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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 8, nº 4 (1981), p. 605-645

<http://www.numdam.org/item?id=ASNSP_1981_4_8_4_605_0>

Remarks about Signorini's Problem in Linear Elasticity.

DAVID KINDERLEHRER

Assume given an elastic body in its natural configuration occupying a region Ω of n-dimensional space $\mathbf{R}^n$. The body is then subjected to assigned body and surface forces in such a manner that, for example, it must remain on or above a portion Γ of the boundary of Ω, $\partial\Omega$. Under these circumstances, we are asked to find the equilibrium configuration of the body, which means we are asked to determine the displacement vector arising from the imposition of the forces with respect to the constraint on Γ. This is a typical example of Signorini's problem in (linear) elastostatics [22]. The existence of a solution and its uniqueness properties have been investigated by Fichera [4]. They are also a consequence of a theorem of Lions and Stampacchia [16] and the problem is discussed at some length in the book of Duvaut and Lions [3].

Here our attention is directed to the smoothness of the solution and the nature of the subset of Γ in contact with the body in the equilibrium configuration. This subset we call the set of coincidence. We confine ourselves to the case where Γ is a smooth finitely connected $n-1$ submanifold of $\partial\Omega$. Special attention will be devoted to the case of plane elasticity. Here we show that the displacement vector is continuous in $\bar{\Omega}$ and continuously differentiable in $\bar{\Omega}$ except perhaps near $\partial\Gamma$ (Theorems 3.5 and 4.2). More generally we are able to prove that the solution is continuous in dimension $n \leqq 4$ except near $\partial\Gamma$ (Theorem 3.6).

As part of our endeavor, we prove that the body is in equilibrium in its deformed state. This means that the equations expressing the balance of forces and moments are valid and may be understood in the classical sense. We also show that the coincidence set has positive $(n-1)$ dimensional measure. Returning to the study of plane elasticity we show that the coincidence set consists of a finite number of intervals and isolated points under suitable hypotheses (Section 6).

Pervenuto alla Redazione il 4 Novembre 1980 ed in forma definitiva il 7 Maggio 1981.

From the physical standpoint, the problem described above is that of a body occupying Ω impressed on a rigid support or punch conforming perfectly to Γ. It is one topic in the theory of contact mechanics (cf. [3] or J. J. Kalker [10]). Or, for example, Villaggio has studied the problem of an elastic body on a soft foundation [24]. Many such questions are within the purview of our method, the well known Hertz problem being another such instance.

The analog for a single equation, the boundary obstacle problem or the thin obstacle problem, was first considered by H. Lewy [14]. We have also found [15] very useful. Our method may be adopted to study the smoothness of solutions of this problem as well [11]. Our reference for the subject of elliptic systems has been Agmon, Douglis and Nirenberg [1].

1. – Complementarity conditions

In this first section we shall define the Signorini problem and give a brief variational analysis of it. Our principal aim is the statement of the complementarity conditions or natural boundary conditions, e.g., (1.11)-(1.13), which will play a role in our regularity proof. Let $\Omega \subset \mathbf{R}^n$ be a bounded region whose boundary $\partial\Omega$ is smooth and contains two smooth (finitely connected and open) $n-1$ dimensional manifolds Γ and Γ' such that $\partial\Omega = \bar{\Gamma} \cup \Gamma' = \Gamma \cup \bar{\Gamma}'$ and $\Gamma \cap \Gamma' = \emptyset$. By $H^{m,s}(\Omega)$ we denote the Sobolev space of distributions in Ω whose derivatives through order m are in $L^s(\Omega)$. Abusing notation, we also let $H^{m,s}(\Omega)$ stand for $(H^{m,s}(\Omega))^n$, the n-fold product of $H^{m,s}(\Omega)$. Also, $H^m(\Omega) = H^{m,2}(\Omega)$ and $H^m_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in H^m-norm.

Let us review a few of the notions of the theory of linear elasticity. Let $a_{ijhk}(x) \in C^\infty(\bar{\Omega})$ satisfy

$$(1.1) \qquad a_{ijhk}(x)\,\xi_{ij}\xi_{hk} \geqq \alpha_0|\xi|^2 \qquad \text{for } \xi \in \mathbf{R}^{n^2} \text{ with } \xi_{ij} = \xi_{ji}$$

and $x \in \bar{\Omega}$ for some $\alpha_0 > 0$. Here $|\xi|^2 = \sum \xi_{ij}^2$ and the usual summation convention is intended on the left side of (1.1). For the a_{ijhk} to represent elastic coefficients the symmetry conditions (1.2) are frequently imposed:

$$(1.2) \qquad a_{ijhk}(x) = a_{jihk}(x) = a_{ijkh}(x)\,, \qquad x \in \bar{\Omega}\,.$$

The linearized strain and stress tensors of $u = (u^1, \dots, u^n) \in H^1(\Omega)$ are given by

$$(1.3) \qquad \varepsilon_{ij} = \varepsilon_{ij}(u) = \tfrac{1}{2}(u^i_{x_j} + u^j_{x_i})\,, \qquad 1 \leqq i,\ j \leqq n\,,$$

and

$$\sigma_{ij} = \sigma_{ij}(u) = a_{ijhk}\varepsilon_{hk}(u)\,, \qquad 1 \leqq i,\ j \leqq n\,. \tag{1.4}$$

From (1.2), the stress matrix $\sigma = (\sigma_{ij})$ is symmetric.

For a point $x \in \Omega$ and a unit vector $\xi \in \boldsymbol{R}^n$, the vector $\sigma(u(x))\,\xi$ is the force per unit area applied at x to the hyperplane whose normal is ξ. The equations (1.4) are Hooke's Law.

Define the bilinear form

$$\begin{aligned} a(u,\zeta) &= \int_\Omega \sigma_{ij}(u)\,\varepsilon_{ij}(\zeta)\,dx \\ &= \int_\Omega a_{ijhk}\,\varepsilon_{hk}(u)\,\varepsilon_{ij}(\zeta)\,dx \end{aligned}\,, \qquad u,\zeta \in H^1(\Omega) \tag{1.5}$$

and note that in view of (1.2),

$$a(u,\zeta) = \int_\Omega \sigma_{ij}(u)\,\zeta^i_{x_j}\,dx\,, \qquad u,\zeta \in H^1(\Omega)\,. \tag{1.6}$$

In equilibrium with respect to body forces $f_1, \ldots, f_n$ the displacement u is a solution of the equations

$$\begin{aligned} &Au = f \qquad \text{in } \Omega \qquad \text{or} \\ &-\frac{\partial}{\partial x_j}\sigma_{ij}(u) = (Au)_i = -\frac{\partial}{\partial x_j}(a_{ijhk}u^h_{x_k}) = f_i \qquad \text{in } \Omega,\ 1 \leqq i \leqq n\,. \end{aligned} \tag{1.7}$$

The conditions (1.1) and (1.2) ensure that (1.7) is an elliptic system in as much as for any $\xi, \eta \in \boldsymbol{R}^n$,

$$\begin{aligned} a_{ijhk}\xi_i\xi_h\eta_j\eta_k &= \tfrac{1}{4}a_{ijhk}(\xi_i\eta_j + \xi_j\eta_i)(\xi_h\eta_k + \xi_k\eta_h) \\ &\geqq \tfrac{1}{4}\alpha_0 \sum_{i,j}(\xi_i\eta_j + \xi_j\eta_i)^2 \\ &= \tfrac{1}{2}\alpha_0\big(|\xi|^2|\eta|^2 + (\xi\cdot\eta)^2\big) \\ &\geqq \tfrac{1}{2}\alpha_0|\xi|^2|\eta|^2. \end{aligned}$$

It is important to keep in mind, however, that the definiteness condition of (1.1) holds only for symmetric tensors $\xi = (\xi_{ij}) \in \boldsymbol{R}^{n^2}$.

Given $f_1, \ldots, f_n \in L^2(\Omega)$ and $g_1, \ldots, g_n \in L^2(\Gamma')$ we define the distribution (of active body and surface forces)

$$(1.8) \qquad \langle T, \zeta \rangle = \int_\Omega f_i \zeta^i \, dx + \int_{\Gamma'} g_i \zeta^i \, dS \,, \qquad \zeta \in H^1(\Omega) \,,$$

where dS denotes the element of surface area on $\partial\Omega$. With $\nu = (\nu_1, \ldots, \nu_n)$ the outward directed normal on $\partial\Omega$ let

$$(1.9) \qquad \boldsymbol{K} = \{v = (v^1, \ldots, v^n) \in H^1(\Omega) \colon v \cdot \nu = v^h \nu_h \leqq 0 \text{ on } \Gamma\} \,.$$

The Signorini problem we consider here is the variational inequality

PROBLEM 1.1. *Find* $u \in \boldsymbol{K}$: $a(u, v - u) \geqq \langle T, v - u \rangle$ *for all* $v \in \boldsymbol{K}$.

The displacement u which resolves Problem 1.1 has least energy

$$\mathcal{E}(v) = \tfrac{1}{2} a(v, v) - \langle T, v \rangle \qquad \text{for } v \in \boldsymbol{K} \,.$$

For example, the reader may wish to recall that in the case of a homogeneous isotropic material, after suitable normalization,

$$(1.10) \qquad \begin{aligned} a_{ijhk} &= (\alpha - 1)\, \delta_{ij} \delta_{hk} + (\delta_{ih} \delta_{jk} + \delta_{ik} \delta_{jh}) \,, \qquad 1 \leqq i, j, h, k \leqq n \,, \\ \sigma_{ij} &= (\alpha - 1)\, \delta_{ij} \varepsilon_{hh} + 2\varepsilon_{ij} \,, \qquad 1 \leqq i,\, j \leqq n \,, \\ (Au)_i &= -\Delta u^i - \alpha \sum_h u^h_{x_i x_h} \,, \qquad 1 \leqq i \leqq n \end{aligned}$$

where the real constant α is chosen so that (1.1) holds. For instance, $\alpha > 0$ for $n = 2$ and $\alpha > \frac{1}{3}$ when $n = 3$ ([13], p. 16), however the system of (1.10) is elliptic if $\alpha > -1$ in any dimension.

Let us derive the complementarity conditions or natural boundary conditions associated to Problem 1.1 which we mentioned earlier. Assuming that $u \in H^2(\Omega)$, an integration by parts yields that

$$a(u, \zeta) = -\int_\Omega \sigma_{ij}(u)_{x_j} \zeta^i \, dx + \int_{\partial\Omega} \sigma_{ij}(u) \nu_j \zeta^i \, dS \,, \qquad \zeta \in H^1(\Omega) \,.$$

First choosing $\zeta \in H^1_0(\Omega)$ so that $u \pm \zeta \in \boldsymbol{K}$ we obtain that $Au = f$ in Ω. Next choosing ζ so that $\zeta = 0$ on Γ, thus again $u \pm \zeta \in \boldsymbol{K}$, we obtain

$$\sigma_{ij} \nu_j = g_i \qquad \text{on } \Gamma', \qquad 1 \leqq i \leqq n \,.$$

Setting this information in the variational inequality,

$$\int_\Gamma \sigma_{ij}(u)\,\nu_j(v^i - u^i)\,dS \geqq 0 \qquad \text{for } v \in \boldsymbol{K}.$$

For $\zeta = (\zeta^1, \ldots, \zeta^n)$ let us write $\zeta = \zeta_\tau + \zeta_\nu \nu$ where ζ_τ is the tangential component of ζ on Γ and $\zeta_\nu = \zeta \cdot \nu$. Thus,

$$\int_\Gamma \sigma_{ij}(u)\,\nu_j(v_\tau - u_\tau)\,dS + \int_\Gamma \sigma_{ij}(u)\,\nu_j\nu_i(v_\nu - u_\nu)\,dS \geqq 0\,,$$

and since v_τ is arbitrary, the first integral vanishes, or

$$\sigma_{ij}(u)\,\nu_j\,\tau_i = 0 \qquad \text{on } \Gamma \text{ whenever } \tau\cdot\nu = 0 \text{ on } \Gamma.$$

Finally,

$$\int_\Gamma \sigma_{ij}(u)\,\nu_j\nu_i(v_\nu - u_\nu)\,dS \geqq 0 \qquad \text{for } v \in \boldsymbol{K}.$$

Whenever $\zeta\cdot\nu \leqq 0$ on Γ, $v = u + \zeta \in \boldsymbol{K}$. This yields that $\sigma_{ij}(u)\,\nu_i\nu_j \leqq 0$ on Γ. On the other hand we may choose $\zeta = 0$ so

$$-\int_\Gamma [\sigma_{ij}(u)\,\nu_i\nu_j]\,u_\nu\,dS \geqq 0\,.$$

But each factor in the integrand is negative (non positive), so the integrand vanishes identically.

Summarizing, if $u \in H^2(\Omega)$ is a solution of Problem 1.1, then

(1.11) $$Au = f \quad \textit{in } \Omega$$
$$\sigma_{ij}\nu_j = g_i \quad \textit{on } \Gamma', \quad 1 \leqq i \leqq n$$

(1.12) $$[u^h\nu_h][\sigma_{ij}\nu_j\nu_i] = 0$$
$$-\sigma_{ij}\nu_j\nu_i \geqq 0 \qquad \textit{on } \Gamma \textit{ and}$$
$$-u^h\nu_h \geqq 0$$

(1.13) $$\sigma_{ij}\tau_i\nu_j = 0 \qquad \textit{on } \Gamma \textit{ for any } \tau \textit{ with } \tau\cdot\nu = 0 \textit{ on } \Gamma.$$

Observe that (1.11)-(1.13) are valid if we assume only that $u \in H^2(\Omega \cap B_r(x))$ for some $B_r(x)$, $r > 0$, whenever $x \in \Omega - \bar{\Gamma} \cap \bar{\Gamma}'$.

To briefly summarize the existence and uniqueness theory of Problem 1.1, let $\mathcal{A}$ denote the set of infinitesimal affine transformations

$$\begin{aligned}&\zeta(x) = c + Bx\,, \quad x \in \boldsymbol{R}^n, \quad \text{where}\\ &c \in \boldsymbol{R}^n \quad \text{and} \quad B = (b_{ij}) \quad \text{is a constant matrix}\\ &\text{with} \quad b_{ij} + b_{ji} = 0\,, \quad 1 \leqq i,\, j \leqq n\,.\end{aligned}$$

It is elementary to check that $a(\zeta, \zeta) = 0$ if and only if $\zeta \in \mathcal{A}$. The result of [4], [16] is that a solution of Problem 1.1 exists provided that

$$\langle T, \zeta\rangle < 0 \qquad \textit{for every } \zeta \in \mathcal{A} \cap \boldsymbol{K} \textit{ with } -\zeta \notin \boldsymbol{K}\,. \tag{1.14}$$

Also, if u, u^* are two solutions, then $u^* = u + \eta$ where $\langle T, \eta\rangle = 0$. A converse also holds, namely, given a solution u of Problem 1.1 and $\eta \in \mathcal{A} \cap \boldsymbol{K}$ with $-\eta \in \mathcal{A} \cap \boldsymbol{K}$, then $u + \eta$ is also a solution.

For technical reasons it will be helpful to consider a variational inequality slightly more general than Problem 1.1. Suppose that

$$a_{ijhk},\, a_{ijh},\, b_{ihk},\, a_{ih} \in C^\infty(\bar{\Omega})\,, \qquad 1 \leqq i, j, h, k \leqq n\,,$$

and that for some $\alpha_0 > 0$ there is a $C > 0$, depending also on Ω, such that

$$a_{ijhk}\xi_i\xi_h\eta_j\eta_k \geqq \alpha_0|\xi|^2|\eta|^2, \qquad \xi, \eta \in \boldsymbol{R}^n,\ x \in \bar{\Omega}\,, \tag{1.15}$$

and

$$\begin{aligned}&\|v_x\|^2_{L^2(\Omega)} \leqq C\big(a_0(v, v) + \|v\|^2_{L^2(\Omega)}\big)\,, \qquad v \in H^1(\Omega)\,,\\ &a_0(v, v) = \int_\Omega a_{ijkh} v^h_{x_k} v^i_{x_j}\, dx\,.\end{aligned} \tag{1.15'}$$

This coerciveness inequality reduces to Korn's inequality when (1.1) holds. Some general conditions pertaining to coerciveness may be found in [5]. We define, for $u, \zeta \in H^1(\Omega)$,

$$a(u, \zeta) = \int_\Omega \{(a_{ijhk}u^h_{x_k} + a_{ijk}u^h)\zeta^i_{x_j} + b_{ihk}u^h_{x_k}\zeta^i + a_{ih}u^h\zeta^i\}\, dx\,. \tag{1.16}$$

Also suppose that Γ, Γ' and Γ'' are mutually disjoint smooth, finitely connected submanifolds of $\partial\Omega$ satisfying $\bar{\Gamma} \cup \bar{\Gamma}' \cup \bar{\Gamma}'' = \partial\Omega$. Let $f_i \in L^2(\Omega)$

and g_i, $\varphi^i \in H^1(\Omega)$, $1 \leqq i \leqq n$, be given, and set

$$\boldsymbol{K} = \{v \in H^1(\Omega) : v \cdot \nu \leqq 0 \text{ on } \Gamma \text{ and } v^i = \varphi^i \text{ on } \Gamma'', \ 1 \leqq i \leqq n\} \tag{1.17}$$

$$\langle T, \zeta \rangle = \int_\Omega f_i \zeta^i \, dx + \int_{\Gamma'} g_i \zeta^i \, dS , \qquad \zeta \in H^1(\Omega) . \tag{1.18}$$

It may be that Γ', Γ'', or both are empty but we always suppose that $\Gamma \neq \emptyset$. We also suppose that $\boldsymbol{K} \neq \emptyset$, a hypothesis made necessary by the introduction of Γ''.

Our more general variational inequality is

PROBLEM 1.2. *To find* $u \in \boldsymbol{K}$: $a(u, v - u) \geqq \langle T, v - u \rangle$ *for* $v \in \boldsymbol{K}$.

Above, $a(\cdot, \cdot)$, $\boldsymbol{K}$, and T are given by (1.16), (1.17), and (1.18).

Analogous to our discussion of the Signorini problem, we set

$$\sigma_{ij} = \sigma_{ij}(u) = a_{ijhk} u^h_{x_k} + a_{ijh} u^h , \qquad 1 \leqq i, j \leqq n , \tag{1.19}$$

and

$$(Au)_i = -\frac{\partial}{\partial x_j} \sigma_{ij}(u) + b_{ihk} u^h_{x_k} + a_{ih} u^h \qquad 1 \leqq i \leqq n . \tag{1.20}$$

The matrix (σ_{ij}) is not necessarily symmetric nor does it generally represent stresses determined by a linear Hooke's law.

Complementarity conditions associated with Problem 1.2 may be derived in the same fashion as (1.11)-(1.13). An integration by parts, assuming that $u \in H^2(\Omega)$, gives that

$$a(u, \zeta) = \int_\Omega \left\{ -\frac{\partial}{\partial x_j} \sigma_{ij}(u) + b_{ihk} u^h_{x_k} + a_{ih} u^h \right\} \zeta^i \, dx + \int_{\partial\Omega} \sigma_{ij}(u) \nu_j \zeta^i \, dS .$$

Thus

$$Au = f \quad \text{in } \Omega \tag{1.21}$$

$$[\sigma_{ij} \nu_i \nu_j][u^h \nu_h] = 0$$

$$\left.\begin{aligned} -\sigma_{ij}\nu_i\nu_j &\geqq 0 \\ -u^h \nu_h &\geqq 0 \end{aligned}\right\} \quad \text{on } \Gamma \tag{1.22}$$

$$\sigma_{ij} \nu_j \tau_i = 0 \quad \text{on } \Gamma \text{ whenever } \tau_h \nu_h = 0 \text{ on } \Gamma \tag{1.23}$$

$$\sigma_{ij} \nu_j = g_i \quad \text{on } \Gamma', \quad 1 \leqq i \leqq n \tag{1.24}$$

$$u^h = \varphi^h \quad \text{on } \Gamma'', \quad 1 \leqq h \leqq n . \tag{1.25}$$

Consider the special case when there is an open subset $\Gamma_0 \subset \Gamma \cap \cap \{x\colon x_n = 0\}$. Then (1.22), (1.23) may be written, assuming $\nu = -e_n = = (0, \ldots, 0, -1)$ on Γ_0,

$$
\begin{aligned}
&u^n \sigma_{nn} = 0 \\
(1.26) \qquad &-\sigma_{nn} \geqq 0\,, \quad u^n \geqq 0 \quad \text{on } \Gamma_0 \\
&\sigma_{\mu n} = 0\,, \quad 1 \leqq \mu \leqq n-1\,.
\end{aligned}
$$

If $\Gamma_0' \subset \Gamma' \cap \{x\colon x_n = 0\}$, then (1.24) may be written

$$
(1.27) \qquad -\sigma_{in} = g_i\,, \quad \text{on } \Gamma_0'\,, \qquad i = 1, \ldots, n\,,
$$

again with $\nu = -e_n$.

If $u \in H^2(\Omega \cap U)$ in a neighborhood U of $x_0 \in \partial\Omega$, one establishes that the conditions (1.21) and (1.22)-(1.25) appropriate to $\partial\Omega \cap U$ hold almost everywhere (and in the sense of distributions). In particular if $\Gamma_0 \subset \subset \{x_n = 0\} \cap \Gamma \cap U$ then $u \in H^1(\Gamma_0)$ and on the set $I_0 = \{x \in \Gamma_0\colon u^n(x) = 0\}$ we know that $u^n_{x_\mu} = 0$ a.e. on I_0, $\mu = 1, \ldots, n-1$. Thus the first equation of (1.26) leads to

$$
(1.28) \qquad u^n_{x_\mu} \sigma_{nn} = 0 \quad \text{on } \Gamma_0\,, \qquad 1 \leqq \mu \leqq n-1\,.
$$

We shall exploit this relation in our proof of regularity.

2. – Local formulation and integrability of the solution.

The object of this section is to show that a solution of the variational inequality Problem 1.2 is in H^2 except perhaps near points of $\partial\Gamma \cup \partial\Gamma' \cup \cup \partial\Gamma'' \subset \partial\Omega$. The solution may fail to be in $H^2(\Omega)$ even if $\partial\Omega$ is unloaded near $\partial\Gamma$, that is, even if $g_i = 0$ on Γ' near $\partial\Gamma$ and $\partial\Gamma'' \cap \partial\Gamma \neq \emptyset$. A simple example is noted at the conclusion of this section. By Sobolev's lemma we then deduce that the solution u is continuous in $\Omega \cup \Gamma$ for $n = 2, 3$.

Problem 1.2 admits a convenient local formulation in a new domain $G_R = \{y \in \boldsymbol{R}^n\colon y_n > 0,\ |y| < R\}$ with $\Gamma \cup \Gamma' \subset \{y_n = 0\}$ and where the significant constraint of $\boldsymbol{K}$ is

$$
v^n(y) \geqq 0 \qquad \text{for } y \in \Gamma.
$$

This will simplify our computations here and in § 3. Given $x_0 \in \partial\Omega$, let U be a neighborhood of $x_0 \in \boldsymbol{R}^n$ with smooth boundary ∂U and set

$$\boldsymbol{K}_0 = \{v \in H^1(\Omega \cap U)\colon v^h \nu_h \leqq 0 \text{ on } \Gamma \cap U \text{ and}$$
$$v^i = u^i \text{ on } \partial(\Omega \cap U) - \Gamma \cup \Gamma',\ i = 1, \ldots, n\}$$

where u is a given solution of Problem 1.2. Clearly, u is a solution of

$$a_0(u, v - u) \geqq \langle T, v - u\rangle_0 \qquad \text{for all } v \in \boldsymbol{K}_0 \tag{2.1}$$

where the subscript 0 indicates that the integrations are restricted to $\Omega \cap U$ and $\Gamma' \cap U$.

Now we shall straighten the portion $\partial\Omega \cap U$ of $\partial\Omega$ and then alter the solution to obtain the simplified constraint. After a rigid motion we may suppose $x_0 = 0 \in \partial\Omega$ and the exterior normal to Ω at $x_0 = 0$ is $\nu = -e_n$. Here and in the sequel, e_i denotes the unit vector in the direction of the x_i-axis, $1 \leqq i \leqq n$. Suppose that $\partial\Omega$ is described by $\partial\Omega\colon x_n = \varphi(x')$, $|x'|$ small, near $x = 0$, with $\varphi(0) = \varphi_{x_\mu}(0) = 0$, $\mu = 1, \ldots, n-1$, and set

$$\begin{aligned} y_\mu &= y_\mu\,, \qquad 1 \leqq \mu \leqq n-1\,, \\ y_n &= x_n - \varphi(x') \end{aligned} \qquad |x| \text{ small}. \tag{2.2}$$

For $R > 0$ sufficiently small, $G_R = \{y \in \boldsymbol{R}^n\colon |y| < R,\ y_n > 0\}$ is the image of $U \cap \Omega$ under (2.2) for some smooth neighborhood U of 0.

Under these circumstances, let $\tau_1(y), \ldots, \tau_n(y)$ denote a frame of smooth orthonormal vectors in $\bar{G}_R$ satisfying

$$\begin{aligned} \tau_\mu(y', 0) \qquad &\text{is tangent to } \partial\Omega \cap U,\ 1 \leqq \mu \leqq n-1\,, \\ -\tau_n(y', 0) = \nu(x) \qquad &\text{is the outward normal to } \partial\Omega \cap U\,. \end{aligned} \tag{2.3}$$

For any vector function $v(x) = (v^1(x), \ldots, v^n(x))$, $x \in \overline{\Omega \cap U}$, define $\tilde{v}(y)$ by

$$\begin{aligned} v(x) &= \sum_1^n \tau_l(y)\tilde{v}^l(y)\,, \qquad v \in H^1(\Omega \cap U)\,, \qquad \text{or} \\ v^i(x) &= \sum_i^n \tau_{il}(y)\tilde{v}^l(y) \qquad \text{where} \\ \tau_l(v) &= (\tau_{1l}(y), \ldots, \tau_{nl}(y))\,. \end{aligned} \tag{2.4}$$

We may calculate a variational inequality for $\tilde{u}(y)$ directly from (2.1). Note especially that

$$\tilde{v}^n(y) = \sum_1^n \tau_l(y)\cdot\tau_n(y)\,\tilde{v}^l(y) = -\,\nu(x)\cdot v(x)$$

so $v^h\nu_h \leqq 0$ on $\Gamma \cap U$ if and only if $\tilde{v}^n(y', 0) \geqq 0$ for $y = (y', 0)$ in the image of Γ.

After an elementary computation, we find that for any $v, \zeta \in H^1(U \cap \Omega)$,

$$a_0(v, \zeta) = \tilde{a}(\tilde{v}, \tilde{\zeta}) \qquad \text{with}$$

$$\tilde{a}(\tilde{v}, \tilde{\zeta}) = \int_{G_R} \{(\tilde{a}_{ijhk}\tilde{v}^h_{y_k} + \tilde{a}_{ijh}\tilde{v}^h)\tilde{\zeta}^i_{y_j} + (\tilde{b}_{ihk}\tilde{v}^h_{y_k} + \tilde{a}_{ih}\tilde{v}^h)\tilde{\zeta}^i\}\,dy \tag{2.5}$$

where $\tilde{a}_{ijhk}(0) = a_{ijhk}(0)$ and all the coefficients smooth. In particular, for R sufficiently small, the coerciveness inequality, that for some $C_0 > 0$,

$$\|v_x\|^2_{L^2(G_R)} \leqq C_0\big(a_0(v, v) + \|v - c\|^2_{L^2(G_R)}\big)\,, \qquad v \in H^1(G_R) \tag{2.6}$$

for any $c \in \boldsymbol{R}^n$, is valid, cf. (1.15′).

Finally suppose that $x = 0 \in \bar{\Gamma}$ and that $\Sigma \subset \{y_n = 0\}$ is the image of $\Gamma \cap U$ and $\Sigma' \subset \{y_n = 0\}$ is the image of $\Gamma' \cap U$ with respect to the change of variables (2.2). Let

$$\boldsymbol{K} = \{\tilde{v} \in H^1(G_R) : \tilde{v}^n \geqq 0 \text{ on } \Sigma,\ \tilde{v}^i = \tilde{u}^i \text{ on } \partial G_R - \Sigma \cup \Sigma',\ 1 \leqq i \leqq n\}. \tag{2.7}$$

Then for suitable functions $\tilde{f}_i \in L^2(G_R)$ and $\tilde{g}_i \in H^1(G_R)$

$$\tilde{u} \in \tilde{\boldsymbol{K}} : \tilde{a}(\tilde{u}, \tilde{v} - \tilde{u}) \geqq \int_{G_R} \tilde{f}_i(\tilde{v}^i - \tilde{u}^i)\,dy + \int_{\Sigma'} \tilde{g}_i(\tilde{v}^i - \tilde{u}^i)\,dy' \qquad \text{for } \tilde{v} \in \boldsymbol{K}\,. \tag{2.8}$$

Thus if u is a solution of Problem 1.2, for each $x_0 \in \partial\Omega$ we may find a smooth linear combination $\tilde{u}$ of u which is the solution of a variational inequality, namely (2.8), whose bilinear form given by (2.5) has the same expression as (1.16) in a domain G_R with convex set given by (2.7). Consequently in our discussion of the smoothness of the solution we may suppose without loss in generality that

$$\begin{gathered} u \quad \textit{is a solution of Problem } 1.2 \textit{ with} \\ \Omega = G_R \quad \textit{for some } R > 0\,, \\ \Gamma \cup \Gamma' \subset \{x_n = 0\}\,, \quad \textit{and} \end{gathered}$$

$$\boldsymbol{K} = \{v \in H^1(G_R) : v^n \geqq 0 \textit{ on } \Gamma \textit{ and } v_i = u^i \textit{ on } \partial G_R - \Gamma \cup \Gamma',\ i = 1, \dots, n\}\,. \tag{2.9}$$

LEMMA 2.1. *Let u be a solution of the variational inequality*

$$u \in \boldsymbol{K}: \quad a(u, v-u) \geqq \langle T, v-u\rangle \quad \text{for } v \in \boldsymbol{K}$$

where $\boldsymbol{K}$ is given by (2.9) *and $a(\cdot,\cdot)$ and T are defined by* (1.16) *and* (1.18). *Let $x_0 \in \Gamma \cup \Gamma'$. Then*

$$u \in H^2(G_R \cap B_\varrho(x_0))$$

for some $\varrho > 0$.

Recall that Γ, Γ', and Γ'' are open in ∂G_R. Before proving the lemma observe that for any vector $\lambda = (\lambda^1, \dots, \lambda^n) \in \boldsymbol{R}^n$

$$a_{inhn}\lambda^i\lambda^h = a_{ijhk}\lambda^i\delta_{jn}\lambda^h\delta_{kn} \geqq \alpha_0|\lambda|^2,$$

thus the system of equations (cf. (1.20))

$$-a_{ijhk}u^h_{x_jx_k} = a_{ijhkx_j}u^h_{x_k} + (a_{ijh}u^h)_{x_j} - b_{ihk}u^h_{x_k} - a_{ih}u^h + f_i\,, \qquad i = 1, \dots, n\,,$$

may be solved for $(u^1_{x_nx_n}, \dots, u^n_{x_nx_n})$ in terms of the right hand side and $u^h_{x_\mu x_j}$, $1 \leqq \mu \leqq n-1$, $1 \leqq h, j \leqq n$. Indeed, for a constant C depending on the operator A,

$$\text{(2.10)} \qquad \sum_1^n |u^h_{x_nx_n}|^2 \leqq C \sum_{h,j=1}^n \left(\sum_{\mu=1}^{n-1}|u^h_{x_\mu x_j}|^2 + |u^h_{x_j}|^2 + |u^h|^2 + |f_h|^2\right).$$

PROOF OF THE LEMMA. Our proof is based on a standard difference quotient technique, cf. Nirenberg [21], Frehse [6], to show that $u^h_{x_\mu x_j} \in L^2(G_R \cap B_\varrho(x_0))$. Then (2.10) is applied.

Confining our attention to the case $x_0 \in \Gamma$, we suppose that $G_R = G_1 = G$ and $x_0 = 0$. Choose ϱ so small that

$$\{x_n = 0,\ |x'| < 4\varrho\} \subset \Gamma, \qquad \varrho < \tfrac{1}{4}\,,$$

and let

$$v_\varepsilon(x) = u(x) + \varepsilon(D_{-t}\eta^2 D_t u)(x), \qquad \varepsilon > 0\,,$$

where $\eta \in C_0^\infty(B_{2\varrho})$, $B_r = B_r(0)$, $0 \leqq \eta \leqq 1$, and $\eta = 1$ on B_ϱ and

$$D_t w(x) = \frac{1}{t}\big(w(x+te_\mu) - w(x)\big)\,, \quad D_{-t}w(x) = \frac{1}{t}\big(w(x) - w(x-te_\mu)\big)\,, \quad t > 0\,,$$

for a fixed $\mu < n$. Thus

$$v_\varepsilon(x) = \left(1 - \frac{\varepsilon}{t^2}\left(\eta(x)^2 + \eta(x - te_\mu)^2\right)\right) u(x) + \\ + \frac{\varepsilon}{t^2}\{\eta(x)^2 u(x + te_\mu) + \eta(x - te_\mu)^2 u(x - te_\mu)\}.$$

Let $x = (x', 0) \in \Gamma$ and consider $v_\varepsilon^n(x', 0)$. If $\eta(x) \neq 0$ and $t \leqq \varrho$, then $|x + te_\mu| \leqq |x| + t \leqq 3\varrho$, so $x + te_\mu \in B_{4\varrho} \cap \partial G_1 \subset \Gamma$. Hence $u^n(x + te_\mu) \geqq 0$. Similarly, $\eta(x - te_\mu)^2 u^n(x - te_\mu) \geqq 0$ for $t \leqq \varrho$. Thus for each $t < \varrho$, $v_\varepsilon^n(x) \geqq 0$ on Γ when $\varepsilon < t^2/2$, or

$$v_\varepsilon \in \boldsymbol{K} \qquad \text{for } \varepsilon < t^2/2 .$$

Now we follow a well established procedure, briefly recounted below. More precise estimates of a similar nature will be given in detail in the next section. Set $v = v_\varepsilon$ in the variational inequality. Since $g_i = 0$ for $|x'| < 4\varrho$, i.e., in the support of $v_\varepsilon - u$, we see that

$$a\big(u, D_{-t}(\eta^2 D_t u)\big) \geqq \int_G f_i(D_{-t}\eta^2 D_t u)\, dx .$$

After a change of variables we obtain that

$$\begin{aligned} &\int_G \eta^2 a_{ijhk} D_t u^h_{x_k} D_t u^i_{x_j} dx \\ &\quad \leqq \left| \int_G \left(D_t(a_{ijhk} u^h_{x_k}) D_t u^i (\eta^2)_{x_j} + \eta^2 D_t a_{ijhk} u^h_{x_k}(x + te_\mu) D_t u^i_{x_j}\right) dx \right| + \\ &\quad + \left| \int_G \Big(D_t(a_{ijh} u^h)(\eta^2 D_t u^i)_{x_j} + b_{ihk} u^h_{x_k}(D_{-t}\eta^2 D_t u^i) + a_{ih} u^h\big(\eta^2 D_t u^i\big)\Big)\, dx \right| + \\ &\quad + \left| \int_G f_i(D_{-t}\eta^2 D_t u^i)\, dx \right| . \end{aligned}$$

We use (2.6) in this fashion. Set $v = \eta D_t u$. Then

$$\begin{aligned} \frac{1}{C_0}\int_G \eta^2 (D_t u^h_{x_k})^2\, dx &\leqq a_0(\eta D_t u, \eta D_t u) + \int_G \eta^2 |D_t u^h|^2\, dx + \int_G (D_t u^h \eta_{x_k})\ dx \\ &\leqq \int_G \eta^2 a_{ijhk} D_t u^h_{x_k} D_t u_{x_j}\, dx + \int_G a_{ijhk} \eta_{x_k} \eta_{x_j} D_t u^h D_t u^i\, dx + \\ &\quad + \int_G a_{ijhk}(\eta_{x_k} D_t u^h + \eta_{x_j} D_t u^i)\, dx + \int \eta^2 |D_t u^h|^2\, dx . \end{aligned}$$

Keeping in mind that

$$\|D_{\pm t}w\|_{L^2(G)} \leqq C\|w_x\|_{L^2(G)}, \qquad \text{for } w \in H^1(G),$$

we find that for some constant C

$$\int_G \sum_{h,j,k} \eta^2 (D_t u^h_{x_k})^2 \, dx \leqq C \int_{G \cap B_{2\varrho}} \sum_h \left(|u^h_{x_k}|^2 + |u^h|^2 + |f_h|^2\right) dx$$

where the Young's Inequality $|ab| \leqq \varepsilon a^2 + (2/\varepsilon) b^2$ has been used. Now we may let $t \to 0$ to conclude that $\eta u^h_{x_\mu x_k} \in L^2(G)$, $\mu < n$; $k, h = 1, \ldots, n$. In view of (2.10), $u \in H^2(G \cap B_\varrho)$.

When $x_0 \in \Gamma'$ the same argument applies noting that $g_i \in H^{\frac{1}{2}}(\Gamma')$. The lemma follows. Q.E.D.

THEOREM 2.2. *Let u be a solution of Problem* 1.2 *with*

$$f_i \in L^2(\Omega), \quad g_i \in H^1(\Omega) \quad \text{and} \quad \varphi^i \in H^2(\Omega), \qquad 1 \leqq i \leqq n.$$

Set $\Omega_\delta = \{x \in \Omega : \text{dist.}(x, \partial\Gamma \cup \partial\Gamma' \cup \partial\Gamma'') > \delta\}$ *for* $\delta > 0$. *Then*

$$u \in H^2(\Omega_\delta) \quad \text{for each } \delta > 0$$

and the complementarity conditions (1.22)-(1.25) *are valid on* $\Gamma \cup \Gamma' \cup \Gamma''$.

PROOF. This is an immediate consequence of the lemma. We know that $u \in H^2_{\text{loc}}(\Omega)$. If $x_0 \in \Gamma \cup \Gamma'$ then the conclusion of the lemma holds in a neighborhood $B_\varrho(x_0) \cap \Omega$ since u is a smooth linear function of $\tilde{u}$. If $x_0 \in \Gamma''$, it follows in an analogous manner that $u \in H^2(\Omega \cap B_\varrho(x_0))$, some $\varrho > 0$, since $\varphi^i \in H^2(\Omega)$. Q.E.D.

From Sobolev's inequality we conclude

COROLLARY 2.3. *Let u be a solution of Problem* 1.2 *with* $f_i \in L^2(\Omega)$, $g_i \in H^1(\Omega)$, *and* $\varphi^i \in H^2(\Omega)$, $1 \leqq i \leqq n$.

(i) *If* $n = 2$, *then* $u \in C^{0,\lambda}(\bar{\Omega}_\delta) \cap H^{1,s}(\Omega_\delta)$ *for* $0 < \lambda < 1$, $1 \leqq s < \infty$,

(ii) *If* $n = 3$, *then* $u \in C^{0,\frac{1}{2}}(\bar{\Omega}_\delta) \cap H^{1,6}(\Omega_\delta)$,

where Ω_δ *is defined in Theorem* 2.2.

A solution of Problem 1.2 or Problem 1.1 may fail to lie in $H^2(\Omega)$ even in the case of a single equation and $g = 0$. To see this let

$$\Omega = \{x = (x_1, x_2) : |x| < 1, \ x_2 > 0\} \subset \mathbf{R}^2,$$

$$\Gamma = (-1, 0), \quad \Gamma' = (0, 1), \quad \Gamma'' = \{|x| = 1, \ x_2 > 0\},$$

and

$$u(x) = -\operatorname{Re} z^{\frac{1}{2}} = -\varrho^{\frac{1}{2}} \cos\theta/2 , \qquad z = x_1 + ix_2 = \varrho \exp[i\theta] ,$$

where (a, b) denotes the segment $\{(x_1, 0): a < x_1 < b\}$ of the real axis.

So $u(x)$ is harmonic and $u(x) = 0$ for $x_1 < 0$, $x_2 = 0$. By the Cauchy-Riemann equations

$$\frac{\partial}{\partial\nu} u(x_1, 0) = -\frac{\partial}{\partial x_2} u(x_1, 0) = -\frac{\partial}{\partial x_1} \operatorname{Im} z^{\frac{1}{2}} = \begin{cases} 0 & \text{if } x_1 < 0 , \\ \frac{1}{2}|x_1|^{-\frac{1}{2}} & \text{if } x_1 < 0 , \end{cases} \qquad x_2 = 0 .$$

Hence $u \in H^1(\Omega)$:

$$-\Delta u = 0$$

$$\begin{cases} u \dfrac{\partial u}{\partial \nu} = 0 \\ u \geqq 0 , \qquad \partial u/\partial\nu \geqq 0 \end{cases} \qquad \text{on } \Gamma$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma'$$

$$u = -\cos\frac{\theta}{2} \qquad \text{on } \Gamma''.$$

It is easy to verify that u is the solution of the variational inequality

PROBLEM 2.4.

$$u \in \boldsymbol{K}: \quad \int_\Omega u_{x_h}(v - u)_{x_h} dx \geqq 0 \qquad \textit{for } v \in \boldsymbol{K} ,$$

where

$$\boldsymbol{K} = \left\{ v \in H^1(\Omega) : v \geqq 0 \textit{ on } \Gamma \textit{ and } v = -\cos\frac{\theta}{2} \textit{ on } \Gamma'' \right\}.$$

3. – Continuity of the first derivatives in two dimensions.

We shall prove that the second derivatives of the solution of Problem 2.1 obey a growth condition which implies continuity of its first derivatives in the two dimensional case. Our method exploits the complementarity conditions to obtain a certain inequality to which Widman's hole filling device may be applied, cf. [9], [25]. The conclusion then follows by a version of Morrey's lemma ([18], p. 79) when $n = 2$. In a brief appendix to this section some elementary technical facts are noted for the reader's convenience.

We employ the notations

$$G_r(x_0) = \{x\colon |x - x_0| < r,\ x_n > 0\} \subset \boldsymbol{R}^n \qquad \text{for } x_0 = (x_0', 0)\,,$$
$$G_r = G_r(0)\,, \qquad G = G_1\,.$$

In the course of the proof, C or const. refers to a constant independent of u and r. The summation convention is understood with respect to $i, j, h, k = 1, \ldots, n$ and $\lambda, \mu = 1, \ldots, n-1$, here as in the previous sections.

Our conclusions will follow from the local integral estimate formulated below.

THEOREM 3.1. *Let u be a solution of Problem 1.2 in G where*

$$\Gamma = \{(x', 0)\colon |x'| < 1\} \subset \partial G \qquad \text{and}$$
$$f_i \in L^\infty(\Omega)\,, \qquad i = 1, \ldots, n\,,$$

then, for each δ, $0 < \delta < \frac{1}{8}$, there are $M > 0$ and λ, $0 < \lambda < 1$, such that

$$\int_{G \cap B_r(x_0)} |u^h_{x_j x_k}|^2\, dx \leqq M^2 r^{2\lambda} \qquad \text{for } x_0 \in G_{1-8\delta} \tag{3.1}$$

and $r \leqq 2\delta$.

Since we are pursuing a local analysis, the regularity properties of the surface forces g_i have not been explicitly mentioned and are not relevant. However we remind the reader that $g_i \in H^1(G)$ according to the hypotheses of Problem 1.2. In the interest of brevity we shall not determine the precise dependence of M and λ on the various parameters.

The principal step in the proof of the theorem is to show that for constants C_0, C_1, and β, $0 < \beta \leqq 1$, depending only on A, G, $\|u\|_{H^2(G_{1-8\delta})}$, and $\|f_i\|_{L^\infty(G)}$,

$$\int_{G_r(x_0)} |u^h_{x_k x_j}|^2\, dx \leqq C_0 \int_{G_{2r}(x_0) - G_r(x_0)} |u^h_{x_k x_j}|^2\, dx + C_1 r^{2\beta}\,, \qquad r \leqq 4\delta\,. \tag{3.2}$$

This will imply (3.1) when $x_0 \in \Gamma$. The estimate (3.2) follows in turn from (3.3) and (3.4) by applying a version of Poincaré's inequality. A companion estimate to (3.2) is available for balls $B_r(x_0)$, $x_0 \in G$, and r suitably restricted. Combining this with the case for $x_0 \in \Gamma$, we shall obtain (3.1).

LEMMA 3.2. *With the hypotheses of Theorem* 3.1 *set*

$$\sigma_{ij} = a_{ijkh} u^h_{x_k} + a_{ijh} u^h, \qquad 1 \leqq i,\ j \leqq n\ ,$$

and for each $x_0 \in \Gamma$, *let* (c_{ij}) *be an* $n \times n$ *constant matrix with* $c_{in} = 0$, $i = 1, \ldots, n$. *Then there are* C_2, $C_3 > 0$ *such that*

$$(3.3) \qquad \int_{G_r(x_0)} |u^h_{x_j x_k}|^2 dx \leqq C_2 \left\{ \frac{1}{r^2} \int_{G_{2r}(x_0) - G_r(x_0)} (\sigma_{ij} - c_{ij})^2 dx + \int_{G_{2r}(x_0) - G_r(x_0)} |u^h_{x_j x_k}|^2 dx \right\} + C_3 F(r)$$

and

$$(3.4) \qquad \int_{G_r(x_0)} |u^h_{x_j x_k}|^2 dx \leqq C_2 \left\{ \frac{1}{r^2} \int_{G_{2r}(x_0) - G_r(x_0)} (|\sigma_{\mu j} - c_{\mu j}|^2 + |u^n_{x_\mu}|^2)\, dx + \int_{G_{2r}(x_0) - G_r(x_0)} |u^h_{x_j x_k}|^2 dx \right\} + C_3 F(r)$$

where

$$(3.5) \qquad F(r) = \int_{G_{2r}(x_0)} (|u^h_{x_k}|^2 + |u^h|^2 + |f_h|^2)\, dx + r^2 \int_{G_{2r}(x_0)} |u^h_{x_j x_k}|^2 dx\ .$$

The use of the coerciveness inequality (2.6) leads us to an unfortunate circumlocution in the course of our estimates. We isolate this step now as a technical observation.

TECHNICAL OBSERVATION 3.3. *Let* $v = (v^1, \ldots, v^n) \in H^1(G)$, $\eta \in C_0^\infty(B_{2r})$, $0 \leqq \eta \leqq 1$, $\eta = 1$ *on* B_r, *and* $|\eta_x| \leqq 2/r$. *Then for a constant* $C > 0$,

$$(3.6) \qquad \int_G (\eta v^h_{x_k})^2 dx \leqq C \left\{ \int_G \eta^2 a_{ijhk} v^h_{x_k} v^i_{x_j} dx + \int_{G_{2r} - G_r} (v^h_{x_k})^2 dx + r^2 \int_G (v^h_{x_k})^2 dx \right\}.$$

The proof of (3.6) is delayed to the appendix.

PROOF OF 3.3. We may suppose that $x_0 = 0$ and $G_R = 0$. Observe first that whenever $\zeta \in H^1(G)$, $\zeta = 0$ on $|x| = 1$,

$$(3.7) \qquad \begin{aligned} a(u, \zeta) &= \int_G \{\sigma_{ij} \zeta^i_{x_j} + (b_{ihk} u^h_{x_k} + a_{ih} u^h)\, \zeta^i\}\, dx \\ &= \int_G \{(\sigma_{ij} - c_{ij})\, \zeta^i_{x_j} + (b_{ihk} u^h_{x_k} + a_{ih} u^h)\, \zeta^i\}\, dx \end{aligned}$$

for any constant matrix (c_{ij}) with $c_{in} = 0$, $1 \leqq i \leqq n$.

Let $\eta \in C_0^\infty(B_{2r})$, $B_\varrho = \{|x| < \varrho\}$, satisfy $0 \leqq \eta \leqq 1$, $\eta = 1$ on B_r and $|\eta_x| \leqq 2/r$. For a given μ, $1 \leqq \mu \leqq n-1$, $t > 0$ and $\varepsilon > 0$ set

$$v_\varepsilon(x) = u(x) + \varepsilon\eta(x)^2 D_{-t} D_t u(x)\,, \qquad x \in G\,,$$

where

$$D_t w(x) = \frac{1}{t}\,(w(x+te_\mu) - w(x)) \qquad \text{and} \qquad D_{-t} w(x) = \frac{1}{t}\,(w(x) - w(x-te_\mu))\,.$$

It is easy to check that for t sufficiently small,

$$v_\varepsilon^n(x) \geqq \left(1 - 2\,\frac{\varepsilon\eta(x)^2}{t^2}\right) u^n(x) \geqq 0\,, \qquad x \in \Gamma,\ 0 < \varepsilon < \tfrac{1}{2}t^2,$$

so $v_\varepsilon \in \boldsymbol{K}$ for ε small, $\varepsilon > 0$. Setting $v = v_\varepsilon$ in the variational inequality and using (3.7) we obtain that

$$\text{(3.8)} \qquad \int_G \{(\sigma_{ij} - c_{ij})(\eta^2 D_{-t} D_t u^i)_{x_j} + (b_{ihk} u^h_{x_k} + a_{ih} u^h)\,\eta^2 D_{-t} D_t u^i\}\,dx \geqq \int_G f_i \eta^2 D_{-t} D_t u^i\,dx\,.$$

Consider the first term on the left in (3.8). Expanding and transposing the D_{-t} we see that

$$\begin{aligned}
&\int_G (\sigma_{ij} - c_{ij})(\eta^2 D_{-t} D_t u^i)_{x_j} dx = \\
&\int_G \{\eta^2(\sigma_{ij} - c_{ij})\,D_{-t} D_t u^i_{x_j} + (\eta^2)_{x_j}(\sigma_{ij} - c_{ij})\,D_{-t} D_t u^i\}\,dx = \\
&\int_G \{-D_t\big(\eta^2(\sigma_{ij} - c_{ij})\big)\,D_t u^i_{x_j} + (\eta^2)_{x_j}(\sigma_{ij} - c_{ij})\,D_{-t} D_t u^i\}\,dx\,.
\end{aligned}$$

Since σ_{ij}, $u^i_{x_j} \in H^1(G_{8\delta})$ by Lemma 2.1, we may let $t \to 0$ in (3.8) to obtain that for each μ, $1 \leqq \mu \leqq n-1$,

$$\begin{aligned}
I_\mu &= \int_G \{\big(\eta^2(\sigma_{ij} - c_{ij})\big)_{x_\mu} u^i_{x_\mu x_j} - (\eta^2)_{x_j}(\sigma_{ij} - c_{ij})\,u^i_{x_\mu x_\mu}\}\,dx \\
&\leqq \int_G (b_{ihk} u^h_{x_k} + a_{ih} u^h - f_i)\,\eta^2\,u^i_{x_\mu x_\mu}\,dx \\
&\leqq \frac{\varepsilon}{n-1} \int_G \eta^2 (u^i_{x_\mu x_\mu})^2\,dx + \frac{C}{\varepsilon} \int_G \eta^2 (|u^h_{x_k}|^2 + |u^h|^2 + |f_h|^2)\,dx
\end{aligned}$$

for any $\varepsilon > 0$. Summing on μ,

$$\int_G \{\eta^2 a_{ijhk} u^h_{x_\mu x_k} u^i_{x_\mu x_j} + \eta^2[a_{ijhkx_\mu} u^h_{x_k} + (a_{ijh} u^h)_{x_\mu}]\} u^i_{x_\mu x_j} + \\ + \{[(\eta^2)_{x_\mu} u^i_{x_\mu x_j} - (\eta^2)_{x_j} u^i_{x_\mu x_\mu}](\sigma_{ij} - c_{ij})\} dx$$
$$= \sum I_\mu$$
$$\leqq \varepsilon \int_G \eta^2 (u^i_{x_\mu x_\mu})^2 dx + \frac{C}{\varepsilon} \int_G \eta^2 (|u^h_{x_k}|^2 + |u^h|^2 + |f_h|^2) dx .$$

We apply Young's Inequality in the second and third terms recalling that

$$\operatorname{supp} \eta_x \subset B_{2r} - B_r \qquad \text{and} \qquad |\eta_x| \leqq \frac{2}{r} .$$

For new constants $\varepsilon > 0$, $C = C(\varepsilon)$ we obtain

$$\int_G \eta^2 a_{ijhk} u^h_{x_\mu x_k} u^i_{x_\mu x_j} dx \leqq \frac{C}{r^2} \int_{G_{2r} - G_r} (\sigma_{ij} - c_{ij})^2 dx + \\ + C \int_G \eta^2 (|u^h_{x_k}|^2 + |u^h|^2 + |f_h|^2) dx + \varepsilon \int_G \eta^2 (u^h_{x_\mu x_k})^2 dx . \tag{3.9}$$

In (3.9) we apply the technical observation (3.6) for each μ with $v = u_{x_\mu} = (u^1_{x_\mu}, \ldots, u^n_{x_\mu})$. This gives that

$$\int_G (\eta u^h_{x_\mu x_k})^2 dx \leqq \frac{C}{r^2} \int_{G_{2r} - G_r} (\sigma_{ij} - c_{ij})^2 dx + C \int_{G_{2r} - G_r} (u^h_{x_\mu x_k})^2 dx + CF(r) + \varepsilon \int_G (\eta u^h_{x_\mu x_k})^2 dx .$$

The conclusion now follows by choosing ε sufficiently small and noting that $\eta = 1$ on B_r. The estimate (2.10) is then employed to account for the remaining second derivatives.

PROOF OF (3.4). On this occasion we choose

$$\begin{aligned} v'_\varepsilon(x) &= u'(x) + \varepsilon \eta(x)^2 D_{-t} D_t u'(x) \\ v^n_\varepsilon(x) &= u^n(x) + \varepsilon D_{-t}(\eta^2 D_t u^n)(x) \end{aligned} \qquad x \in G$$

with the notations as before. As in the proof of Lemma 2.1, one checks that

$$v^n_\varepsilon(x) \geqq 0 \qquad \text{for } x \in \Gamma,\ \varepsilon > 0 \text{ small},$$

so $v_\varepsilon \in \boldsymbol{K}$ for small $\varepsilon > 0$. Writing $v_\varepsilon = u + \varepsilon\zeta$, by (3.7) and the variational inequality

$$(3.10)\qquad \frac{1}{\varepsilon}\, a(u, v - u_\varepsilon) = a(u, \zeta)$$
$$= \int_G (\sigma_{ij} - c_{ij})\, \zeta^i_{x_j} dx + \int_G (b_{ihk}\, u^h_{x_k} + a_{ijh}\, u^h)\, \zeta^i\, dx$$
$$= \int_G \{(\sigma_{\lambda j} - c_{\lambda j})\, \zeta^\lambda_{x_j} + (\sigma_{nj} - c_{nj})\, \zeta^n_{x_j}\}\, dx + \int_G (b_{ihk}\, u^h_{x_k} + a_{ih}\, u^h)\, \zeta^i dx$$
$$\geqq \int_G f_i\, \zeta^i\, dx\,.$$

In particular,

$$\int_G (\sigma_{nj} - c_{nj})\, \zeta^n_{x_j} dx = \int_G (\sigma_{nj} - c_{nj})(D_{-t}\, \eta^2 D_t\, u^n)_{x_j} dx$$
$$= -\int_G D_t\, \sigma_{nj} (\eta^2 D_t\, u^n)_{x_j} dx\,.$$

Here we may let $t \to 0$, again because σ_{ij}, $u^i_{x_j} \in H^1(G_{8\delta})$. The terms involving $\sigma_{\lambda j} - c_{\lambda j}$, i.e., ζ^λ for $\lambda < n$, may be treated exactly as in the proof of (3.3), so we may let $t \to 0$ in (3.10). This gives for each $\mu = 1, \dots, n-1$.

$$(3.11)\qquad II_\mu = \int_G \{\eta^2 (\sigma_{\lambda j} - c_{\lambda j})_{x_\mu} u^\lambda_{x_\mu x_j} - (\eta^2)_{x_j} (\sigma_{\lambda j} - c_{\lambda j})\, u^\lambda_{x_\mu x_\mu}\}\, dx + \int_G \sigma_{nj x_\mu} (\eta^2\, u^n_{x_\mu})_{x_j} dx$$
$$\leqq \int_G \eta^2 (b_{\lambda hk}\, u^h_{x_k} + a_{\lambda h}\, u^h - f_\lambda)\, u^\lambda_{x_\mu x_\mu} dx +$$
$$+ \int_G (b_{nhk}\, u^h_{x_k} + a_{nh}\, u^h - f_n)(\eta^2\, u^n_{x_\mu})_{x_\mu} dx\,.$$

We first calculate that

$$(3.12)\qquad \int_G \sigma_{nj x_\mu} (\eta^2\, u^n_{x_\mu})_{x_j} dx = \int_G \sigma_{nj x_\mu} (\eta^2\, u^n_{x_\mu x_j} + (\eta^2)_{x_j} u^n_{x_\mu})\, dx$$
$$= \int_G \{\eta^2\, a_{njhk}\, u^h_{x_k x_\mu}\, u^n_{x_j x_\mu} + \eta^2 [a_{njhk x_\mu} u^h_{x_k} + (a_{ijh}\, u^h)_{x_\mu}]\, u^n_{x_\mu x_j} +$$
$$+ (\eta^2)_{x_j} \sigma_{nj x_\mu}\}\, u^n_{x_\mu}\, dx\,.$$

Observe that for any $0 < \varepsilon \leqq \varepsilon_1$,

$$(3.13)\qquad \left|\int_G (\eta^2)_{x_j}\sigma_{njx_\mu}u^n_{x_\mu}\,dx\right| \leqq \varepsilon_1\int_G \eta^2(\sigma_{njx_\mu})^2\,dx + \frac{C}{\varepsilon_1}\int_G |\eta_x|^2|u^n_{x_\mu}|^2\,dx \leqq \varepsilon\int_G \eta^2(u^h_{x_\mu x_k})^2\,dx + \frac{C}{\varepsilon}\int_G |\eta_x|^2|u^n_{x_\mu}|^2\,dx + C\varepsilon\int_G \eta^2(|u^h_{x_k}|^2 + |u^h|^2)\,dx\,.$$

The terms involving $\zeta^\lambda_{x_j}$ in (3.10) are treated exactly as in the proof of (3.3) (viz. the passage to (3.9)). Summing (3.11) on μ and using (3.13) to control (3.12), we deduce, analogous to (3.9), that

$$(3.14)\qquad \int_G \eta^2 a_{ijhk}u^h_{x_\mu x_k}u^i_{x_\mu x_j}dx \leqq \frac{C}{r^2}\int_{G_{2r}-G_r}\{(\sigma_{\lambda j}-c_{\lambda j})^2 + (u^n_{x_\mu})^2\}\,dx + C\int_G \eta^2(|u^h_{x_k}|^2 + |u^h|^2 + |f_h|^2)\,dx + \varepsilon\int_G \eta^2(u^h_{x_\mu x_k})^2\,dx\,,$$

where the term involving $f_n(\eta^2)_{x_\mu}u^n_{x_\mu}$ on the right is treated analogously to (3.13).

The desired estimate follows from (3.14) by employing the technical observation (3.6) and (2.10) precisely as in the proof of (3.3).

PROOF OF ESTIMATE (3.2). We apply the Poincaré type inequality of Lemma 3.7 to (3.3) or (3.4). According to (1.28),

$$(3.15)\qquad u^n_{x_\mu}\sigma_{nn} = 0 \text{ on } \Gamma,\ 1 \leqq \mu \leqq n-1\,.$$

Assuming as before that $G_R = G$ and $x_0 = 0$, for each $r \leqq 4\delta$, one of two cases occurs:

(i) $\mathrm{meas}_{n-1}\{(x', 0)\colon \sigma_{nn} = 0\} \cap T_r \geqq \frac{1}{2}\,\mathrm{meas}_{n-1}T_r$

or

(ii) $\mathrm{meas}_{n-1}\{(x', 0)\colon u^n_{x_\mu} = 0,\ \mu = 1, \ldots, n-1\} \cap T_r \geqq \frac{1}{2}\,\mathrm{meas}_{n-1}T_r$,

$T_r = \{(x', 0)\colon r < |x'| < 2r\}\,.$

Suppose that for a given r, $0 < r \leqq 4\delta$, (i) holds. Then we consider (3.3).

Keeping in mind that $c_{in} = 0$,

$$\int_{G_{2r}-G_r} \sigma_{nn}^2\,dx \leqq Cr^2 \int_{G_{2r}-G_r} |\sigma_{nnx}|^2\,dx$$
$$\leqq Cr^2 \int_{G_{2r}-G_r} |(a_{nnhk} u^h_{x_k} + a_{nhk} u^h)_x|^2\,dx$$
$$\leqq \text{const.}\, r^2 \int_{G_{2r}-G_r} |u^h_{x_k x_j}|^2\,dx + \text{const.}\, r^2 \int_{G_{2r}} (|u^h_{x_k}|^2 + |u^h|^2)\,dx\,.$$

Now $\sigma_{\mu n} = 0$ on Γ, $1 \leqq \mu \leqq n-1$, so again we obtain

$$\int_{G_{2r}-G_r} \sigma_{\mu n}^2\,dx \leqq \text{const.}\, r^2 \int_{G_{2r}-G_r} |u^h_{x_n x_j}|^2\,dx + \text{const.}\, r^2 \int_{G_{2r}-G_r} (|u^h_{x_n}|^2 + |u^h|^2)\,dx\,.$$

Finally, if $j < n$, the ordinary Poincaré inequality may be used since the c_{ij} may be chosen to our convenience. Thus from (3.3) we obtain the estimate

$$\int_{G_r} |u^h_{x_j x_k}|^2\,dx \leqq (1 + C)\,C_2 \int_{G_{2r}-G_r} |u^h_{x_j x_k}|^2\,dx + (CC_2 + C_3)\,F(r)\,. \tag{3.16}$$

Now $u \in H^2(G_{8\delta})$ and $f_i \in L^\infty(G)$, thus by the Sobolev and Hölder inequalities,

$$F(r) \leqq C(\delta)\|u\|^2_{H^2(G_{8\delta})}(\text{meas}\, G_{2r})^{1-2/2^*} + Cr^n \sum_1^n \|f_i\|_{L^\infty(G)} + r^2\|u\|^2_{H^2(G_{8\delta})}$$
$$\leqq C_2 r^2,$$

where $1/2^* = \frac{1}{2} - 1/n$ if $n > 2$ and 2^* is any finite number if $n = 2$. This gives (3.2).

If on the other hand (ii) holds, we turn to (3.4) applying our variation of Poincaré's lemma to the term

$$\int_{G_{2r}-G_r} |u^n_{x_\mu}|^2\,dx$$

and the ordinary Poincaré inequality to the remaining terms. Again (3.16) and thus (3.2) follows.

PROOF OF THEOREM 3.1. Adding

$$C_0 \int_{G_r(x_0)} |u^h_{x_k x_j}|^2\,dx$$

to both sides of (3.2) and dividing by $1 + C_0$, we obtain, for a different C_1,

$$\int_{G_r(x_0)} |u^h_{x_j x_k}|^2 \, dx \leqq \theta \int_{G_{2r}(x_0)} |u^h_{x_j x_k}|^2 \, dx + C_1 r^{2\beta} \qquad r \leqq 4\delta \,, \tag{3.17}$$

$$\theta = \frac{C_0}{1 + C_0} < 1 \,.$$

It is well known that (3.1) follows from (3.17) by iteration. Indeed, if $\omega(r)$, $r \leqq 4\delta$, is an increasing function which satisfies

$$\omega(r) \leqq \theta\omega(2r) + Cr^{2\beta}, \qquad r \leqq 4\delta, \text{ for a fixed } \theta < 1 \tag{3.18}$$

then

$$\omega(r) \leqq K(\omega(4\delta) + C) \left(\frac{r}{\delta}\right)^{2\alpha}, \qquad r \leqq 2\delta \,, \tag{3.19}$$

where K and $\alpha \leqq \beta$ depend only on θ and β (cf. Stampacchia [23] or [12], p. 81).

Thus we obtain for some M_0 and λ,

$$\int_{G_r(x_0)} |u^h_{x_k x_j}|^2 \, dx \leqq M_0^2 r^{2\lambda}, \qquad x_0 \in \Gamma, \ |x_0| < 1 - 8\delta \,, \tag{3.1$'$}$$

and $r \leqq 2\delta$.

By the argument just given for $x_0 \in \Gamma$ we may establish a similar inequality for $x_0 \in G$. This is

$$\int_{B_r(x_0)} |u^h_{x_j x_k}|^2 \, dx \leqq N^2 \left(\frac{r}{d}\right)^{2\lambda}, \qquad 0 < r \leqq 2d, \ x_0 \in G_{1-8\delta} \,, \tag{3.20}$$

$4d = \text{dist.}(x_0, \partial G) \leqq 1 - 8\delta$, where $0 < \lambda \leqq 1$ and

$$N^2 = N_1^2 \left\{ \int_{B_{2d}(x_0)} |u^h_{x_j x_k}|^2 \, dx + \tilde{F}(2d) \right\}, \tag{3.21}$$

$$\tilde{F}(r) = \int_{B_{2r}(x_0)} (|u^h_{x_k}|^2 + |u^h|^2 + |f_h|^2) \, dx + r^2 \int_{B_{2r}(x_0)} |u^h_{x_j x_k}|^2 \, dx \,,$$

for some constant $N_1 > 0$. To prove this, one merely notes that for $x_0 \in G$ and r sufficiently small,

$$\int_{B_r(x_0)} |u^h_{x_j x_k}|^2 \, dx \leqq C_5 \left\{ \frac{1}{r^2} \int_{B_{2r}(x_0) - B_r(x_0)} (\sigma_{ij} - c_{ij})^2 \, dx + \int_{B_{2r}(x_0) - B_r(x_0)} |u^h_{x_k x_j}|^2 \, dx + \tilde{F}(r) \right\}, \tag{3.22}$$

for an arbitrary $n\times n$ matrix of constants (c_{ij}), which is analogous to (3.3). One now applies the Poincaré inequality and the hole filling technique from which (3.20) follows with the estimate of (3.21).

To complete the proof of Theorem 3.1 we combine (3.20) and (3.1) for the case $x_0\in\Gamma$. This is elementary but involves the examination of several cases. Let $x_0\in G_{1-8\delta}$ and $r\leqq 2\delta$ and suppose first that $x_{0n}=\text{dist.}(x_0, \partial G)=4d$.

Case 1. $x_{0n}\leqq 2\delta$ and $x_{0n}\leqq r$. Then

$$B_r(x_0)\cap G\subset G_{2r}(x_0') , \qquad x_0'=(x_{01}, \dots, x_{0n-1}, 0) ,$$

whence

$$\int_{B_r(x_0)\cap G} |u^h_{x_jx_k}|^2\,dx \leqq \int_{G_{2r}(x_0')} |u^h_{x_jx_k}|^2\,dx \leqq M_0^2(2r)^{2\lambda} .$$

Case 2. $r\leqq x_{0n}\leqq 4\delta\leqq 2\delta$ and $d\leqq \delta/2$. Here we have that

$$B_r(x_0)\subset G_{8d}(x_0') \qquad \text{and} \qquad 8d\leqq 2\delta;$$

so applying (3.21),

$$N^2\leqq N_1^2\,Cd^{2\lambda} \qquad \text{for some } C>0 .$$

Thus

$$\int_{B_r(x_0)\cap G}|u^h_{x_jx_k}|^2\,dx=\int_{B_r(x_0)}|u^h_{x_jx_k}|^2\,dx\leqq N_1^2\,Cd^{2\lambda}\left(\frac{r}{d}\right)^{2\lambda}=N_1^2\,Cr^{2\lambda} .$$

Case 3. $r\leqq x_{0n}=4d$ and $d\geqq \delta/2$. Now we have that

$$B_r(x_0)\subset G \qquad \text{and} \qquad \frac{1}{d}\leqq\frac{2}{\delta}$$

so,

$$\int_{B_r(x_0)\cap G}|u^h_{x_jx_k}|^2\,dx\leqq N_0^2\left(\frac{2}{\delta}\right)^{2\lambda} r^{2\lambda} ,$$

where we may take,

$$N_0^2=N_1^2\Big\{\int_{G_{1-8\delta}} |u^h_{x_jx_k}|^2\,dx+F(\tfrac{1}{4}(1-8\delta))\Big\} ,$$

for example.

Finally suppose that $x_{0n} > 4d = \text{dist.}(x_0, \partial G)$. The set of such points in $G_{1-8\delta}$ satisfy $x_{0n} > \frac{1}{2}(1 - |x_0'|^2)$. For any such x_0, $4d \geqq 8\delta$ or $d \geqq 2\delta$, and thus

$$\int_{G \cap B_r(x_0)} |u^h_{x_j x_k}|^2 \, dx = \int_{B_r(x_0)} |u^h_{x_j x_k}|^2 \, dx \leqq N_0^2 (2\delta)^{-2\lambda} r^{2\lambda} .$$

This establishes (3.1) with

$$M^2 = \max\left(2^{2\lambda} M_0^2, \; C N_1^2, \; N_0^2 \left(\frac{2}{\delta}\right)^{2\lambda}\right). \qquad \text{Q.E.D.}$$

We now study the two dimensional problem

LEMMA 3.4. *Let $n = 2$. Let u be a solution of Problem 1.2 in G where*

$$\Gamma = \{(x_1, 0) \colon |x_1| < 1\} \quad \textit{and} \quad f_h \in L^\infty(G) , \quad h = 1, 2 .$$

Then $u \in C^{1,\lambda}(G \cup \Gamma)$ for some $\lambda > 0$.

PROOF. This is an immediate consequence of Morrey's growth lemma, [18] p. 79, or more properly, a slight variant of it.

THEOREM 3.5. *Let $n = 2$. Let u be a solution of Problem 1.2 in Ω where*

$$f_i \in L^\infty(\Omega) , \quad g_i \in H^{1,\infty}(\Omega) , \quad \textit{and} \quad \varphi^i \in H^{2,\infty}(\Omega) , \qquad i = 1, 2 .$$

Set

$$\Omega_\delta = \{x \in \Omega \colon \text{dist.}(x, \partial\Gamma \cup \partial\Gamma' \cup \partial\Gamma'') > \delta\} , \qquad \delta > 0 .$$

Then

$$u \in C^{1,\lambda}(\bar{\Omega}_\delta) \qquad \textit{for each } \delta > 0 \textit{ and some } \lambda = \lambda(\delta) > 0 .$$

PROOF. Given $x_0 \in \Gamma \cup \Gamma' \cup \Gamma''$, we may assume after local transformations that $x_0 = 0$ and that u is a solution of Problem 1.2 in G with the interval $(-1, 1)$ corresponding to an arc of $\partial\Omega$. If $x_0 \in \Gamma$, then $u \in C^{1,\lambda}$ near x_0 by the preceding lemma. The same method applies to points $x_0 \in \Gamma' \cup \Gamma''$.

Suppose $x_0 \in \Gamma'$, for example. After changing variables as indicated, so $\Gamma' \subset (-1, 1)$ near x_0,

$$a(u, v - u) \geqq \int_G f_i (v^i - u^i) \, dx + \int_{\Gamma'} g_i (v^i - u^i) \, dx_1 , \qquad v \in \boldsymbol{K} .$$

For any $\zeta \in H^1(G)$ with $\zeta = 0$ for $|x| \geqq \frac{1}{2}\operatorname{dist}(0, \partial\Gamma')$, we may write

$$a(u, \zeta) = \int_G \{(\sigma_{ij} - c_{ij})\zeta^i_{x_j} + (b_{ihk}u^h_{x_k} + a_{ih}u^h)\zeta^h\}\, dx - \int_G g_i \zeta^i_{x_2} dx\,.$$

where $c_{i2} = -g_i$ and c_{i1} is arbitrary.

Now choose as a test function

$$v_\varepsilon = u + \varepsilon D_{-t}\big(\eta^2(D_t u - c)\big)\,, \qquad x \in G,\ c \in \mathbf{R}^2,$$

and proceed as before. Q.E.D.

THEOREM 3.6. *Let $n \leqq 4$. Let u be a solution of Problem 1.2 in Ω where $f_i \in L^\infty(\Omega)$, $i = 1, \ldots, n$. Then*

$$u \in C^{0,\mu}(\Omega \cup \Gamma) \qquad \textit{for some } \mu > 0\,.$$

PROOF. As before, given a point $x_0 \in \Gamma$ we may assume that $x_0 = 0$ and that u is a solution of Problem 1.2 in G with $\Gamma = \{|x| < 1,\ x_n = 0\}$. Thus Theorem 3.1 may be applied. The condition (3.1) implies that $u^i_{x_j} \in L^s(G_R)$, $R = 1 - 16\delta$, for $s = 4(2-\lambda)/(1-\lambda) > 4$ when $n = 4$ by a theorem of Meyers [17] or Campanato [2]. Utilizing Sobolev's inequality we obtain that

$$u \in C^{0,\mu}(\bar{G} \cap B_R)\,, \qquad \mu = 1 - 4/\delta\,. \qquad \text{Q.E.D.}$$

Although we have adopted the technique of Morrey spaces to deduce this last result, it is also possible to employ Gehring's « reverse Hölder inequality », Gehring [7] or Giaquinta and Modica [8], Prop. 5.1. Here one argues directly from the inequalities (3.3), (3.4) and (3.22) and concludes that

$$u \in H^{2,s}(G_R) \qquad \text{for some } s > 2 \text{ and } R = 1 - 16\delta\,.$$

Appendix.

LEMMA 3.7. *Let $\zeta \in H^1(G_{2r} - G_r)$ and suppose that for $\delta > 0$*

$$\operatorname{meas}_{n-1}\{x' : \zeta(x', 0) = 0\} \geqq \delta r^{n-1}\,.$$

Then

$$\int_{G_{2r}-G_r} \zeta^2\, dx \leqq C r^2 \int_{G_{2r}-G_r} \zeta^2_{x_j} dx$$

where $C = C(\delta)$.

LEMMA 3.8. *Let $\zeta \in H^1(G_{2r})$. Then*

(i)
$$\int_{G_{2r}} \zeta^2\, dx \leqq K_1 r^2 \int_{G_{2r}} \zeta_{x_j}^2 dx + K_2\Big(\int_{G_{2r}-G_r} \zeta\, dx\Big)^2$$

and

(ii)
$$\int_{G_{2r}-G_r} \zeta^2\, dx \leqq K_1 r^2 \int_{G_{2r}-G_r} \zeta_{x_j}^2 dx + K_2\Big(\int_{T_r^+} \zeta\, dx'\Big)^2,$$

where $T_r^+ = \{(x', 0): r < |x'| < 2r,\ x_{n-1} > 0\}$ and $K_1, \dots, K_4$ depend only on the dimension, n.

The proofs of these statements are elementary.

PROOF OF TECHNICAL OBSERVATION 3.3. Apply the coerciveness inequality (2.6) to $w = \eta(v - c)$ where

$$c^i = (\operatorname{meas}(G_{2r} - G_r))^{-1} \int_{G_{2r}-G_r} v^i\, dx\,.$$

To the term involving $\|w\|^2_{L^2(G_{2r})}$ apply Lemma 3.8 (i).

4. – The global continuity of the solution in two dimensions.

According to Theorem 2.2, a solution of Problem 1.2 in $H^2(\Omega_\delta) \cap C^{0,\lambda}(\bar{\Omega}_\delta)$ for any λ, $0 < \lambda < 1$, in the two dimensional case. Indeed, it is even in $C^{1,\lambda}(\bar{\Omega}_\delta)$ by Theorem 3.5. However it is not necessarily in the class $H^2(\Omega)$, as the example of Problem 2.4 illustrates. Nonetheless, the Dirichlet integral of the solution satisfies a growth condition which implies continuity in $\bar{\Omega}$.

THEOREM 4.1. *Let u be a solution of Problem 1.2 in $\Omega \subset \mathbf{R}^2$ where*

$$f_i \in L^\infty(\Omega)\,, \quad g_i \in H^{1,\infty}(\Omega)\,, \quad \text{and} \quad \varphi_i \in H^{2,\infty}(\Omega)\,, \qquad i = 1, 2\,.$$

Then there are $\delta > 0$, $M > 0$, and $\lambda > 0$ such that

(4.1)
$$\int_{\Omega \cap B_r(x_0)} |u^h_{x_k}|^2\, dx \leqq M^2 r^{2\lambda} \qquad \text{for } r \leqq \delta \text{ and } x_0 \in \bar{\Omega}\,.$$

PROOF. The proof is merely a simplified version of that of Theorem 3.1. Given $x_0 \in \bar{\Gamma}$, suppose that $x_0 = 0$ and that, after local transformations, a portion $\Omega \cap B_{16\delta}(x_0)$ contains $G_{8\delta}(x_0) = G_{8\delta}$ with $(-8\delta, 8\delta)$ corresponding to an arc of $\partial\Omega$.

We may suppose that the segment $(0, 8\delta) \subset \Gamma$, $0 \in \Gamma \cap \bar{\Gamma}'$. We now briefly describe the derivation of the estimate analogous to (3.3), (3.4). Let $\eta \in C_0^\infty(B_{2r})$, $0 \leqq \eta \leqq 1$, $\eta = 1$ on B_r and $|\eta_x| \leqq 2/r$, as usual, and set

$$v = u - \zeta = u - \eta^2(u - c)$$

where $c = (c^1, c^2) \in \mathbf{R}^2$ with $c^2 \geqq 0$. Thus

$$v^2(x) = \big(1 - \eta(x)^2\big)\, u^2(x) + c^2 \eta(x)^2 \geqq 0 \qquad \text{for } x \in \Gamma$$

so $v \in \mathbf{K}$. Setting this v in the variational inequality gives

$$a(u, \zeta) \leqq \int_G f_i \zeta^i \, dx + \int_{\Gamma'} g_i \zeta^i \, dx_1 . \tag{4.2}$$

After some elementary manipulations, we find that for any $\varepsilon > 0$, there is a $C = C(\varepsilon) > 0$ such that

$$\int_G \eta^2 a_{ijhk} u^h_{x_k} u^i_{x_j} dx \leqq \varepsilon \int_G (\eta u^h_{x_k})^2 \, dx + a(u, \zeta) + \frac{C}{r^2} \int_{G_{2r} - G_r} (u^h - c^h)^2 \, dx + C F_0(r) \tag{4.3}$$

where

$$F_0(r) = \int_{G_{2r}} \big(|u^h|^2 + |u^h - c^h|^2\big) \, dx .$$

Turning to the right hand side of (4.2), let us choose

$$c^h = \frac{1}{r} \int_r^{2r} u^h(x_1, 0) \, dx , \qquad h = 1, 2 ,$$

so in particular $c^2 \geqq 0$ since $(0, 8\delta) \subset \Gamma$ and

$$|c^h| \leqq r^{-\frac{1}{2}} \|u^h\|_{L^2(T_r)} , \qquad T_r = (-2r, -r) \cup (r, 2r) , \qquad h = 1, 2 .$$

Thus

$$\begin{aligned} \left| \int_{\Gamma'} g_h \eta^2 (u^h - c^h) \, dx_1 \right| &\leqq \int_{T_r} |g_h| |u^h - c^h| \, dx_1 \\ &\leqq \|g_h\|_{L^2(T_r)} \|u^h - c^h\|_{L^2(T_r)} \\ &\leqq \|g_h\|_{L^2(T_r)} \big(2 \|u^h\|_{L^2(T_r)}\big) \\ &\leqq 2\sqrt{2}\, r^{\frac{1}{2}} \|g_h\|_{L^\infty(G)} \|u^h\|_{L^2(T_r)} \\ &\leqq C r^{\frac{1}{2}} \|g\|_{L^\infty(G)} \|u\|_{H^1(G)} \end{aligned}$$

since $u \in H^1(G)$. Also,

$$\left|\int_G f_h \zeta^h \, dx\right| \leqq Cr^2 + \int_{G_{2r}} (u^h - c^h)^2 \, dx .$$

Thus, after applying Poincaré's Inequality in the form of Lemma 3.7 (ii),

$$a(u, \zeta) \leqq Cr^{\frac{1}{2}} \|u\|_{H^1(G)} + Cr^2 .$$

Placing this in (4.3) and again using Poincaré's inequality leads to the estimate

$$\int_{G_r} (u^h_{x_k})^2 \, dx \leqq C_1 \int_{G_{2r} - G_r} (u^h_{x_k})^2 \, dx + C_2 r^{\frac{1}{2}} , \qquad r \leqq 4\delta . \tag{4.4}$$

Thus

$$\omega(r) = \int_{G_r} (u^h_{x_k})^2 \, dx , \qquad r \leqq 4\delta$$

satisfies (3.18). The conclusion follows in this case.

The other case is when $0 \in \bar{\Gamma} \cap \bar{\Gamma}''$. The argument here follows the same lines with

$$v(x) = u(x) - \zeta(x) = u(x) - \eta(x)^2\big(u(x) - \varphi(x)\big) \qquad x \in G , \tag{4.5}$$

as the test variation. Here it is important to note that we may assume that

$$\varphi^2(x_1, 0) \geqq 0 \qquad \text{for } x = (x_1, 0) \in \Gamma$$

since $\varphi^2(0) < 0$ implies that the convex $\boldsymbol{K}$ is empty so no solution of Problem 1.2 exists. Note that the estimate (4.1) also holds for $x_0 \in \bar{\Gamma}' \cap \bar{\Gamma}''$, which may be shown by taking $v(x)$ as in (4.5).

Once again an estimate is available for points $x_0 \in G$. Specifically we have

$$\int_{B_r(x_0)} |u^h_{x_k}|^2 \, dx \leqq N^2 \left(\frac{r}{d}\right)^{2\lambda}, \qquad 0 < r \leqq d, \ x_0 \in G_{1-8\delta} , \tag{4.6}$$

$4d = \text{dist.}(x_0, \partial G \leqq 1 - 8\delta$, for some λ, $0 < \lambda \leqq 1$, with an appraisal for N similar to (3.21) where $u^h_{x_j x_k}$ is replaced by $u^h_{x_j}$. Combining (4.1) in the case $x_0 = (x_{10}, 0)$ with (4.6) we obtain (4.1) for any $x_0 \in G_{1-8\delta}$. The theorem follows.

THEOREM 4.2. *Let u be a solution of Problem* 1.2 *in* $\Omega \subset \mathbf{R}^2$ *where*

$$f_i \in L^\infty(\Omega)\,, \quad g_i \in H^{1,\infty}(\Omega)\,, \quad \text{and} \quad \varphi^i \in H^{2,\infty}(\Omega)\,, \qquad i = 1, 2\,.$$

Then

$$u \in C^{0,\lambda}(\bar{\Omega}) \qquad \text{for some } \lambda > 0\,.$$

PROOF. The proof follows from Morrey's lemma [18], p. 79.

5. – First applications.

Can any information of a mechanical nature be derived from our mathematical analysis of the Signorini problem, Problem 1.1? Let us illustrate how our integrability lemma, Lemma 2.1, or Theorem 2.2 may be used to verify that the deformed body is in equilibrium and retains substantial contact with its rigid support. Let Ω and Γ be as in § 1 and let u be a solution of Problem 1.1. The set

$$I = \{x \in \Gamma \colon u \cdot \nu = 0\} \tag{5.1}$$

which does not undergo normal displacement under the imposition of the forces T, is called the set of coincidence of u. It is defined up to a set of measure zero in Γ (and for $n \leq 4$ is a closed subset of Γ.) The normal pressure, defined on $\partial\Omega$ except for $x \in \partial\Gamma$ by virtue of Theorem 2.2, is given by

$$\sigma_\nu = \sigma_\nu(u) = \sigma_{ij}(u)\,\nu_i \nu_j \qquad x \in \partial\Omega - \partial\Gamma \tag{5.2}$$

where ν is the outward normal to $\partial\Omega$. Moreover, $\sigma_\nu \in L^2_{\text{loc}}(\partial\Omega - \partial\Gamma)$, that is, σ_ν is square integrable on compact subsets of $\partial\Omega - \partial\Gamma$. Recall that $\sigma_\nu \leq 0$ on Γ by (1.12).

THEOREM 5.1. *Let u be a solution of Problem* 1.1 *and let I and σ_ν be defined by* (5.1) *and* (5.2). *Then $\sigma_\nu \in L^1(\Gamma)$ and*

$$\int_I \sigma_\nu(\zeta \cdot \nu)\, dS = \int_\Gamma \sigma_\nu(\zeta \cdot \nu)\, dS = a(u, \zeta) - \langle T, \zeta \rangle \tag{5.3}$$

for any $\zeta = (\zeta^1, \ldots, \zeta^n) \in C^1(\bar{\Omega})$.

In general, σ_ν does not belong to $L^2(\Gamma)$ as the examples of sections 2 and 6 suggest.

PROOF. This is elementary. For the sake of clarity we first show that

$$\int_\Gamma \sigma_\nu(\zeta\cdot\nu)\,dS = a(u,\zeta) - \langle T,\zeta\rangle\,, \qquad \zeta\in C^1(\bar\Omega)\,. \tag{5.4}$$

Let $\varphi\in C^1(\bar\Omega)$ vanish in a neighborhood of $\partial\Gamma$ in Ω. In this case we may integrate by parts in $a(u,\varphi)$ and (5.4) follows immediately.

For $0<r<1$ let $\eta_r(x)$ be a scalar valued Lipschitz function in $\boldsymbol{R}^n$ satisfying

$$\eta_r(x) = \begin{cases} 1 & \text{dist.}\,(x,\partial\Gamma)\geqq 2r \\ 0 & \text{dist.}\,(x,\partial\Gamma)\leqq r \end{cases}\,,$$

$$0\leqq\eta_r\leqq 1\,,\qquad |\eta_{rx}|\leqq K/r\,,\qquad \text{and}\qquad \eta_r\leqq\eta_\varrho \quad\text{for } \varrho\leqq r\,.$$

Since $\partial\Gamma$ is a compact $n-2$ dimensional manifold,

$$\begin{aligned}\int_{\boldsymbol{R}^n}|\eta_{rx}(x)|^2\,dx &\leqq \frac{K^2}{r^2}\,\text{meas}\,\{x\in\boldsymbol{R}^n\colon r\leqq\text{dist.}\,(x,\partial\Gamma)\leqq 2r\} \\ &\leqq \frac{K^2}{r^2}\,r^2\,C\leqq C_0\end{aligned}$$

for some constants C, C_0.

Now choose $\zeta\in C^1(\bar\Omega)$ so that $\zeta=\nu$ on $\partial\Omega$ and set $\varphi=\eta_r\zeta$ in (5.4). Recalling that $\sigma_\nu\leqq 0$ on Γ, we see that

$$-\sigma_\nu(\zeta\eta_r\cdot\nu) = -\eta_r\sigma_\nu \qquad \text{increases to } -\sigma_\nu \text{ as } r\to 0 \text{ on } \Gamma\,.$$

Meanwhile in (5.4) we have

$$\int_\Gamma\eta_r\sigma_\nu\,dS = \int_\Omega\sigma_{ij}(u)\,\zeta^i\,\eta_{rx_j}dx + \int_\Omega\sigma_{ij}(u)\,\zeta^i_{x_j}\eta_r\,dx - \langle T,\eta_r\zeta\rangle\,.$$

Now $u_x\in L^2(\Omega)$ so

$$\begin{aligned}\left|\int_\Omega\sigma_{ij}(u)\,\zeta^i\,\eta_{rx_j}dx\right| &\leqq C\|\eta_{rx}\|_{L^2(\boldsymbol{R}^n)}\|u_x\|_{L^2(\operatorname{supp}\eta_{rx}\cap\Omega)} \\ &\leqq CC_0\|u_x\|_{L^2(\operatorname{supp}\eta_{rx}\cap\Omega)}\to 0\end{aligned}$$

as $r\to 0$. Since $\eta_r\to 1$ pointwise a.e., by the monotone convergence theorem $\sigma_\nu\in L^1(\Gamma)$ and

$$\int_\Gamma\sigma_\nu\,dS = \int_\Omega\sigma_{ij}(u)\,\zeta^i_{x_j}dx - \langle T,\zeta\rangle\,.$$

For an arbitrary $\zeta \in C^1(\bar{\Omega})$, (5.4) is an easy consequence of the bounded convergence theorem and the preceeding argument, using, of course, that $\sigma_\nu \in L^1(\Gamma)$.

Finally, $\sigma_\nu = 0$ in $\Gamma - I$ by the complementarity conditions, which demonstrates (5.3). Q.E.D.

COROLLARY 5.2 (Balance of forces). *Let u be a solution of Problem* 1.1 *and let I and σ_ν be defined by* (5.1) *and* (5.2). *Then*

$$\int_I \sigma_\nu(\zeta \cdot \nu)\, dS + \langle T, \zeta \rangle = 0 \tag{5.5}$$

for any affine rigid motion ζ and

$$\operatorname{meas}_{n-1} I > 0 . \tag{5.6}$$

PROOF. For any affine rigid motion, $a(u, \zeta) = 0$, so (5.5) holds. According to the existence hypothesis (1.14), there is an $\eta \in \mathcal{A} \cap \boldsymbol{K}$ with $\langle T, \eta \rangle < 0$. Thus

$$\int_I \sigma_\nu(\eta \cdot \nu)\, dS = - \langle T, \eta \rangle > 0$$

which implies (5.6). Q.E.D.

A particular consequence of the corollary is that the body, known to be an equilibrium in its deformed state, cannot be supported by a stress distribution on the boundary $\partial\Gamma$ of its rigid support. For otherwise $I \subset \partial\Gamma$ but $\operatorname{meas}_{n-1} \partial\Gamma = 0 < \operatorname{meas}_{n-1} I$, or (5.6) could not hold.

A situation of special interest is when Γ is contained in a plane, say,

$$\Gamma \subset \{x_n = 0\} \qquad \text{and} \qquad \nu = - e_n ,$$

in agreement with our conventions in § 1. In this case $\sigma_\nu = \sigma_{nn}(u)$. Let

$$F_i = \langle T, e_i \rangle , \qquad i = 1, \dots, n ,$$

be the external force in the i-th direction. It follows from (5.5) that

$$F_\mu = 0 , \quad \mu = 1, \dots, n-1 , \qquad \text{and} \qquad F_n = \int_I \sigma_{nn}(u)\, dx' < 0 , \tag{5.7}$$

since $e_n \in \boldsymbol{K}$ and $-e_n \notin \boldsymbol{K}$. Analogous expressions hold for the various moments.

6. – The coincidence set in Signorini's problem.

Our efforts here will concern the simplest case, plane elasticity for a homogeneous isotropic body with Γ a segment of the x_1-axis. In this section it will be convenient to use complex notation. Let $\Omega \subset \boldsymbol{R}^2$ be a domain with smooth boundary and assume that

$$(6.1) \qquad \Gamma = (-c, c) \subset \tilde{\Gamma} = (-\tilde{c}, \tilde{c}) \subset \partial\Omega \cap \{x_2 = 0\} \qquad \textit{for some } \tilde{c} > c > 0$$

where (a, b) stands for the interval (a, b) of the real axis. Suppose that $\nu = (0, -1)$ is the outward pointing unit normal to Ω on $\tilde{\Gamma}$. Let $\alpha > 0$ be given and introduce the stress tensor, bilinear form, and corresponding second order operator

$$(6.2) \qquad \begin{aligned} &\sigma_{hk} = (\alpha - 1)\,\delta_{hk}\,\varepsilon_{jj} + 2\varepsilon_{hk}\,, && 1 \leqq h,\ k \leqq 2\,,\\ &a(v, \zeta) = \int_\Omega \sigma_{hk}(v)\,\zeta^h_{x_k}\,dx\,, && \text{and}\\ &(Av)_h = -\Delta v^h - \alpha \sum_k v^k_{x_h x_k}\,, && h = 1, 2\,. \end{aligned}$$

Let us summarize our information about the solution of Problem 1.1. The distribution of surface and body forces

$$\langle T, \zeta \rangle = \int_\Omega f_h \zeta^h\,dx + \int_{\Gamma'} g_h \zeta^h\,ds\,, \qquad \zeta \in H^1(\Omega)\,,$$

is assumed to satisfy the condition

$$(6.3) \qquad \langle T, \zeta \rangle < 0 \qquad \text{whenever } \zeta \in \mathcal{A} \cap \boldsymbol{K},\ -\zeta \notin \mathcal{A} \cap \boldsymbol{K}\,,$$

where $\boldsymbol{K}$ is the convex set of admissible functions

$$\boldsymbol{K} = \{v \in H^1(\Omega) \colon v^2 \geqq 0 \ \text{ on } \Gamma\}$$

and $\mathcal{A}$ is the collection of infinitesimal affine motions (cf. (1.14)). With these hypotheses, i.e. (6.1)-(6.3), let u be a solution of the Signorini problem.

PROBLEM 6.1. $u \in \boldsymbol{K}$: $a(u, v - u) \geqq \langle T, v - u \rangle$ *for* $v \in \boldsymbol{K}$.

Provided that $f_h \in L^\infty(\Omega)$ and $g_h \in H^{1,\infty}(\Omega)$, $h = 1, 2$, we know that

$$\begin{aligned} &u \in C^{0,\lambda}(\bar{\Omega}) \qquad \textit{for some } \lambda > 0 \qquad \textit{and}\\ &u \in H^2(\Omega_\delta) \cap C^{1,\lambda}(\bar{\Omega}_\delta) \qquad \textit{for each } \delta > 0 \qquad \textit{where}\\ &\lambda = \lambda(\delta) > 0 \qquad \textit{and} \qquad \Omega_\delta = \{z \in \Omega \colon |z - c| > \delta \textit{ and } |z + c| > \delta\}\,. \end{aligned}$$

The complementarity conditions are valid; those of interest may be written

$$
\begin{aligned}
&(Au)_h = -(\sigma_{h1x_1} + \sigma_{h2x_2}) = f_h && \text{in } \Omega,\ h = 1, 2\\
&-\sigma_{22} \geqq 0\,, \qquad u^2 \geqq 0 && \\
&u^2\sigma_{22} = 0 && \text{on } \Gamma\\
&\sigma_{12} = 0 && \\
&-\sigma_{12} = g_1 \quad \text{and} \quad -\sigma_{22} = g_2 && \text{on } \tilde{\Gamma} - \bar{\Gamma}\,.
\end{aligned}
\tag{6.4}
$$

In addition

$$u^2_{x_1}\sigma_{22} = 0 \qquad \text{on } \Gamma\,. \tag{6.5}$$

The coincidence set

$$I = \{z \in \Gamma\colon u^2(z) = 0\} \tag{6.6}$$

is a relatively closed non empty subset of Γ by Corollary 5.2.

To facilitate our investigation, introduce the complex valued functions

$$
\begin{aligned}
&w(z) = \sigma_{22} + \sigma_{11} + i\varkappa(u^2_{x_1} - u^1_{x_2})\,, &&\\
&w^*(z) = \sigma_{22} - \sigma_{11} + 2i\sigma_{12}\,, && z = x_1 + ix_2 \in \Omega\\
&\varkappa = 2\alpha/1 + \alpha &&
\end{aligned}
\tag{6.7}
$$

which satisfy $w, w^* \in L^2(\Omega) \cap H^1(\Omega_\delta) \cap C^{0,\lambda}(\bar{\Omega}_\delta)$ for any $\delta > 0$.

LEMMA 6.2. *Let u be a solution of Problem 6.1 with $f_1 = f_2 = 0$ in Ω and define w, w^* by (6.7). Then*

(i) *$w(z)$ is holomorphic in Ω and*

(ii) *there is a holomorphic $\varphi_0(z)$ such that*

$$w^*(z) = \tfrac{1}{2}\bar{z}w'(z) + \varphi_0(z)\,.$$

In other words, the system of (6.4) may be written

$$\begin{pmatrix} w_{\bar{z}} \\ w^*_{\bar{z}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} w_z \\ w^*_z \end{pmatrix}.$$

PROOF. The proof of (i) is a rearrangement of the equations in (6.4). To prove (ii) note that

$$\frac{\partial}{\partial z} w = 2 \frac{\partial}{\partial z} (\sigma_{11} + \sigma_{22}) = 2 \frac{\partial}{\partial \bar{z}} w^* \quad \text{in } \Omega . \tag{6.8}$$

In fact, $Au = 0$ in Ω, thus

$$\begin{aligned} 2\frac{\partial w^*}{\partial \bar{z}} &= \sigma_{22x_1} - \sigma_{11x_1} - 2\sigma_{12x_2} + i(\sigma_{22x_2} - \sigma_{11x_2} + 2\sigma_{12x_1}) \\ &= \sigma_{22x_1} + \sigma_{11x_1} - i(\sigma_{22x_2} + \sigma_{11x_2}) \\ &= 2\frac{\partial}{\partial z}(\sigma_{22} + \sigma_{11}) . \end{aligned}$$

The first equality in (6.8) follows since $\operatorname{Re} w = \sigma_{22} + \sigma_{11}$. The assertion (ii) now follows by integration. Q.E.D.

In view of the lemma

$$f(z) = \tfrac{1}{2} z w'(z) + \varphi_0(z) , \qquad z \in \Omega , \tag{6.9}$$

is holomorphic in Ω. In addition, since $z = \bar{z}$ on $\tilde{\Gamma}$, we deduce that formally

$$f(z) = w^*(z) \qquad \text{for } z \in \tilde{\Gamma} - \partial\Gamma, \tag{6.10}$$

$\partial\Gamma = \{-c, c\}$. Consequently, although w^* is not holomorphic in Ω, its boundary values on $\tilde{\Gamma} - \partial\Gamma$ coincide with those of a holomorphic function. This is the property of w^* which we shall exploit.

THEOREM 6.3. *Let u be a solution of Problem* 5.1 *under the hypotheses* (6.1)-(6.3). *Suppose that*

$$f_h = 0 \quad \text{in } \Omega \qquad \text{and} \qquad g_h = 0 \quad \text{on } \tilde{\Gamma} \text{ for } h = 1, 2 , \tag{6.11}$$

and set

$$I = \{z \in \Gamma : u^2(z) = 0\} .$$

Then I is the union of a finite number of intervals and finitely many isolated points.

LEMMA 6.4. *With the hypotheses of the theorem, define $f(z)$ by* (6.9). *Then $f(z)$ is analytically extensible into a full neighborhood of $\bar{\Gamma}$ in the z-plane.*

Assuming this lemma we give a proof of the theorem.

PROOF. Due to (6.11), and (6.4), the shear stress

$$\sigma_{12} = 0 \qquad \text{on } \tilde{\Gamma} - \partial\Gamma$$

so $u^2_{x_1} = -u^1_{x_2}$ on $\tilde{\Gamma} - \partial\Gamma$. Thus

$$w(z) = \sigma_{11} + \sigma_{22} + 2i\varkappa u^2_{x_1} \qquad \text{on } \tilde{\Gamma} - \partial\Gamma \qquad \text{while}$$

$$\sigma_{11} = \sigma_{22} - f \qquad \text{on } \tilde{\Gamma} - \partial\Gamma .$$

Recall that w is continuous in $\bar{\Omega} - \partial\Gamma$ and $f \in C(\Omega \cup \tilde{\Gamma})$ so the statements above make sense. Combining these gives for $q(z) = \frac{1}{2}(w(z) + f(z))$,

$$q(z) = \tfrac{1}{2}(w(z) + f(z)) = \sigma_{22} + i\varkappa u^2_{x_1} \qquad z \in \tilde{\Gamma} - \partial\Gamma . \tag{6.12}$$

Thus, $\sigma_{22} + i\varkappa u^2_{x_1}$ is the boundary value of an analytic function.

According to the complementarity conditions (6.5) and (6.11),

$$u^2_{x_1}\sigma_{22} = 0 \quad \text{on } \Gamma \qquad \text{and} \qquad \sigma_{22} = 0 \quad \text{on } \tilde{\Gamma} - \bar{\Gamma},$$

so we have in particular that

$$\operatorname{Im} q(z)^2 = 2\varkappa u^2_{x_1}\sigma_{22} = 0 \qquad \text{on } \tilde{\Gamma} - \partial\Gamma .$$

Thus $q(z)^2$ admits an analytic extension into a neighborhood U of $\bar{\Gamma}$ in the z-plane, indeed,

$$q(z)^2 = \overline{q(\bar{z})^2} \qquad \text{for } \operatorname{Im} z < 0 ,$$

with possible isolated singularities at $z = c, -c$.

Since $w \in L^2(\Omega)$ and f is smooth near Γ by Lemma 6.4, $q(z)^2 \in L^1(\Omega)$. Thus the singularities of $q(z)^2$ are at worst poles of first order, so

$$\Phi(z) = -(z - c)(z + c)\, q(z)^2 \tag{6.13}$$

is holomorphic in the neighborhood U of $\bar{\Gamma}$ and real valued on $\bar{\Gamma}$. In particular, $\operatorname{Re}\Phi(z)$ is a real analytic function of $z = x_1$ on a segment containing $\bar{\Gamma}$, say $\tilde{\Gamma}$, so it has only finitely many zeros there. Indeed we may express Γ as the union of disjoint open intervals $\Gamma_1^+, \ldots, \Gamma_k^+, \Gamma_1^-, \ldots, \Gamma_l^-$, and a finite number of points $a_1, \ldots, a_m$ such that

$$\{z \in \Gamma\colon \operatorname{Re}\Phi(z) > 0\} = \Gamma_1^+ \cup \ldots \cup \Gamma_k^+ ,$$
$$\{z \in \Gamma\colon \operatorname{Re}\Phi(z) < 0\} = \Gamma_1^- \cup \ldots \cup \Gamma_l^- ,$$
$$\operatorname{Re}\Phi(a_j) = 0 , \qquad j = 1, \ldots, m .$$

Since $u^2_{x_1} = 0$ on I, (cf. (6.5)),

$$\operatorname{Re}\Phi(z) = (c^2 - z^2)\,\sigma_{22}^2 \geqq 0 \qquad \text{on } I$$

thus

$$I \subset \Gamma_1^+ \cup \ldots \cup \Gamma_k^+ \cup \{a_1, \ldots, a_m\} .$$

On the other hand, $u^2(z) > 0$ in $\Gamma - I$ and $u^2\sigma_{22} = 0$ on Γ imply $\sigma_{22} = 0$ on $\Gamma - I$. Thus

$$\Gamma_s^+ \subset I , \qquad s = 1, \ldots, k , \qquad \text{or}$$
$$\Gamma_1^+ \cup \ldots \cup \Gamma_k^+ \subset I \subset \Gamma_1^+ \cup \ldots \cup \Gamma_k^+ \cup \{a_1, \ldots, a_m\} . \qquad \text{Q.E.D.}$$

This proof was motivated by H. Lewy's theorem [15].

PROOF OF LEMMA 6.4. First we justify (6.10). Since w^* is continuous any $z_0 \in \bar{\Gamma} - \partial\Gamma$, it will suffice to show that

$$\lim_{z \to z_0} \left(f(z) - w^*(z)\right) = 0 \qquad z_0 \in \bar{\Gamma} - \partial\Gamma . \tag{6.14}$$

Let φ be holomorphic in $B_r(z)$ for some $z \in \Omega$, $r > 0$ so

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial B_\varrho(z)} \frac{\varphi(t)}{t - z}\, dt , \qquad \varrho < r .$$

Thus

$$2\pi\varrho\,|\varphi(z)| \leqq \int_0^{2\pi} |\varphi(t)|\, d\theta , \qquad \varrho < r .$$

Integrating and applying the Schwarz inequality, we obtain the elementary estimate

$$r|\varphi(z)| \leqq \frac{1}{\sqrt{\pi}} \|\varphi\|_{L^2(B_r(z))} . \tag{6.15}$$

Thus for $z \in \Omega$ and near z_0,

$$f(z) - w^*(z) = \tfrac{1}{2}(z - \bar{z})\, w'(z) = i x_2 w'(z)$$

so by (6.15),

$$|f(z) - w^*(z)| \leqq \frac{1}{\sqrt{\pi}} \|w'\|_{L^2(B_{x_2}(z))} \to 0$$

as $x_2 = \operatorname{Im} z \to 0$, since $w' \in L^2(\Omega_\delta)$. This gives (6.14).

Hence $f(z)$ is continuous in $\Omega \cup \tilde{\Gamma} - \{c, -c\}$ and by virtue of (6.11) is real valued on $\tilde{\Gamma} - \{c, -c\}$. Hence we may extend f to a neighborhood U of $\bar{\Gamma}$ as a holomorphic function in U which may admit isolated singularities at the points $z = -c, c$. To show that these singularities are removable, we shall prove that for $z = c$ (or $-c$),

$$f \in L^2(T_\delta)\,, \qquad T_\delta = \left\{ z \in \Omega \colon \left| \arg(z - c) - \frac{\pi}{2} \right| < \frac{\pi}{4}\,,\ |z| < \delta \right\}$$

for a $\delta > 0$. In view of this, the Laurent expansion of f at $z = c$ admits no negative powers, so f is holomorphic in a neighborhood of $z = c$. The estimate of Theorem 4.1 will serve us here.

In general, suppose that φ is holomorphic in $B_\varrho(z)$ for some $\varrho > 0$. Then, analogous to (6.15),

$$\varrho^3 |\varphi'(z)| \leqq C \|\varphi\|_{L^1(B_\varrho(z))} \leqq C_1 \varrho \|\varphi\|_{L^2(B_\varrho(z))}\,. \tag{6.16}$$

Now for each $z = c + r\exp[i\theta] \in T_\delta$, δ small, $B_{r/2}(z) \subset \Omega$ so (5.18) may be applied to w with $\varrho = r/2$. Furthermore, since $B_r(z) \subset G_{3r/2}(c)$, by Theorem 4.1,

$$\|w\|_{L^2(B_r(z))} \leqq C r^\lambda\,, \qquad r \leqq \delta\,.$$

Thus for some $\lambda > 0$,

$$r^3 |w'(z)| \leqq C_2 r^{1+\lambda} \qquad \text{or} \qquad |w'(z)| \leqq C_2 |z - c|^{\lambda - 2}\,, \qquad z \in T_\delta\,.$$

Consequently

$$\int_{T_\delta} |f(z) - w^*(z)|^2\, dx = \int_{T_\delta} |x_2 w'(z)|^2\, dx$$

$$\leqq M \int_0^\delta r^{2\lambda - 1}\, dr < \infty\,.$$

Since $w^* \in L^2(\Omega)$, we conclude that $f \in L^2(T_\delta)$. Thus c, $-c$ are removable singularities. Q.E.D.

Poles may indeed occur in $q(z)^2$ at $z = c$, $z = -c$ or both. This is equivalent to the presence of infinite stress points or the failure of the solution to lie in $H^2(\Omega)$. However when a pole arises at $z = c$, say, the stress tensor there has a prescribed singularity. From the formulas w, w^* of Lemma 6.2,

$$(6.17)\qquad \begin{aligned} w(z) &= 2q(z) - f(z) \\ w^*(z) &= -ix_2\big(2q'(z) - f'(z)\big) + f(z) \end{aligned} \qquad z \in \Omega .$$

If $q(z)^2$ has a first order pole as $z = c$, that is, for an $A \neq 0$,

$$q(z)^2 = \frac{A}{z-c} + \Phi_0(z) ,$$

$$\Phi_0(z) \qquad \text{holomorphic in a neighborhood of } c ,$$

then

$$q(z) = \frac{B}{\sqrt{z-c}}\, q_0(z) \qquad \text{near} \qquad z = c , \qquad z \in \Omega$$

where q_0 is holomorphic near $z = c$ and $q_0(c) \neq 0$. On the other hand $f(z)$ is holomorphic near $z = c$; thus, we have

COROLLARY 6.5. *With the hypotheses of Theorem* 6.3, *the stress tensor* (σ_{hk}) *is continuous on* $\tilde{\Gamma}$ *with the possible exception of* $z = c, -c$. *If the stress tensor fails to be continuous at* $z = c$, *say, then*

$$|\sigma_{hk}| \leqq M|z-c|^{-\frac{1}{2}} , \qquad z \in \Omega,\ |z-c| \ \textit{small},$$

for some $M > 0$.

In Corollary 5.2 we showed that the coincidence set, or contact set, of a Signorini problem is not empty provided (6.3) is satisfied. We then devoted considerable effort to its analysis, leaving open the possibility that the noncoincidence set might be empty. When it is, that is, when $I = \Gamma$, there seems to be little more to say. Let us offer now two examples, one with $I = \Gamma$ and the other with I properly contained in Γ. According to (6.17) it is sufficient to specify the functions $q(z)$ and $f(z)$ provided that we fix $u^h(z)$ at some point.

Let Ω be any smooth domain in the z-plane whose boundary contains the segment $\tilde{\Gamma} = (-1, 1)$ with $\nu = (0, -1)$ the outward directed normal to Ω on $\tilde{\Gamma}$. Let $0 < c < 1$ and set $\Gamma = (-c, c)$.

EXAMPLE 6.6. Set

$$q(z) = \frac{-i}{\sqrt{z^2 - c^2}}, \qquad z \in \Omega,$$

so

$$\sigma_{22} + 2i\varkappa u^2_{x_1} = q(z) = -\frac{1}{|c^2 - z^2|^{\frac{1}{2}}} < 0 \qquad \text{for } z \in \Gamma,$$

i.e., for $-c < z < c$. Here $\sqrt{z^2 - c^2}$, for real c, denotes the branch holomorphic in the complex plane slit along $(-c, c)$ which behaves like z for large values of $|z|$. Thus in the expression above, for « $z \in \Gamma$ » we intend

$$\lim_{\varepsilon \to 0^+} q(z + i\varepsilon), \qquad -c < z < c.$$

Note in particular that $\sqrt{z^2 - c^2} = -|z^2 - c^2|^{\frac{1}{2}}$ for $z < -c$ (and real.)

Let f be a holomorphic function in Ω, smooth in $\bar{\Omega}$, which is real valued on $\tilde{\Gamma}$. From the above, $u^2 = \text{const} = 0$ on Γ and take $u^1(0) = 0$. In this case $I = \Gamma$.

EXAMPLE 6.7. Let $0 < a < c$ and set

$$q(z) = \frac{-iz}{\sqrt{z^2 - c^2}} \sqrt{z^2 - a^2}, \qquad z \in \Omega.$$

In this case

$$\sigma_{22} + 2i\varkappa u^2_{x_1} = q(z) = -iz \left| \frac{a^2 - z^2}{c^2 - z^2} \right|^{\frac{1}{2}}, \qquad |z| \leqq a,\ z \in \Gamma$$

is imaginary whereas

$$q(z) = \begin{cases} z \left| \dfrac{z^2 - a^2}{z^2 - c^2} \right|^{\frac{1}{2}} & -c < z < -a \\[2ex] -z \left| \dfrac{z^2 - a^2}{z^2 - c^2} \right|^{\frac{1}{2}} & a < z < c \end{cases}$$

which is negative.

Again we choose f to be an arbitrary holomorphic in Ω, smooth in $\bar{\Omega}$, and real valued on $\tilde{\Gamma}$. Fix $u^2(c) = 0$ and, say, $u^1(0) = 0$.

In this case, it follows that

$$I = (-c, a] \cup [a, c) \subset \Gamma.$$

As $a \to 0$, q tends to a function much like the q of the first example. As $a \to c$, $q(z) = q_a(z) \to -iz$, which is imaginary on the real axis. In this case $F_2 = 0$ so the applied forces are equilibrated.

Although these examples have infinite stress at $z = c, -c$, examples assigned with finite stress may be found in the same way.

The questions of plane elastostatics have been discussed by Muskhelishvili [19], [20] from the viewpoint of integral equations. A typical contact problem in this theory is the problem of the indentation of an elastic body, usually a half plane, by a rigid stamp. However the technique of [19] is to assume that I consists of a connected interval and then to solve equations for its endpoints. Our example 5.7 is the solution of such a problem. In general, once the solution is obtained in this fashion, one must check *a posteriori* that

$$-\sigma_{hk}\nu_h\nu_k \geqq 0 \qquad \text{on } \Gamma .$$

The author would like to thank Professor John Athanosopoulos for his assistance in constructing the examples.

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