma(x) \cdot a_i(x)\, a_j(x)\, a_k(x)\Big\} \cdot Z_l\ (\mathrm{mod}\ \mathcal{A}')\,.$$

Thus the condition $\mathcal{F}'_X \subseteq \mathcal{A}'$ is satisfied if and only if

$$(21)\quad \begin{cases} \sum_{i,j,k,\alpha=1}^{N} c^l_{k\alpha j}(x)\cdot J^\alpha_i(x)\cdot a_i(x)\,a_j(x)\,a_k(x) = 0 \quad \text{for } l = 1, \ldots, q\,, \\ \text{and} \\ \sum_{i,j,k,\beta,\gamma=1}^{N} c^l_{\beta\gamma k}(x)\cdot J^\beta_i(x)\cdot J^\gamma_j(x)\cdot a_i(x)\,a_j(x)\,a_k(x) = 0 \quad \text{for } l = 1, \ldots, q\,. \end{cases}$$

These equations define for each $x \in M$ a (real) algebraic set $\mathcal{M}_x \subset H_x(M)$.

Then we have to take the condition $[JX, X] \notin \mathcal{A}$ into account. By (20), $[JX, X] \in \mathcal{A}$ if and only if

$$(22)\qquad \sum_{i,j,k=1}^{N} c^l_{ji}(x)\cdot J^j_i(x)\cdot a_i(x)\,a_k(x) = 0 \quad \text{for } l = 1, \ldots, p\,,$$

and these equations define likewise a (real) algebraic set $\mathcal{N}_x$ in each $H_x(M)$.

Next we set $\mathcal{V}_x := \mathcal{M}_x \setminus \mathcal{N}_x$. Then for each $x \in M$, $\mathcal{V}_x$ is a locally closed real analytic set in $H_x(M)$ such that for $(a_1(x), \ldots, a_N(x)) \in \mathcal{V}_x$, $x \in M$, the corresponding $X = \sum_{i=1}^{N} a_i(x)\cdot X_i$ satisfies $\mathcal{F}'_X \subseteq \mathcal{A}'$ and $[JX, X] \notin \mathcal{A}$. Of course it can happen that the sets $\mathcal{V}_x$ are empty, and then there will be no $X \in \mathrm{C}^*_1$. However, suppose that it is not so.

To proceed further we have to assume that each $\mathcal{V}_x$ is locally smooth, and that the family $\{\mathcal{V}_x\}_{x\in M}$ varies smoothly in the parameter x. More precisely, fix a point $x_0 = (x^0_1, \ldots, x^0_m) \in M$ and a point $(a^0_1(x_0), \ldots, a^0_N(x_0)) \in \mathcal{V}_{x_0}$, and make the following three assumptions:

1. *Local regularity of* $\mathcal{V}_{x_0}$: $\mathcal{V}_{x_0}$ is a real analytic manifold of dimension r near $(a^0_1(x_0), \ldots, a^0_N(x_0))$ in $H_{x_0}(M)$.

This means (perhaps after having made a change of the indices) that $\mathcal{V}_{x_0}$ can be presented in the form

$$a_j(x_0) = F^{x_0}_j(a_1(x_0), \ldots, a_r(x_0))\,, \qquad j = r+1, \ldots, N\,,$$

for $(a_1(x_0), \ldots, a_N(x_0))$ near $(a^0_1(x_0), \ldots, a^0_N(x_0))$, where the functions $F^{x_0}_j$ are real analytic.

2. *Regularity of* $\{\mathcal{V}_x\}_{x\in M}$ *near* $(x_0, a^0(x_0))$: Let $\pi\colon H(M) \to M$ be the canonical projection. Then there is a neighbourhood U of $(x^0_1, \ldots, x^0_m; a^0_1(x_0), \ldots, a^0_N(x_0))$ in $H(M)$ such that for each $x \in \pi(U)$, $\mathcal{V}_x \cap U$ is a real analytic r-dimensional manifold.

3. *Coherence.* There are real analytic functions

$$F_j(x_1, \dots, x_m; a_1(x), \dots, a_r(x)), \qquad j = r+1, \dots, N,$$

defined on U such that for all $x \in \pi(U)$ the r-dimensional real analytic manifolds $\mathcal{V}_x \cap U$ can be represented by

$$a_j(x) = F_j(x_1, \dots, x_m; a_1(x), \dots, a_r(x)), \qquad j = r+1, \dots, N.$$

In particular, $F_j^{x_0}(a(x_0)) = F_j(x_0; a(x_0))$. Hence with these assumptions the conditions $\mathcal{F}'_X \subseteq \mathcal{A}'$ and $[JX, X] \notin \mathcal{A}$ are satisfied for X of the form

$$X = \sum_{i=1}^{r} a_i(x) \cdot X_i + \sum_{i=r+1}^{N} F_i(x; a_1(x), \dots, a_r(x)) \cdot X_i. \tag{23}$$

Next we have to determine the functions $a_1(x), \dots, a_r(x)$ so that $\mathcal{F}'_X = \mathcal{F}_X$. To do this we consider $a_1, \dots, a_r$ no longer as functions of x, but as new independent variables, and prolong the problem to the $(x_1, \dots, x_m; a_1, \dots, a_r)$-space. That is, we regard $X_1, \dots, X_N$ as vector fields $\hat{X}_1, \dots, \hat{X}_N$ on this space, which are independent of the a-coordinates. Then we get the prolonged vector field

$$\hat{X} = \sum_{i=1}^{r} a_i \cdot \hat{X}_i + \sum_{i=r+1}^{N} F_i(x; a) \cdot \hat{X}_i$$

corresponding to (23). Note that $\hat{X}$ is a completely determined vector field on the $(x; a)$-space. From $\hat{X}$ the vector fields $J\hat{X}$ and $[J\hat{X}\ \hat{X}]$ are constructed, where J is still defined by (19), *i.e.* is independent of the a-variables.

If the variables a_i are replaced by determined functions $a_i(x)$, then of course $\hat{X}$ and $J\hat{X}$ go over into the corresponding X and JX. However, $[J\hat{X}, \hat{X}]$ does not necessarily turn into $[JX, X]$ (by (20) it does modulo $\mathcal{A}$ though, and this is of some importance), since for instance $\hat{X}_j a_1 = 0$, while $X_j a_1(x)$ need not vanish.

Let $\hat{\mathcal{F}}_X$ be the vector field system on the $(x; a)$-space generated by $\hat{X}$, $J\hat{X}$, $[J\hat{X}, \hat{X}]$ and $\partial/\partial a_i$ for $i = 1, \dots, r$. Setting $x = x_0$ and $a_i = a_i^0(x_0)$, $i = 1, \dots, r$, for a moment, we get vectors $\hat{X}_0$, $(J\hat{X})_0$ and $[J\hat{X}, \hat{X}]_0$ in $T_{x_0}(M)$ that together span a 3-dimensional subspace of $T_{x_0}(M)$. Then we choose local coordinates $(x_1, \dots, x_m)$ near $x_0 \in M$ such that $dx_1 \wedge dx_2 \wedge dx_3|_{T_{x_0}(M)}$ does not vanish on this subspace. Near $(x_0; a^0(x_0))$ in the $(x; a)$-space it is then possible to find a basis for $\hat{\mathcal{F}}_X$ in the following form:

$$\begin{cases} V_i = \dfrac{\partial}{\partial x_i} + \sum\limits_{k=4}^{N} b_{ik}(x; a) \cdot \dfrac{\partial}{\partial x_k}, & i = 1, 2, 3, \\ \partial/\partial a_j, & j = 1, \ldots, r. \end{cases}$$

The prolonged vector field system that we are looking for then has a basis of the form

$$W_i = V_i + \sum_{k=1}^{N} w_{ik}(x; a) \cdot \frac{\partial}{\partial a_k}, \qquad i = 1, 2, 3, \tag{24}$$

where the functions $w_{ik}(x; a)$ are to be determined so that (W_1, W_2, W_3) is complete.

Now, at last, our problem has been formulated in such a way that the Vessiot theory can be applied. If (24) is involutive of order 3, then by means of the methods in section 3 the coefficients $w_{ik}(x; a)$ are determined (up to a set of arbitrary functions in a certain number of variables) so that the vector field system (W_1, W_2, W_3) is complete. Otherwise we have to make further prolongations (now in the standard form given in [15], section 15), until we either arrive at a solution, or obtain an incompatible system of linear equations.

Suppose now that we have found a solution in the form of a complete vector field system $\mathcal{W} = (W_1, W_2, W_3)$. Then $\mathcal{W}$ defines a foliation $\mathcal{L}^*$ of the $(x; a)$-space in a neighbourhood of $(x_0; a^0(x_0))$ with 3-dimensional leaves. Since we have made sure that the coordinate functions x_1, x_2 and x_3 are functionally independent on these leaves, we can find $m + r - 3$ invariants $\xi_k(x; a)$, $k = 4, \ldots, m + r$, of $\mathcal{W}$ such that

i) $\xi_k(x_0; a^0(x_0)) = 0$, $\quad k = 4, \ldots, m + r$;

ii) $\{x_1, x_2, x_3, \xi_4, \ldots, \xi_{m+r}\}$ form a system of local coordinates in a neighbourhood Ω of $(x_0; a^0(x_0))$ in the $(x; a)$-space;

iii) $\dfrac{\partial(\xi_4, \ldots, \xi_m)}{\partial(x_4, \ldots, x_m)}(x_0; a^0(x_0)) \neq 0$.

If $\omega := \{(x; a) \in \Omega : \xi_k(x; a) = 0 \text{ for } k = m + 1, \ldots, m + r\}$, then by iii) there is a neighbourhood V of $(x_0; a^0(x_0))$ in ω which is diffeomorphic to an open neighbourhood U of x_0 in M under the projection mapping ϱ defined by $\varrho(x; a) := x$. Furthermore, we can assume that on V

$$a_i = A_i(x_1, \ldots, x_m), \qquad i = 1, \ldots, r,$$

and

$$\xi_k = t_{k-3}(x_1, \dots, x_m), \qquad k = 4, \dots, m,$$

where $A_i(x)$ and $t_{k-3}(x)$ are real analytic functions defined on U.

Because $\xi_{m+1}, \dots, \xi_{m+r}$ are invariants for $\mathcal{W}$, $\mathfrak{L}^*$ can be restricted to V to give a foliation $\mathfrak{L}'$ there with the invariants $\xi_4, \dots, \xi_m$. By means of the diffeomorphism ϱ, $\mathfrak{L}'$ defines a foliation $\mathfrak{L} = \{L_t\}_{t\in T}$ of U, with the invariants $t_1(x), \dots, t_{m-3}(x)$. Hence the parameter space $T \simeq \mathbb{R}^{m-3}$. We claim that $\mathfrak{L}$ is a Lewy foliation of U.

As a matter of fact, by the very construction $\mathcal{W}$ contains vector fields of the form

$$\tilde{X} = \sum_{i=1}^{r} a_i \cdot X_i + \sum_{i=r+1}^{N} F_i(x; a) \cdot X_i + \sum_{k=1}^{r} c_k(x; a) \cdot \partial/\partial a_k,$$

and

$$J\tilde{X} = \sum_{i=1}^{r} a_i \cdot JX_i + \sum_{i=r+1}^{N} F_i(x; a) \cdot JX_i + \sum_{k=1}^{r} c_k(x; a) \cdot \partial/\partial a_k.$$

$\tilde{X}$ and $J\tilde{X}$ have restrictions $\check{X}$ and $J\check{X}$ to V, which are tangent to $\mathfrak{L}'$. Under the projection mapping ϱ, $\check{X}$ is mapped to

$$X := \varrho_*(\check{X}) = \sum_{i=1}^{r} A_i(x) \cdot X_i + \sum_{i=r+1}^{N} F_i(x; A_1(x), \dots, A_r(x)) \cdot X_i,$$

and $\varrho_*(J\check{X}) = JX$. Obviously X and JX are tangent to the foliation $\mathfrak{L}$. Because (22) is excluded, $[JX, X] \not\equiv 0 \pmod{\mathcal{A}}$, so in particular X, JX and $[JX, X]$ are linearly independent. Hence they together span the tangent spaces of each 3-dimensional leaf of $\mathfrak{L}$. Consequently $(X, JX, [JX, X])$ is complete, and $\mathfrak{L}$ is a Lewy foliation.

It should be remarked that not all regular and transversal Lewy foliations near $x_0 \in M$ are found in this way, but only those for which $X(x_0)$ is near the vector $X_0 = \sum_{i=1}^{N} a_i^0(x_0) \cdot X_i(x_0)$. To find further ones we have to repeat this procedure for other neighbourhoods in $\mathcal{V}_{x_0}$.

To each transversal Lewy foliation $\{L_t\}$ there corresponds an analytic continuation $\{\tilde{L}_t\}$, and also a nonvanishing normal vector field N defined as the image of $J[JX, X]$ under the projection mapping

$$T(\mathbb{C}^n)|_M \to T(\mathbb{C}^n)|_M / T(M).$$

If u is a CR function on M, then for each t in the parameter space $u|_{L_t}$ will have a one-sided local analytic continuation defined on that side of L_t in $\tilde{L}_t$ into which N points.

To each $X \in \mathcal{A}$ is associated the $\bar{\partial}_M$-operator $Z = X + iJX$. If $X \in \mathcal{C}_0$, $\bar{Z}$ is called a Cauchy-Riemann operator, since this is what $\bar{Z}$ is on the leaves of the foliation defined by (X, JX). If $X \in \mathcal{C}_1^*$, the corresponding $\bar{Z}$ is called a Lewy operator. Now, if we have found enough Cauchy-Riemann and Lewy operators to form a basis for the $\bar{\partial}_M$-equations, we may stop at this point. Otherwise we continue to determine the classes $\mathcal{C}_i$ for $i > 1$. However, since this is analogous (although a bit more complicated) to what is done above, we do not give the details here.

8. – Hypersurface foliations.

DEFINITION. *A $(2q+1)$-dimensional transversal and nondegenerate hypersurface foliation of M in $\mathbf{C}^n$ is a foliation $\{H_s\}_{s\in S}$ (where S is a parameter space) so that each $(2q+1)$-dimensional leaf H_s can be extended to a complex $(q+1)$-dimensional manifold $\tilde{H}_s$ defined near M in such a way that*

i) *H_s is a real nondegenerate hypersurface in $\tilde{H}_s$, and*

ii) *$\tilde{H}_s$ intersects M transversally along H_s, for each $s \in S$.*

Let $\{H_s\}_{s\in S}$ be such a hypersurface foliation of M. Then remark 2 in section 2 shows that each H_s, regarded as a hypersurface in $\tilde{H}_s$, locally admits q independent Lewy foliations $\{L^i_{s,t}\}_{t\in T}$, $i = 1, \ldots, q$. It can then happen that these behave nicely with respect to the parameter s, so that they define foliations $\{L^i_r\}_{r\in R}$ of M, with $R = S \times T$, depending real analytically on r, and thus making $\{H_s\}$ locally generated by q Lewy foliations.

Now we want to turn this around and construct hypersurface foliations by means of triples $\{X, JX, [JX, X]\}$ with $X \in \mathcal{A}$, using methods similar to those in the preceding section.

Hence we want to find q N-tuples $a^\alpha(x) = (a^\alpha_1(x), \ldots, a^\alpha_N(x))$, $\alpha = 1, \ldots, q$, of real analytic functions so that if

$$X^\alpha := \sum_{i=1}^{N} a^\alpha_i(x) \cdot X_i \quad \text{for } \alpha = 1, \ldots, q, \tag{25}$$

then the vector field system

$$\mathcal{H} := (X^1, JX^1, [JX^1, X^1], \ldots, X^q, JX^q, [JX^q, X^q])$$

defines a $(2q+1)$-dimensional transversal and nondegenerate hypersurface foliation. It is easy to see that this is the case if and only if the following conditions are satisfied:

1) The vector fields $X^1, JX^1, \dots, X^q, JX^q$ are linearly independent.
2) $[JX^\alpha, X^\alpha] \not\equiv 0 \pmod{\mathcal{A}}$ for $\alpha = 1, \dots, q$.
3) Let $\pi\colon \mathcal{A}' \to \mathcal{A}'/\mathcal{A}$, and set $Y^\alpha := \pi([JX^\alpha, X^\alpha])$.

   Then rank $\{Y^1, \dots, Y^q\} = 1$.
4) $\mathcal{H}$ is complete.
5) rank $\mathcal{H} = 2q + 1$.

The leaves H_s of the corresponding foliation will have tangent spaces of the form $\mathbf{C}^q \times \mathbf{R}$, where $\mathbf{C}^q$ is formed by $X^1, JX^1, \dots, X^q, JX^q$. Thus by the Cartan-Kähler theory these leaves can locally be continued to complex $(q+1)$-dimensional manifolds $\tilde{H}_s$. 2) assures that H_s will be a nondegenerate hypersurface in $\tilde{H}_s$.

As a first step in the search for the N-tuples $a^\alpha(x)$, condition 4) is replaced by the weaker

4′) $\mathcal{H}' \subseteq \mathcal{A}''$.

Also introduce the complementary conditions to 1) and 2), *i.e.*,

1′) $X^1, JX^1, \dots, X^q, JX^q$ are linearly dependent, and

2′) $[JX^\alpha, X^\alpha] \equiv 0 \pmod{\mathcal{A}}$ for $\alpha = 1, \dots, q$.

1′) means that a number of determinants involving the functions $a_i(x)$ vanish, thus giving algebraic conditions on the $a_i(x)$ for each x.

By (22), 2′) is equivalent to the algebraic equations

$$\sum_{i,j,k=1}^{N} c^l_{ij}(x) \cdot J^j_i(x) \cdot a^\alpha_i(x) a^\alpha_k(x) = 0 \qquad \text{for } l = 1, \dots, p \text{ and } \alpha = 1, \dots, q,$$

in $a_i(x)$.

From (20) it follows that 3) is satisfied if and only if

$$\Big(\sum_{i,j,k=1}^{N} c^l_{ij}(x) \cdot J^j_i(x) \cdot a^\alpha_i(x) a^\alpha_k(x)\Big) \cdot \Big(\sum_{i,j,k=1}^{N} c^m_{ij}(x) \cdot J^j_i(x) \cdot a^\beta_i(x) a^\beta_k(x)\Big) =$$
$$= \Big(\sum_{i,j,k=1}^{N} c^m_{ij}(x) \cdot J^j_i(x) \cdot a^\alpha_i(x) a^\alpha_k(x)\Big) \cdot \Big(\sum_{i,j,k=1}^{N} c^l_{ij}(x) \cdot J^j_i(x) \cdot a^\beta_i(x) a^\beta_k(x)\Big)$$

for $l, m = 1, \dots, p$, and $\alpha, \beta = 1, \dots, q$.

Finally, from (20) it follows that 4′) is equivalent to a number of homogeneous equations of fourth order in the functions $a_i(x)$.

For each $x \in M$ we regard $(a_1^1(x), \ldots, a_N^1(x), \ldots, a_1^q(x), \ldots, a_N^q(x))$ as the coordinates of the qN-dimensional fiber $H_x^q(M) \cong (\mathbf{R}^{qN})_x$. Then 3) and 4′) together define an algebraic set $\mathcal{S}_x$ in each $H_x^q(M)$. Similarly, each of 1′) and 2′) defines an algebraic set in $H_x^q(M)$ for each $x \in M$, and we let $\mathcal{I}_x \subset H_q^q(M)$ be the union of these two sets. Then $\mathcal{R}_x := \mathcal{S}_x \setminus \mathcal{I}_x$ is a locally closed real analytic set in $H_x^q(M)$ for each $x \in M$, such that the conditions 1), 2), 3) and 4′) are satisfied for those $a^\alpha(x)$ which belong to $\mathcal{R}_x$.

Next fix a point $x_0 \in M$ for which $\mathcal{R}_{x_0}$ is nonempty (supposing that such a point exists), and let $a^0(x_0)$ be a regular point in $\mathcal{R}_{x_0}$, so that $\mathcal{R}_{x_0}$ is a real analytic manifold of dimension r, say, near $a^0(x_0)$. With the same kind of regularity and coherence assumptions as in the last section, the sets $\mathcal{R}_x$ can be described by equations of the form

$$a_i^\alpha(x) = A_i^\alpha(x_1, \ldots, x_m; u_1, \ldots, u_r) \tag{26}$$

near the point $(x_0; a^0(x_0)) \in H^q(M)$, where $(u_1, \ldots, u_r)$ is a set of r real variables, and the functions $A_i^\alpha(x; u)$ are real analytic.

Replacing $a_i(x)$ in (25) by the right-hand side of (26) we obtain determined vector fields $\check{X}^\alpha$, $J\check{X}^\alpha$ and $[J\check{X}^\alpha, X^\alpha]$, $\alpha = 1, \ldots, q$, on the $(x; u)$-space, such that the conditions 1), 2), 3) and 4′) are satisfied for the corresponding vector fields on the x-space obtained by regarding the u_i, $i = 1, \ldots, r$, as parameters. By 1) and 2) the vector field system generated by these $3q$ vector fields on the $(x; u)$-space have a rank which is at least equal to $2q + 1$. To satisfy 5) we want it to be exactly $2q + 1$. This is achieved by setting a number of determinants, involving $A_i^\alpha(x; u)$, $J_i^j(x)$, $c_{ij}^l(x)$ and certain of their x-derivatives, equal to zero, and hence defines a real analytic variety $\mathcal{V}$ in the $(x; u)$-space. In order to obtain a hypersurface foliation of a whole neighbourhood in the x-space in the end, we have to assume that the x-coordinates locally are functionally independent on $\mathcal{V}$. Moreover we suppose that for some integer $s \leqslant r$, $\mathcal{V}$ is an $(m + s)$-dimensional real analytic manifold near a regular point $P \in \mathcal{V}$. Then $\mathcal{V}$ can be defined by equations of the form

$$u_i = U_i(x_1, \ldots, x_m; w_1, \ldots, w_s), \quad i = 1, \ldots, r, \tag{27}$$

near P, where $w_1, \ldots, w_s$ are real variables, and the functions $U_i(x; w)$ are real analytic. Substituting (27) in (26) gives

$$a_i^\alpha(x) = F_i^\alpha(x_1, \ldots, x_m; w_1, \ldots, w_s), \tag{28}$$

with $F_i^\alpha(x; w)$ real analytic.

Using (28) in the expression (25) we get determined vector fields

$$\hat{X}^{\alpha}=\sum_{k=1}^{s} F_{i}^{\alpha}(x;w)\cdot X_{i}\,,\qquad \alpha=1,\dots,q\,,$$

on the $(x;w)$-space, such that the conditions 1), 2), 3), 4′) and 5) are satisfied for the corresponding vector fields on the x-space, which are parameterized by $w_1,\dots,w_s$. Let $\{V_i\}_{i=1}^{2q+1}$ be a resolved basis for the vector field system generated by $\hat{X}^{\alpha}$, $J\hat{X}^{\alpha}$, $[J\hat{X}^{\alpha},\hat{X}^{\alpha}]$, $\alpha=1,\dots,q$, on the $(x;w)$-space. Then the prolonged vector field system that we are looking for has a basis which can be written in the form

$$W_i=V_i+\sum_{k=1}^{s} v_{ik}(x;w)\cdot\frac{\partial}{\partial w_k}\,,\qquad i=1,\dots,2q+1\,.$$

To satisfy 4), the functions $v_{ik}(x;w)$ are to be determined so that $(W_1,\dots,W_{2q+1})$ is complete. Now this problem falls within the realm of the Vessiot theory, which therefore shows the existence (or maybe nonexistence) of a family of foliations of the $(x;w)$-space with $(2q+1)$-dimensional leaves associated to a complete vector field system of the form $(W_1,\dots,W_{2q+1})$. Projecting such a foliation down to the x-space (*i.e.*, M) in a similar way to what was done in the last section, we then get a transversal and nondegenerate hypersurface foliation defined locally on M.

Suppose now that $\{H_s\}$ is such a hypersurface foliation of M with associated analytic continuation $\{\tilde{H}_s\}$, and let u be a *CR* function on M. If H_s is strictly pseudoconvex in $\tilde{H}_s$, then $u_s:=u|_{H_s}$ can be extended to a holomorphic function defined on a one-sided neighbourhood of H_s in $\tilde{H}_s$. However, if the Levi form of H_s (as a hypersurface in $\tilde{H}_s$) has both positive and negative eigenvalues, then u_s will be the restriction to H_s of a holomorphic function defined in a full neighbourhood of H_s in $\tilde{H}_s$ (see e.g. [14] for the case of hyperfunctions). In particular, u will then be real analytic along H_s.

REFERENCES

[1] A. Andreotti - C. D. Hill, *Complex characteristic coordinates and tangential Cauchy-Riemann equations*, Ann. Scuola Norm. Sup. Pisa (3), **26** (1972), pp. 299-324.

[2] A. Andreotti - C. D. Hill, *E. E. Levi convexity and the Hans Lewy problem, Parts I and II*, Ann. Scuola Norm. Sup. Pisa (3), **26** (1972), pp. 325-363, 767-806.

[3] E. Cartan - A. Einstein, *Lettres sur le parallélisme absolu 1929-1932*, Acad. Roy. de Belgique et Princeton Univ. Press, Bruxelles, 1979.
[4] G. B. Folland - E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), pp. 429-522.
[5] M. Freeman, *Local complex foliation of real submanifolds*, Math. Ann. **209** (1974), pp. 1-30.
[6] R. Hermann, *Convexity and pseudoconvexity for complex manifolds*, II, J. Math. Mech. **13** (1964), pp. 1065-1070.
[7] L. R. Hunt - R. O. Wells, *Extension of CR-functions*, Amer. J. Math., **98** (1976), pp. 805-820.
[8] E. Kähler, *Einführung in die Theorie der Systeme von Differentialgleichungen*, Chelsea Publ. Co., New York, 1949.
[9] M. Kuranishi, *Lectures on exterior differential systems*, Tata Institute of Fundamental Research, Bombay, 1962.
[10] L. Nirenberg, *A complex Frobenius theorem*, Seminar on Analytic Functions, Institute for Advanced Study (1957), pp. 172-189.
[11] L. Nirenberg, *Lectures on linear partial differential equations*, Regional Conf. Series in Math., no. 17, Amer. Math. Soc., Providence, R.I., 1973.
[12] J. Polking - R. O. Wells, *Hyperfunction boundary values and a generalized Bochner-Hartogs theorem*, Several complex variables (Proc. Sym. Pure Math., Vol. XXX, Part 1), pp. 187-193, Amer. Math. Soc., Providence, R.I., 1977.
[13] F. Sommer, *Komplex-analytische Blätterung reeller Mannigfaltigkeiten im $\mathbf{C}^n$*, Math. Ann. **136** (1958), pp. 111-133.
[14] O. Stormark, *A note on a paper by Andreotti and Hill concerning the Hans Lewy problem*, Ann. Scuola Norm. Sup. Pisa (4), **2** (1975), pp. 557-569.
[15] E. Vessiot, *Sur une théorie nouvelle des problèmes généraux d'intégration*, Bull. Soc. Math. France, **52** (1924), pp. 336-395.

The Royal Institute of Technology
Department of Mathematics
S-100 44 Stockholm 70, Sweden