ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

R. E. L. TURNER

Internal waves in fluids with rapidly varying density

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 8, nº 4 (1981), p. 513-573

<http://www.numdam.org/item?id=ASNSP_1981_4_8_4_513_0>

Internal Waves in Fluids with Rapidly Varying Density (*).

R. E. L. TURNER

Introduction.

We consider an incompressible, inviscid fluid completely filling the region between two horizontal planes at $y = 0$ and $y = 1$. An acceleration due to gravity of magnitude g is assumed to act in the negative y direction. We choose orthogonal (x, z) axes in the plane at $y = 0$ and assume henceforth that all quantities are independent of z; that is, we examine two dimensional flows. Our aim will be to show that, with density variation present, the fluid will support traveling waves of permanent form; i.e. flow patterns which appear steady to an observer moving with a fixed velocity in the x direction. If the moving observer sees a steady flow field at $x = \pm\infty$ which is horizontal with constant velocity c, then an observer for whom the fluid at $x = \pm\infty$ is at rest will see a wave traveling to the left, with the crest having velocity c. In this paper we deal with steady flows and merely mention at one point the substitution of variables needed to produce a traveling wave.

One possible steady flow is that which is everywhere horizontal with velocity c. We call this a trivial flow. Naturally, such a flow observed in a moving frame so that it is at rest at $x = \pm\infty$, will be at rest everywhere. We shall see that to have a nontrivial wave we will have to have a nonconstant density. It is the interaction of gravity and the variable density that makes wave propagation possible. We set down one further assumption before proceeding to a discussion of a model. We will assume that the density is nondiffusive; i.e., that the gradient of the density $\varrho = \varrho(x, y)$ at each point is orthogonal to the velocity vector at that point. Density stratification due to temperature variation, salinity, or dissimilar

(*) Research supported by National Science Foundation Grant MCS-7904426.
Pervenuto alla Redazione il 4 Agosto 1980 ed in forma definitiva il 9 Maggio 1981.

layers of fluid persists for time scales which are long compared to time scales for wave propagation so nondiffusivity is often assumed. A consequence of this last assumption is that ϱ is constant along streamlines. We'll see in § 1 that we can thus assume that $\varrho = \varrho(\psi)$ where ψ is a « stream function ». Our analysis will depend on the explicit form of $\varrho(\psi)$. A method for choosing ϱ will be explained in § 1, but can be roughly described as follows: consider a flow pattern for which the deviation from a trivial flow is essentially confined to a bounded region (one expects this behavior for a « solitary » wave). At $x = \pm\infty$ the flow is horizontal and the variable ψ measuring the « flux » between 0 and y is an increasing function $\psi = \psi(y)$. Thus, giving a density $\varrho = \varrho_\infty(y)$ produces a function $\varrho(\psi) = \varrho_\infty(y(\psi))$. Taking $\varrho(\psi)$ as given one can ask for divergence free solutions of the Euler equation consistent with having the given $\varrho = \varrho(\psi)$ along streamlines. This is the point of departure for an analytical treatment.

Before describing our results we give some background to the problem, but only in briefest terms, for the article of Benjamin [1] includes a thorough discussion of the mechanics underlying internal waves and describes the principal results. The earliest work on permanent waves in stratified flow was that of Keulegan [2] and Long [3] who treated a two fluid system with fixed boundaries and exhibited the characteristics of solitary waves through perturbation analysis. Peters and Stoker [4] treated a similar problem but with a free upper surface and showed that such a system supported surface as well as internal solitary waves. Benjamin's article [1] unifies the earlier results and exhibits both cnoidal (periodic) and solitary waves in a variety of flow situations. Benjamin's point of departure is Long's equation [5] (cf. also Dubreil-Jacotin [6]), but incorporating a simplification due to Yih [7]. The equation is (1.5) in the present paper.

The first results in the exact theory, that is, finding solutions of the Euler equations or equivalently, of the Long-Yih equation, are due to Ter-Krikorov [8] who considers a smoothly varying density with a free boundary. He shows the existence of small amplitude periodic waves which, in the limit of increasing period, become solitary waves. His work employs methods analogous to those of Friedrichs and Hyers [9] who treated the problem of the surface solitary wave for a fluid of constant density. In a paper with J. Bona and D. Bose [10] we analyzed the Long-Yih equation in the strip $0 < y < 1$, for a substantial class of densities $\varrho_\infty(y)$ and showed the existence of periodic and solitary waves of finite amplitude. This we did for both fixed « energy » and fixed velocity. The allowable class of densities included, for example, $\varrho_\infty(y) = \alpha(1 - \beta y)^n$ with α, β, and n positive, though as n approaches the value 4 the waves are restricted in amplitude. The method used was basically variational, but involved the use of symmetric rearrangements of functions as well. The method appears to break down if the density ϱ_∞ is not convex or concave, in a rough sense. In the present paper we treat a density which undergoes a rapid transition between two constant values and which in a suitable limit approaches a density which is piecewise constant. As noted, the formulation in [10] seems ill-suited to the problem at hand in which we consider a density which rapidly changes from concave to convex. Instead, as is done in [1] and [8] we formulate a problem for y with x and ψ as independent variables. Ter-Krikorov analyzed an equation for $y(x, \psi)$ directly, not using a variational formulation. Benjamin's perturbation scheme is based on a certain order of vanishing of a flow force integral and elucidates the behavior of the lowest order terms in a perturbation expansion of solutions of the Euler equations. Here we use a variational method to arrive at solutions of the Euler equation. There are many parallels with the work in [10], but as we must deal with a singular quasilinear elliptic expression rather than the Laplacian, the technical problems are completely different.

To describe the results we obtain we must introduce some further notation. We assume that the density $\varrho_\infty(y)$ is a constant ϱ_0 for $0 < y < 1 - \eta_1$ $(0 < \eta_1 < 1)$, decreases smoothly to a constant ϱ_1 $(0 < \varrho_1 < \varrho_0)$ over a transition region $1 - \eta_1 < y < 1 - \eta_1 + \delta$ and remains ϱ_1 for larger y. The methods would apply to more general densities, but to simplify an already lengthy analysis, we restrict the density as indicated. In all of the results the transition width δ and the « wave energy » R (roughly an H^1 norm measuring the deviation from a trivial flow) must be suitably small. We first show the existence of waves which are periodic in x with period $2k > 0$ and have energy $R > 0$. There can, of course, be flows with no vertical component of velocity—« conjugate » flows in the sense of Benjamin [11]. In fact the period must be sufficiently large to insure a vertical component (cf. Corollary 7.1). A basic feature of the flow is determined by the sign of $e = \varrho_0(1-\eta_1)^{-2} - \varrho_1\eta_1^{-2}$. For $e > 0$ and k large we find waves of elevation while for $e < 0$, they are waves of depression. These results follow from Corollary 7.1 as well. The distinction, based on the sign of e, is consistent with earlier perturbation analyses. The wave amplitude is even in x, about the crest, while the amplitude and velocity decay exponentially over the interval $0 \leqslant x \leqslant k$ independently of k (cf. § 4 and § 7). The estimates obtained allow us to let δ approach zero and obtain periodic waves for a discontinuous density. The existence of small amplitude periodic waves for two fluids can be reduced to a problem of bifurcation from a simple eigenvalue (cf. Zeidler [12] and references there). However, the result obtained that way appears not to allow a nontrivial limit as the period increases to infinity. We can let $k \to \infty$ with $\delta > 0$ or with $\delta = 0$ and obtain solitary waves with exponential decay characteristics over the entire interval $0 \leqslant x < \infty$. These results are proven in the last section. By proving results which are independent of the transition width δ we show that the models with $\delta > 0$ (which one might believe are more physically reasonable) and the model with $\delta = 0$ are essentially interchangeable.

The outline of the paper is the following: In § 1 we give a formulation of the problem. In § 2 we exhibit the variational nature of the problem and state the main results of the paper in Theorem 2.1. The original variational problem is singular, but can be approached by means of a regularized problem. The nonsingular problem is solved in § 2 and some preliminary estimates done. To show that the solution obtained in § 2 is a solution of the original problem we require the regularity estimates of § 3. In § 4 we show that the solution can be assumed to have certain symmetry properties. The bounds in § 5 show that there are waves with speeds above the speed of « infinitesimal long waves »; i.e., that we have super-critical wave speeds. In § 6 we obtain lower bounds on the amplitude of waves, independently of the period $2k$. This is a prerequisite to obtaining nontrivial waves in the limit of increasing periods. The exponential decay characteristics are shown in § 7. In § 8 we recapitulate the various results obtained and state Theorem 8.1 which covers the assertions of Theorem 2.1 when $\delta > 0$ and $k < \infty$. The remainder of § 8 is occupied with the limiting forms of solutions corresponding to having a transition width $\delta = 0$ or a limit $k = \infty$ of increasing periods. Theorems 8.2-8.4 detail the limiting behavior and include the remaining cases of Theorem 2.1.

1. – Formulation of the problem.

We begin with a description of a problem in spatial independent coordinates, i.e. in $S = \{(x, y) | -\infty < x < \infty,\ 0 < y < 1\}$ and later transform it so that x and a stream variable ψ are independent coordinates. As we will ultimately state results in the latter formulation we will proceed formally, postponing regularity requirements for the second formulation. We let $U(x, y)$ and $V(x, y)$ be the horizontal and vertical components, respectively, of a vector field $\boldsymbol{q} = (U, V)$ defined in S. Let $p = p(x, y)$ denote the pressure in a fluid occupying the region S; $\varrho(x, y)$, its density; and g, the acceleration of gravity in the negative y direction. We ask that the steady state Euler equations

$$(1.1)\qquad \begin{cases} \varrho(UU_x + VU_y) = -p_x \\ \varrho(UV_x + VV_y) = -p_y - \varrho g \end{cases}$$

be satisfied and that the steady continuity equation

$$\varrho \operatorname{div} \boldsymbol{q} + \boldsymbol{q} \cdot \operatorname{grad} \varrho \tag{1.2}$$

be satisfied. Here we use subscripts for partial derivatives; we will also use other standard notation for derivatives in what follows. If we have a solution of (1.1), (1.2), then $\tilde{U}(x, y, t) = U(x + ct, y) - c$, $\tilde{V} = V(x + ct, y)$, $\tilde{p} = p(x + ct, y)$, and $\tilde{\varrho} = \varrho(x + ct, y)$ satisfy the time dependent Euler equation and continuity equation (cf. [13], Chapter 1). The $(\tilde{U}, \tilde{V})$ flow field represents the (U, V) field as seen by an observer moving in the positive x direction with velocity c. In particular, if (U, V) approaches $(c, 0)$ as $x \to \pm\infty$ then to an observer at $x = 0$ $(\tilde{U}, \tilde{V})$ represents a wave of permanent shape traveling in the negative x direction with velocity c, in fluid which is « at rest » at $x = \pm\infty$.

As noted in the introduction, we will use a formulation of the steady flow problem introduced by Long [5] (cf. also [6]) and simplified by Yih [7]. It involves a further supposition that the density ϱ is nondiffusive; i.e. $\boldsymbol{q} \cdot \operatorname{grad} \varrho = 0$. As a consequence ϱ will be constant on each streamline. Further, if $l(\varrho)$ is any differentiable function of ϱ, then

$$\operatorname{div}\big(l(\varrho)\, \boldsymbol{q}\big) = l(\varrho) \operatorname{div} \boldsymbol{q} + l'(\varrho)\, \boldsymbol{q} \cdot \operatorname{grad} \varrho = 0$$

and thus $l(\varrho)\, \boldsymbol{q}$ has an associated pseudo-stream-function. In particular, if $l(\varrho) = \varrho^{1/2}$ then there is a pseudo-stream-function $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial y} = \varrho^{1/2} U , \qquad \frac{\partial \psi}{\partial x} = -\varrho^{1/2} V \tag{1.3}$$

and since $\boldsymbol{q} \cdot \operatorname{grad} \varrho = 0$, ϱ is a function of ψ, $\varrho(\psi)$. Further, from Bernoulli's theorem, along each streamline the stagnation pressure or « total head » H is constant. Thus

$$H = p + \tfrac{1}{2}\varrho(U^2 + V^2) + \varrho g y = H(\psi) . \tag{1.4}$$

If one eliminates p from (1.1) using (1.2)-(1.4) one finds that

$$\Delta \psi(x, y) + g y \varrho'(\psi) = H'(\psi) \tag{1.5}$$

where $\Delta \psi = \psi_{xx} + \psi_{yy}$ and primes denote derivatives. We call (1.5) the Long-Yih equation. Conversely, if one chooses arbitrary functions $\varrho(\psi)$ and $H(\psi)$ and can solve (1.5) for $\psi = \psi(x, y)$, then with (U, V) defined by (1.3) and p defined using (1.4), the equations (1.1) and (1.2) will be satisfied.

Our ultimate aim is to obtain a solution of (1.1), (1.2) which will correspond to a solitary wave; that is, for which the flow at $x = \pm\infty$ approaches a horizontal flow of velocity $(c, 0)$ in a fluid with a given density stratification « at ∞ » depending only on the vertical coordinate, say $\varrho = \varrho_\infty(y)$. In choosing these conditions at $x = \pm\infty$ we are effectively choosing the functions $\varrho(\psi)$ and $H(\psi)$. To see this we examine what ϱ and H would be in a totally horizontal flow with $(U, V) \equiv (c, 0)$. Since $V \equiv 0$, ψ in (1.3) is a function of y alone; we'll denote it by $\mathring{\psi}(y)$. Then (1.3) becomes

$$\frac{\partial \mathring{\psi}}{\partial y} = \varrho_\infty^{1/2}(y)\, c\,. \tag{1.6}$$

We initially assume that ϱ_∞ is positive and infinitely differentiable. If we assume $\mathring{\psi}(0) = \psi_0$, then

$$\mathring{\psi}(y) = \psi_0 + \int_0^y \varrho_\infty^{1/2}(s)\, c\, ds\,, \tag{1.7}$$

a strictly increasing function of y with a strictly increasing inverse function $\mathring{y}(\psi)$. Then ϱ as a function of the stream parameter ψ is

$$\varrho(\psi) = \varrho_\infty(\mathring{y}(\psi))\,. \tag{1.8}$$

From the second Euler equation (cf. (1.1)) we see that p must be hydrostatic:

$$p(y) = -\int_0^y \varrho_\infty(s)\, g\, ds \tag{1.9}$$

and hence

$$H(\psi) = p(\mathring{y}(\psi)) + \tfrac{1}{2}\varrho(\psi)c^2 + \varrho(\psi) g \mathring{y}(\psi)\,. \tag{1.10}$$

Naturally, the function $\psi(x, y) = \mathring{\psi}(y)$ is a solution of (1.5) for any choice of c and thus we have a one-parameter family of solutions of (1.5). We call a flow corresponding to $\mathring{\psi}(y)$ and a given c a trivial flow. In studying the Long-Yih equation (1.5) we will use the functions ϱ and H described in (1.8) and (1.10), and eventually ask that $\psi(x, y) - \mathring{\psi}(y) \to 0$ as $|x| \to \infty$, the idea being that for a solitary wave the flow for large x should look like a trivial flow. As a step toward obtaining solitary waves we also treat waves which are periodic in x. For these the trough between waves will approximate a trivial flow. To obtain nontrivial flows we'll have to impose conditions on the density ϱ_∞. We can see that if ϱ_∞ is a constant, $\varrho(\psi)$ and

$H(\psi)$ are both constants and (1.5) reduces to Laplace's equation. If we specify $\psi(x,0)=\psi_0$ and $\psi(x,1)=\psi_1=\psi_0+\int_0^1\varrho_\infty^{1/2}(s)\,c\,ds$, then the only harmonic function which is bounded in the strip S and satisfies the boundary conditions is a linear function of y corresponding to a trivial flow. Thus a necessary condition for nontrivial flows is a variable density $\varrho_\infty(y)$.

It should be remarked that the functions $\varrho(\psi)$ and $H(\psi)$ are defined by (1.8) and (1.10), respectively, only in the range $\psi_0\leqslant\psi\leqslant\psi_1$. In a treatment of the problem using ψ as a dependent variable (as in [10]) it is common to extend the functions outside $[\psi_0,\psi_1]$ and eventually show that any solution obtained has range in $[\psi_0,\psi_1]$. An extension will not be necessary here as we will use ψ as an independent variable.

We'll assume that no reversal of flow occurs in any vertical section; i.e., that $U>0$ or, equivalently, that ψ is strictly increasing in y for each x. For such a ψ the inverse function $y=y(x,\psi)$ exists and it is this function we'll work with. If we assume that ψ has two continuous derivatives and $\psi_y>0$, then from (1.5), and the relations $\psi_x+\psi_y y_x=0$ and $\psi_y y_\psi=1$ we see that $y(x,\psi)$ satisfies

$$-\frac{\partial}{\partial x}\left[\frac{y_x}{y_\psi}\right]+\frac{1}{2}\frac{\partial}{\partial\psi}\left[\frac{1+y_x^2}{y_\psi^2}\right]+g\varrho' y=H'. \tag{1.11}$$

Our basic problem is finding solutions of (1.11) satisfying

$$y(x,\psi_0)=0\,,\qquad y(x,\psi_1)=1\,. \tag{1.12}$$

To describe the regularity required of solutions and to have notation for use in the sequel we introduce some function spaces. Let $\mathfrak{D}_k=\{(x,y)|\,|x|<k,\ \psi_0<\psi<\psi_1\}$ and denote $\mathfrak{D}_\infty$ by $\mathfrak{D}$. The space $C_k^0(\mathfrak{D})$ consists of continuous functions on $\mathfrak{D}$ which are $2k$ periodic in x with the norm

$$\|y\|_{C_k^0}=\sup_{\mathfrak{D}_k}|y(x,\psi)|\,;$$

$C_k^l(\mathfrak{D})$ consists of those $y\in C_k^0$ having continuous derivatives through order l with

$$\|y\|_{C_k^l}=\sup_{|\beta|\leqslant l}|D^\beta y|$$

where $\beta=(\beta_1,\beta_2)$ is a pair of nonnegative integers, $|\beta|=\beta_1+\beta_2$, and D^β represents the partial derivative $\partial^{|\beta|}/\partial x^{\beta_1}\partial\psi^{\beta_2}$. For $0<\alpha<1$ we let $|y|_\alpha$

be the smallest constant C for which

$$|y(x, \psi) - y(\tilde{x}, \tilde{\psi})| \leq C\left(|x - \tilde{x}|^2 + |\psi - \tilde{\psi}|^2\right)^{\alpha/2}$$

and let $C_k^{l+\alpha}(\mathfrak{D})$ consist of the $y \in C_k^\alpha$ for which $|D^\beta y|_\alpha < \infty$ for $|\beta| \leq l$ with the norm

$$\|y\|_{C_k^{l+\alpha}} = \|y\|_{C_k^l} + \sup_{|\beta|=l} |D^\beta y|_\alpha .$$

Functions in $C_k^{l+\alpha}(\mathfrak{D})$ have natural extensions to the closure $\overline{\mathfrak{D}}$, preserving the norm and we write $C_k^{l+\alpha}(\overline{\mathfrak{D}})$ for the space of extended functions. The space $C_k^l(\overline{\mathfrak{D}})$ denotes that subset of $C_k^l(\mathfrak{D})$ having continuous extensions to $\overline{\mathfrak{D}}$. We use $L_k^2(\mathfrak{D})$ to denote the space of measurable functions y, $2k$ periodic in x, which are square integrable on $\mathfrak{D}_k$ and define a norm

$$\|y\|_{L_k^2} = \left(\int_{\mathfrak{D}_k} y^2 \, dx \, d\psi\right)^{1/2} .$$

We let $\mathring{C}_k^1(\mathfrak{D}) = \{y \in C_k^1(\mathfrak{D}) \,|\, y(x, \psi_0) = y(x, \psi_1) = 0\}$ and let H_k^1 be the completion of $\mathring{C}_k^1(\mathfrak{D})$ in the norm

$$\|y\|_{H_k^1} = \|y\|_{L_k^2} + \left(\int_{\mathfrak{D}_k} |\nabla y|^2\right)^{1/2} .$$

For an interval I, $\mathring{H}^1(I)$ denotes the standard space of functions which vanish at the endpoints and have one derivative in L^2.

A space without a subscript k will denote the corresponding function space on the whole strip $\mathfrak{D}$. If we are considering functions on another 2 dimensional strip, such as S above, we use $C_k^0(S)$, etc. to denote the analogous spaces.

For any arbitrary open set $\mathfrak{D}' \subset \mathfrak{D}$ we use $L^\infty(\mathfrak{D}')$ to denote the space of measurable, essentially bounded functions, with the standard norm denoted $\| \;\|_{L^\infty(\mathfrak{D}')}$. We let $W^{1,\infty}(\mathfrak{D}')$ denote the collection of continuous functions on $\mathfrak{D}'$ with L^∞ distribution derivatives and let

$$\|y\|_{W^{1,\infty}(\mathfrak{D}')} = \|y\|_{L^\infty} + \|y_x\|_{L^\infty} + \|y_\psi\|_{L^\infty} .$$

For a density $\varrho_\infty(y)$ which is continuously differentiable we'll find a solution y of (1.11), (1.12) in $C_k^{2+\alpha}(\overline{\mathfrak{D}})$ or in $C^{2+\alpha}(\overline{\mathfrak{D}}) \cap L^2(\mathfrak{D})$. In § 8 we'll consider a limiting case of a discontinuous ϱ_∞ and in that case we need a weaker formulation. If $\tilde{\mathfrak{D}}$ is a subset of $\mathfrak{D}$ on which ϱ' and H' are continuous we call $y(x, \psi) \in C^1(\tilde{\mathfrak{D}})$ (with $y_\psi > 0$) a weak solution of (1.11) on $\tilde{\mathfrak{D}}$ if for all C^∞ test functions φ with support in $\tilde{\mathfrak{D}}$,

$$\int_{\tilde{\mathfrak{D}}} \left[\frac{y_x}{y_\psi}\varphi_x - \frac{1}{2}\left(\frac{1+y_x^2}{y_\psi^2}\right)\varphi_\psi + g\varrho' y\varphi - H'\varphi\right] dx\, d\psi = 0 . \tag{1.13}$$

Let $\mathfrak{D}^+ = \{(x, \psi) \in \mathfrak{D} | 0 < \psi < \psi_1\}$ and $\overline{\mathfrak{D}} = \mathfrak{D} - \overline{\mathfrak{D}^+}$, the bar denoting the closure. Suppose $\varrho'(\psi)$ and $H'(\psi)$ are continuous on $\mathfrak{D}^\pm$, with a possible discontinuity at $\psi = 0$.

DEFINITION 1.1. *If $y \in C^\beta(\overline{\mathfrak{D}}) \cap C^{1+\alpha}(\overline{\mathfrak{D}^+}) \cap C^{1+\alpha}(\overline{\mathfrak{D}^-})$ for some $\alpha > 0$, $\beta > 0$; $y_x \in C^0(\mathfrak{D})$; and $y_\psi > 0$; we call y a solution of* (1.11) *if the following two conditions are satisfied*:

i) *y is a weak solution of* (1.11) *in $\mathfrak{D}^+$ and $\mathfrak{D}^-$*;

ii) *the pressure $p(x, \psi)$ computed from* (1.4) *in each of $\mathfrak{D}^+$, $\mathfrak{D}^-$ has a continuous extension to $\mathfrak{D}$.*

The notion of solution incorporates what is physically expected at a streamline where the density is discontinuous; that is, continuity of the slope y_x on a streamline and of the pressure. One could equally well introduce the notion of a weak solution on all of $\mathfrak{D}$ from which the continuity of pressure would follow.

2. – A variational problem.

At the outset we focus attention on solutions of (1.11) which are $2k$ periodic in x. In section 8 we will allow k to approach ∞ to obtain « solitary » waves, Let $S_k = \{(x, y) \big| |x| < k,\ 0 < y < 1\}$ and $S_\infty = S$. The Long-Yih equation (1.5) is formally the Euler equation for the functional

$$M(\psi) = \int_{S_k} [\tfrac{1}{2}|\nabla\psi|^2 - gy\varrho(\psi) + H(\psi)]\, dx\, dy \tag{2.1}$$

defined for $\psi \in H_k^1$. In this section we assume ϱ is smooth. Then M is Frechet differentiable snd

$$\langle M'(\psi), \chi\rangle = \int_{S_k} [\nabla\psi\nabla\chi - gy\varrho'(\psi)\chi + H'(\psi)\chi]\, dx\, dy \tag{2.2}$$

where $\langle\ ,\ \rangle$ represents the pairing between H_k^1 and its dual. At a critical point of M, i.e. where $M'(\psi) = 0$, we see that (2.2) yields the weak form of the Long-Yih equation. A functional which formally has equation (1.11) as its Euler equation is

$$\Phi(y) = \int_{\mathfrak{D}_k} \left[\frac{1}{2}\frac{1+y_x^2}{y_\psi} + g\varrho'(\psi)\frac{y^2}{2} - H'(\psi)\, y\right] dx\, d\psi \tag{2.3}$$

in which the terms correspond (up to an added constant in the case of the second and third terms) to those in M, respectively, under the change of variables. Choosing a domain for Φ presents difficulties due to the presence of y_ψ in a denominator in the integrand. We will accommodate this by altering the functional, but only after some further reformulation. The set $\mathcal{A} = \{y \in C_k^1(\mathfrak{D})\,|\, y_\psi > 0,\ y(x, \psi_0) = 0,\ y(x, \psi_1) = 1\}$ is a domain in which Φ' exists, though not one suited to invoking variational principles. The function $\mathring{\psi}(y)$ (cf. (1.7)) is the pseudo-stream-function for a trivial flow and is a solution of (1.5) as one easily verifies. Its inverse function $\mathring{y}(\psi) \in \mathcal{A}$ is a solution of (1.11) and thus a critical point of Φ; i.e. $\langle \Phi'(\mathring{y}), u\rangle = 0$ where $\langle\ ,\ \rangle$ denotes the pairing of C_k^1 with its dual; here the pairing takes the form $\iint \Phi'(\mathring{y})\, u = 0$ for any $u \in C_k^1$ which vanishes on $\psi = \psi_0, \psi_1$. Consider $y \in \mathcal{A}$ and let $u = y - \mathring{y}$. Defining

$$N(y) = \int_{\mathfrak{D}_k} \frac{1}{2}\frac{1+y_x^2}{y_\psi} \tag{2.4}$$

we find

$$N(\mathring{y} + u) = \int_{\mathfrak{D}_k} \left[\frac{1}{2}\frac{1}{\mathring{y}_\psi} - \frac{1}{2}\frac{u_\psi}{\mathring{y}_\psi^2} + \frac{1}{2}\left\{\frac{u_\psi^2 + \mathring{y}_\psi^2 u_x^2}{\mathring{y}_\psi^2(\mathring{y}_\psi + u_\psi)}\right\}\right] = \tag{2.5}$$

$$= N(\mathring{y}) + \langle N'(\mathring{y}), u\rangle + \int_{\mathfrak{D}_k} \frac{1}{2}\left\{\frac{u_\psi^2 + \mathring{y}_\psi^2 u_x^2}{\mathring{y}_\psi^2(\mathring{y}_\psi + u_\psi)}\right\}.$$

It is then easy to see that

$$\Phi(y) = \Phi(\mathring{y} + u) = \Phi(\mathring{y}) + \langle \Phi'(\mathring{y}), u\rangle + E(u) \tag{2.6}$$

where

$$E(u) = \int_{\mathfrak{D}_k} \left[\frac{1}{2}\frac{\mathring{y}_\psi^2 u_x^2 + u_\psi^2}{\mathring{y}_\psi^2(\mathring{y}_\psi + u_\psi)} + g\varrho'(\psi)\frac{u^2}{2}\right] dx\, d\psi\,. \tag{2.7}$$

Since $\Phi'(\mathring{y}) = 0$, y is a critical point of Φ if and only if u is a critical point of E, the domain of E being naturally inherited from that of Φ. A further change of variables will reduce E to a more convenient form. As is done in [8] we let

$$\eta(\psi) = \mathring{y}(\psi) - \mathring{y}(0) = \int_0^{\psi} \frac{ds}{\varrho_\infty^{1/2}(\mathring{y}(s))\,c} \tag{2.8}$$

and

$$w(x, \eta) = u(x, \psi(\eta)) = y(x, \psi(\eta)) - \mathring{y}(\psi(\eta)); \qquad \tilde{\varrho}(\eta) = \varrho(\psi(\eta)) \tag{2.9}$$

where $\psi(\eta)$ is inverse to $\eta(\psi)$. Then E corresponds to

$$G(w) = \int_{\Omega_k} \left[\frac{1}{2} c^2 \tilde{\varrho}(\eta) \frac{w_x^2 + w_\eta^2}{1 + w_\eta} + g\tilde{\varrho}'(\eta) \frac{w^2}{2}\right] dx\, d\eta \tag{2.10}$$

where $\Omega_k = \{(x, \eta)\big|\,|x| < k,\ \eta(\psi_0) = \eta_0 < \eta < \eta_1 = \eta(\psi_1)\}$ and $\Omega_\infty = \Omega$. Since the derivative η_ψ is bounded above and below, one shows easily that u is a critical point of E if and only if w is a critical point of G, the domain of G now consisting of $w \in C_k^1(\Omega)$ for which $w_\eta > -1$.

From (2.8), η is related to the original y scale by a shift of size $-\mathring{y}(0) = \eta_0 < 0$. However, η is a *stream* coordinate. Note that from (1.8) and (2.9)

$$\tilde{\varrho}(\eta) = \varrho_\infty\big(\mathring{y}(\psi(\eta))\big) = \varrho_\infty(\eta - \eta_0)$$

so $\tilde{\varrho}$ has no dependence on c. In our subsequent analysis of the problem we will use x and η as independent coordinates and will omit the tilda on ϱ. When we have occasion to refer back to E we will make clear the distinction between the variables η and ψ. For notational convenience we will sometimes replace the pair (x, η) by (x_1, x_2).

We have still not specified the function ϱ and do it at this point. From the last paragraph we see that it suffices to describe its dependence on η. For given constants ϱ_0 and $0 < \varrho_1 < \varrho_0$ we let $\varrho = \varrho(\eta)$ be a non-increasing C^∞ function such that

$$\begin{cases} \varrho(\eta) = \varrho_0; & \eta_0 \leqslant \eta \leqslant 0\,, \\ \varrho_0 \geqslant \varrho(\eta) \geqslant \varrho_1; & 0 < \eta \leqslant \delta\,, \\ \varrho(\eta) = \varrho_1; & \delta < \eta \leqslant \eta_1\,. \end{cases} \tag{2.11}$$

The particular form of ϱ on $0 < \eta \leqslant \delta$ is not important. In fact the methods we use would allow some smooth variation in density outside of the interval $[0, \delta]$. However, for simplicity we treat the case described here, our main objective being the description of internal waves with a rapid variation in density, the extreme case being a discontinuity. Note that the description (2.11) corresponds to a rapid density change at height $y = |\eta_0| = 1 - \eta_1$ in the original spatial variable.

The functional G is defined for any $w \in C_k^1$ satisfying $w_\eta > -1$. If we let $\lambda = g/c^2$ and define

$$(2.12) \qquad \begin{cases} F(w) = \displaystyle\int_{\Omega_k} \frac{1}{2}\,\varrho(\eta)\,\frac{w_x^2 + w_\eta^2}{1 + w_\eta}\,dx\,d\eta \\ B(w) = -\displaystyle\int_{\Omega_k} \varrho'(\eta)\,\frac{w^2}{2}\,dx\,d\eta \end{cases}$$

then a critical point of G corresponds to a solution of the nonlinear eigenvalue problem

$$(2.13) \qquad F'(w) = \lambda B'(w)$$

which one can view as arising from a constrained variational problem. Of course the integrand in F inherits the defect of having a denominator $1 + w_\psi$ which can be zero. On the other hand we'll see that it is the source of solitary waves; in its absence, (2.13) would be linear and, as some elementary Fourier analysis will show (cf. § 7), cannot have a nontrivial solution w with ∇w in $L^2(\Omega)$.

We'll see (cf. (2.21)) that the linearization of (2.13) at $w = 0$:

$$\frac{\partial}{\partial x}\,\varrho\,\frac{\partial w}{\partial x} + \frac{\partial}{\partial \eta}\,\varrho\,\frac{\partial w}{\partial \eta} = \lambda \varrho' w\,,$$

$w = 0$ at $\eta = \eta_0, \eta_1$, has a lowest eigenvalue $\mu = \mu_\delta$ $(\delta \geqslant 0)$ corresponding to a velocity $c_\delta = (g/\mu)^{1/2}$, a so-called « critical » velocity. With this notation and noting that $w(x, \eta)$ represents the deviation from horizontal of the streamline having the label η we now state the main results of the paper.

THEOREM 2.1. *Let* $e = \varrho_0 \eta_0^{-2} - \varrho_1 \eta_1^{-2}$ *and suppose* $e \neq 0$. *Then there are positive numbers (all constants appearing depend on* $\varrho_0^{\cdot}$, $0 < \varrho_1 < \varrho_0$, *and* η_0 *in addition to indicated parameters)* $\tilde{\delta}$, $\tilde{R}$, *and* $\tilde{k} = \tilde{k}(R)$ *such that for* $0 \leqslant \delta < \tilde{\delta}$, $0 < R \leqslant \tilde{R}$ *and* $\tilde{k}(R) < k \leqslant +\infty$ *the problem* (1.11)-(1.12) *has a nontrivial solution* (λ, y) *(*λ *is implicit) with the following properties*

1) $0 < \lambda < \mu_\delta(1 - \tilde{C}R^{4/3})$; *i.e., the associated speed* $c = (g/\lambda)^{1/2}$ *is « supercritical ».*

2) *y has period $2k$ in x (for $k < \infty$).*
The streamline displacement

$$w(x, \eta) = y(x, \psi(\eta)) - \mathring{y}(\psi(\eta)) \quad (cf.\ (1.7),\ (2.8),\ (2.9))$$

satisfies.

3) $F(w) = R^2$.

4) *The associated flow is a wave of elevation ($w > 0$) if $e > 0$ and depression ($w < 0$) if $e < 0$.*

5) $w(x, \eta) = w(-x, \eta)$ *and for* $0 \leqslant x_1 \leqslant x_2 \leqslant k$, $w(x_1, \eta) - w(x_2, \eta) \geqslant 0$ $(\leqslant 0)$ *if* $e > 0$ (< 0).

6) $|w| \leqslant C_1 \exp[-px]$ *and* $|\nabla w| \leqslant C_2 \exp[-px]$, *on* $0 \leqslant x \leqslant k$ *for a* $p > 0$, $C_i = C_i(R, p)$, $i = 1, 2$.

The vertical component of velocity V is $\not\equiv 0$ and satisfies $|V| \leqslant C \exp[-px]$ on $0 \leqslant x \leqslant k$.

Theorem 2.1 is included in a series of results stated in § 8 which also detail the regularity properties of solutions and the convergence properties as $\delta \to 0$ or $k \to \infty$. The intervening sections contain the substance of the proofs.

All the results in Theorem 2.1 will follow from a study of equation (2.13) to which we now turn. To circumvent the unboundedness in F we alter its integrand to remove the singularity, solve a nonsingular problem, and show that by restricting the « size » of the solution we obtain a solution of the original problem. Let $\xi = \xi(s)$ be a C^∞ function, defined for $s \geqslant 0$, which is 1 for $0 \leqslant s \leqslant 1$, decreases to zero at $s = 2$, and is zero for $s \geqslant 2$. Let

$$f(p_1, p_2) = \frac{1}{2} \frac{p_1^2 + p_2^2}{1 + p_2}$$

so that $\varrho f(\nabla w)$ is the integrand in F. We define

$$a(p_1, p_2) = \xi_r f(p_1, p_2) + (1 - \xi_r) \frac{p_1^2 + p_2^2}{2} \tag{2.14}$$

where $\xi_r = \xi((p_1^2 + p_2^2)/r^2)$ and $0 < r^2 < \frac{1}{2}$. We will see shortly that $a(p_1, p_2)$ has a positive Hessian for suitably restricted r. Suppose we replace F by

$$A(w) = \int_{\Omega_k} \varrho(\eta)\, a(w_x, w_\eta)\, dx\, d\eta \tag{2.15}$$

for $w \in H_k^1$. One shows easily that A is Frechet differentiable. If we find a solution of the quasilinear elliptic problem

$$A'(w) = \lambda B'(w) \tag{2.16}$$

with $|\nabla w| < r$, then w will be a solution of $F'(w) = \lambda B'(w)$ since A' and F' coincide for such a w. We'll see that the existence problem for (2.16) is not difficult. The main work in the paper will be to obtain estimates on w and its derivatives. A different convex extension of f might allow for sharper estimates and a larger value of $\tilde{R}$ in Theorem 2.1. We have, however, chosen not to carry explicit constants through all the elliptic estimates.

We begin with technical lemmas regarding $a(p_1, p_2)$. Denote $\partial a/\partial p_i$ by a_i, $\partial^2 a/\partial p_i \partial p_j$ by a_{ij}, and third derivatives by a_{ijk} in obvious notation. Similarly, denote derivatives of f by f_i and f_{ij}. We use the convention that repeated indices are summed from 1 to 2 and let $O(r)$ be a term bounded by a constant times r, the constant being a computable numerical constant.

LEMMA 2.1. *There exists an r_0, $0 < r_0 < 1/\sqrt{2}$ such that for $0 < r \leqslant r_0$ there are positive constants σ_i, $i = 1, 2, 3, 4, 5$ and $\nu > 0$ such that $a = a(p_1, p_2)$ defined by* (2.14) *satisfies*

1) $\frac{1}{2}\sigma_1(p_1^2 + p_2^2) \leqslant a(p_1, p_2) \leqslant \frac{1}{2}\sigma_2(p_1^2 + p_2^2)$;
2) $\sigma_3(p_1^2 + p_2^2) \leqslant a_1 p_1 + a_2 p_2 \leqslant \sigma_4(p_1^2 + p_2^2)$;
3) $a_1^2 + a_2^2 \leqslant \sigma_5(a_1 p_1 + a_2 p_2)$;
4) $a_{ij}\zeta_i\zeta_j \geqslant \nu(\zeta_1^2 + \zeta_2^2)$ *for all* p_i, p_j;
5) $\operatorname{grad} a_2 = (0, 1) + O(r)$;
6) $a_{ijk}(p_1, p_2) = a_{ijk}(0, 0) + O(r)$; *at* $(0, 0)$, $a_{111} = 0$, $a_{112} = 2$, $a_{122} = 0$, *and* $a_{222} = 6$, *the other derivatives being equal to one of these by symmetry.*

PROOF. The function a can be written

$$a(p_1, p_2) = \frac{p_1^2 + p_2^2}{2}\left(1 - \frac{\xi_r p_2}{1 + p_2}\right)$$

and so part 1) follows provided $|\xi_r p_2/(1 + p_2)| < 1$. Since $\xi_r = 0$ for $p_1^2 + p_2^2 \geqslant 2r^2$ we can assume $|p_2| < r\sqrt{2}$ and then any choice of $r < 1/2\sqrt{2}$ will suffice. Of course the σ_i depend on the choice of r. For part 2) we write a as

$$a(p_1, p_2) = \frac{p_1^2 + p_2^2}{2} - \frac{1}{2}\frac{(p_1^2 + p_2^2)p_2}{1 + p_2}\xi_r$$

from which we calculate

$$(2.17)\qquad \begin{cases} a_1 = p_1 - \dfrac{p_1 p_2 \xi_r}{1+p_2} - \dfrac{(p_1^2+p_2^2)p_1 p_2}{(1+p_2)} \dfrac{\xi_r'}{r^2}, \\ a_2 = p_2 - \dfrac{p_2^2 \xi_r}{1+p_2} - \dfrac{1}{2} \dfrac{(p_1^2+p_2^2)\xi_r}{(1+p_2)^2} - \dfrac{(p_1^2+p_2^2)p_2^2}{1+p_2} \dfrac{\xi_r'}{r^2}. \end{cases}$$

Thus

$$a_1 p_1 + a_2 p_2 = (p_1^2+p_2^2)\left[1 - \frac{p_2 \xi_r}{1+p_2} - \frac{1}{2} \frac{p_2 \xi_r}{(1+p_2)^2} - \frac{(p_1^2+p_2^2)p_2 \xi_r'}{(1+p_2)r^2}\right].$$

From this last expression it is an easy matter to see that the inequalities in 2) hold for a sufficiently small r. Item 3) follows similarly.

As regards the second derivatives, they deviate from the Kronecker δ_{ij} only for $p_1^2+p_2^2 \leqslant 2r^2$ and then by at most a finite sum of terms, each a constant times an expression $p_1^{m_1} p_2^{m_2}(1+p_2)^{-k} r^{-2l}$ with $k=1$ or 2; $l=0$, 1 or 2; $m_1 \geqslant 0$, $m_2 \geqslant 0$, and $m_1+m_2-2l \geqslant 1$. That is, $a_{ij} = \delta_{ij} + O(r)$ and hence 4) and 5) hold. Part 6) is shown similarly.

REMARK 1. Since $a = f$ for $p_1^2+p_2^2 < r^2$ properties 1)-4) of the lemma hold for f in that range. In fact,

$$f_{ij} = \begin{pmatrix} \dfrac{1}{1+p_2} & \dfrac{-p_1}{(1+p_2)^2} \\ \\ \dfrac{-p_1}{(1+p_2)^2} & \dfrac{1+p_1^2}{(1+p_2)^3} \end{pmatrix}$$

which has a determinant $(1+p_2)^{-4}$ and a positive trace for $p_2 > -1$.

REMARK 2. We assume henceforth that the cutoff value r satisfies $0 < r \leqslant r_0$. Note that the constants $\sigma_1, \ldots, \sigma_5$, and ν in the previous lemma can all be made arbitrarily close to 1 by choosing r_0 sufficiently small. For this reason we will ignore the dependence of estimates on the constants $\sigma_1, \ldots, \sigma_5$ and will eventually omit the ν dependence as well. For now we prefer to exhibit the ellipticity constant ν in the proofs.

COROLLARY 2.1. *With A as defined by* (2.15) *we have*

1) $\dfrac{\sigma_1}{2}\displaystyle\int_{\Omega_k} \varrho |\nabla w|^2 \leqslant A(w) \leqslant \dfrac{\sigma_2}{2}\int_{\Omega_k} \varrho |\nabla w|^2$

2) $\sigma_3 \int\limits_{\Omega_k} \varrho |\nabla w|^2 \leqslant \langle A'(w), w \rangle \leqslant \sigma_4 \int\limits_{\Omega_k} \varrho |\nabla w|^2$;

3) $2\dfrac{\sigma_3}{\sigma_2} A(w) \leqslant \langle A'(w), w\rangle \leqslant 2\dfrac{\sigma_4}{\sigma_1} A(w)$.

PROOF. The inequalities 1) and 2) follow immediately from parts 1) and 2) of the preceding lemma since $\varrho > 0$. Part 3) is a consequence of 1) and 2).

For use in the next lemma we introduce $w^+(x, \eta) = \max(0, w(x, \eta))$ and $w^- = w - w^+$. The map $w \to w^+$ (or w^-) is continuous from L^2_k into itself or into $L^2(\Omega_k)$ with the weight ϱ' as can be seen by a simple computation (cf. [14], p. 41).

THEOREM 2.2. *For any $R > 0$ the problem* (2.16)

$$A'(w) = \lambda B'(w)$$

has solutions (λ_i, w_i), $i = 1, 2$ with $\lambda_i > 0$, $w_i \in H^1_k \cap C^\infty(\bar{\Omega})$ and $A(w_i) = R^2$. The solution w_1 satisfies $w_1 > 0$ in Ω_k and is characterized by

$$B(w_1) = \sup_{\substack{w \in H^1_k \\ A(w) = R^2}} B(w^+) . \tag{2.18}$$

The solution w_2 satisfies $w_2 < 0$ and is similarly characterized with $B(w^-)$ replacing $B(w^+)$.

PROOF. From part 1) of Corollary 2.1 we see that the set $\{w | A(w) = R^2\}$ is bounded in H^1_k and thus is compactly embedded in $L^2_k(\Omega)$. Let c^+ be the supremum in (2.18) and let v_n, $n = 1, 2, 3, \ldots$, be a maximizing sequence. There is a subsequence of v_n converging weakly in H^1_k and strongly in L^2_k to a $w_1 \in H^1_k$. Assume we've renamed the functions so that v_n is the convergent sequence. Denote $B(w^+)$ by $B_+(w)$. The map $w \to B_+(w)$ is continuous on H^1_k and differentiable (cf. [14], p. 41) with

$$\langle B'_+(w), \chi \rangle = -\int \varrho' w^+ \chi = \langle B'(w^+), \chi \rangle .$$

Since v_n converges to w_1 in L^2_k, v_n^+ converges to w_1^+ and by continuity, $B_+(w_1) = c^+$. Clearly $c^+ > 0$ so $w_1 \neq 0$. Since A has a convex integrand and v_n converges weakly in H^1_k, $A(w_1) \leqslant R^2$. We claim that $A(w_1) = R^2$. For if not, then for some $t > 1$, $A(tw_1) = R^2$, since as $t \to +\infty$, $A(tw_1) \to$

$\to +\infty$. But for $t > 1$, $B_+(tw_1) = t^2 c^+ > c^+$, contradicting the characterization of c^+.

Since $A(w_1)$ and $B_+(w_1)$ are Frechet differentiable and w_1 is a constrained maximum it follows that $A'(w_1) = \lambda_1 B'_+(w_1)$ for a real λ_1. That is

$$\int_{\Omega_k} (\varrho a_1(\nabla w_1)\varphi_x + \varrho a_2(\nabla w_1)\varphi_\eta) = -\lambda_1 \int_{\Omega_k} \varrho' w_1^+ \varphi$$

for all $\varphi \in H_k^1$. Choosing $\varphi = w_1$ and using part 2) of Corollary 2.1 we see that $\lambda_1 > 0$. The function w_1 is a weak solution of the equation

$$\frac{\partial}{\partial x_i} \varrho \tilde{a}_{ij} \frac{\partial w_1}{\partial x_j} = \lambda_1 \varrho' w_1^+ \tag{2.19}$$

where $\tilde{a}_{ij} = \int_0^1 a_{ij}(t\nabla w_1)\,dt$ and thus we can apply a weak maximum principle ([15], p. 168) to show $w_1 \geqslant 0$ (cf. [16] for a similar application). In this case, since w_1 is periodic, the boundary consists of points where $\eta = x_2 = 0$ or 1 and on that set $w_1 = 0$ in the H^1 sense (cf. the definition of H_k^1). We conclude that $w_1 \geqslant 0$ and thus $w_1^+ = w_1$. Then w_1 is a weak solution of the quasilinear elliptic equation

$$\frac{\partial}{\partial x_i} \varrho a_i(\nabla w_1) = \lambda_1 \varrho' w_1 \tag{2.20}$$

in which $a(p_1, p_2)$ and $\varrho(x_2)$ are C^∞ functions. As a consequence of results in Chapter 4 of [17] the function $w(x_1, x_2)$ is a C^∞ function on $\bar{\Omega}$. The strong maximum principle ([15], p. 33) applied to (2.19) shows that $w_1 > 0$ in Ω_k. The case of (λ_2, w_2) is treated analogously, completing the proof of the theorem.

Before pursuing estimates on w independent of δ (cf. § 3) we compile some additional technical lemmas regarding the problems (2.13) and (2.16). Note that the formal linearization of (2.13) about $w = 0$ is

$$\frac{\partial}{\partial x} \varrho \frac{\partial w}{\partial x} + \frac{\partial}{\partial \eta} \varrho \frac{\partial w}{\partial \eta} = \lambda \varrho' w\,. \tag{2.21}$$

The lowest eigenvalue of (2.21) can be obtained by separation of variables and, since ϱ is independent of x, will correspond to a function of η alone. It is just the lowest eigenvalue of the Sturm-Liouville problem

$$\begin{cases} \dfrac{\partial}{\partial \eta} \varrho \dfrac{\partial w}{\partial \eta} = \tilde{\lambda} \varrho' w \\ w(\eta_0) = w(\eta_1) = 0\,. \end{cases} \tag{2.22}$$

For later use we note that the lowest eigenvalue μ of (2.21) is characterized by

$$\mu^{-1} = \sup_{z\in H_k^1} \frac{-\int_{\Omega_k} \varrho' z^2}{\int_{\Omega_k} \varrho |\nabla z|^2} . \tag{2.23}$$

This is a standard result and follows from the same type of arguments as were given in the last proof. An obvious consequence of (2.23) is that

$$\int_{\Omega_k} \varrho |\nabla z|^2 \geqslant \mu \int_{\Omega_k} (-\varrho') z^2 \tag{2.24}$$

for all $z \in H_k^1$. Naturally, the eigenvalue μ viewed as the lowest eigenvalue of (2.22) has an analogous characterization with z in (2.23) being $z = z(\eta)$ and the integrals being taken from η_0 to η_1. As in the nonlinear problem, if $\xi = \xi(\eta)$ denotes an eigenfunction for μ, it has one sign in $\eta_0 < \eta < \eta_1$. Of course μ depends on δ and could be denoted μ_δ. If $\delta = 0$ one can define a weak form of (2.22) by requiring

$$\int_{\eta_0}^{\eta_1} \varrho(\eta) \frac{dw}{d\eta} \frac{d\varphi}{d\eta} d\eta = -\tilde{\lambda} \int_{\eta_0}^{\eta} \varrho(\eta) \frac{d}{d\eta} (w(\eta)\varphi(\eta))\, d\eta \tag{2.25}$$

for arbitrary φ in $\mathring{H}^1([\eta_0, \eta_1])$ with w a fixed element of the same space. One easily verifies that $w = \xi^0$, defined by

$$\xi^0(\eta) = \begin{cases} 1 - \eta/\eta_0; & \eta_0 \leqslant \eta < 0 \\ 1 - \eta/\eta_1; & 0 \leqslant \eta \leqslant \eta_1 , \end{cases} \tag{2.26}$$

is a solution of (2.25) with $\tilde{\lambda} = \mu_0$ defined by

$$\mu_0^{-1} = \frac{\varrho_0 - \varrho_1}{\varrho_0/|\eta_0| + \varrho_1/\eta_1} . \tag{2.27}$$

In the literature treating waves in fluids the velocity $c_\delta = (g/\mu_\delta)^{1/2}$ is called the *critical velocity* and is regarded as the velocity with which « infinitesimal long waves » propagate.

We next obtain some quantitative estimates on ξ and μ when $\delta > 0$. These will be used in this section and in Section 5.

LEMMA 2.2. *Let μ be the lowest eigenvalue of* (2.22) *and $\xi(\eta)$ the associated eigenfunction, normalized so that* $\max \xi = 1$. *Then*

$$\frac{1}{|\eta_0| + \varrho_0 \delta/\varrho_1} \leqslant \xi_\eta \leqslant \frac{1}{|\eta_0|} \qquad \textit{on } \eta_0 \leqslant \eta < 0$$

$$\frac{-1}{\eta_1 - \delta} \leqslant \xi_\eta \leqslant \frac{-1}{\eta_1 - \delta + \varrho_1 \delta/\varrho_0} \qquad \textit{on } \delta < \eta < \eta_1$$

$$\frac{\varrho_1}{\varrho_0} \frac{-1}{\eta_1 - \delta} \leqslant \xi_\eta \leqslant \frac{\varrho_0}{\varrho_1} \frac{1}{|\eta_0|} \qquad \textit{on } 0 \leqslant \eta \leqslant \delta$$

and

$$|\xi - \xi^0| \leqslant \delta \left[\bar{s} + \frac{1}{\eta_1} \right] \qquad \textit{for } \eta_0 \leqslant \eta \leqslant \eta_1$$

where $\bar{s} = \max |\xi_\eta|$ *on* $0 \leqslant \eta \leqslant \delta$.

PROOF. From the equation (2.22) satisfied by (ξ, μ) we conclude that ξ_η is constant on $[\eta_0, 0]$ and on $[\delta, \eta_1]$. Let $s_0 > 0$ and $s_1 < 0$ be the slopes on the respective intervals. We can integrate the equation to obtain

$$\varrho(\eta)\xi_\eta(\eta) - \varrho(\bar{\eta})\xi_\eta(\bar{\eta}) = \mu \int_{\bar{\eta}}^{\eta} \varrho' \xi \, . \tag{2.28}$$

If we let $\bar{\eta} = 0$, then since $\varrho' \xi < 0$

$$\xi_\eta \leqslant \frac{\varrho_0}{\varrho(\eta)} s_0 \leqslant \frac{\varrho_0}{\varrho_1} s_0$$

for $\eta \geqslant 0$. Since $\xi(\eta_0) = 0$, $\xi(0) = -s_0 \eta_0$ and so

$$\xi(\eta) \leqslant -s_0 \eta_0 + \frac{\varrho_0}{\varrho_1} s_0 \eta \qquad \text{for } \eta \geqslant 0 \, .$$

Since $\xi_\eta < 0$ for $\eta > \delta$, the maximum of ξ occurs for some $\bar{\eta}$ in $[0, \delta]$. It follows that

$$-s_0 \eta_0 \leqslant 1 = \xi(\bar{\eta}) \leqslant -s_0 \eta_0 + \frac{\varrho_0}{\varrho_1} s_0 \delta$$

or that

$$\frac{1}{|\eta_0| + (\varrho_0/\varrho_1)\delta} \leqslant s_0 \leqslant \frac{1}{|\eta_0|}$$

which is the first claim of the lemma. Similarly from (2.28) one shows $\xi_\eta \geqslant (\varrho_1/\varrho_0)s_1$ and proceeds to the second inequality in the lemma. The third inequality follows immediately from the bounds on ξ_η. For the final claim we note that $\max|\xi(\eta) - \xi^0(\eta)|$ must occur for $0 \leqslant \eta \leqslant \delta$ since $\xi - \xi^0$ is linear outside that interval. Since ξ and ξ^0 each assume the value 1 in $[0, \delta]$ a simple estimate using the derivative bounds for ξ and ξ^0 yields the final assertion.

LEMMA 2.3. *The lowest eigenvalue $\mu = \mu_\delta$ of* (2.22) *satisfies*

$$\frac{(\varrho_0 - \varrho_1)(1 - \delta/\eta_1)^2}{\varrho_0/|\eta_0| + \varrho_1/\eta_1 + (\delta/\eta_1^2)(\varrho_0 - \varrho_1)} \leqslant \mu_\delta^{-1} \leqslant \frac{\varrho_0 - \varrho_1}{\dfrac{\varrho_0|\eta_0|}{(|\eta_0| + (\varrho_0/\varrho_1)\delta)^2} + \dfrac{\varrho_1(\eta_1 - \delta)}{(\eta_1 - \delta + (\varrho_1/\varrho_0)\delta)^2}}$$

and consequently $\lim_{\delta\to 0} \mu_\delta = \mu_0$. *Further* $\mu_\delta \leqslant \mu(\varrho_0, \varrho_1, \eta_0)$ *if* $0 < \delta \leqslant \eta_1/2$.

PROOF. We use the variational characterization analogous to (2.23). The quotient equals μ^{-1} with $z = \xi$ and we can use the last lemma to estimate the quotient from above. In particular the first two inequalities of the lemma yield a lower bound for $\int \varrho \xi_\eta^2$ with $\max \xi = 1$ and the numerator is estimated above by $\varrho_0 - \varrho_1$. For a lower bound on μ^{-1} we merely use ξ^0 as a trial function. The final assertions are clear from the inequalities.

We next obtain crude bounds on the value λ_i occurring in Theorem 2.1. This is an intermediate step to obtaining the more precise bounds given in Section 5.

LEMMA 2.4. *Let* $\delta_0 = \min(\varrho_1|\eta_0|/4\varrho_0, \varrho_0\eta_1/8\varrho_1, \eta_1/2)$. *Then there are positive constants* l_0, l_1 *such that for* $0 < \delta \leqslant \delta_0$, λ_i *in Theorem* 2.1 *satisfies*

$$l_0 \leqslant \frac{\lambda_i}{\mu} \leqslant l_1; \qquad i = 1, 2 \tag{2.29}$$

where μ is the lowest eigenvalue of (2.2).

PROOF. We let (λ, w) stand for either solution found in Theorem 2.1. After applying the two sides of (2.16) to the vector w one can solve for λ and use Corollary 2.1 to obtain

$$\lambda = \frac{\langle A'(w), w\rangle}{\langle B'(w), w\rangle} \geqslant \sigma_3 \frac{\int_{\Omega_k} \varrho|\nabla w|^2}{-\int_{\Omega_k} \varrho' w^2}. \tag{2.30}$$

However, the Rayleigh quotient of integrals in (2.30) is at least as large as the first eigenvalue of

$$\sigma_3 \operatorname{div}(\varrho \operatorname{grad} w) = \tilde{\lambda}\varrho' w \tag{2.31}$$

with w vanishing at $\eta = \eta_0$, η_1 and $2k$ periodic in x. Comparing (2.31) with (2.21) we see that the lowest eigenvalue is $\tilde{\lambda} = \sigma_3\mu$, μ being the lowest eigenvalue of (2.22). Thus $l_0 = \sigma_3$ will suffice in (2.29).

For an upper bound on λ we use Corollary 2.1 again to obtain

$$\lambda = \frac{\langle A'(w), w\rangle}{\langle B'(w), w\rangle} \leqslant \frac{2\sigma_4\sigma_1^{-1}A(w)}{B(w)} = \frac{2\sigma_4\sigma_1^{-1}R^2}{B(w)} .$$

Now suppose $\lambda = \lambda_1$, so that $w > 0$. If $z = z(x, \eta) \geqslant 0$ is any function in H_k^1 for which $A(z) = R^2$, then since $B(z) \leqslant B(w)$, by virtue of the variational characterization, we see that $\lambda \leqslant 2\sigma_4\sigma_1^{-1}R^2B^{-1}(z)$. We let $z = \alpha\xi(\eta)$ with ξ the function from Lemma 2.2 and $\alpha > 0$ a free parameter. Let $\bar{s}$ denote the largest value of $|\xi_\eta|$ for $0 \leqslant \eta \leqslant \delta$; from Lemma 2.2 one can estimate $\bar{s}$. Since

$$\int_{\Omega_k} \varrho|\nabla z|^2 = 2k\alpha^2 \int_{\eta_0}^{\eta_1} \varrho\xi_\eta^2 = -2k\alpha^2\mu \int_{\eta_0}^{\eta_1} \varrho'\xi^2 \leqslant 2k\alpha^2\mu(\varrho_0 - \varrho_1) ,$$

if we let $\alpha^2 = \alpha_0^2 = R^2(\sigma_2 k\mu(\varrho_0 - \varrho_1))^{-1}$, then according to part 1) of Corollary 2.1 we'll have $A(z) \leqslant R^2$. Hence for an $\alpha \geqslant \alpha_0$ we'll have $A(\alpha\xi) = R^2$ and $B(z) = 2k\alpha^2\int(-\varrho')\xi^2 \geqslant 2k\alpha_0^2(\varrho_0 - \varrho_1)(1 - 2\bar{s}\delta)$. From Lemma 2.2 and the choice of δ, $1 - 2\bar{s}\delta \geqslant \frac{1}{2}$. Finally, using the lower bound for $B(z)$ in the expression estimating λ we find that $\lambda \leqslant 2\mu\sigma_1^{-1}\sigma_2\sigma_4$. One obtains the same estimate for λ_2 using $-z$.

3. – Regularity of w.

The results of this section apply to any solution of $A'(w) = \lambda B'(w)$, equation (2.16). To further simplify notation we'll let $h(x_2) = \lambda(\varrho_0 - \varrho(x_2))$ and have $h'(x_2) \geqslant 0$. We'll use $\bar{h}$ for $\max h = \lambda(\varrho_0 - \varrho_1)$. The equation satisfied by $w = w(x_1, x_2)$ is:

$$-\frac{\partial}{\partial x_1}\varrho(x_2)\, a_1(w_{x_1}, w_{x_2}) - \frac{\partial}{\partial x_2}\varrho(x_2)\, a_2(w_{x_1}, w_{x_2}) = h'w . \tag{3.1}$$

As noted earlier, since ϱ and a are assumed to be C^∞ functions, w is C^∞ on $\bar{\Omega}$. Of course the estimates on the derivatives depend on the size of the derivative of ϱ; i.e., they depend upon δ. Our aim in this section is to establish estimates which do not depend on δ or on the period $2k$.

Let σ, x_1', x_1'' be real numbers with $x_1' < x_1''$ and $\sigma > 0$. Suppose ζ is a C^∞ function of x_1 such that $\zeta = 1$ for $x_1' \leqslant x_1 \leqslant x_1''$; $\xi = 0$ for $x_l \leqslant x_1'' - 1/\sigma$ or $x_1 \geqslant x_1'' + 1/\sigma$; and $0 < \zeta < 1$ for all other x_1. We can assume $|\zeta_{x_1}| \leqslant 2\sigma$. Let $\Omega'' = \{(x_1, x_2) \in \Omega | \zeta = 1\}$ and $\Omega' = \{(x_1, x_2) \in \Omega | \zeta > 0\}$. A typical estimate in this section will be of interior type; i.e., an estimate of a function on Ω'' in terms of data on Ω'.

It will be crucial for our arguments to fully utilize a divergence structure for (3.1). To that end we write the equation as

$$-\frac{\partial}{\partial x_i}\varrho(x_2)\,a_i(w_{x_1}, w_{x_2}) = \frac{\partial}{\partial x_2}(hw) - hw_{x_2}, \tag{3.2}$$

using the summation convention. Some of the estimates we make would simplify if Ω'' were taken to be a period region Ω_k. However, it will be useful later to have estimates on arbitrary regions of the type Ω''. We'll let $b = \max a_{ij}(p_1, p_2)$ where the maximum is taken over all i, j, p_1, p_2. Recall that $a_{ij}\xi_i\xi_j \geqslant \nu\xi_i\xi_i$ with ν independent of p_1, p_2. We use C, C_1, etc. to denote constants, differing from one context to another, and indicate what parameters they depend on.

LEMMA 3.1. *Suppose $w \in H_k^1 \cap C_k^2(\Omega)$ is a solution of (3.2). Then*

$$\int_{\Omega''} |\nabla w_{x_1}|^2 \leqslant C \int_{\Omega'} |\nabla w|^2 \tag{3.3}$$

where $C = C(\nu, b, \bar{h}, \varrho_0, \varrho_1, \sigma)$. As $\sigma \to 0$ C approaches $\bar{h}^2/\varrho_1\nu^2$.

PROOF. Using the cutoff function ζ we conclude from (3.2) that

$$\int_{\Omega'} \frac{\partial}{\partial x_i}\left(\varrho(x_2)\,a_i\right)\frac{\partial}{\partial x_1}\zeta^2 w_{x_1} = -\int_{\Omega'}\left(\frac{\partial}{\partial x_2}(hw) - \varrho w_{x_2}\right)\frac{\partial}{\partial x_1}\zeta^2 w_{x_1}. \tag{3.4}$$

In integrating a term such as $(\partial/\partial x_2)(\varrho a_2)(\partial/\partial x_1)\zeta^2 w_{x_1}$ two integrations by parts will interchange the indices in the derivatives. Two benefits result. We avoid applying an x_2 derivative to $\varrho(x_2)$ and can express the resulting integrand as $\zeta^2\varrho a_{2j}(\partial/\partial x_j)w_{x_1}(\partial/\partial x_2)w_{x_1}$. Effecting such interchanges and carrying out the resulting derivatives we can write (3.4) as

$$\int_{\Omega'} \zeta^2 \varrho a_{ij} \frac{\partial}{\partial x_i} w_{x_1} \frac{\partial}{\partial x_j} w_{x_1} = \int_{\Omega'} \Big[\zeta^2 h \left(w_{x_2} \frac{\partial}{\partial x_1} w_{x_1} - w_{x_1} \frac{\partial}{\partial x_2} w_{x_1} \right) + \\ + 2\zeta\zeta_{x_1} w_{x_1} \left(h w_{x_2} - \varrho a_{11} \frac{\partial}{\partial x_1} w_{x_1} - \varrho a_{12} \frac{\partial}{\partial x_2} w_{x_1} \right) \Big].$$

Using the standard inequality $ab \leqslant \varepsilon a^2 + (4\varepsilon)^{-1} b^2$ and the ellipticity we obtain

$$\varrho_1 \nu \int_{\Omega'} \zeta^2 |\nabla w_{x_1}|^2 \leqslant \varepsilon(\bar{h} + 4b\sigma\varrho_0) \int_{\Omega'} \zeta^2 |\nabla w_{x_1}|^2 + \left(\frac{\bar{h}}{4\varepsilon} + \frac{b\sigma\varrho_0}{\varepsilon} + 2\bar{h}\sigma \right) \int_{\Omega'} |\nabla w|^2.$$

Now for ε appropriately chosen we find

$$\int_{\Omega'} \zeta^2 |\nabla w_{x_1}|^2 \leqslant C \int_{\Omega'} |\nabla w|^2 \tag{3.5}$$

and since $\zeta^2 = 1$ on Ω'', we obtain (3.3) with C depending on the indicated parameters. The choice $\varepsilon = \varrho_1 \nu / 2\bar{h}$ gives the desired limiting behavior as $\sigma \to 0$.

We now have an L^2 bound on ∇w_{x_1} in terms of a controllable quantity; i.e., if we fix the size of A, say $A(w) = R^2$, then according to Corollary (2.1), $\int_{\Omega_k} |\nabla w|^2$ is at most a constant times R^2. Next let $v = w_{x_1}$ and apply $\partial/\partial x_1$ to both sides of (3.2) to obtain

$$\frac{\partial}{\partial x_i} \varrho(x_2)\, a_{ij}(w_{x_1}, w_{x_2}) \frac{\partial v}{\partial x_j} = -\frac{\partial}{\partial x_2}(hv) + h v_{x_2}. \tag{3.6}$$

It follows from Theorem 8.29 of [15] that v satisfies a Hölder condition. More precisely

$$\|v\|_{C^\alpha(\Omega'')} \leqslant C\big(\|v\|_{L^2(\Omega')} + \|hv\|_{L^4(\Omega')} + \|h v_{x_2}\|_{L^2(\Omega')}\big) \tag{3.7}$$

where $\alpha > 0$ and $C > 0$ depend on ν, b, ϱ_0, ϱ_1, and σ, and where Ω' is defined as above. We have

$$\|hv\|_{L^4(\Omega')} \leqslant \bar{h} \|v\|_{L^4(\Omega')} \leqslant \bar{h} C \|v\|_{H^1(\Omega')}$$

with C independent of the length of Ω' in the x_1 direction ([18], Lemma 5.14). Poincare's inequality gives

$$\pi^2 \int_{\Omega'} v^2 \leqslant \int_{\Omega'} |\nabla v|^2$$

and hence from (3.7) and Lemma 3.1 we obtain

LEMMA 3.2. *If $w \in H_k^1$ is a solution of* (3.2) *and $v = w_{x_1}$ then*

$$\|v\|_{C^\alpha(\Omega'')} \leq C\left(\int_{\Omega'} |\nabla w|^2\right)^{1/2} \tag{3.8}$$

with $C = C(\nu, b, \bar{h}, \varrho_0, \varrho_1, \sigma)$.

We are now in a position to obtain a preliminary L^∞ bound on w_{x_2}.

LEMMA 3.3. *Let w be a solution of* (3.2), *let $\Omega' = \{(x_1', x_2) \in \Omega \,|\, |x_1' - x_1| < 2\}$, and let $R' = \left(\int_{\Omega'} |\nabla w|^2\right)^{1/2}$. Let r be the cutoff parameter in Lemma 2.2. Then there is a constant C, depending on η_0 and the parameters entering Lemma 3.2, and there is a constant $M \leq \sqrt{2}r + CR'$ such that*

$$|w_{x_2}(x_1, x_2)| \leq M. \tag{3.9}$$

PROOF. For $|x_2| \geq \min(|\eta_0|, \eta_1)$, $\varrho' = 0$ and (3.9) follows as did the previous lemma. For the remaining x_2 we can suppose that the point at which we want to estimate w_{x_2} is the origin of new coordinates $(\tilde{x}_1, \tilde{x}_2)$, but for ease of writing omit the tildas and use (x_1, x_2) in the proof. We let $\Omega'' = \{(x_1, x_2) \in \Omega' \,|\, |x_1| < 1\}$. Using a cutoff function ζ with $\zeta(x_1) = 1$, $\zeta(s_0) = 0$ we have

$$w(x_1, x_2) = \int_{\eta_0}^{x_2}\int_{s_0}^{x_1} \frac{\partial}{\partial x_1} \zeta w_{x_2}$$

and a simple estimate using Lemma 3.1 shows that

$$m = \|w\|_{L^\infty(\Omega'')} \leq CR'. \tag{3.10}$$

From lemma 3.2 $|v(0, 0)|$ and $\|v\|_{C^\alpha(\Omega'')}$ are likewise bounded by C^1R'.

We will use a comparison theorem in $\tilde{\Omega} = \Omega'' \cap \{x_2 > 0\}$ for the quasilinear operator Q defined by

$$Qw = \frac{\partial}{\partial x_i} \hat{\varrho}(x_2)\, a_i(w_{x_1}, w_{x_2})$$

where $\hat{\varrho}$ is the density based at the new origin. We'll use only the property $\hat{\varrho}_{x_2} \leq 0$ so a more precise description of $\hat{\varrho}$ seems unnecessary. Since w is a solution of (3.1), $Qw = \lambda \hat{\varrho}_{x_2} w$. Consider a function u defined by

$$u(x_1, x_2) = w(0, 0) + A_1 x_1 + A_\alpha \operatorname{Re}(x_2 + ix_1)^{1+\alpha} + A_2 x_2 \tag{3.11}$$

where α is the Hölder exponent from (3.8), $A_1 = w_{x_1}(0, 0) = v(0, 0)$, and the constants A_α, A_2 are to be determined so that Theorem 9.2 of [15] is applicable.

Note that with $\theta = \tan^{-1} x_1/x_2$,

$$\frac{\partial}{\partial x_2} \operatorname{Re}(x_2 + ix_1)^{1+\alpha} = (1+\alpha)(x_2^2 + x_1^2)^{(\alpha-1)/2} x_2 \cos\big((1+\alpha)\,\theta\big) - \\ - (1+\alpha)(x_2^2 + x_1^2)^{(\alpha-1)/2} x_1 \sin\big((1+\alpha)\,\theta\big)$$

and so

$$\left|\frac{\partial}{\partial x_2} \operatorname{Re}(x_2 + ix)^{1+\alpha}\right| \leqslant (1+\alpha)\, d^\alpha \tag{3.12}$$

where $d^2 = x_1^2 + x_2^2$. In $\tilde{\Omega}$, $d \leqslant \sqrt{2}$. Hence if

$$|A_2| - (1+\alpha)\,|A_\alpha|\,2^{\alpha/2} > \sqrt{2}r \tag{3.13}$$

then $|u_{x_2}| > \sqrt{2}r$ and from (2.14) we see that Qu takes the simple form

$$Qu = \hat{\varrho}\,\Delta u + \hat{\varrho}_{x_2} u_{x_2} = \hat{\varrho}_{x_2}\left(A_\alpha \frac{\partial}{\partial x_2} \operatorname{Re}(x_2 + ix_1)^{1+\alpha} + A_2\right). \tag{3.14}$$

Here we make essential use of the fact that u is harmonic and must have $|\nabla u| > \sqrt{2}r$ to exploit this property.

Since $\hat{\varrho}_{x_2} \leqslant 0$, to have $Qu \geqslant Qw$ we require

$$A_\alpha \frac{\partial}{\partial x_2} \operatorname{Re}(x_2 + ix_1)^{1+\alpha} + A_2 - \lambda w \leqslant 0\,,$$

which will be satisfied if

$$(1+\alpha)\,|A_\alpha|\,2^{\alpha/2} + A_2 + \lambda m \leqslant 0\,. \tag{3.15}$$

Next, consider $u - w$ on the boundary $\partial\tilde{\Omega}$ starting where $x_2 = 0$ and $0 \leqslant x_1 \leqslant 1$. Using the C^α estimate (3.8) on $v = w_{x_1}$ we have

$$w_{x_1}(x_1, 0) - w_{x_1}(0, 0) \geqslant -CR' x_1^\alpha \tag{3.16}$$

and, since $w_{x_1}(0, 0) = A_1$,

$$u_{x_1}(x_1, 0) - w_{x_1}(0, 0) = A_\alpha(1+\alpha)\,\cos\left((1+\alpha)\frac{\pi}{2}\right) x_1^\alpha.$$

Then

$$w_{x_1}(x_1, 0) - u_{x_1}(x_1, 0) \geqslant \left[- CR' - A_\alpha(1+\alpha)\cos(1+\alpha)\frac{\pi}{2}\right] x_1^\alpha .$$

But $\cos(1+\alpha)(\pi/2) < 0$ so for

$$A_\alpha \geqslant \frac{CR'}{(1+\alpha)|\cos(1+\alpha)(\pi/2)|}, \tag{3.17}$$

$w_{x_1} - u_{x_1} \geqslant 0$ for $0 \leqslant x_1 \leqslant 1$. Since $w = u$ at $(0, 0)$, (3.17) will guarantee $w \geqslant u$ where $x_2 = 0$ and $0 \leqslant x_1 \leqslant 1$. Similarly, (3.17) implies $w \geqslant u$ for $x_2 = 0$ and $-1 \leqslant x_1 \leqslant 0$.

The set $\tilde{\Omega}$ has the form $(-1, 1) \times (0, D)$ for some $D \leqslant 1$. We choose $\bar{x}_2 \in (0, D)$ so that if $\bar{\theta} = \tan^{-1}(\bar{x}_2^{-1})$, then $\cos((1+\alpha)\bar{\theta}) = \hat{c} < 0$. We will choose $A_2 \leqslant 0$ in (3.11) and hence if

$$A_\alpha \geqslant \frac{2m + |A_1|}{|\hat{c}|}, \tag{3.18}$$

then $u \leqslant -m \leqslant w$ where $0 \leqslant x_2 \leqslant \bar{x}_2$ and $x_1 = 1$ (or -1 by symmetry). One verifies that on the remainder of the boundary of the rectangle,

$$d^{1+\alpha}\cos(1+\alpha)\theta \leqslant D^{1+\alpha}.$$

Hence $u \leqslant w$ provided

$$2m + A_1 + A_\alpha D^{1+\alpha} + A_2 \bar{x}_2 \leqslant 0$$

or, since $D \leqslant 1$, provided

$$-A_2 \geqslant (\bar{x}_2)^{-1}(2m + A_1 + A_\alpha) . \tag{3.19}$$

First we choose A_α to satisfy (3.17) and (3.18). Then we choose $A_2 < 0$ so that (3.13), (3.15) and (3.19) are satisfied.

Then $Qu \geqslant Qw$ in $\tilde{\Omega}$ and $u \leqslant w$ on the boundary. The remaining hypotheses of the comparison theorem ([15], p. 207) regard the form of Q and are readily verified. It follows that $u \leqslant w$ and hence $w_{x_2}(0, 0) \geqslant u_{x_2}(0, 0) = A_2$. In a completely analogous way one shows $w_{x_2}(0, 0) \leqslant |A_2|$ and thus $M = |A_2|$ serves in the lemma. One readily sees that M depends only on the parameters indicated in the statement of the lemma, recalling that $\lambda = \bar{h}(\varrho_0 - \varrho_1)^{-1}$. Moreover as R' approaches 0, so do m and A_1, allowing the choice of A_α to approach zero and the choice of A_2 to approach zero in proportion with R', except in (3.13) where $|A_2| \leqslant \sqrt{2}r + CR'$ will be compatible with satisfying (3.13) for $R' > 0$. This completes the proof of the lemma.

Again using ζ, Ω' and Ω'' introduced at the beginning of this section we show

LEMMA 3.4. *There exist positive constants r_1 and R_0 such that if A is defined using a cutoff at $r \leqslant r_1$ and $w \in H_k^1 \cap C_k^3(\bar{\Omega})$ is a solution of* (3.1) *with $A(w) = R^2$, $R < R_0$, then $v = w_{x_1}$ satisfies*

$$\int_{\Omega''} |\nabla v_{x_1}|^2 \leqslant C \int_{\Omega'} |\nabla w|^2 \tag{3.20}$$

where $\Omega'' \subset \Omega' \subset \Omega_k$. Here R_0 and C depend on ν, b, $\bar{h}$, ϱ_0, ϱ_1, η_0, and σ. For all sufficiently small σ one can take $C = 4\bar{h}^4/\varrho_0 \varrho_1 \nu^4$.

PROOF. Recall, it is assumed that $r_1 \leqslant r_0$ from Lemma 2.1. We multiply equation (3.6) by $(\partial/\partial x_1)\zeta^2 v_{x_1}$ and integrate (all integrals will be over Ω' unless otherwise stated) to obtain

$$\int \frac{\partial}{\partial x_i}\left[\varrho(x_2)\, a_{ij}(w_{x_1}, w_{x_2}) \frac{\partial v}{\partial x_j}\right] \frac{\partial}{\partial x_1} \zeta^2 v_{x_1} = \int \left[-\frac{\partial}{\partial x_2}(hv) \frac{\partial}{\partial x_1} \zeta^2 v_{x_1} + hv_{x_2} \frac{\partial}{\partial x_1} \zeta^2 v_{x_1}\right].$$

As in the proof of Lemma 3.1 we integrate by parts to obtain

$$\int \frac{\partial}{\partial x_1}\left[\varrho(x_2)\, a_{ij}(w_{x_1}, w_{x_2}) \frac{\partial v}{\partial x_j}\right] \cdot \frac{\partial}{\partial x_i} \zeta^2 v_{x_1} = \int \left[-\frac{\partial}{\partial x_1}(hv) \frac{\partial}{\partial x_2} \zeta^2 v_{x_1} + hv_{x_2} \frac{\partial}{\partial x_1} \zeta^2 v_{x_1}\right]$$

or

$$\begin{aligned} \int \zeta^2 \varrho(x_2)\, a_{ij} \frac{\partial}{\partial x_j} v_{x_1} \frac{\partial}{\partial x_i} v_{x_1} = \\ = -\int \varrho(x_2)\left(a_{ijk}\, w_{x_k x_1} v_{x_j} \frac{\partial}{\partial x_i} \zeta^2 v_{x_1}\right) - \int \varrho(x_2)\, a_{1j}\, v_{x_1 x_j} \cdot 2 v_{x_1} \zeta \zeta_{x_1} + \\ + \int [-\zeta^2 h v_{x_1} v_{x_1 x_2} + \zeta^2 h v_{x_2} v_{x_1 x_1} + 2 h v_{x_2} \zeta \zeta_{x_1} v_{x_1}] \end{aligned} \tag{3.21}$$

where we have used a_{ijk} to denote a third derivative of the function a. Using ellipticity we see that the left side of (3.21) is at least νJ, where $J = \int \zeta^2 \varrho |\nabla v_{x_1}|^2$. We will show that the right side of (3.21) can be bounded by $\frac{1}{2}\nu J$ plus a multiple of $\int |\nabla w|^2$. Then (3.20) will follow easily.

We will use the form of a_{ijk} given in Lemma 2.1. That is, if we let $a_{ijk} = a_{ijk}(0, 0) + \alpha_{ijk}$, we know the values at $(0, 0)$ and that α_{ijk} is of order r. We let α be a bound for $|\alpha_{ijk}|$ for any choice of indices. Separating out the expressions $a_{ijk}(0, 0)$ which do not vanish we find that the right side of (3.21) is at most

$$(3.22)\qquad 2\left|\int \varrho(x_2)\cdot 2v_{x_2}v_{x_1}\frac{\partial}{\partial x_1}\zeta^2 v_{x_1}\right| + \left|\int \varrho(x_2)\,2v_{x_1}v_{x_1}\frac{\partial}{\partial x_2}\zeta^2 v_{x_1}\right| + \\ + \left|\int \varrho(x_2)\,6v_{x_2}v_{x_2}\frac{\partial}{\partial x_2}\zeta^2 v_{x_1}\right| + \alpha\int \zeta^2\varrho\sum_{i,j,k}|v_{x_k}v_{x_j}v_{x_1x_i}| + \\ + 2\alpha\int \zeta\varrho\sum_{j,k}|\zeta_{x_1}v_{x_k}v_{x_j}v_{x_1}| + 2b\int \zeta\varrho\sum_{j}|\zeta_{x_1}v_{x_1x_j}v_{x_1}| + \\ + \bar{h}\int \zeta^2(v_{x_1}v_{x_1x_2}| + |v_{x_2}v_{x_1x_1}|) + 2\bar{h}\int \zeta|\zeta_{x_1}v_{x_1}v_{x_2}|\,.$$

Let us denote the eight terms in (3.22) by $T_1, \ldots, T_8$ respectively. To estimate T_1 we write

$$\int \varrho(x_2)\,v_{x_2}v_{x_1}\frac{\partial}{\partial x_1}\zeta^2 v_{x_1} = \int \varrho(x_2)\,v_{x_2}\left[\frac{\partial}{\partial x_1}\left(\frac{\zeta^2 v_{x_1}^2}{2}\right) + \zeta\zeta_{x_1}v_{x_1}^2\right]$$

from which, after one integration by parts, we find

$$(3.23)\qquad T_1 \leqslant 2\left|\int \varrho(x_2)\,v_{x_1x_2}\zeta^2 v_{x_1}^2\right| + 4\left|\int \varrho(x_2)\,v_{x_2}\zeta\zeta_{x_1}v_{x_1}^2\right|\,.$$

Since T_2 is equal to the first term on the right in (3.23)

$$(3.24)\qquad T_1 + T_2 \leqslant \varepsilon_1\int \zeta^2\varrho v_{x_2x_1}^2 + \frac{4}{\varepsilon_1}\int \zeta^2\varrho v_{x_1}^4 + 4\int \zeta^2\varrho v_{x_1}^4 + \sigma^2\int v_{x_2}^2\,.$$

Here and below we use the inequality $ab \leqslant \varepsilon a^2 + (1/4\varepsilon)\,b^2$. To estimate T_3 we write

$$\int \varrho(x_2)\,v_{x_2}^2\frac{\partial}{\partial x_2}\zeta^2 v_{x_1} = \frac{1}{3}\int \zeta^2\varrho\frac{\partial}{\partial x_1}v_{x_2}^3 = -\frac{2}{3}\int \zeta\zeta_{x_1}\varrho v_{x_2}^3$$

and so

$$(3.25)\qquad T_3 \leqslant 4\left|\int \zeta\zeta_{x_1}\varrho v_{x_2}^3\right| \leqslant \varepsilon_2\int \zeta^2\varrho v_{x_2}^4 + \frac{4\sigma^2}{\varepsilon_2}\int \varrho v_{x_2}^2\,.$$

If we estimate the terms T_4 through T_8 in a straightforward way and combine the result with (3.24) and (3.25) we find

$$\begin{aligned}(3.26)\qquad \nu J \leqslant \left(\varepsilon_1 + \varepsilon_3\alpha + \varepsilon_4 b\sigma + \varepsilon_5\frac{\bar h}{\varrho_0}\right)J + \\ + \left(4\sigma + \frac{4}{\varepsilon_1} + \frac{2\alpha}{\varepsilon_3} + 4\alpha\sigma\right)\int \zeta^2\varrho v_{x_1}^4 + \\ + \left(\varepsilon_2 + \frac{2\alpha}{\varepsilon_3} + 4\alpha\sigma\right)\int\zeta^2\varrho v_{x_2}^4 + \\ + \left(\sigma + \frac{4\sigma^2}{\varepsilon_2}\varrho_0 + 2\alpha\sigma\varrho_0 + \frac{b\sigma}{\varepsilon_4} + \frac{\bar h}{4\varepsilon_5} + \bar h\sigma\right)\int |\nabla v|^2.\end{aligned}$$

We allow distinct ε_i's in the inequalities to maintain flexibility and to « label » terms. Continuing, we have

$$\begin{aligned}\int\zeta^2\varrho v_{x_1}^4 &= -\int v\frac{\partial}{\partial x_1}(\zeta^2\varrho v_{x_1}^3) = \\ &= -\int 2v\zeta\zeta_{x_1}\varrho v_{x_1}^3 + 3v\zeta^2\varrho v_{x_1}^2 v_{x_1x_1}.\end{aligned}$$

So, supposing $\max|v| \leqslant \bar v$ on Ω' (cf. (3.8)), we obtain

$$\int\zeta^2\varrho v_{x_1}^4 \leqslant \varepsilon_6\bar v\int\zeta^2\varrho v_{x_1}^4 + \frac{\bar v\sigma^2}{\varepsilon_6}\int\varrho v_{x_1}^2 + 3\varepsilon_7\bar v\int\zeta^2\varrho v_{x_1}^4 + \frac{3\bar v}{4\varepsilon_7}\int\zeta^2\varrho v_{x_1x_1}^2$$

and thus

$$(3.27)\qquad (1 - \varepsilon_6\bar v - \varepsilon_7\bar v)\int\zeta^2\varrho v_{x_1}^4 \leqslant \frac{\bar v\sigma^2}{\varepsilon_6}\int\varrho v_{x_1}^2 + \frac{3\bar v}{4\varepsilon_7}\int\zeta^2\varrho v_{x_1x_1}^2.$$

Next, we have

$$\begin{aligned}\int\zeta^2\varrho v_{x_2}^4 &= \int\zeta^2\varrho w_{x_2x_1}^3\cdot w_{x_2x_1} = \\ &= -\int w_{x_2}\frac{\partial}{\partial x_1}(\zeta^2\varrho w_{x_2x_1}^3) = \\ &= -\int w_{x_2}(2\zeta\zeta_{x_1}\varrho w_{x_2x_1}^3 + 3\zeta^2\varrho w_{x_2x_1}^2\cdot w_{x_2x_1x_1}).\end{aligned}$$

Letting M be an upper bound for $|w_{x_2}|$ (cf. Lemma 3.3) we see that

$$\int\zeta^2\varrho v_{x_2}^4 \leqslant \varepsilon_8\sigma M\int\zeta^2\varrho w_{x_1x_2}^4 + \frac{\sigma M}{\varepsilon_8}\int\varrho w_{x_1x_2}^2 + 3\varepsilon_9 M\int\zeta^2\varrho w_{x_1x_2}^4 + \frac{3M}{4\varepsilon_9}\int\zeta^2\varrho w_{x_1x_1x_2}^2$$

and thus

$$(3.28)\qquad (1-\varepsilon_8\sigma M-3\varepsilon_9 M)\int\zeta^2\varrho v_{x_2}^4\leqslant\frac{\sigma M}{\varepsilon_8}\int\varrho v_{x_2}^2+\frac{3M}{4\varepsilon_9}\int\zeta^2\varrho v_{x_1x_2}^2 .$$

We let $\varepsilon_6=\varepsilon_7=(8\bar{v})^{-1}$, $\varepsilon_8=(8\sigma M)^{-1}$, and $\varepsilon_9=(8M)^{-1}$. Then

$$(3.29)\qquad \int_{\Omega'}\zeta^2\varrho v_{x_1}^4\leqslant 16\sigma^2\bar{v}^2\int_{\Omega'}\varrho v_{x_1}^2+12\bar{v}^2\int_{\Omega'}\zeta^2\varrho v_{x_1x_1}^2$$

and

$$(3.30)\qquad \int_{\Omega'}\zeta^2\varrho v_{x_2}^4\leqslant 16\sigma^2 M^2\int_{\Omega'}\varrho v_{x_2}^2+12M^2\int_{\Omega'}\zeta^2\varrho v_{x_1x_2}^2 .$$

Since $\Omega'\subset\Omega_k$, if $A(w)\leqslant R_0^2$, then by Corollary 2.1 $\|w\|^2_{H_k^1}\leqslant 2\sigma_1^{-1}R_0^2$ and by Lemma 3.2 applied to Ω' and Ω_k, $\bar{v}\leqslant CR_0$. Similarly, we can apply Lemma 3.3 to estimate $|w_{x_2}|$. For a suitable R_0, $\sqrt{2}r+CR'\leqslant\sqrt{2}r+\tilde{C}R_0\leqslant 2r$. Then if r is restricted to satisfy $r\leqslant r_1$, the bound M for $|w_{x_2}|$ becomes $2r_1$. To simplify matters we will assume R_0 is chosen so that the upper bound $\bar{v}=2r_1$ as well. If (3.29) and (3.30) are used to estimate $\int\zeta^2\varrho v_{x_i}^4$ $(i=1,2)$ in (3.26) and the choices $\varepsilon_3=4\sqrt{6}r_1$, $\varepsilon_5=\varrho_0\nu_k 8\bar{h}$, $\varepsilon_1=8\sqrt{3}r_1$ are made, then an inequality

$$(3.31)\qquad \left(\nu-\frac{\nu}{8}-16\sqrt{3}r_1-8\sqrt{6}r_1\alpha+O(r_1^2\sigma)\,O(\varepsilon_4\sigma)\right)J\leqslant \leqslant\left(\frac{2\bar{h}^2}{\varrho_0\nu}+O(\sigma)+O\left(\frac{\sigma}{\varepsilon_4}\right)\right)\int_{\Omega'}|\nabla v|^2$$

results. Since $\alpha=O(r)$, if we choose r_1 and ε_4 appropriately (cf. Remark 2 following Lemma 2.1), we obtain a bound on J which, combined with Lemma 3.1, yields (3.20). In the limit $\sigma\to 0$ we can obtain

$$\frac{\nu}{2}J\leqslant\frac{2\bar{h}^2}{\varrho_0\nu}\int_{\Omega'}|\nabla v|^2$$

from (3.31) which combined with the limiting constant in Lemma 3.1, produces a constant $C=4\bar{h}^4/\varrho_0\varrho_1\nu^4$ in (3.20). This completes the proof of Lemma 3.4.

We now have L^2 bounds on $D_{x_1}^iD_{x_2}^jw$ for $i+j\leqslant 3$ and $j=0$ or 1 and can make the bounds small on Ω'' by making $\int|\nabla w|^2$ small over $\Omega'\supset\Omega''$. We next obtain sharper information regarding w_{x_2}. We'll use b' for an upper bound on $|a_{ijk}|$ over all indices and over all p_1, p_2.

LEMMA 3.5. *There are positive constants $\tilde{r}$ and R_0 (cf. Lemma 3.4) such that if the cutoff r satisfies $r = \tilde{r}$ and $w \in H_k^1 \cap C_k^3(\bar{\Omega})$ is a solution of (3.1) with $A(w) = R^2$, $R < R_0$, then*

$$\|w_{x_2}\|_{L^\infty(\Omega'')} \leqslant C\left(\int_{\Omega'} |\nabla w|^2\right)^{1/2} \tag{3.32}$$

for $\Omega'' \subset \Omega' \subset \Omega_k$. Further if $\tilde{\Omega}$ denotes either component of $\{(x_1, x_2) \in \Omega'' \,|\, \varrho_{x_2} = 0\}$ then with α from Lemma 3.2

$$\|w_{x_2}\|_{C^\alpha(\tilde{\Omega})} \leqslant C'\left(\int_{\Omega'} |\nabla w|^2\right)^{1/2}. \tag{3.33}$$

Here R_0, C, and C' depend on ν, b, b', $\bar{h}$, ϱ_0, ϱ_1, η_0, and σ.

PROOF. Assume $r \leqslant r_1$ so that Lemma 3.4 applies. We integrate form (3.2) of the equation between points (s, t) and $(s, t + \tau)$ in Ω'' obtaining

$$-\int_t^{t+\tau} \frac{\partial}{\partial x_1} \varrho(x_2)\, a_1(\nabla w)|_{x_1 = s}\, dx_2 - \varrho(x_2)\, a_2\big(w_{x_1}(s, x_2),\, w_{x_2}(s, x_2)\big)\big|_t^{t+\tau} = \tag{3.34}$$
$$= h(x_2)\, w(s, x_2)\big|_t^{t+\tau} - \int_t^{t+\tau} h(x_2)\, w_{x_2}(s, x_2)\, dx_2 .$$

Using the function ζ introduced at the outset of this section and assuming $\zeta(s) = 1$, $\zeta(s_0) = 0$ we have

$$\int_t^{t+\tau} \frac{\partial}{\partial x_1} \varrho a_1(\nabla w)|_{x_1 = s}\, dx_2 = \int_t^{t+\tau}\int_{s_0}^{s} \frac{\partial}{\partial x_1} \zeta^2 \frac{\partial}{\partial x_1} \varrho(x_2)\, a_1(\nabla w)\, dx_1\, dx_2 = \tag{3.35}$$
$$= \int_t^{t+\tau}\int_{s_0}^{s} 2\zeta\zeta_{x_1} \varrho a_{1j}\, w_{x_1 x_j} dx_1\, dx_2 + \int_t^{t+\tau}\int_{s_0}^{s} \zeta^2 \varrho(a_{1j}\, w_{x_1 x_1 x_j} + a_{1jk}\, w_{x_1 x_j} w_{x_1 x_k})\, dx_1\, dx_2 .$$

In terms of $v = w_{x_1}$ we obtain

$$\left|\int_t^{t+\tau} \frac{\partial}{\partial x_1} \varrho a_1(\nabla w)\Big|_{x_1 = s} dx_2\right| \leqslant \tag{3.36}$$
$$\leqslant \varrho_0 \int_t^{t+\tau}\int_{s_0}^{s} \left[2\sqrt{2}\sigma b |\nabla v| + \sqrt{2} b |\nabla v_{x_1}| + 2b'\zeta^2 |\nabla v|^2\right] dx_1\, dx_2 .$$

However $|\nabla v|$, $|\nabla v_{x_1}|$, and $|\nabla v|^2$ are all in L^2 on $\Omega'' = [t, t+\tau] \times [s_0, s]$ with L^2 norms bounded by a multiple of $R' = \left(\int_{\Omega'} |\nabla w|^2\right)^{1/2}$ for $\Omega'' \subset \Omega' \subset \Omega_k$; cf. (3.3), (3.20), (3.29), and (3.30) (we assume $R \leqslant R_0$, so $(R')^2 \leqslant CR_0R'$). Using the Schwarz inequality on the double integral in (3.36) we conclude that

$$\left| \int_t^{t+\tau} \frac{\partial}{\partial x_1} \varrho a_1(\nabla w)\Big|_{x_1 = s} dx_2 \right| \leqslant \tau^{1/2} C_1 R' \tag{3.37}$$

where C_1 arises from the earlier estimates just listed and the constants in (3.36). In an analogous way one shows

$$\left| \int_t^{t+\tau} h(x_2)\, w_{x_2}(s, x_2)\, dx_2 \right| \leqslant \tau^{1/2} C_2 R' \tag{3.38}$$

and hence from (3.34)

$$\left| [\varrho(x_2)\, a_2(w_{x_1}(s, x_2), w_{x_2}(s, x_2)) + h(x_2)\, w(s, x_2)]\big|_{x_2 = t}^{t+\tau} \right| \leqslant \tau^{1/2} C_3 R'. \tag{3.39}$$

In (3.39) we evaluate derivatives at two points on the line where $x_1 = s$. At one such point, say $(s, t+\tau)$, we may assume $w_{x_2} = 0$ for w has a maximum (or minimum) on the line. From (2.17) we see that at $(s, t+\tau)$ $|a_2(w_{x_1}, w_{x_2})| \leqslant \frac{1}{2} w_{x_1}^2$. As in the proof of Lemma 3.3 $|w_{x_1}| \leqslant C_4 R'$ and $|w| \leqslant C_5 R'$ for all points on the line $x_1 = s$. Thus from (3.39)

$$|a_2(w_{x_1}(s, t), w_{x_2}(s, t))| \leqslant \frac{1}{\varrho_1} \left[\frac{\varrho_0}{2} (C_4 R')^2 + 2\bar{h} C_5 R' + \tau^{1/2} C_3 R' \right]. \tag{3.40}$$

Since $\xi_r = 0$ for $p_1^2 + p_2^2 \geqslant 2r^2$ we see from (2.17) that

$$a_2(p_1, p_2) = p_2(1 + O(r)) + O(p_1^2) \tag{3.41}$$

and by restricting r, if necessary, to $r = \tilde{r} \leqslant r_1$, one can assume $|O(r)| < \frac{1}{2}$. Setting $p_1 = w_{x_1}(s, t)$, one can assume $|O(p_1^2)| < C_6 (R')^2$, according to Corollary 2.1 and Lemma 3.2. Then, from (3.40) and (3.41)

$$|w_{x_2}(s, t)| \leqslant CR',$$

that is, (3.32) holds.

Now consider a region $\tilde{\Omega} = \Omega'' \cap \{x_2 < 0\}$ or $\Omega'' \cap \{x_2 > \delta\}$ where $\varrho_{x_2} = 0$ (it would be enough to have ϱ_{x_2} bounded for this argument). Since

$h_{x_2} = 0$ in $\tilde{\Omega}$, starting with the form (3.1) of the equation for w and proceeding as above we arrive at

$$\left| a_2\big(w_{x_1}(s, x_2),\, w_{x_2}(s, x_2)\big)\Big|_{x_2=t}^{t+\tau} \right| \leqslant \tau^{1/2} C_1 R'. \tag{3.42}$$

Let $w_{x_i}(s, t) = q_2$ and $w_{x_i}(s, t + \tau) = q_i + p_i$. From Lemma 3.2 we conclude that $|p_1| \leqslant \tau^{\alpha} C_2 R'$. We would like to conclude that

$$|w_{x_2}(s, t + \tau) - w_{x_2}(s, t)| = |p_2| \leqslant \tau^{\alpha} C' R'.$$

From part 5) of Lemma 2.1 it follows that

$$a_2(q_1 + p_1, q_2 + p_2) - a_2(q_1, q_2) = p_2 + O(r) z \cdot (p_1, p_2) \tag{3.43}$$

where z is a unit 2-vector. From (3.42) and (3.43) we find

$$p_2 + O(r) p_2 = O(r) \tau^{\alpha} C_2 R' + \tau^{1/2} C_1 R' \tag{3.44}$$

and with $|O(r)| < \frac{1}{2}$, as before, conclude that $|p_2| < \tau^{\alpha} C' R'$ (assume $\alpha \leqslant \frac{1}{2}$). Thus w_{x_2} is Hölder continuous with respect to changes in x_2.

To show that w_{x_2} is Hölder continuous as x_1 changes let

$$T(x_1, x_2) = a_2\big(w_{x_1}(x_1, x_2),\, w_{x_2}(x_1, x_2)\big)\,.$$

Consider (3.1) in a region $\tilde{\Omega}$ where $h' = 0$ and ϱ is constant. Integrate the equation from t to $t + \tau$ in x_2 for $x_1 = s$ and $s + \gamma$, subtract the results, and express the difference as

$$T(s + \gamma, t + \tau) - T(s + \gamma, t) - \big(T(s, t + \tau) - T(s, t)\big) = \int_t^{t+\tau} \int_s^{s+\gamma} \frac{\partial^2}{\partial x_1^2} a_1\,. \tag{3.45}$$

The type of integral occurring in (3.45) has already been estimated (cf. (3.35)-(3.37)). One finds that

$$|T(s + \gamma, t + \tau) - T(s, t + \tau) - [T(s + \gamma, t) - T(s, t)]| \leqslant \tau^{1/2} \gamma^{1/2} C_3 R'. \tag{3.46}$$

Since

$$\frac{\partial}{\partial x_i} a_{ij}(w_{x_1}, w_{x_2}) \frac{\partial}{\partial x_j} w_{x_2} = 0$$

in $\tilde{\Omega}$, i.e. where $\varrho = $ constant, an interior estimate (cf. [15], Theorem 8.29) yields

$$\|w_{x_2}\|_{C^\alpha(\Omega''')} \leqslant C_4 \|w_{x_2}\|_{L^2(\tilde{\Omega})}, \tag{3.47}$$

where Ω''' is a subdomain of $\tilde{\Omega}$. One can choose Ω''' so that the distance from Ω''' to $\eta = 0$ is larger than some positive number δ', independently of δ. Then the constant C_4 in (3.47) will depend only on the parameters listed in the lemma. Since the cutoff parameter r is bounded, $|\text{grad}\, a_2| \leqslant \tilde{C}$ for some constant $\tilde{C}$. It then follows from (3.47) that

$$|T(s+\gamma, t) - T(s, t)| \leqslant \gamma^\alpha C_4 \|w_{x_2}\|_{L^2(\Omega'')}, \tag{3.48}$$

provided the segment from (s, t) to $(s+\gamma, t)$ is in Ω'''. Now let $(s, t+\tau)$ and $(s+\gamma, t+\tau)$ be unrestricted points in $\tilde{\Omega}$ to obtain

$$|T(s+\gamma, t+\tau) - T(s, t+\tau)| \leqslant \gamma^\alpha C_4 R' + \tau^{1/2}\gamma^{1/2} C_3 R'$$

from (3.46) and (3.48). Having a bound on the variation of $a_2(w_{x_1}, w_{x_2})$ as x_1 varies, one can obtain an equation analogous to (3.44), but with γ playing the role that τ played. The Hölder continuity in x_1 follows. Hence

$$\|w_{x_2}\|_{C^\alpha(\Omega'')} \leqslant C' R',$$

completing the proof of Lemma 3.5.

We will use the Hölder estimates in Section 8. In the next section we will use the following corollary of Lemma 3.5.

COROLLARY 3.2. *Let $\tilde{r}$ be as in Lemma 3.5. Then there is a positive constant R_1 depending on ν, b, b', h, ϱ_0, ϱ_1, η_0, and σ such that if w is a solution of*

$$A'(w) = \lambda B'(w)$$

coming from Theorem 2.2, with $0 < R < R_1$, then

$$A(w) = F(w)$$

and

$$F'(w) = \lambda B'(w)$$

(cf. (2.12), (2.13), (2.15)); *that is*

$$\frac{\partial}{\partial x}\varrho(\eta)\frac{w_x}{1+w_\eta} + \frac{\partial}{\partial \eta}\varrho(\eta)\frac{(2w_\eta + w_\eta^2 - w_x^2)}{2(1+w_\eta)^2} = \lambda \varrho' w\,. \tag{3.49}$$

PROOF. Assume $R_1 \leqslant R_0$ so that Lemma 3.5 applies. From Corollary 2.1 and Lemmas 3.2 and 3.5 R_1 can, if necessary, be reduced further to guarantee $|\nabla w| < \tilde{r}$. Then the functions a and f in (2.14) agree near ∇w and the conclusions follow easily.

For use in § 7 we include the following estimate for w, continuing to use $\Omega'' \subset \Omega'$ introduced early in this section.

LEMMA 3.6. *Let $w \in C_k^2(\bar{\Omega})$ be a solution of $A'(w) = \lambda B'(w)$. Then*

$$\int_{\Omega''} |\nabla w|^2 \leqslant C \max_{\Omega'} w^2 \tag{3.50}$$

where C depends on $\bar{h}$, σ, and the measure of Ω'.

PROOF. Using a cutoff function introduced at the beginning of this section we start with

$$\int_{\Omega'} \frac{\partial}{\partial x_i} \varrho a_i(w_{x_1}, w_{x_2}) \zeta^2 w = \lambda \int_{\Omega'} \varrho' \zeta^2 w^2$$

or

$$\int_{\Omega'} \zeta^2 \varrho a_i(w_{x_1}, w_{x_2}) w_{x_i} = \lambda \int_{\Omega} (-\varrho') \zeta^2 w^2 - \int_{\Omega} \varrho a_i 2\zeta\zeta_{x_i} w \leqslant$$

$$\leqslant \lambda \int_{\Omega} (-\varrho') \zeta^2 w^2 + \varepsilon \int_{\Omega} \varrho \zeta^2 (a_1^2 + a_2^2) + \frac{\sigma^2}{\varepsilon} \int w^2 .$$

Using parts 2) and 3) of Lemma 2.1 we have

$$\sigma_3(1 - \varepsilon\sigma_5) \int_{\Omega'} \zeta^2 \varrho |\nabla w|^2 \leqslant \int_{\Omega'} \zeta^2 \varrho (1 - \varepsilon\sigma_5) a_i(w_{x_1}, w_{x_2}) w_{x_i} \leqslant \tag{3.51}$$

$$\leqslant \max_{\Omega'} w^2 \left(\lambda(\varrho_0 - \varrho_1) + \frac{\sigma^2}{\varepsilon} \right) |\Omega'|$$

where $|\Omega'| =$ measure of Ω'. The desired inequality follows from (3.51). Recall that $\bar{h} = \lambda(\varrho_0 - \varrho_1)$.

4. – Symmetrization.

Our aim in this section will be to show that the solutions of $A'(w) = \lambda B'(w)$ arising in Theorem 2.2 can be assumed to lie in a class of functions with a particular « symmetry » property, provided $A(w) = R^2$ is restricted as in Corollary 3.2. We begin with a brief description of Steiner symmetrization (also called decreasing rearrangement) of a function. The reader is directed to [19] for further information. We use the notation $\mathfrak{D}_k$, $\mathfrak{D}$ from Section 1 and consider a class P_k of functions defined on $\mathfrak{D}$. Each function $z = f(x, \psi)$ in P_k should be continuous and $2k$ periodic in x. Further it should be piecewise affine: i.e. for each f there is a triangulation of $\mathfrak{D}_k$ such that on each triangle f has the form $c_0 + c_1 x + c_2 \psi$, the c_i's changing from one triangle to another (to avoid ambiguity below, we'll assume $c_1 \neq 0$). Further each f is assumed constant for $\psi = \psi_0$ and for $\psi = \psi_1$. The class P_k is dense in H^1_k (the periodic H^1 functions) and will be large enough for our application. To define the symmetrization we consider a fixed ψ and let $\mu(f, \psi, t) = \text{meas}\{x | f > t\}$ where the measure is Lebesgue measure on $-k \leqslant x \leqslant k$ and t is a real number. We seek a function $\hat{f}(x, \psi)$ such that for each ψ, $\hat{f}(-x, \psi) = \hat{f}(x, \psi)$ on $|x| \leqslant k$, $\hat{f}$ is nonincreasing in x for $0 \leqslant x \leqslant k$, and the distribution functions satisfy $\mu(\hat{f}, \psi, t) = \mu(f, \psi, t)$. If $t(\mu, \psi)$, $0 \leqslant \mu \leqslant k$ is the function inverse to the decreasing function of t, $\frac{1}{2}\mu(f, \psi, t)$, then one verifies that the function f defined on $0 \leqslant x \leqslant k$ by $\hat{f}(x, \psi) = t(x, \psi)$, extended evenly to $-k \leqslant x \leqslant 0$ and periodically outside $\mathfrak{D}_k$, is a function with the desired properties. We call $\hat{f}$ the symmetrization of f. One shows easily that the function $\hat{f}$ will again be in P_k and that

$$\widehat{g + \gamma f} = g + \gamma \hat{f} \tag{4.1}$$

if $g = g(\psi)$ and $\gamma > 0$. The function $\hat{f}$ has the further properties:

$$\int_{\mathfrak{D}_k} G(\psi, f(x, \psi))\, dx\, d\psi = \int_{\mathfrak{D}_k} G(\psi, \hat{f}(x, \psi))\, dx\, d\psi \tag{4.2}$$

for any continuous $G(\psi, z)$ and

$$\int_{\mathfrak{D}_k} |\nabla f|^2\, dx\, d\psi \geqslant \int_{\mathfrak{D}_k} |\nabla \hat{f}|^2\, dx\, d\psi \tag{4.3}$$

where the integrals are taken in the H^1 sense. (The proofs given in [19], Note A, can be applied over one period.)

We will also want to consider the space of continuous, piecewise affine, $2k$ periodic functions on $S = \{(x, y) | x \in R,\ 0 < y < 1\}$ and denote this space by P_k as well. In particular we'll use the property corresponding to (4.2) for a function $\psi = \psi(x, y)$ with integration over $S_k = S \cap \{|x| < k\}$.

While our aim is to « symmetrize » w we do it through a symmetrization of $y = y(x, \psi)$ defined by

$$y = \mathring{y}(\psi) + w(x, \eta(\psi)) \tag{4.4}$$

where $\mathring{y}$ corresponds to a trivial flow with speed $c = (g/\lambda)^{1/2}$ (cf. (2.9)). We will define symmetrization for a $w = w(x, \eta)$ which is $2k$ periodic in x on Ω; which is 0 on $\eta = \eta_0, \eta_1$; and which gives rise to an element $y(x, \psi) \in P_k$ through the correspondence (4.4). We further require that $w_\eta > -1$ and that $w \geqslant 0$ (or $\leqslant 0$) in Ω_k. We denote the collection of such functions w by P'_k.

DEFINITION 4.1. *To each $w = w(x, \eta) \in P'_k$ we associate a function $\hat{w}$ as follows:*

1) *for $w \geqslant 0$, let $y = \mathring{y}(\psi) + w(x, \eta(\psi))$*
2) *form $\hat{y}$ and $\hat{u}(x, \psi) = \hat{y} - \mathring{y}$*
3) *let $\hat{w} = \hat{u}(x, \psi(\eta))$*
4) *for $w \leqslant 0$, $\hat{w} = -\widehat{(-w)}$.*

Note that for fixed η, $w(x, \eta)$ is piecewise linear in x and $\hat{w}$ could easily be defined directly. However it is simplest to symmetrize piecewise affine functions. They have the added feature that the class is preserved in the transformation taking $y(x, \psi)$ to the inverse function $\psi(x, y)$. To achieve this simplifying feature we introduce the slightly awkward definition of P'_k and $\hat{w}$.

Our aim is to show that the functional F in (2.12) cannot increase if w is replaced by $\hat{w}$. We'll require some lemmas

LEMMA 4.1. *Suppose $y = y(x, \psi) \in P_k$ and that for a given ψ and all x*

$$s_1 \Delta\psi \leqslant y(x, \psi + \Delta\psi) - y(x, \psi) \leqslant s_2 \Delta\psi \tag{4.5}$$

for any real increment $\Delta\psi$ satisfying $0 \leqslant \Delta\psi \leqslant \varepsilon_0$, $\varepsilon_0 > 0$. Then (4.5) *holds with $\hat{y}$ replacing y.*

PROOF. Letting $\mu(y, \psi, t) = \mu(\psi, t)$ we conclude from (4.5) that

$$\{x | y(x, \psi + \Delta\psi) > t + s_1 \Delta\psi\} \subset \{x | y(x, \psi) > t\}$$

so

$$\mu(\psi + \Delta\psi, t + s_1 \Delta\psi) \geqslant \mu(\psi, t) .$$

By the definition of $\hat{y}$, $\hat{y}(\frac{1}{2}\mu(\psi,t), \psi+\Delta\psi) \geqslant t+s_1\Delta\psi$. Since $t=\hat{y}(\frac{1}{2}\mu(\psi,t),\psi)$,

$$\hat{y}(\tfrac{1}{2}\mu(\psi,t), \psi+\Delta\psi) - \hat{y}(\tfrac{1}{2}\mu(\psi,t), \psi) \geqslant s_1\Delta\psi$$

and as μ takes all values in $[0, 2k]$ as t varies, we conclude that the first inequality in (4.5) holds for $\hat{y}$. The second is shown similarly.

LEMMA 4.2. *Suppose $y \in P_k$, $y \geqslant \mathring{y}$ and $y_\psi \geqslant s_1 > 0$ a.e. then $y(x, \psi)$ and $\hat{y}(x, \psi)$ give rise to inverse functions $\psi(x, y)$ and $\tilde{\psi}(x, y)$, respectively, which are in P_k and satisfy $\hat{\psi}(x, y) = \tilde{\psi}(x-k, y)$.*

PROOF. From Lemma 4.1 one easily concludes $\hat{y}_\psi \geqslant s_1$ a.e. and so the inverse functions are defined. Note that if $y(x, \psi(x, y)) = y$, $y_x + y_\psi \cdot \psi_x = 0$ and $y_\psi \cdot \psi_y = 1$ so ψ_x and $\psi_y > 0$ are constant on a triangle in the (x, y) space corresponding to an (x, ψ) triangle, so one easily sees that $\psi \in P_k$. Note, however, that y_x and ψ_x have opposite signs. Now suppose (y', ψ') is in the graph of $\psi(x_0, y)$ for some x_0 in $[-k, k]$. Then we claim that

$$\{x|y(x, \psi') < y'\} = \{x|\psi(x, y') > \psi'\} . \tag{4.6}$$

To see this, suppose $y_1 = y(x, \psi') < y'$. Then $\psi(x, y_1) = \psi'$ and since ψ is increasing in y, $\psi(x, y') > \psi'$, showing an inclusion in one direction. The reverse is similar. Now using the definition of $\hat{\psi}$ and $\hat{y}$ together with (4.6) and its analogue for $\hat{y}$ we see that

$$\begin{aligned} \operatorname{meas}\{x|\hat{\psi}(x, y') > \psi'\} &= \operatorname{meas}\{x|\psi(x, y') > \psi'\} \\ &= \operatorname{meas}\{x|y(x, \psi') < y'\} \\ &= \operatorname{meas}\{x|\hat{y}(x, \psi') < y'\} \\ &= \operatorname{meas}\{x|\tilde{\psi}(x, y') > \psi'\} \end{aligned} \tag{4.7}$$

where we use the fact that

$$\operatorname{meas}\{x|y(x, \psi') < y'\} = 2k - \operatorname{meas}\{x|y(x, \psi') > y'\}$$

and a similar equality for $\hat{y}$, valid since their x derivatives are nonzero. Since $\hat{\psi}$ and $\tilde{\psi}$ have the same measure distribution and $\tilde{\psi}$ is increasing on $0 \leqslant x \leqslant k$ and periodic, it follows that $\hat{\psi}(x, y) = \tilde{\psi}(x-k, y)$.

LEMMA 4.3. *Suppose $w = w(x, \eta) \in P_k'$ and $|\nabla w| \leqslant \tilde{r}$, $\tilde{r}$ being the cutoff parameter in (2.14). Then $|\nabla \hat{w}| \leqslant \tilde{r}$ and*

$$F(\hat{w}) \leqslant F(w) . \tag{4.8}$$

PROOF. Consider a value $\eta = \bar{\eta}$ such that $w_\eta(x, \eta)$ exists, except for a finite set of x's on the boundaries of a triangulation. The function $u(x, \psi) = w(x, \eta(\psi))$ will satisfy $|u_\psi| \leqslant \tilde{r}\bar{s}$ at $\psi = \bar{\psi} = \psi(\bar{\eta})$ a.e. in x with $\bar{s} = \eta_\psi(\bar{\psi})$. The associated $y = \mathring{y} + u$ satisfies $\mathring{y}_\psi - \tilde{r}\bar{s} \leqslant y_\psi \leqslant \mathring{y}_\psi + \tilde{r}\bar{s}$ for $\psi = \bar{\psi}$ and hence a.e. in x, $(\mathring{y}_\psi - \tilde{r}\bar{s} - o(1))\Delta\psi \leqslant y(\bar{\psi} + \Delta\psi) - y(\bar{\psi}) \leqslant (\mathring{y}_\psi + \tilde{r}\bar{s} + o(1))\Delta\psi$ for an increment $\Delta\psi$. From Lemma 4.1 it follows that $\hat{y}_\psi$, existing a.e. in x for $\psi = \bar{\psi}$, has the same upper and lower bound as y_ψ. Reversing steps we let $\hat{u} = \hat{y} - \mathring{y}$ and $\hat{w}(x, \eta) = \hat{u}(x, \psi(\eta))$ and readily see that $|\hat{w}_\eta| \leqslant \tilde{r}$ at $\eta = \bar{\eta}$ a.e. in x. In fact the bound holds except on the boundaries of a triangulation.

To show (4.8) we relate $F(w)$ to the functionals introduced in Section 2. Reintroducing $\tilde{\varrho}$ we see from (2.5), (2.7) and (2.10) that

$$c^2F(w) = G(w) - \int_{\Omega_k} g\tilde{\varrho}' \frac{w^2}{2} = E(u) - \int_{\mathfrak{D}_k} g\varrho' \frac{u^2}{2} = \tag{4.9}$$

$$= N(\mathring{y} + u) - N(\mathring{y}) - \langle N'(\mathring{y}), u\rangle$$

with a corresponding identity for $\hat{w}$ and $\hat{u}$. Thus $F(\hat{w}) \leqslant F(w)$ will follow if we show

$$N(\hat{y}) - N(\mathring{y}) - \langle N'(\mathring{y}), \hat{u}\rangle \leqslant N(y) - N(\mathring{y}) - \langle N'(\mathring{y}), u\rangle .$$

Since $u = 0$ for $\psi = \psi_0$ and $\psi = \psi_1$ we see looking at (2.5) that

$$\langle N'(\mathring{y}), u\rangle = \int_{\mathfrak{D}_k} \frac{1}{2} \frac{u_\psi}{\mathring{y}_\psi^2} = -\int_{\mathfrak{D}_k} \frac{1}{2} \frac{\partial}{\partial\psi}\left(\frac{1}{\mathring{y}_\psi^2}\right) u$$

with an analogous expression for $\langle N'(\mathring{y}), \hat{u}\rangle$. From (4.2) we can conclude that $\langle N'(\mathring{y}), u\rangle = \langle N'(\mathring{y}), \hat{u}\rangle$ and thus it will suffice to show $N(\hat{y}) \leqslant N(y)$. Recall that $N(y)$ is merely the Dirichlet integral in new variables (cf. (2.1), (2.3), (2.4)). Using Lemma 4.2 together with inequality (4.3) we have

$$N(\hat{y}) = \int_{S_k} |\nabla\tilde{\psi}|^2 \, dx\, dy = \int_{S_k} |\nabla\hat{\psi}|^2 dx\, dy \leqslant \int_{S_k} |\nabla\psi|^2 \, dx\, dy = N(y)$$

completing the proof of the lemma.

In the next result we combine the last lemma with Theorem 2.2 and elements of its proof to improve on Corollary 3.2. In stating subsequent results we will suppress the dependence of constants on quantities related to a choice of cutoff parameter r. Assuming that the choice $r = \tilde{r}$ is made (cf. Lemmas 2.1, 3.4, and 3.5) we take the values $\sigma_1, \ldots, \sigma_5$ from Lemma 2.1 and the bounds ν, b, b' associated with $a(p_1, p_2)$ as given; ν will be approximately 1, and so on. We also suppress σ, coming from the cutoff function ζ, since the bounds in Section 3 are uniform in σ for σ small. The constant $\bar{h}$ occurring in Section 3 is merely $\lambda(\varrho_0 - \varrho_1)$ and from Lemmas 2.3 and 2.4 we see that λ is uniformly bounded provided $0 < \delta < \delta_0$, $\delta_0(\varrho_0, \varrho_1, \eta_0, \eta_1)$ coming from Lemma 2.4. Further since $\eta_1 - \eta_0 = 1$ we henceforth focus on the parameters ϱ_0, ϱ_1, and η_0 as regards the dependence of various bounds on parameters.

THEOREM 4.1. *There are positive constants δ_0, R_1, Λ^- and Λ^+ depending on ϱ_0, $\varrho_1 < \varrho_0$, and η_0 such that for $0 < \delta \leqslant \delta_0$, $0 < R \leqslant R_1$ and $k > 0$ the equation* (2.13)

$$F'(w) = \lambda B'(w)$$

has solutions (λ_i, w_i), $i = 1, 2$ with $\Lambda^- \leqslant \lambda_i \leqslant \Lambda^+$, $w_i \in H_k^1(\Omega) \cap C^\infty(\Omega)$, and $F(w_i) = R^2$. The solution w_1 satisfies $w_1 = \hat{w}_1 > 0$ in Ω_k and has the variational characterization (2.18). *The solution w_2 satisfies $w_2 = \hat{w}_2 < 0$ and is characterized by* (2.18) *with $B(w^-)$ replacing $B(w^+)$. Each solution has the regularity shown in Lemma* 3.5.

PROOF. Let δ_0 and R_1 be the constants occurring in Lemma 2.4 and Corollary 3.2, respectively. Let (λ, w) with $w > 0$ be obtained from Theorem 2.2. As noted, Lemmas 2.3 and 2.4 yield $\Lambda^- \leqslant \lambda \leqslant \Lambda^+$ and $\bar{h} = \lambda(\varrho_0 - \varrho_1) \leqslant \Lambda^+(\varrho_0 - \varrho_1)$ with Λ^- and Λ^+ depending on ϱ_0, ϱ_1, and η_0. Thus in Corollary 3.2, $R_2 = R_2(\varrho_0, \varrho_1, \eta_0)$.

From Corollary 3.2, $|\nabla w| < \tilde{r}$, so A and F agree near w. The function w can be approximated arbitrarily closely in $W^{1,\infty}(\Omega_k)$ by functions in P_k'. This can easily be seen by going over to $y(x, \psi) = \bar{y} + w(x, \eta(\psi))$ where one can approximate y (in this case $y \in C^1$) arbitrarily closely by piecewise affine functions on a triangulation. The transformation back to w preserves $W^{1,\infty}$ proximity. For $n = 3, 4, 5, \ldots$ we choose $w_n \in P_k'$ so that $w_n \geqslant 0$, $|\nabla w_n| < \tilde{r}$, $|F(w_n) - R^2| < 1/n$, and $\|w_n - w\|_{W^{1,\infty}} < 1/n$. One computes that $F(tw_n)$ has a t derivative

$$\frac{1}{2}\int_{\Omega_k} \varrho |\nabla w_n|^2 \frac{t(2 + t(\partial w_n/\partial\eta))}{(1 + t(\partial w_n/\partial\eta))^2}$$

and thus there is a $t = t_n$ with $|t_n - 1|$ of order $1/n$ such that $F(t_n w_n) = R^2$.

For large n we'll have $|t_n \nabla w_n| < \tilde{r}$ so $F(t_n w_n) = A(t_n w_n)$. If $B(w) = -\int_{\Omega_k} \varrho'(w^2/2) = s$, then since $t_n w_n$ converges to w in L_k^2, $s_n = B(t_n w_n)$ converges to s as n approaches ∞. From (4.1) and Lemma 4.3, $F(t_n w_n) \geqslant F(t_n \hat{w}_n) = A(t_n \hat{w}_n)$. Since $A(t\hat{w}_n)$ approaches ∞ as t increases to ∞ and is continuous in t, there is a $\hat{t}_n \geqslant t_n$ such that $A(\hat{t}_n \hat{w}_n) = R^2$. The property (4.2) holds for w_n and $\hat{w}_n$ and so $B(\hat{t}_n \hat{w}_n) = B(\hat{t}_n w_n) \geqslant s_n$. Hence $\hat{t}_n \hat{w}_n$ is a maximizing sequence for the original problem (2.18). Referring to the proof of Theorem 2.2 we see that a subsequence will converge in H_k^1 to a solution w_1 of $A'(w) = \lambda B'(w)$ with a corresponding λ_1. Since $\hat{t}_n \hat{w}_n$ is nonnegative and symmetrized, w_1 inherits these properties; that is, $(\partial/\partial x) w_1 \leqslant 0$ for $0 \leqslant x \leqslant k$ in the L^2 sense. But since it satisfies (3.1), we see as before that $w_1 > 0$ and that it is smooth so $(\partial/\partial x) w_1 \leqslant 0$ pointwise. The estimate on λ_1 follows as before and the regularity in Lemma 3.5 holds since w_1 is a solution of (3.1) and $R \leqslant R_1$. Likewise, $|\nabla w_1| < \tilde{r}$ as before and so $F'(w_1) = \lambda_1 B'(w_1)$. The treatment of (λ_2, w_2) is completely analogous. This completes the proof.

5. – Precise bounds on λ.

One of the eigenvalues λ_1 or λ_2 occurring in Theorem 4.1 can be shown to be strictly less than the « critical » eigenvalue μ. Which eigenvalue it is depends on the sign of

$$e = \frac{\varrho_0}{\eta_0^2} - \frac{\varrho_1}{\eta_1^2}. \tag{5.1}$$

If $e > 0$ we can estimate λ_1 and if $e < 0$, λ_2. We'll see in Section 8 that the corresponding function, w_1 if $e > 0$ and w_2 if $e < 0$, will have a non-trivial limit as the period approaches ∞.

LEMMA 5.1. *Let μ be the lowest eigenvalue of problem* (2.21) *and let (λ_i, w_i) $i = 1, 2$ be the solutions of $F'(w) = \lambda B'(w)$ from Theorem* 4.1. *Then if $z_i \in H_k^1$, $|\nabla z_i| < \tilde{r}$, $F(z_i) = R^2$, $z_1 \geqslant 0$, and $z_2 \leqslant 0$ it follows that*

$$\lambda_i \leqslant \frac{R^2}{2B(z_i)} \left(3 + \frac{(CR)^2}{1 - CR}\right) - \frac{\mu}{2} \tag{5.2}$$

where $C = C(\varrho_0, \varrho_1, \eta_0)$.

PROOF. Suppose $i = 1$; the case $i = 2$ is similar. Letting $\lambda_1 = \lambda$,

$w_1 = w$, and $z_1 = z$ we have

$$\text{(5.3)}\qquad \langle F'(w), w\rangle = \int\limits_{\Omega_k} \varrho(\eta)\left[\frac{w_x}{1+w_\eta}w_x + \frac{w_\eta}{1+w_\eta}w_\eta - \frac{1}{2}\frac{w_x^2+w_\eta^2}{(1+w_\eta)^2}w_\eta\right] =$$

$$= \int\limits_{\Omega_k} \varrho(\eta)|\nabla w|^2\left[\frac{3}{2}\frac{1}{1+w_\eta} - \frac{1}{2} + \frac{1}{2}\frac{w_\eta^2}{(1+w_\eta)^2}\right] =$$

$$= 3F(w) - \int\limits_{\Omega_k} \frac{1}{2}\varrho(\eta)|\nabla w|^2 + P(w)$$

in obvious notation. From Corollary 2.1 and Lemma 3.5 $|w_\eta| \leqslant CR$ so

$$P(w) \leqslant \frac{(CR)^2}{1-CR}\iint\limits_{\Omega_k} \frac{1}{2}\varrho(\eta)\frac{|\nabla w|^2}{1+w_\eta} = \frac{(CR)^2}{1-CR}F(w)\,.$$

Equation (2.13) yields $\langle F'(w), w\rangle = \lambda\langle B'(w), w\rangle = 2\lambda B(w)$ so

$$\text{(5.4)}\qquad \lambda = \frac{\langle F'(w), w\rangle}{2B(w)} \leqslant \frac{F(w)}{2B(w)}\left(3 + \frac{(CR)^2}{1-CR}\right) - \frac{1}{2B(w)}\int\limits_{\Omega_k}\frac{1}{2}\varrho(\eta)|\nabla w|^2.$$

If $F(z) = R^2$, then $A(z) = R^2$ and $B(z) \leqslant B(w)$, by the characterization in (2.18). The quotient in the second term on the right side of (5.4) is the Rayleigh quotient for a linear eigenvalue problem which has a lowest eigenvalue $\mu/2$ (cf. (2.21)). Since $F(w) = R^2$, the inequality (5.2) follows from (5.4).

LEMMA 5.2. *Let (λ_i, w_i), $i = 1, 2$, be the solutions from Theorem 4.1. Suppose e defined by (5.1) satisfies $e \neq 0$ and let $\lambda(e) = \lambda_1$ if $e > 0$, λ_2 if $e < 0$. Then there are positive constants $\tilde{\delta}$, $\tilde{R}$, $k_1 = k_1(R)$, and C_1 depending on ϱ_0, ϱ_1, and η_0 such that if $0 < \delta \leqslant \tilde{\delta}$, $0 < R \leqslant \tilde{R}$, and $k > k_1$,*

$$\text{(5.5)}\qquad \lambda(e) \leqslant \mu(1 - C_1 R^{4/3})$$

where μ is the lowest eigenvalue of (2.22).

PROOF. We'll suppose $e > 0$, the case $e < 0$ being similar. The inequality (5.5) will follow from (5.2) with the use of a suitable function z. We let $z \in H_k^1$ be defined by

$$z(x, \eta) = \alpha\xi(\eta)\exp[-\beta|x|]$$

for $(x, \eta) \in [-k, k]\times[\eta_0, \eta_1]$ where ξ is the eigenfunction from Lemma 2.2

and α, β are constants to be determined. Using $\int(...)\,d\eta$ to denote an integral over $[\eta_0, \eta_1]$ and denoting $1-\exp[-j\beta k]$ by $t_j = t_j(\beta, k)$ we find

$$B(z) = -\alpha^2\int_{\Omega_k}\varrho'\,\frac{\xi^2}{2}\exp[-2\beta|x|] = -\frac{\alpha^2 t_2}{2\beta}\int\varrho'\,\frac{\xi^2}{2}\,d\eta = \frac{\alpha^2 t_2}{2\beta\mu}\int\varrho\xi_\eta^2\,d\eta \tag{5.6}$$

and

$$\begin{aligned} F(z) &= \frac{1}{2}\int_{\Omega_k}\varrho\left(\alpha^2\xi_\eta^2\exp[-2\beta|x|] + \alpha^2\beta^2\xi^2\exp[-2\beta|x|]\right)\cdot \\ &\quad\cdot\left(1-\alpha\xi_\eta\exp[-\beta|x|] + \frac{\alpha^2\xi_\eta^2\exp[-2\beta|x|]}{1+\alpha\xi_\eta\exp[-\beta|x|]}\right) = \frac{\alpha^2 t_2}{2\beta}\int\varrho\xi_\eta^2\,d\eta + \\ &\quad + \frac{\alpha^2\beta t_2}{2}\int\varrho\xi^2\,d\eta - \frac{\alpha^3 t_3}{3\beta}\int\varrho\xi_\eta^3\,d\eta - \frac{\alpha^3\beta t_3}{3}\int\varrho\xi^2\xi_\eta\,d\eta + \\ &\quad + \frac{\alpha^4 t_4}{4\beta}\int\varrho\xi_\eta^4\,d\eta(1+O(\alpha)) + \frac{\alpha^4\beta t_4}{4}\int\varrho\xi^2\xi_\eta^2\,d\eta(1+O(\alpha))\,. \end{aligned} \tag{5.7}$$

We introduce the further notation $\int\varrho\xi^i\xi_\eta^j\,d\eta = m_{ij}$. Of particular interest here is m_{03}. Using Lemma 2.2 we obtain the estimate

$$\int\varrho\xi_\eta^3\,d\eta \geqslant \frac{\varrho_0|\eta_0|}{(|\eta_0|+\varrho_0\,\delta/\varrho_1)^3} - \left(\frac{\varrho_1}{\varrho_0(\eta_1-\delta)}\right)^3\delta - \frac{\varrho_0(\eta_1-\delta)}{(\eta_1-\delta)^3}\,. \tag{5.8}$$

For $\delta=0$ the right side of (5.8) is just e in (5.1). As $e>0$, $m_{03}>0$ for a range of δ, $0<\delta<\tilde{\delta}\,(\varrho_0, \varrho_1, \eta_0)\leqslant\delta_0$, δ_0 from Lemma 2.4. For $e<0$ one obtains an upper bound for m_{03} which reduces to e when $\delta=0$.

Now suppose $\alpha>0$ and let $\beta=\gamma\sqrt{\alpha}$ where γ and α are to be determined (for $m_{03}<0$ choose $\alpha<0$ and $\beta=\gamma\sqrt{|\alpha|}$). With the new notation

$$F(z) = \frac{\alpha^{3/2}}{2\gamma}t_2 m_{02} + \frac{\alpha^{5/2}\gamma}{2}t_2 m_{20} - \frac{\alpha^{5/2}}{3\gamma}t_3 m_{03} - \frac{\alpha^{7/2}\gamma}{3}t_3 m_{21} + \varepsilon(\alpha,\gamma) \tag{5.9}$$

where $\varepsilon(\alpha,\gamma)\leqslant\text{const}\,\alpha^{7/2}$ for $\alpha<\alpha_0$, α_0 a positive constant chosen so that $\alpha_0|\xi_\eta|<\frac{1}{2}$. We assume $k>k_1=(\ln 2)/3\beta$ so that, $1\geqslant t_j\geqslant\frac{1}{2}$ for $j=2,3$ and choose γ so that $\gamma t_2 m_{20}-\frac{2}{3}\gamma^{-1}t_3 m_{03} = -t_2 m_{02}$ (γ will depend on k in an inessential way). The condition $F(z)=R^2$, i.e.

$$R^2 = \frac{\alpha^{3/2}}{2\gamma}t_2 m_{02}(1-\gamma\alpha+O(\alpha^2))\,, \tag{5.10}$$

will determine $\alpha(R)=(2\gamma/t_2 m_{02})^{2/3}R^{4/3}+O(R^{8/3})$ in a range $0<R\leqslant\tilde{R}\leqslant R_1$, R_1 from Theorem 4.1. If necessary, $\tilde{R}$ can be reduced to guarantee $|\nabla z|<\tilde{r}$.

In the expression $R^2/2B(z)$ occurring in (5.2) we use (5.10) in the numerator and (5.6) in the denominator (with $\beta = \gamma\sqrt{\alpha}$) obtaining

$$(5.11) \qquad \lambda \leqslant \frac{\mu}{2}\left(1 - \gamma\alpha + O(\alpha^2)\right)\left(3 + \frac{(CR)^2}{1 - CR}\right) - \frac{\mu}{2} = \\ = \mu\left(1 - \frac{3}{2}\gamma\alpha(R) + \frac{1}{2}(CR)^2 + O(R^{8/3})\right).$$

Choosing $\tilde{R}$ smaller, if necessary, we obtain the inequality (5.5) from (5.11). Note that since $\beta = \gamma\sqrt{\alpha}$, the lower bound k_1 is of order $R^{-2/3}$. We have noted some differences in the case $e < 0$; otherwise it is done similarly, completing the proof.

6. – A lower bound for the amplitude.

Up to this point we have been concerned with solutions of $F'(w) = \lambda B'(w)$ which ostensibly were functions of two variables. It may be that the solutions obtained are really functions of η alone. That such solutions exist can be seen by restricting the variational procedure to functions of η alone or, what amounts to the same thing, treating the Sturm-Liouville problem for $w = w(\eta)$:

$$(6.1) \qquad \begin{cases} \dfrac{\partial}{\partial\eta}\varrho(\eta)\dfrac{1}{2}\dfrac{(2w_\eta + w_\eta^2)}{(1 + w_\eta)^2} = \lambda\varrho' w \\ w(\eta_0) = 0\,, \qquad w(\eta_1) = 0 \end{cases}$$

by variational methods. One could also treat (6.1) as bifurcation from the simple eigenvalue μ in (2.22). Taking this approach one can show that for k small the solutions w_1, w_2 in Theorem 4.1 are both functions of η alone.

If $w = w(\eta)$ is a solution of (6.1) for which $\int \frac{1}{2}\varrho w_\eta^2/(1 + w_\eta) = N_1^2$, then viewed as an element of H_k^1 it satisfies $F(w) = 2kN_1^2$. Thus if we fix $F(w) = R^2$ and let $k \to \infty$, the corresponding N_1 must approach zero. That is, the functions of η, normalized by $F(w) = R^2$, approach the function $w = 0$ in H_1^k as $k \to \infty$. The next lemma shows that one of the solutions w_1 or w_2 does not « collapse » in this way as $k \to \infty$.

LEMMA 6.1. *Assume $0 < \delta \leqslant \tilde{\delta}$, $0 < R \leqslant \tilde{R}$, and $k > k_1$, from Lemma 5.2. Let w stand for a solution from Theorem 4.1 which has the same sign as e (cf. 5.1). Then there is a constant $C = C(\varrho_0, \varrho_1, \eta_0)$ such that*

$$\|w\|_{L^\infty(\Omega_k)} \geqslant CR^{4/3}.$$

PROOF. Fix R and let $m = \|w\|_{L^\infty(\Omega_k)}$. From Lemma 3.6 with $\Omega'' \subset \Omega_k$ one obtains

$$\int_{\Omega''} |\nabla w|^2 \leqslant C_0 m^2$$

and from Lemma 3.5 with $\Omega''' \subset \Omega''$

$$\|w_\eta\|_{L^\infty(\Omega''')} \leqslant \tilde{C} m\,.$$

The same estimate will hold on any translate of Ω''' in the x direction so $|w_\eta| \leqslant \tilde{C}m$ on Ω_k. Referring to the proof of Lemma 5.1 and equality (5.3) in particular, we see that

$$\lambda = \frac{\langle F'(w), w\rangle}{2B(w)} \geqslant \frac{\int_{\Omega_k} \varrho(\eta)|\nabla w|^2}{2B(w)}\left(1 - \frac{3}{2}\tilde{C}m\right) \geqslant \mu\left(1 - \frac{3}{2}\tilde{C}m\right).$$

Comparing this last inequality with the upper bound (5.5) one concludes that $C_1 R^{4/3} \leqslant \frac{3}{2}\tilde{C}m$ and so $m(R) = \frac{2}{3} C_1 \tilde{C}^{-1} R^{4/3}$ is a lower bound for $\|w\|_{L^\infty}$.

7. – Exponential decay.

In this section we show that the solution w of $F'(w) = \lambda B'(w)$ which has the same sign as e will exhibit exponential decay in x for $0 \leqslant x \leqslant k$. Moreover, the constants describing the decay will be independent of k and δ and so the decay will persist when we consider limits in k and δ. We will need some properties of the Green's function for the equation

$$(7.1) \qquad -\frac{\partial}{\partial x}\varrho(\eta)\frac{\partial z}{\partial x} - \frac{\partial}{\partial \eta}\varrho(\eta)\frac{\partial z}{\partial \eta} + \lambda\varrho' z = g(x, \eta)$$

on the strip $\Omega = \mathbf{R} \times [\eta_0, \eta_1]$ with $z = z(x, \eta)$ vanishing at $\eta = \eta_0$ and $\eta = \eta_1$. We write (7.1) as $Lz = g$.

LEMMA 7.1. *Suppose* $0 < \delta \leqslant \eta_1/2$ *and let* λ *be fixed, with* $\lambda < \mu$, μ *the lowest eigenvalue of* (2.22). *Let* (w_n, γ_n) *be eigenfunctions and eigenvalues of*

$$(7.2) \qquad \begin{cases} -\dfrac{d}{d\eta}\varrho\dfrac{dw}{d\eta} + \lambda\varrho' w = \gamma\varrho w \\ w(\eta_0) = w(\eta_1) = 0 \end{cases}$$

with $\int_{\eta_0}^{\eta_1} \varrho w_n^2\, d\eta = 1$ *and* $\gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \ldots$. *Then there are constants* C_i,

$i = 0, 1, 2, 3$ *depending on* ϱ_0, ϱ_1 *and* η_0 *such that*

$$(7.3) \qquad \begin{cases} \gamma_1 \geqslant C_0(\mu - \lambda) \\ \displaystyle\int \left|\frac{dw_n}{d\eta}\right|^2 d\eta \leqslant C_1 n^2 \\ |w_n|_{L^\infty} \leqslant C_2 n \\ \displaystyle\left|\frac{dw_n}{d\eta}\right|_{L^\infty} \leqslant C_3 n^2. \end{cases}$$

PROOF. The eigenvalues are characterized by

$$(7.4) \qquad \gamma_n = \min_{\mathcal{E}_n} \max_{\substack{w \in \mathcal{E}_n \\ w \neq 0}} \frac{\int(\varrho|dw/d\eta|^2 + \lambda\varrho' w^2)\, d\eta}{\int \varrho w^2\, d\eta}$$

for $n = 1, 2, \ldots$ where $\mathcal{E}_n$ is any n dimensional subspace of $\mathring{H}^1([\eta_0, \eta_1])$. Since μ is the lowest eigenvalue of the problem (2.22)

$$(7.5) \qquad \frac{1}{\mu}\int \varrho \left|\frac{dw}{d\eta}\right|^2 d\eta \geqslant \int (-\varrho')\, w^2\, d\eta\,.$$

Note that, since $\delta \leqslant \eta_1/2$, Lemma 2.3 yields a bound for μ in terms of ϱ_0, ϱ_1, and η_0. Using (7.5) and the fact that $\varrho' \leqslant 0$ we get

$$(7.6) \qquad \int \varrho \left|\frac{dw}{d\eta}\right|^2 d\eta \geqslant \int \left(\varrho \left|\frac{dw}{d\eta}\right|^2 + \lambda\varrho' w^2\right) d\eta \geqslant \left(1 - \frac{\lambda}{\mu}\right)\int \varrho \left|\frac{dw}{d\eta}\right|^2.$$

Suppose we let $\mathring{\gamma}_n$, $n = 1, 2, 3, \ldots$ denote the eigenvalues in (7.2) when $\lambda = 0$. Then the characterization (7.4) together with (7.6) yields $\mathring{\gamma}_n \geqslant \gamma_n \geqslant$ $\geqslant (1 - \lambda/\mu)\mathring{\gamma}_n$. Using the bounds on $\varrho(\eta)$ one easily shows $\varrho_1 \varrho_0^{-1}(n\pi)^2 \leqslant \mathring{\gamma}_n \leqslant$ $\leqslant \varrho_0 \varrho_1^{-1}(n\pi)^2$, using the variational characterization corresponding to $-w_{\eta\eta} =$ $= \tilde{\lambda} w$ and recalling that $\eta_1 - \eta_0 = 1$. The first inequality in (7.3) follows, setting $n = 1$.

We can estimate the numerator in (7.4) by

$$\begin{aligned}(7.7) \qquad \int \varrho \left|\frac{dw}{d\eta}\right|^2 d\eta + \lambda \int \varrho' w^2\, d\eta &= \int \varrho \left|\frac{dw}{d\eta}\right|^2 d\eta - \lambda \int \varrho\, 2w \frac{dw}{d\eta}\, d\eta \geqslant \\ &\geqslant \int \varrho \left|\frac{dw}{d\eta}\right|^2 d\eta - \frac{1}{2}\int \varrho \left|\frac{dw}{d\eta}\right|^2 d\eta - 2\lambda^2 \int \varrho w^2\, d\eta = \\ &= \frac{1}{2}\int \varrho \left|\frac{dw}{d\eta}\right|^2 d\eta - 2\lambda^2 \int \varrho w^2\, d\eta\,.\end{aligned}$$

Since the quotient in (7.4) equals γ_n with $w = w_n$ and since $\int \varrho |w_n|^2 d\eta = 1$, (7.7) provides a bound

$$\varrho_1 \int \left| \frac{dw_n}{d\eta} \right|^2 \leqslant \int \varrho \left| \frac{dw_n}{d\eta} \right|^2 \leqslant 2\gamma_n + 4\lambda^2$$

which easily gives the second claim in (7.3) since $\lambda < \mu$ and μ is bounded. Next,

$$w_n(\eta) = \int_{\eta_0}^{\eta} \frac{dw_n}{d\eta}\, d\eta \leqslant \left(\int \left| \frac{dw_n}{d\eta} \right|^2 d\eta \right)^{1/2}$$

providing an L^∞ bound. Finally, we integrate (7.2) in the form

$$-\frac{d}{d\eta}\left(\varrho \frac{dw_n}{d\eta} \right) + \lambda \left[\frac{d}{d\eta} (\varrho w_n) - \varrho \frac{dw_n}{d\eta} \right] = \gamma_n \varrho w_n$$

from a point $\bar{\eta}$ where $dw_n/d\eta = 0$ to obtain

$$\left| \varrho(\eta) \frac{dw_n}{d\eta} \right| = \left| \lambda \varrho w_n |_{\bar{\eta}}^{\eta} - \lambda \int_{\bar{\eta}}^{\eta} \varrho \frac{dw_n}{d\eta} - \gamma_n \int_{\bar{\eta}}^{\eta} \varrho w_n \right| \leqslant$$
$$\leqslant 2\lambda \varrho_0 C_2 n + \lambda \varrho_0 \left(\int \left(\frac{dw_n}{d\eta} \right)^2 \right)^{1/2} + \gamma_n \varrho_0^{1/2} \leqslant \tilde{C}_3 n^2$$

from which the last of (7.3) follows, completing the proof.

The estimates in the last lemma are rather crude but will be sufficient for use in estimating the Green's function

$$G(x - x', \eta, \eta') = \sum_{n=1}^{\infty} \frac{\exp[-\gamma_n^{1/2}|x - x'|]}{2\gamma_n^{1/2}} w_n(\eta) w_n(\eta') \tag{7.8}$$

for the operator L in (7.1).

The next lemma is a technical one required in estimating the decay of w. It is tailored to the problem at hand, but could clearly be extended to cover a variety of decay problems for nonlinear elliptic equations in conjunction with estimates like those in the remainder of this section.

LEMMA 7.2. *Let $b = \{b_j\}$, $n \leqslant j \leqslant 2k - n$ be a sequence of nonnegative real numbers satisfying*:

1) $b_{k+i} = b_{k-i}$ $i = 1, 2, \ldots, k - n$
2) $b_j \leqslant d_0 j^{-2/3}$ *for* $n \leqslant j \leqslant k$

and

3) $b_j \leqslant \tau_j(b)$ *for* $n+1 \leqslant j \leqslant k$ *where*

$$\tau_j(b) = d_1(b_{j-1} + b_j + b_{j+1})^2 + d_2\left[\sum_{i=n+1}^{2k-n-1} \exp[-p|i-j|]\, b_i + q\exp[-p(j-n)]\right]^2$$

and

$$q = [d_3 \exp[-pn] + d_4 n^{-2/3}]\,.$$

Here $d_0, \ldots, d_4$ *and* p *are positive constants.*

Then for n *sufficiently large*

$$b_j \leqslant C\exp[-2pj] \tag{7.9}$$

$n+1 \leqslant j \leqslant k$ *where* C *depends on* $d_0, \ldots, d_4$, p *and* n.

PROOF. Consider the space $\boldsymbol{R}^{k-n}$ and let the norm of an element $g = \{g_{n+1}, \ldots, g_k\}$ be $|g| = \max|g_i|$, $n+1 \leqslant i \leqslant k$. Let K be the cone of vectors g with all $g_i \geqslant 0$. We have a natural order relation on $\boldsymbol{R}^{k-n}$: $g \geqslant \bar{g}$ if and only if $g - \bar{g} \in K$. We define an order-preserving map T from K into itself as follows: we extend g to a sequence indexed on $[n, 2k-n]$ by setting $g_n = d_0 n^{-2/3}$ and then let $g_i = g_{2k-i}$ for $i > k$. Using g again to denote the extended sequence we define $Tg = h \in K$ by the formula

$$h_j = \tau_j(g)\,; \qquad j = n+1, n+2, \ldots, k\,,$$

τ_j being defined in part 3) of the lemma. If we restrict b to the index set $[n+1, k]$ (we'll continue to call it b) then from the hypotheses 1)-3) of the lemma we conclude $b \leqslant Tb$. We will show that T is a contraction in a certain invariant order interval and has a unique fixed point in a smaller order interval characterized by exponential decay. It will then follow that b has exponential decay, for otherwise the iterates of T starting at b would yield a second fixed point.

Let $I_1 = \{g \in K \,|\, g_j \leqslant d_0 j^{-2/3}\}$ and consider a pair of elements $g, \tilde{g} \in I_1$. A straightforward estimate shows that

$$|Tg - T\tilde{g}| \leqslant \left[\left(9d_1 + \frac{d_2}{(1-\exp[-p])^2}\right)|g + \tilde{g}| + \frac{2d_2 q\exp[-p]}{1-\exp[-p]}\right]|g - \tilde{g}|\,.$$

Since $|g + \tilde{g}|$ and q decrease to zero as n increases, T will be a contraction for n sufficiently large. To see that T maps I_1 into itself we estimate the

j-th component of Tg by

$$(7.10)\qquad 9d_1 d_0^2 (j-1)^{-4/3} + \\ + d_2\left[2d_0 \sum_{i=n+1}^{k} \exp[-p|i-j|]\, i^{-2/3} + q\exp[-p(j-n)]\right]^2$$

using the fact that g_i is even about k and that, since $j \leqslant k$, the discrete convolution over $[k, 2k-n-1]$ is at most that over $[n+1, k]$. To estimate the sum in (7.10) we note that the sum over $n+1 \leqslant i \leqslant j-1$ is at most $\tilde{d} j^{-2/3}$ provided

$$\sum_{i=n+1}^{j-1} \frac{\exp[-p(j-i)]}{(i/j)^{2/3}} \leqslant \tilde{d}\,.$$

However, a simple integral comparison shows the last inequality to be true with $\tilde{d} = \tilde{d}(p)$ independent of j. As regards the sum in (7.10) over $j \leqslant i \leqslant k$, it is easily seen to be at most $j^{-2/3}(1-\exp[-p])^{-1}$. Since $q \to 0$ as n increases it is easy to see that the expression in (7.10) is at most $d_0 j^{-2/3}$ for $n+1 \leqslant j \leqslant k$ provided n is sufficiently large.

Next consider $I_2 = \{g \in K \,|\, g_j \leqslant q^{3/2}\exp[-2p(j-n)],\ n+1 \leqslant j \leqslant k\}$. To see that I_2 is invariant under T for n large, we'll need estimates on some sums. Clearly

$$\sum_{i=n+1}^{j-1} \exp[-p(j-i)]\exp[-2p(i-n)] = \exp[-p(j-n)] \sum_{i=n+1}^{j-1} \exp[-p(i-n)]$$

and

$$\sum_{i=j}^{k} \exp[-p(i-j)]\exp[-2p(i-n)] = \\ = \exp[-p(j-n)] \sum_{i=j}^{k} \exp[-2p(i-j)-p(i-n)]\,.$$

Since $\exp[-2p] < \exp[-p]$, each sum is at most $\exp[-p(j-n)]\cdot$ $\cdot(\exp[p]-1)^{-1}$. Using the symmetry of g_j about k and these estimates it is easy to show that for $n+2 \leqslant j \leqslant k$

$$(Tg)_j \leqslant 9d_1 q^3 \exp[4p]\exp[-4p(j-n)] + \\ + d_2 q^2 \exp[-2p(j-n)][4q^{1/2}(\exp[p]-1)^{-1}+1]^2.$$

Choosing n sufficiently large and thus q sufficiently small we see that $(Tg)_j \leqslant q^{3/2}\exp[-2p(j-n)]$. When $j = n+1$, $(Tg)_j$ involves $g_n = d_0 n^{-2/3}$.

Specifically

$$(Tg)_{n+1} \leqslant d_1(d_0 n^{-2/3} + q^{3/2}\exp[-2p] + q^{3/2}\exp[-4p])^2 + \\ + d_2 q^2 \exp[-2p][4q^{1/2}(\exp[p]-1)^{-1}+1]^2$$

which is less than $q^{3/2}\exp[-2p]$ for n sufficiently large, since q is of order $n^{-2/3}$ for n large. Thus $T(I_2) \subset I_2$. Moreover, for n large $I_2 \subset I_1$.

There must be a fixed point of T in I_2. Suppose b in the statement of the lemma (restricted to $[n+1, k]$) is not in I_2. By hypothesis $Tb \geqslant b$ and thus the order interval $I_3 = (b+K) \cap I_1$ is invariant under T. Hence T has a fixed point in I_3. But $I_2 \cap I_3 = \emptyset$ and $I_2 \cup I_3 \subset I_1$ where T is a contraction and hence has a unique fixed point. The contradiction implies that $b \in I_2$ which means (7.9) holds for a suitable C.

Before making use of Lemma 7.2 we'll need a crude bound on the decay of the solution w occurring in Lemma 6.1.

LEMMA 7.3. *Let $w = w_1$ or w_2 in Theorem 4.1. Then*

$$|w(x,\eta)| \leqslant \frac{\bar{C}R^{2/3}}{x^{1/3}} \tag{7.11}$$

on $0 \leqslant x \leqslant k$ with $\bar{C} = \bar{C}(\varrho_0, \varrho_1, \eta_0)$.

PROOF. Let $m(x) = \max|w(x,\eta)|$ for $\eta_0 \leqslant \eta \leqslant \eta_1$. Since $w_\eta \leqslant CR$ from Lemma 3.5, $|w(x,\eta)| \geqslant m(x)/2$ on an η interval of length at least m/CR. Since $|w|$ is nonincreasing in x for $0 \leqslant x \leqslant k$,

$$\|w\|^2_{L^2(\Omega_k)} \geqslant 2x\left(\frac{m}{2}\right)^2 \cdot \frac{m}{CR}. \tag{7.12}$$

But from the Poincaré inequality and Corollary 2.1

$$\int_{\Omega_k} w^2 \leqslant \pi^{-2}\int_{\Omega_k} |\nabla w|^2 \leqslant \frac{2\pi^{-2}}{\sigma_1 \varrho_1} R^2. \tag{7.13}$$

The inequality (7.11) follows from (7.12) and (7.13).

If we combine Lemmas 6.1 and 7.3 we easily obtain

COROLLARY 7.1. *Suppose $0 < \delta \leqslant \tilde{\delta}$ and $0 < R \leqslant \tilde{R}$ from Lemma 5.2. Let w stand for a $2k$ periodic solution from Theorem 4.1, which has the same sign as e* (cf. 5.1). *Then there is a constant $\tilde{k}(R)$ such that for $k > \tilde{k}(R)$, w is not a function of η alone. Here $\tilde{k}(\tilde{R}) \leqslant CR^{-2}$ with $C = C(\varrho_0, \varrho_1, \eta_0)$.*

PROOF. We assume $\tilde{k} \geqslant k_1$ from Lemma 5.2 and recall from the proof of that lemma that $k_1 \leqslant C' R^{-2/3}$. If w is independent of x then from Lemmas 6.1 and 7.3, $CR^{4/3} \leqslant \bar{C} R^{2/3} k^{-1/3}$ so $k \leqslant k_2(R) \equiv (\bar{C}/C)^3 R^{-2}$. The choice $\tilde{k}(R) = \max\{k_1(R), k_2(R)\}$ meets the requirements for $\tilde{k}$.

LEMMA 7.4. *The solution w from Theorem 4.1 having the same sign as e satisfies*

$$\left\{\begin{array}{l} |w(x,\eta)| \leqslant C \exp[-px] \\ |\nabla w(x,\eta)| \leqslant C' \exp[-px] \end{array}\right. \tag{7.14}$$

for $0 \leqslant x \leqslant k$ where p is any real constant satisfying $p < \gamma_1^{1/2}$ (cf. (7.2)) and C, C' are constants depending on ϱ_0, ϱ_1, η_0, R and p.

PROOF. We use the equation (3.49) for w and write it as

$$Lw = \operatorname{div} V \tag{7.15}$$

where L is the elliptic operator in (7.1) and

$$V = -\varrho\left(\frac{w_x w_\eta}{1+w_\eta}, \frac{4w_x w_\eta + 2w_x w_\eta^2 + w_\eta^2 - w_x^2}{2(1+w_\eta)^2}\right). \tag{7.16}$$

Let $\zeta = \zeta(x)$ be a C^∞ function taking values in $[0,1]$ which is supported on $[-\frac{1}{2}, \frac{3}{2}]$ and which equals 1 on $[-\frac{1}{4}, \frac{5}{4}]$. For $0 \leqslant j \leqslant 2k$, let ζ_j denote the extension of $\zeta(x-j)$ to a $2k$ periodic function of x and let S_j denote the support of $\zeta(x-j)$ as a function on Ω_k.

In analogy with (7.6) one sees from (2.24) that for $w \in H_k^1 \cap C_k^2(\Omega)$

$$\int_{\Omega_k} (Lw)\, w \geqslant \left(1 - \frac{\lambda}{\mu}\right) \int_{\Omega_k} \varrho |\nabla w|^2. \tag{7.17}$$

Since L is coercive and uniformly elliptic we can uniquely solve

$$L\theta = \zeta_j \operatorname{div} V \tag{7.18}$$

and

$$L\chi = (1-\zeta_j) \operatorname{div} V \tag{7.19}$$

for θ, χ in $H_k^1 \cap C_k^2$ (cf. [20]). By uniqueness, $w = \theta + \chi$ and we can estimate w by estimating θ and χ.

We begin with

$$\int_{\Omega_k} (L\theta)\theta = \int_{\Omega_k} (\zeta_j \operatorname{div} V)\theta$$
$$= \int_{\Omega_k} V \cdot \operatorname{grad} \zeta_j \theta .$$

Using (7.17) and the form of V together with standard inequalities we find

$$\left(1 - \frac{\lambda}{\mu}\right) \int_{\Omega_k} \varrho |\nabla\theta|^2 \leqslant C_\varepsilon \int_{S_j} |\nabla w|^4 + \varepsilon \int_{S_j} |\nabla\theta|^2 \tag{7.20}$$

where C_ε depends on ϱ_0 (we are assuming $|\nabla w| < \tilde{r} < 1$). Choosing ε to be $(\mu - \lambda)/2\mu$ in (7.20) we get

$$\int_{\Omega_k} \varrho |\nabla\theta|^2 \leqslant \frac{2\mu C_\varepsilon}{\mu - \lambda} \int_{S_j} |\nabla w|^4. \tag{7.21}$$

Let $B_j = \{(x, \eta) \in \Omega | j \leqslant x \leqslant j + 1\}$. According to Lemmas 3.2 and 3.5 we can estimate $|\nabla w|$ on S_j in terms of $\int |\nabla w|^2$ over the set $B_{j-1} \cup B_j \cup B_{j+1} \supset S_j$. We define

$$b_j = \int_{B_j} |\nabla w|^2 \tag{7.22}$$

and conclude from (7.21) that

$$\int_{\Omega_k} |\nabla\theta|^2 \leqslant \tilde{C}_1 (b_{j-1} + b_j + b_{j+1})^2. \tag{7.23}$$

To estimate χ we write

$$\chi(x, \eta) = \int_{\Omega} G(x - x', \eta, \eta')(1 - \zeta_j(x')) \operatorname{div} V(x', \eta')\, dx' d\eta'$$

and restrict x to the interval $[j, j + 1]$. Due to the presence of the factor $1 - \zeta_j$, we will have $|x - x'| \geqslant \frac{1}{4}$ when the integrand is nonzero. Given the estimates of Lemma 7.1 it is an easy matter to see from (7.8) that the sum for G as well as the sums corresponding to any compound derivative of G containing at most one each with respect to η and η' converge absolutely for $|x - x'| \geqslant \frac{1}{4}$. If we integrate by parts to remove derivatives from V, we can show simply that

$$|\nabla\chi| \leqslant \tilde{C}_2 \sum_{\substack{i=-\infty \\ j \neq 1}}^{\infty} \exp[-p|i - j|]\, b_i . \tag{7.24}$$

The periodicity of w allows the further estimate

$$\sum_{i\neq j}\exp[-p|i-j|]b_i\leqslant\sum_{i=1}^{2k}\sum_{N=-\infty}^{\infty}\exp[-p|i+2kNd-j|]b_i= \tag{7.25}$$
$$=\frac{1}{1-\exp[-2kp]}\sum_{i=1}^{2k}\exp[-p|i-j|]b_i\,.$$

In the sequel we will absorb the factor $(1-\exp[-2kp])^{-1}$ into the constants appearing.

From Lemmas 7.3 and 3.6 we know that $b_i\leqslant\min(C_0'm^2, C_0''i^{-2/3})$ where $m=\|w\|_{L^\infty}$ and C_0', C_0'' depend on ϱ_0, ϱ_1, η_0, and R. Suppose $C_0''n_0^{-2/3}\leqslant C_0'm^2$ and that $n\geqslant n_0$. Then

$$\beta_j\equiv\sum_{i=0}^{n}\exp[-p(j-i)]b_i\leqslant \tag{7.26}$$
$$\leqslant C_0\Big[m^2\sum_{i=1}^{n_0-1}\exp[-p(j-i)]+\sum_{i=n_0}^{n}\exp[-p(j-i)]i^{-2/3}\Big]$$
$$\leqslant C_0\exp[-p(j-n)][m^2\exp[-p(n-n_0+1)]+\tilde{C}_0n^{-2/3}]$$

where we have estimated the second term in the square bracket using a comparison with an integral as in the proof of Lemma 7.2. Since w is even in x and $2k$ periodic, $b_i=b_{2k-i}$. Thus for $j\leqslant k$,

$$\sum_{i=2k-n}^{2k}\exp[-p(j-i)]b_i\leqslant\beta_j\,. \tag{7.27}$$

If we combine (7.23)-(7.27), recalling that $b_j\leqslant 2\Big(\int_{B_j}|\nabla\theta|^2+\int_{B_j}|\nabla\chi|^2\Big)$ we obtain

$$b_j\leqslant C_1(b_{j-1}+b_j+b_{j+1})^2+C_2\Big[\sum_{i=n+1}^{2k-n-1}\exp[-p|i-j|]b_i+q\exp[-p(j-n)]\Big]^2$$

where $q=[C_3\exp[-pn]+C_4n^{-2/3}]$.

We can now appeal to Lemma 7.2 to conclude that there is a constant $\tilde{C}=\tilde{C}(\varrho_0,\varrho_1,\eta_0,R,p)$ such that

$$\int_{B_j}|\nabla w|^2=b_j\leqslant\tilde{C}\exp[-2pj]$$

for $n<j\leqslant k$ and n sufficiently large. The estimates (7.14) now follow from inequality (3.10) and from Lemmas 3.2 and 3.5.

8. – Steady waves and limiting forms.

To avoid excessive length in the statements of the theorems in this section we briefly summarize what we know at this point and recall notation that will be used in the statements. A density stratification (2.11) is given which is ϱ_0 for y below $|\eta_0|$ and drops to a value $\varrho_1 > 0$ over a transition width δ. With this density the problem (2.16) $A'(w) = \lambda B'(w)$, $A(w) = R^2$ has $2k$ periodic solutions $w_1 > 0$ and $w_2 < 0$ with corresponding eigenvalues λ_1, λ_2. The restrictions $0 < \delta < \delta_0$, $0 < R < R_1$ in Theorem 4.1 guarantee that the pairs (w_i, λ_i) are actually solutions of the «physical» problem $F'(w) = \lambda B'(w)$ and that they are symmetrized. Letting $e = \varrho_0 \eta_0^{-2} - \varrho_1 \eta_1^{-2}$(cf. (5.1)) we distinguished the solution w having the same sign as e. With the restrictions $0 < \delta \leqslant \tilde{\delta}$, $0 < R \leqslant \tilde{R}$, and $k > \tilde{k}(R)$ from Lemma 5.2 and Corollary 7.1, we saw in §§ 5-7 that the associated λ is strictly below the critical value $\mu = \mu_\delta$, that $\|w\|_{L^\infty}$ has a positive lower bound, that w has nontrivial dependence on x, and, indeed, that it has exponential decay in x as does its gradient. All of these estimates were independent of δ and k for fixed R. Of course, $\tilde{\delta}$, $\tilde{R}$, and $\tilde{k}$ depend on ϱ_0, $\varrho_1 < \varrho_0$, and η_0.

With $c^2 = g/\lambda$, w is a critical point of G in (2.10) or equivalently $u = u(x, \psi) = w(x, \eta(\psi))$ is a critical point of $E(u)$ in (2.7) where $\mathring{y}(\psi)$ is the inverse function to $\mathring{\psi}(y)$ defined by (1.7) using ϱ_∞ and c. The functions $\varrho(\psi)$ and $H(\psi)$ are defined by (1.8)-(1.10). Since $\mathring{y}$ is a critical point of Φ defined by (2.3), according to (2.6) $y = \mathring{y}(\psi) + u(x, \psi)$ is a critical point of Φ; that is, y satisfies equation (1.11). Since ϱ_∞ is C^∞ for $\delta > 0$, all solutions are C^∞. The solution y corresponds to a periodic internal wave. According to Corollary 7.1 it is a wave with vertical component $V \not\equiv 0$. For $e > 0$ it is a wave of elevation; for $e < 0$ it is a wave of depression. This is easily seen since for $e > 0$ (< 0) and $0 < x < k$ we have $y_x \leqslant 0$ $(\geqslant 0)$ on a streamline. Alternatively, w measures the spatial deviation of the streamline with label η.

We can briefly summarize the results so far obtained as follows

THEOREM 8.1. *If the hypotheses of Theorem 2.1 are further restricted to require $\delta > 0$ and $k < \infty$ then the problem (1.11)-(1.12) has a nontrivial solution (λ, y), with $y \in C^\infty$, satisfying properties 1)-6) of Theorem 2.1.*

We'll use Ω_k^- and Ω_k^+ to denote the subsets of Ω_k where $\eta < 0$ or $\eta > 0$, respectively. We let $\Omega^\pm = \Omega_\infty^\pm$ and define $\mathfrak{D}_k^\pm$, $\mathfrak{D}^\pm$ analogously using $\psi < 0$ or $\psi > 0$.

THEOREM 8.2. *Suppose $e = \varrho_0 \eta_0^{-2} - \varrho_1 \eta_1^{-2} \neq 0$, $0 < R \leqslant \tilde{R}$, and $k > \tilde{k}(R)$.*

Then for $\delta = 0$ *there is a solution* (λ, y) *of problem* (1.11)-(1.12) *satisfying properties* 1)-6) *of Theorem* 2.1 *with* $|\nabla w|$ *taken in the* $W^{1,\infty}(\Omega_k)$ *sense. Here the base flow corresponds to*

$$\mathring{y}(\psi) = \begin{cases} \dfrac{\psi - \psi_0}{\varrho_0^{1/2} c}; & \psi_0 \leqslant \psi \leqslant 0 \\ \dfrac{\psi}{\varrho_0^{1/2} c} - \dfrac{\psi_0}{\varrho_0^{1/2} c}; & 0 \leqslant \psi \leqslant \psi_1 . \end{cases} \tag{8.1}$$

Also, w *satisfies*

7) *For each* $\beta < 1$ *and some* $\alpha > 0$, w *is a limit in* $H_k^1 \cap C_k^\beta(\Omega)$ *and in* $C^{1+\alpha}$ *on each compact subset of* $\Omega^\pm$ *of solutions of* $F'(w) = \lambda B'(w)$ *with* δ*'s approaching* 0. *Further* $w \in C_k^{1+\alpha}(\bar{\Omega}^\pm)$ *and* $\partial w/\partial x \in C_k^\alpha(\Omega)$ *with bounds independent of* k.

PROOF. We suppose $e > 0$ (the case $e < 0$ being similar), fix R in $(0, \tilde{R})$ and fix $k > \tilde{k}(R)$. Let δ_j be a sequence of transition widths converging to zero and (λ_j, w_j) the associated solutions in Theorem 4.1 with $w_j > 0$. To avoid any confusion suppose $j = 3, 4, 5, \ldots$. We will return to the associated $y_j(x, \psi)$ below. First we establish limit properties for a subsequence of w_j. By the results earlier in this section properties 1)-6) of Theorem 2.1 are satisfied by (λ_j, w_j) uniformly in j. By Corollary 2.1 and Lemma 3.5 w_j is uniformly bounded in $C^{1+\alpha}$ on $\Omega_k - \Sigma_j$ where $\Sigma_j = \{(x, \eta) | 0 < \eta < \delta_j\}$ and uniformly bounded in $W^{1,\infty}$. Thus a subsequence $w_{1,1}, w_{1,2}, w_{1,3}, \ldots$ converges to w weakly in H_k^1, strongly in L_k^2 and by Arzela-Ascoli, uniformly in $C^1(\Omega_k - \Sigma_1)$ and in $C^\beta(\Omega_k)$ for a chosen $\beta < 1$. We can also assume that the corresponding eigenvalues converge to λ. A further subsequence $w_{2,1}$, $w_{2,2}$, etc. converges in $C^1(\Omega_k - \Sigma_2)$ as well, and continuing we find a diagonal sequence $w_{j,j}$ which, in addition to the given convergence, converges in $C^{1+\alpha}$ on every compact subset of Ω_k^+ or Ω_k^- (cf. [17], p. 283). For notational convenience suppose the original w_j is the sequence with these convergence properties. Since w_j has a uniform $C^{1+\alpha}$ bound on $\Omega_k - \Sigma_j$, the limit w has an extension to $C^{1+\alpha}(\overline{\Omega_k^\pm})$ with a norm independent of k. Since $(\partial/\partial x) w_j$ has a C^α bound uniformly in j and k in all of Ω_k (cf. Lemma 3.2) the limit function w will have a derivative $\partial w/\partial x$ in $C_k^\alpha(\Omega)$ with the same bound. We use this w in part 7) and form $y = \mathring{y} + w(x, \eta(\psi))$ from it. To complete 7) we note that the convergence of w_j in H_k^1 follows from the C^1 convergence of w_j on each compact subset of $\Omega_k^\pm$ and from the uniform boundedness of ∇w_j on all of Ω_k (cf. Lemmas 3.2, 3.5). In fact, properties 1), 3), 5), and 6) follow easily from the nature of the convergence and the bounds existing for λ_j and w_j. Likewise, 2) will be satisfied independently of $\mathring{y}$ since w inherits periodicity.

To see that part 4) holds we first note that w satisfies the linear equation (2.19) $(\partial/\partial x_i)\tilde{a}_{ij}(\partial w/\partial x_j) = 0$ in $\Omega^\pm$ with $\tilde{a}_{ij}$ smooth (cf. the proof of Theorem 2.2). We're assuming $e > 0$ so the limiting w satisfies $w \geqslant 0$. Since $w = 0$ where $\eta = \eta_0$ or η_1, and is periodic in x, the maximum of w in each of $\Omega^\pm$ must occur where $\eta = 0$. Since $w = \hat{w}$, the maximum occurs at $x = 0$ and must be positive, for $w \not\equiv 0$. From the strong maximum principle it follows that $w > 0$ in both $\Omega^\pm$. We'll show that $w > 0$ where $\eta = 0$, as well, so that 2) holds. If $w(x_0, 0) = 0$ for some $x_0 > 0$ then since $w = \hat{w}$, $w_x(x_0, 0) = 0$ follows. To see that w and w_x cannot both vanish at $(x_0, 0)$ we integrate the equation $F'(w) = \lambda B'(w)$ (equivalently (3.49)) over a rectangle $D = \{(x, \eta) \,|\, |x - x_0| < \varepsilon_1,\ |\eta| < \varepsilon_2\}$, assuming for the moment that w is smooth on D (i.e. consider $w = w_j$). We find

$$(8.2)\qquad \int_{-\varepsilon_2}^{\varepsilon_2}\left(\varrho\,\frac{w_x}{1+w_\eta}\right)\Bigg|_{x_0-\varepsilon_1}^{x_0+\varepsilon_1} d\eta + \int_{x_0-\varepsilon_1}^{x_0+\varepsilon_1}\left(\varrho\,\frac{1}{2}\,\frac{(2w_\eta + w_\eta^2 - w_x^2)}{(1+w_\eta)^2}\right)\Bigg|_{-\varepsilon_2}^{\varepsilon_2} dx = \lambda\int_{x_0-\varepsilon_1}^{x_0+\varepsilon_1}(\varrho w)\Big|_{-\varepsilon_2}^{\varepsilon_2} - \lambda\iint_D \varrho w_\eta\,.$$

The identity (8.2), which holds for (λ_j, w_j) will hold for the limiting quantities (λ, w) described above since all integrands are bounded and converge uniformly in closed subsets of $D \cap \Omega^\pm$. Since the integrands in (8.2) are continuous in $\bar{\Omega}^+$ and $\bar{\Omega}^-$ separately, we can let $\varepsilon_2 \to 0$ and then $\varepsilon_1 \to 0$ to obtain

$$(8.3)\qquad \left[\varrho\left(\frac{w_\eta + \frac{1}{2}w_\eta^2}{1+w_\eta}\right)\right]_{x_0} = \lambda[\varrho w]_{x_0}$$

where $[g]_{x_0} = g(x_0, 0+) - g(x_0, 0-)$. In general the term w_x would appear in (8.3), but we're assuming it vanishes at $(x_0, 0)$. Since we are assuming $w(x_0, 0) = 0$, a further consequence of the strong maximum principle is that $w_\eta(x_0, 0+) > 0$ and $w_\eta(x_0, 0-) < 0$. Of course, $|w_\eta| \leqslant \tilde{r} < \frac{1}{2}\sqrt{2}$. Since $w(x_0, 0) = 0$, $[\varrho w]_{x_0} = 0$ and the equation resulting from (8.3) cannot be satisfied. Thus $w(x, 0) > 0$ for $0 \leqslant x \leqslant k$.

We must still show that we obtain a weak solution of (1.11) by adding to w a suitable $\mathring{y}$, i.e. the limit of the trivial flow solutions associated with w_j, λ_j. Letting $c_j = (g/\lambda_j)^{1/2}$ and letting $\mathring{y}_j(\psi)$ be the function $\mathring{y}$ corresponding to a trivial flow with speed c_j and transition width δ_j, we know that $y_j(x, \psi) = \mathring{y}_j(\psi) + w_j(x, \eta_j(\psi))$, with $\eta_j(\psi) = \mathring{y}_j(\psi) - \mathring{y}_j(0)$, is a solution of (1.11)-(1.12). Letting $c = \lim_{j\to\infty} c_j$ one shows easily using (1.7) that $\mathring{y}_j$ converge uniformly to $\mathring{y}$ given in (8.1) and that the convergence is in C^1 on any closed set not containing $\psi = 0$. The functions $\eta_j(\psi)$ converge in the same manner to

$$\eta(\psi) = \begin{cases} \dfrac{\psi}{\varrho_0^{1/2} c}; & \psi_0 \leqslant \psi \leqslant 0 \\ \dfrac{\psi}{\varrho_1^{1/2} c}; & 0 < \psi \leqslant \psi_1 . \end{cases} \tag{8.4}$$

Since the $C^{1+\alpha}$ bounds on y_j are uniform in j on each compact subset of $\Omega^\pm$, y_j converges to a function $y = y(x, \psi)$ in C^1 on such subsets and the resulting y has extensions to $C^{1+\alpha}$ of the closed regions $\overline{\Omega_k^\pm}$. Similarly $y \in C_k^\beta(\Omega)$ for any $\beta < 1$ and $\partial y/\partial x \in C_k^\alpha(\Omega)$. Clearly (1.12) holds. We have $\partial y_j/\partial\psi \geqslant (\partial \mathring{y}_j/\partial\psi)(1 - \tilde{r})$ (cf. (2.8)) so $\partial y/\partial\psi > 0$ for $\psi \neq 0$. Since the associated functions ϱ_j' and H_j' converge uniformly on compact subsets of $\Omega^\pm$, it follows that $y = y(x, \psi)$ is a weak solution of (1.11) in $\mathfrak{D}^\pm$. In fact, since ϱ_j', H_j' and their limits are zero wherever ϱ is constant, w satisfies an analytic equation and thus is an analytic function of x and ψ (cf. [21], p. 505). Since $\mathring{y}$ is linear in $\Omega_k^\pm$, y is also analytic in $\mathfrak{D}^\pm$. One can also see this from the fact that the inverse function $\psi(x, y)$ is harmonic where ϱ is constant (cf. (1.5)) and hence ψ is analytic for $y \neq y(x, 0)$.

What remains to be shown is the continuity of pressure (item ii) in Definition (1.1)) which is the condition connecting the two regimes on either side of $\psi = 0$. Suppose for the moment that we have a solution $y = y(x, \psi)$ of (1.11) which is in $C_k^2(\Omega)$. The pressure is defined by equation (1.4) and according to (1.3) can be expressed as

$$\begin{aligned} p &= H(\psi) - \frac{1}{2}|\nabla\psi|^2 - gy\varrho(\psi) = \\ &= H(\psi) - \frac{1}{2}\frac{1 + y_x^2}{y_\psi^2} - gy\varrho(\psi) . \end{aligned} \tag{8.5}$$

Let $\tilde{D} = \{(x, \psi) \big| |x - x_0| < \varepsilon_1,\ |\psi| < \varepsilon_2\}$. In analogy with the derivation of (8.3) from (3.49) one can integrate (1.11) over $\tilde{D}$, pass to the limit using smooth solutions and then let $\varepsilon_2 \to 0$ and $\varepsilon_1 \to 0$ to obtain

$$[p]_{x_0} = 0 \tag{8.6}$$

for all x_0, using (8.5). Thus p is continuous and the proof is complete.

Referring again to the discussion at the beginning of this section for notation we have the following result, showing that a system of two layers of constant but differing density will support a solitary wave.

THEOREM 8.3. *For $\delta = 0$, each $e \neq 0$, and each R, $0 < R < \tilde{R}$ there is a solution (λ, y) of problem* (1.11)-(1.12) *satisfying properties* 1), 3), 4), 5), *and* 6) *of Theorem* 2.1 *with $k = +\infty$ and ∇w taken in the sense of $W^{1,\infty}$. The base flow corresponds to $\mathring{y}$ in* (8.1) *and w satisfies*

7) *For each $\beta < 1$ and each bounded set $\tilde{\Omega}$, w is a limit in $C^{\beta}(\tilde{\Omega}) \cap C^1(\tilde{\Omega}^{\pm})$ of periodic solution with periods increasing to ∞. Further $w \in C^{1+\alpha}(\Omega^{\pm})$ and $\partial w/\partial x \in C^{\alpha}(\Omega)$ for some $\alpha > 0$.*

PROOF. As before, we do the case $e > 0$ and fix an R, $0 < R \leqslant \tilde{R}$. For $k = 3, 4, 5, \ldots$ let w_k denote the solution from Theorem 8.2 having period $2k$. From part 7) of that result and the Arzela-Ascoli Theorem it follows that a subsequence $w_{1,1}, w_{2,2}, \ldots$ converges in $C^{\beta}(\Omega_1) \cap C^1(\overline{\Omega}_1^{\pm})$ to a function w. A further subsequence has the same type of convergence on Ω_2, and so on. The diagonal sequence $w_{k,k}$ will converge to a function w on Ω in $C^{\beta}(\tilde{\Omega}) \cap C^1(\overline{\Omega^{\pm}})$ for each bounded set $\tilde{\Omega}$. By restricting k to some subset of the integers we can call the convergent sequence w_k and can suppose a sequence is chosen so that λ_k converges to a number λ. The bounds on λ_k persist, so λ satisfies property 1) with $\delta = 0$. Likewise, properties 5), 6) and the remainder of 7) follow from the convergence and uniform estimates. Of course, all the properties shown so far are shared by the solution $w = 0$. That w is nontrivial follows from the lower bound in Lemma 6.1. Alternatively, since the exponential decay of ∇w_k is uniform in k, the contribution to the integral giving $F(w_k)$, coming from $\Omega_k - \Omega_N$, can be made arbitrarily small for a suitably large N and all $k > N$. Then since the gradient converges in L^{∞} on each Ω_N, it follows that $F(w) = R^2$, property 3), where the integral is taken on all of Ω. Property 4) is shown as in the previous proof.

The associated functions $\mathring{y}_k(\psi)$ and $\eta_k(\psi)$ are given by (8.1) and (8.4), respectively, and change with k only in that the speed $c = c_k = (g/\lambda_k)^{1/2}$ changes. The convergence of $y_k = \mathring{y}_k + w_k(x, \eta_k(\psi))$ to function $y(x, \psi)$ in $\tilde{\mathfrak{D}}$, in $C^{\beta}(\tilde{\mathfrak{D}}) \cap C^1(\tilde{\mathfrak{D}}^{\pm})$ for each bounded $\tilde{\mathfrak{D}}$, follows from the known convergence of the component functions. The property of being a weak solution of (1.11) in $\mathfrak{D}^{\pm}$ is maintained by the C^1 convergence as is the pressure continuity. The boundary condition (1.12) is clearly satisfied by y so the proof is complete.

Rather than let $\delta \to 0$ and then let $k \to \infty$ one can first let $k \to \infty$ for a fixed $\delta > 0$, obtaining a solitary wave corresponding to a smooth density. We state a theorem dealing with this limit below. A subsequent limit in δ can be taken and might produce a solution of (1.11) different from that in Theorem 8.3 (we do not show uniqueness) but we conjecture that it yields the same solution. Consequently we don't state a theorem covering the latter limit.

THEOREM 8.4. *For each* $e \neq 0$, *for each* δ, $0 < \delta < \tilde{\delta}$ *and for each* R, $0 < R < \tilde{R}$ *there is a solution* (λ, y) *of problem* (1.11)-(1.12) *satisfying properties* 1), 3), 4), 5), *and* 6) *of Theorem* 2.1 *with* $k = \infty$. *Also*

7) *w is in C^∞ and is the limit in C^∞ on each bounded set, of solutions from Theorem 8.1, with periods increasing to ∞. The function y is C^∞ in $\mathfrak{D}$.*

PROOF. Most of the proof is analogous to that of Theorem 8.3 and we don't repeat the similar parts. The fact that w is in C^∞ follows from standard regularity results (cf. [16], Chapter 4). The $C^{l+\alpha}$ bounds on w on bounded sets depend on the smoothness of $f(p_1, p_2)$ and ϱ', on the ellipticity constant and on λ and hence will be independent of the period $2k$. We can thus use a diagonal process to choose a sequence w_k, k in some subset of the positive integers, so that w_k converges in $C^l(\Omega_n)$ for each $l > 0$ and $n > 0$. Since η is C^∞, y is C^∞ in $\mathfrak{D}$, completing the proof.

We conclude this section with a few comments on the results. The restriction on energy $0 < R < \tilde{R}$ in the results of this section is largely due to the regularization of $f(p_1, p_2)$ by $a(p_1, p_2)$ in § 2. Otherwise the techniques are capable of producing « finite amplitude » solutions as is done in [10]. In none of the permanent wave problems can one expect to have solutions of arbitrarily large energy. In the recent work of Amick and Toland [22] (cf. also [23]) one can reasonably assume that the finite range of possible energies is exhausted along a branch of solutions they obtain. One end of the branch is the bifurcation point $c = c_0$, $w = 0$ in our notation and the other « end » is the limiting case of zero particle velocity at the crest of the wave, the Stoke's wave.

We saw in Theorems 8.1 and 8.2 that the wave is one of elevation if $\varrho_0 \eta_0^{-2} > \varrho_1 \eta_1^{-2}$ and is of depression if the reverse inequality holds. This condition was obtained by Keulegan [2], Long [3], and Benjamin [1] in asymptotic analyses of the problem. The exponential behavior of the wave « tails » is observed in the asymptotic analyses just cited and in the exact treatment ([8], [9]) of both internal and surface waves. In the works [8] and [9] which use an asymptotic analysis as the basis for an existence result using the implicit function theorem, the exponent in the exponential decay is quite precise. In our notation it is a multiple of $\Delta = c_0^{-2} - c^2$, measuring the deviation from the critical wave speed. Then waves are asymptotically equal to $\Delta \exp[-\sqrt{\Delta}|x|]$ multiplied by a function not depending on Δ. In Lemma 7.4 we showed the exponential decay to be of order $\exp[-p|x|]$

where p was chosen to satisfy $p < \gamma_1^{1/2}$, γ_1 being the first eigenvalue of (7.2). The eigenvalue γ_1 is bounded below by a multiple of $\mu - \lambda = g(c_0^{-2} - c^{-2})$, so we obtain the correct order of decay.

It is interesting to examine the velocity (U, V) near the fluid interface corresponding to $\psi = 0$ (equivalently $\eta = 0$) in the case $\delta = 0$. Since $y(x, 0)$ is the height of the streamline where $\psi = 0$ and $y_x = w_x$, the continuity of w_x expresses the fact that fluid particles on either side of the streamline are moving in the same direction. Underlying all the estimates in Section 3 is the idea of controlling the L^∞ norm of w_η. No stronger estimate is possible, for when $\delta = 0$ the strong maximum principle shows $w_\eta(0, 0-) > 0$ and $w_\eta(0, 0+) < 0$ when $w > 0$, and thus no further regularity could have been attained, uniformly in δ. The discontinuity in w_η corresponds to a discontinuity in U across the separating streamline. To see this we recall that

$$\varrho^{1/2} U = \frac{\partial \psi}{\partial y} = \frac{1}{\partial y/\partial \psi} = \frac{1}{\mathring{y}_\psi + w_\eta \mathring{y}_\psi} = \frac{\varrho^{1/2} c}{1 + w_\eta}$$

using (1.3), (1.6) and (2.8), and so $U = c(1 + w_\eta)^{-1}$. Thus for a wave of elevation, corresponding to $w > 0$, we see that at the crest, U is smaller than c just below the streamline $\psi = 0$ and larger than c just above it.

Internal waves which exhibit the characteristics of solitary waves are observed, both in the laboratory and in the field. Walker [24] has generated such waves in a two-fluid system, though with a free surface rather than a rigid top. His article also contains references to field observations in geophysical two-layer systems.

REFERENCES

[1] T. B. Benjamin, *Internal waves of finite amplitude and permanent form*, J. Fluid Mech., **25** (1966), pp. 241-270.

[2] G. H. Keulegan, *Characteristics of internal solitary waves*, J. Res. Nat. Bur. Standards, **51** (1953), pp. 133-140.

[3] R. R. Long, *Solitary waves in one- and two-fluid systems*, Tellus, **8** (1956), pp. 460-471.

[4] A. S. Peters - J. J. Stoker, *Solitary waves in liquids having non-constant density*, Comm. Pure Appl. Math., **13** (1960), pp. 115-164.

[5] R. R. Long, *Some aspects of the flow of stratified fluids*. Part I: *A theoretical investigation*, Tellus, **5** (1953), pp. 42-57.

[6] M. L. Dubreil-Jacotin, *Sur les théorèmes d'existence relatifs aux ondes permanentes periodiques à deux dimensions dans les liquides hétérogènes*, J. Math. Pures Appl. (9), **19** (1937), pp. 43-67.

[7] C.-S. YIH, *Exact solutions for steady two-dimensional flow of a stratified fluid*, J. Fluid Mech., **9** (1960), pp. 161-174.

[8] A. M. TER-KRIKOROV, *Théorie exacte des ondes longues stationnaires dans un liquide hétérogène*, J. Mécanique, **2** (1963), pp. 351-376.

[9] K. O. FRIEDRICHS - D. H. HYERS, *The existence of solitary waves*, Comm. Pure Appl. Math., **7** (1954), pp. 517-550.

[10] J. BONA - D. BOSE - R. E. L. TURNER, *A global theory of steady waves in continuously stratified fluids*, Math. Res. Center Tech. Rep., University of Wisconsin, to appear.

[11] T. B. BENJAMIN, *A unified theory of conjugate flows*, Philos. Trans. Roy. Soc. London, *A* **269** (1971), pp. 587-643.

[12] E. ZEIDLER, *Bifurcation theory and permanent waves*, in *Applications of Bifurcation Theory*, P. Rabinowitz ed., Academic Press, New York, 1977.

[13] J. J. STOKER, *Water Waves*, Interscience Publishers, Inc., New York, 1957.

[14] J. B. MCLEOD - R. E. L. TURNER, *Bifurcation for non-differentiable operators with an application to elasticity*, Arch. Rational Mech. Anal., **63** (1976), pp. 1-45.

[15] D. GILBARG - N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der mathematische Wissenschaften 224, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[16] A. AMBROSETTI - P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Functional Analysis, **14** (1973), pp. 349-381.

[17] O. A. LADYZHENSKAYA - N. N. URAL'TSEVA, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[18] R. A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.

[19] G. POLYA - G. SZEGO, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematical Studies 27, Princeton University Press, Princeton, 1950.

[20] S. AGMON - A. DOUGLIS - L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math., **12** (1959), pp. 623-727.

[21] R. COURANT - D. HILBERT, *Methods of Mathematical Physics*, vol. II, Interscience Publishers, New York, 1962.

[22] C. J. AMICK - J. F. TOLAND, *On Solitary Water-Waves of Finite Amplitude*, Arch. Rat. Mech. Anal., **76** (1981), pp. 9-95.

[23] G. KEADY - J. NORBURY, *On the existence theory for irrotational water waves*, Math. Proc. Cambridge Philos. Soc., **83** (1978), pp. 137-157.

[24] L. R. WALKER, *Interfacial solitary waves in a two-fluid medium*, The Physics of Fluids, **16** (1973), pp. 1796-1804.

University of Wisconsin
Department of Mathematics
Madison, Wisconsin 53706