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A Classification of Strictly Pseudoconcave Homogeneous Manifolds.

A. T. HUCKLEBERRY - D. SNOW

Dedicated to the memory of Aldo Andreotti

1. – Introduction.

A non-compact complex manifold X is *strictly pseudoconcave* if it can be exhausted by relatively compact open sets whose boundaries are smooth and strictly pseudoconcave. In [9], under the further assumption that a complex Lie group acts holomorphically and transitively on X, we show that X is the total space of a positive line bundle over a compact homogeneous rational manifold Q. The bundle map is given by the natural map of coset spaces $\pi\colon X = G/H \to G/RH = Q$ where R is the radical of G. Furthermore, since every positive line boundle over a compact homogeneous rational manifold is homogeneous and ample, [8], X can be equivariantly imbedded as a homogeneous cone in complex projective space as follows. Let $\{s_0, \ldots, s_n\}$ be a basis of the vector space of sections and let z be a local fiber coordinate. Then the map $\mu\colon X \to \boldsymbol{P}^{n+1}$ defined by $\mu(p) = [z(p):s_0(\pi(p)):\ldots:s_n(\pi(p))]$ realizes X as the union of projective lines connecting the hyperplane section $\mu(X) \cap \{z = 0\} \cong Q$ to the point $[1:0:\ldots:0]$, with the vertex $[1:0:\ldots:0]$ itself removed. Any such cone over a compact homogeneous rational manifold is clearly strictly pseudoconcave and homogeneous under a complex Lie group.

In this paper we extend the above classification to strictly pseudoconcave manifolds X which are homogeneous in the classical sense, i.e. given two points $p, q \in X$, there exists an automorphism $g \in \operatorname{Aut}(X)$ such that $g(p) = q$. (If $\dim_{\boldsymbol{C}} X = 2$ we must also assume that X has a compactification to a

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complex space.) Note that this is a priori a much weaker assumption since a non-compact complex manifold can be homogeneous in this sense and not be homogeneous under a Lie group (see e.g. [10]). However, the first step of our classification is to show that $G := \text{Aut}\,(X)$ *is* in fact a Lie group (§ 2). Thus, the major part of this paper is devoted to the case in which X is homogeneous via the group G, but where *no complex* Lie group acts holomorphically and transitively on X. Under these hypotheses, we show (§ 4) that X has an equivariant compactification V which is a homogeneous rational manifold under the action of a complex semi-simple Lie group S. The Lie group G is simple and is a real form of S (§ 5).

If $A := V \setminus X$ has interior, then, due to the strict pseudoconcavity of X, A is a non-compact hermitian symmetric space and V is its compact dual. In addition, X can be realized as a tube neighborhood in the normal bundle of a codimension 1 *complex* orbit of a maximal compact subgroup K_G of G (§ 6). A theorem of E. Oeljeklaus, [16], then implies that $V = \boldsymbol{P}^n$ and X is the complement of a closed euclidean ball $\boldsymbol{B}^n$. It seems worth noting that this is the only way that a homogeneous Stein manifold can be imbedded as a domain in a connected compact complex manifold so that the complement of its closure is also homogeneous.

If A has no interior, then the methods of analytic continuation along with the algebraic techniques of J. Wolf, [25], show (§ 6, § 7, § 8) that A is a totally real submanifold of V with $\dim_{\boldsymbol{R}} A = \dim_{\boldsymbol{C}} V$. Moreover, both G and K_G act transitively on A and the generic K_G-orbit in V is a strictly pseudoconvex hypersurface contained in a Stein submanifold of V. Due to the work of A. Morimoto and T. Nagano, [13], and T. Nagano, [15], this situation is well-understood. The following is a typical example: V is complex projective space $\boldsymbol{P}^n$, with $G = PSL(n+1, \boldsymbol{R})$ acting on V in the usual way, and X is the complement of $A := \boldsymbol{R}P^n \subset \boldsymbol{P}^n$. In § 3 we discuss all of the possible examples. A detailed statement of our classification is contained in § 9, Theorem 9.2.

2. – Preliminaries.

Let X be a complex manifold with $\dim_{\boldsymbol{C}} X \geqslant 2$, and let Ω be a relatively compact open subset of X with smooth boundary, $\partial\Omega$. We say that $\partial\Omega$ is *strictly pseudoconcave* if, for every point $p \in \partial\Omega$, there is a smooth local defining function ψ for $\partial\Omega$ in some coordinate neighborhood U containing p, such that $\Omega \cap U = \{x \in U \mid \psi(x) > 0\}$ and such that the Levi-form of ψ at p,

$$\mathfrak{L}(\psi)_p := \sum \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j}(p)\, dz_i \otimes d\bar{z}_j\,,$$

is positive definite on the unique maximal complex subspace of the (real) tangent space of $\partial\Omega$ at p. This definition is easily shown to be independent of both the choice of ψ and the choice of local coordinates, so that strict pseudoconcavity is a holomorphic invariant of $\partial\Omega$. It is well known that the defining function ψ may be chosen so that $\mathfrak{L}(\psi)_p$ is positive definite on the full complex tangent space of X at p. In this case, ψ is said to be *strictly plurisubharmonic* at p. It is clear that there exists a smooth function φ defined in an open neighborhood W of the boundary $\partial\Omega \subset W$, such that $\Omega \cap W = \{x \in W | \varphi(x) > 0\}$, and such that φ is strictly plurisubharmonic at every point of W. Consequently, for almost all values of $\varepsilon > 0$ sufficiently near 0, the relatively compact open sets

$$\Omega_\varepsilon := \Omega \setminus \{x \in W | \varphi(x) < \varepsilon\} \quad \text{and} \quad \Omega_{-\varepsilon} := \Omega \cup \{x \in W | \varphi(x) > -\varepsilon\}$$

have strictly pseudoconcave boundaries.

A complex manifold X with $\dim_{\boldsymbol{C}} X \geqslant 2$ is defined to be *strictly pseudoconcave* if there exists open subsets X_ν of X, $\nu \in \boldsymbol{N}$, such that

1) X_ν is relatively compact in $X_{\nu+1}$, for all $\nu \in \boldsymbol{N}$,
2) $X = \bigcup \{X_\nu | \nu \in \boldsymbol{N}\}$, and
3) ∂X_ν is strictly pseudoconcave, for all $\nu \in \boldsymbol{N}$.

The collection of open sets X_ν, $\nu \in \boldsymbol{N}$, is called a *strictly pseudoconcave exhaustion* for X. From the remarks in the preceding paragraph, we see immediately that this definition is equivalent to the following: There exists a smooth function $\varphi\colon X \to \boldsymbol{R}$ and a decreasing sequence of numbers $c_\nu \in \boldsymbol{R}$ such that

1) $X_\nu := \{x \in X | \varphi(x) > c_\nu\}$ is relatively compact in $X_{\nu+1}$, for all $\nu \in \boldsymbol{N}$,
2) $X = \bigcup \{X_\nu | \nu \in \boldsymbol{N}\}$, and
3) φ is strictly plurisubharmonic in an open neighborhood of ∂X_ν.

The function φ is called a *strictly pseudoconcave exhaustion function* for X. The advantage of this latter definition is that the boundaries, ∂X_ν, are not required to be smooth. This is convenient for proving that *a closed complex submanifold Y of a strictly pseudoconcave manifold X is also strictly pseudoconcave, whenever* $\dim_{\boldsymbol{C}} Y \geqslant 2$. One merely restricts the above exhaustion function φ to Y and notes that the restricted function is strictly plurisubharmonic in a neighborhood of ∂Y_ν. Throughout this paper we will make use of the above notation for a strictly pseudoconcave exhaustion (function) without further comment.

For a strictly pseudoconcave manifold X of dimension greater than 2, it is known that X has a minimal normal compactification, V. That is, X can be realized as an open submanifold of a normal compact complex space V, where V is minimal in the sense that $V \setminus X$ contains no positive dimensional compact complex analytic sets. The minimal compactification V is essentially obtained by taking the « Stein completion » of the manifold $X_{\nu+1} \setminus \bar{X}_\nu$, and gluing this completion back onto X (see [1], [21]). *Thus,* $V \setminus \bar{X}_\nu$ *is a Stein space,* which, for appropriate choice of ν, can be realized as a bounded subspace of $\boldsymbol{C}^N$. We shall make frequent use of this fact. Indeed, it is the primary way in which we apply the notion of strict pseudoconcavity in this paper. Compactifications may not exist in general for a strictly pseudoconcave manifold X of dimension 2, [21]. However, we will only consider the case when, in fact a compactification V does exist for X. We can take this compactification V to be minimal and normal by blowing down any exceptional sets in $V \setminus X$, and then taking its normalization.

We now prove an extension lemma for strictly pseudoconcave manifolds.

LEMMA 2.1. *Let X be a strictly pseudoconcave manifold and let V be a minimal normal compactification of X. Let $g\colon X \to X$ be a biholomorphic map of X onto itself. Then there exists a biholomorphic map $\tilde{g}\colon V \to V$ such that $g = \tilde{g} \circ \iota$, where $\iota\colon X \to V$ is the inclusion map.*

PROOF. Choose ν such that $V \setminus \bar{X}_\nu$ can be realized as a bounded subspace of $\boldsymbol{C}^N$. Let $j\colon V \setminus \bar{X}_\nu \to \boldsymbol{C}^N$ be the imbedding. We note that there exists a $\mu > \nu$ such that $g(X \setminus \bar{X}_\mu) \subset X \setminus \bar{X}_\nu$. Otherwise, we could construct a sequence of points $x_n \in X$ such that $v := \lim (x_n) \in V \setminus X$ and such that $g(x_n) \in \bar{X}_\nu$. But then $g(x_n)$ contains a subsequence which converges to a point $x \in \bar{X}_\nu$. By continuity, $v = g^{-1}(x) \in X$, a contradiction. Now, let $g' := j \circ g\colon X \setminus \bar{X}_\mu \to \boldsymbol{C}^N$. Then g' has components $g'_i\colon X \setminus \bar{X}_\mu \to \boldsymbol{C}$. By a general Hartogs' Theorem [20], each g'_i extends as a holomorphic function to $V \setminus \bar{X}_\mu$, since $V \setminus \bar{X}_\mu$ is Stein. Thus, we obtain a holomorphic map $\tilde{g}'\colon V \setminus \bar{X}_\mu \to \boldsymbol{C}^N$, and it is clear that $\tilde{g}'(V \setminus \bar{X}_\mu) \subset j(V \setminus \bar{X}_\nu)$. Define the holomorphic map $\tilde{g}\colon V \to V$ to be $j^{-1} \circ \tilde{g}'$ on $V \setminus \bar{X}_\mu$ and g on X. Doing the same for g^{-1} (which can also be arranged to map $X \setminus \bar{X}_\mu$ into $X \setminus \bar{X}_\nu$), we obtain another holomorphic map $\tilde{g}^{-1}\colon V \to V$. Since $\tilde{g} \circ \tilde{g}^{-1} = \mathrm{id}$ on the open set X, we see that $\tilde{g} \circ \tilde{g}^{-1} = \mathrm{id}$ on V. Thus, $\tilde{g}$ is invertible and therefore is the desired biholomorphic map of V onto itself extending g. □

A biholomorphic map of a complex manifold (or space) X onto itself is called an *automorphism*. The group of all automorphisms of X is denoted by $\mathrm{Aut}(X)$. We give $\mathrm{Aut}(X)$ the *compact-open topology* by declaring the open sets of Aut (X) to be generated by sets of the form

$$\{g \in \mathrm{Aut}\,(X) | g(K) \subset U\}\ ,$$

for compact subsets $K \subset X$ and open subsets $U \subset X$. If X is *compact* it is well-known that Aut (X) with the compact-open topology has the structure of a complex Lie group ([2], [11]). With the help of Lemma 2.1 we show that a similar result holds for strictly pseudoconcave manifolds.

THEOREM 2.2. *Let X be a strictly pseudoconcave manifold and let V be a minimal normal compactification of X. Then* Aut (X) *with the compact-open topology is homeomorphically isomorphic to a closed Lie subgroup of* Aut (V).

PROOF. Let G be the stabilizer in Aut (V) of the closed set $A := V \setminus X$, $G := \{g \in \mathrm{Aut}\,(V) | g(A) \subset A\}$. The subgroup G is clearly closed and therefore is a closed Lie subgroup of Aut (V), [6]. Furthermore, we obtain from Lemma 2.1 a monomorphism Aut $(X) \to$ Aut (V), $g \to \tilde{g}$, which is easily seen to be surjective onto G. Thus, it remains to show that the isomorphism of groups Aut $(X) \to G$ is also a homeomorphism. Since any compact (resp. open) subset of X is again compact (resp. open) in V, this isomorphism is open with respect to the compact-open topology. To show that it is also continuous, we need only prove that if $\{g_n\}$ is a sequence of automorphisms of V which converges uniformly on X to an automorphism g of V, then $\{g_n\}$ also converges uniformly to g on V. We choose X_ν such that $V \setminus \bar{X}_\nu$ is a bounded Stein space as in Lemma 2.1, so that all of the automorphisms g_n can be expressed on $V \setminus \bar{X}_\nu$ in terms of bounded holomorphic functions. Therefore, since these functions converge uniformly on $X_\mu \setminus \bar{X}_\nu$, $\mu > \nu$, they converge uniformly on $V \setminus \bar{X}_\nu$ by the maximum principle. The limit map which these functions define together with $g \circ \iota$ (where $\iota: X \to V$ is the inclusion) defines an automorphism of V which is clearly equal to g. Therefore, $\{g_n\}$ converges uniformly to g on V. $\square$

Thus, if X is a homogeneous strictly pseudoconcave manifold, we can identify the underlying real analytic manifold of X with the coset space G/H where $G = \mathrm{Aut}\,(X)$ and H is the isotropy subgroup of some point $x \in X$, $H := \{g \in G | g(x) = x\}$.

Let G be a connected real Lie subgroup of a complex Lie group S. We define $G^{\mathbf{C}}$, *the complexification of G in S*, to be the smallest, not necessarily closed, connected complex Lie subgroup of S which contains G. Equivalently, if $\mathfrak{g}$ and $\mathfrak{s}$ are the Lie algebras of G and S repectively, then $\mathfrak{g}$ is a real linear subspace of $\mathfrak{s}$. We define $\mathfrak{g}^{\mathbf{C}}$ to be the complex Lie algebra $\mathfrak{g} + J\mathfrak{g}$ where J is the real linear transformation defining the complex structure of $\mathfrak{s}$. Then $G^{\mathbf{C}}$

is the connected complex Lie subgroup of S canonically associated to g^{C}. The group G is said to be a *real form* of S if $\mathfrak{s} = \mathfrak{g} + J\mathfrak{g}$ as a direct sum. Note that if G and S are as above, and if both act holomorphically on a complex manifold (or space) V, then G^{C} acts holomorphically on V in the sense that the map $G^{C} \times V \to V$ is holomorphic. We refer the reader to [6] for more details on Lie groups and group actions.

We now prove a useful fibration lemma for homogeneous strictly pseudoconcave manifolds.

LEMMA 2.3. *Let G be a connected Lie group and H a closed subgroup of G such that $X := G/H$ is a strictly pseudoconcave manifold. (If $\dim_{C} X = 2$, we also assume that X has a compactification.) Let J be any closed subgroup of G containing H such that $Y := G/J$ is a complex manifold and the canonical map of coset spaces $G/H \to G/J$ is holomorphic. If the fiber J/H has positive dimension, then Y is compact.*

PROOF. We may clearly assume that G is a closed subgroup of $\mathrm{Aut}^0(X)$. Let V be a minimal normal compactification of X. By Theorem 2.2 we may identify G with a closed Lie subgroup of $\mathrm{Aut}(V)$. Choose an exhaustion set X_ν such that $V \setminus \bar{X}_\nu$ is a bounded Stein space. We claim that the fiber J/H, and hence every fiber, must intersect the fixed compact set $\bar{X}_\nu$, showing that Y is compact. For suppose $J(x) = J/H \subset X$ did not intersect $\bar{X}_\nu$. Then, since $J^0(x) = (J^0)^{C}(x) \cap X$ we have $(J^0)^{C}(x) \subset V \setminus \bar{X}$ where $(J^0)^{C}$ is the complexification of J^0 in $\mathrm{Aut}(V)$. However, $(J^0)^{C}$ is positive dimensional, so we obtain non-constant bounded holomorphic functions on the complex Lie algebra of $(J^0)^{C}$ (which is biholomorphic to C^k), a contradiction. □

3. – Examples.

In this section we present examples of non-compact strictly pseudoconcave homogeneous manifolds X of a real Lie group G. The main purpose of this paper is to prove that the following list of examples exhausts all possibilities.

(1) The first example is $X = P^n \setminus B^n$, $n \geq 2$, where

$$B^n := \{[1:z] \in P^n | {}^t\bar{z}z < 1\} .$$

It is clear that X is strictly pseudoconcave, e.g. by defining the exhaustion function to be $\varphi(z) = {}^t\bar{z}z - 1$ inside the ball of radius 2 (say) in $C^n = P^n \setminus P^{n-1}_\infty$. (We always take P^{n-1}_∞ to be $\{z_0 = 0\}$ unless otherwise noted.)

To see that X is homogeneous under a Lie group we consider the cone over X in $\boldsymbol{C}^{n+1}$, $\tilde{X} := \boldsymbol{P}^{-1}(X)$, where $\boldsymbol{P}\colon \boldsymbol{C}^{n+1}\setminus\{0\} \to \boldsymbol{P}^n$ is the defining map of $\boldsymbol{P}^n$. Note that $\tilde{X} = \{(z_0, z) \in \boldsymbol{C}^{n+1}\setminus\{0\} | {}^t\bar{z}z > \bar{z}_0 z_0\}$. Thus, the stabilizer of X in $SL(n+1, \boldsymbol{C})$ is simply the stabilizer of $\partial\tilde{X} = \{(z_0, z) \in \boldsymbol{C}^{n+1}\setminus\{0|\}{}^t\bar{z}z = \bar{z}_0 z_0\}$ in $SL(n+1, \boldsymbol{C})$. This group coincides with the stabilizer of $\boldsymbol{P}^{-1}(\boldsymbol{B}^n)$, and can be defined as follows. Let E be the matrix $\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$. Then E defines a hermitian form on $\boldsymbol{C}^{n+1}$, $\langle z, w\rangle_E := {}^t\bar{w}Ez$. The stabilizer of $\tilde{X}$ is then the group of isometries with respect to this form,

$$\begin{aligned} G &= \{A \in SL(n+1, \boldsymbol{C}) | Az \in \partial\tilde{X}, \text{ for all } z \in \partial\tilde{X}\} \\ &= \{A \in SL(n+1, \boldsymbol{C}) | \langle Az, Az\rangle_E = 0, \text{ for all } z \in \partial\tilde{X}\} \\ &= \{A \in SL(n+1, \boldsymbol{C}) | {}^t\bar{A}EA = E\} \\ &= \{A \in SL(n+1, \boldsymbol{C}) | \langle Az, Az\rangle_E = \langle z, z\rangle_E, \text{ for all } z \in \boldsymbol{C}^{n+1}\} . \end{aligned}$$

Since G is obviously closed in $SL(n+1, \boldsymbol{C})$, it is itself a Lie group. Given any two points $\tilde{z}_0$, $\tilde{w}_0 \in \tilde{X}$, we can use the Gram-Schmidt method to construct orthonormal (with respect to E) bases for $\boldsymbol{C}^{n+1}$, $\{z^0, z^1, \dots, z^n\}$ and $\{w^0, w^1, \dots, w^n\}$, such that $\tilde{z}_0 = \|\tilde{z}^0\|_E z^0$ and $\tilde{w}^0 = \|\tilde{w}^0\|_E w^0$. Let A be the change of basis matrix such that $A(z^i) = w^i$, $i = 0, \dots, n$. Then $A \in G$, and A takes the complex line containing z^0 onto the complex line containing w^0. Thus, X is homogeneous under the Lie group G. Note that this proof shows that G also has $\boldsymbol{B}^n$ and the $(2n-1)$-sphere $S^{2n-1} = \partial\boldsymbol{B}^n$ as homogeneous manifolds. The group G is often denoted by $SU^1(n+1)$ and called the indefinite unitary group. It is a real form of $SL(n+1, \boldsymbol{C})$.

In this example, X can be realized as a real analytic homogeneous fiber bundle over $\boldsymbol{P}^{n-1}_\infty$ with fiber isomorphic to the 1-dimensional disk, D. This is seen by restricting the projection $p\colon \boldsymbol{P}^n\setminus\{0\} \to \boldsymbol{P}^{n+1}$ to X and observing that p is equivariant under $SU^1(n+1)$. In this way X can be thought of as a tube neighborhood of the zero-section of the hyperplane section bundle over $\boldsymbol{P}^{n-1}_\infty$. This bundle structure on X cannot be holomorphically locally trivialized because otherwise we would obtain a holomorphic local trivialization for $\boldsymbol{B}^n\setminus\{0\}$ as a punctured disk bundle over $\boldsymbol{P}^{n-1}$ which is impossible ([7]).

The remaining examples all share a common property: They are defined as the complement in a compact homogeneous rational manifold of a totally real imbedded symmetric space of rank 1. For each symmetric space of rank 1 (i.e. S^n, $\boldsymbol{RP}^n$, $\boldsymbol{P}^n$, quaternionic projective space $\boldsymbol{QP}^n$, and the Cayley projective plane), we give this construction. Wherever the geometry is transparent, we show explicitly that the complement is in fact a strictly pseudoconcave homogeneous manifold of a Lie group.

(2) Consider the n-dimensional projective quadric hypersurface, $Q^n = := \{z \in \boldsymbol{P}^{n+1} | {}^t zz = 0\}$, with the n-sphere $S^n := \{[i:x_1:\ldots:x_{n+1}] \in \boldsymbol{P}^{n+1} | x_j \in \boldsymbol{R}, {}^t xx = 1\}$ imbedded as a totally real submanifold. Note that $S^n = Q^n \cap V_{\boldsymbol{R}}$, where $V_{\boldsymbol{R}}$ is the totally real projective subspace $\{[ix_0:x_1:\ldots:x_{n+1}] \in \boldsymbol{P}^{n+1} | x_j \in \boldsymbol{R}\}$, and that $S^n = Q^n \cap S^{2n+1}$ where $S^{2n+1} = \{z \in \boldsymbol{P}^{n+1} | {}^t\bar{z}z = \bar{z}_0 z_0\}$ is the unit sphere in $\boldsymbol{C}^{n+1} = \boldsymbol{P}^{n+1} \setminus \boldsymbol{P}^n_\infty$. In addition, S^n is the set of fixed points in Q^n of the involution $[z] \mapsto [\bar{z}]$.

Now define $X = Q^n \setminus S^n$. We claim that X is strictly pseudoconcave and homogeneous under a Lie group. To see that X is strictly pseudoconcave, we first consider the real analytic function $\tilde{\varphi}: \boldsymbol{C}^{n+1} \setminus \boldsymbol{P}^n_\infty \to \boldsymbol{R}$ defined by $\tilde{\varphi}(z) = {}^t\bar{z}z - 1$. Then, $\mathfrak{L}(\tilde{\varphi})_p = I$ for all $p \in \boldsymbol{C}^{n+1}$, so φ is strictly plurisubharmonic. Let $Y = Q^n \cap \boldsymbol{C}^{n+1} = Q^n \setminus \boldsymbol{P}^n_\infty$ and $\iota: Y \to \boldsymbol{C}^{n+1}$ be the inclusion. If we define $\varphi = \tilde{\varphi} \circ i$, then $\varphi(z) = {}^t\bar{z}z - 1 = 2({}^t\mathrm{Im}\,(z) \cdot \mathrm{Im}\,(z))$. In addition, φ is strictly plurisubharmonic since $\tilde{\varphi}$ is strictly plurisubharmonic on $\boldsymbol{C}^{n+1}$. Now smooth φ to a constant function outside the ball of radius 2 (say) in $\boldsymbol{C}^{n+1} = \boldsymbol{P}^{n+1} \setminus \boldsymbol{P}^n_\infty$. Then $\varphi: X \to \boldsymbol{R}$ is a strictly pseudoconcave exhaustion for X. Note that $S^n = \{[z] \in Q^n | \varphi([z]) = 0\}$.

To see that X is a homogeneous manifold of a Lie group, we define G to be the stabilizer of X in $SO(n+2, \boldsymbol{C})$. Then we have

$$\begin{aligned} G &= \{A \in SO(n+2, \boldsymbol{C}) | [Az] \in X, \text{ for all } [z] \in X\} \\ &= \{A \in SO(n+2, \boldsymbol{C}) | [Az] \in S^n, \text{ for all } [z] \in S^n\} \\ &= \{A \in SO(n+2, \boldsymbol{C}) | {}^t(\overline{Az}) E(Az) = 0, \text{ for all } [z] \in S^n\} \\ &= \{A \in SO(n+2, \boldsymbol{C}) | {}^t\bar{A}EA = E\} \\ &= SO(n+2, \boldsymbol{C}) \cap SU^1(n+2) =: SO^1(n+2)\,. \end{aligned}$$

Here we have, as before, that $E = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$ and $SU^1(n+2)$ is the indefinite unitary group in $SL(n+2, \boldsymbol{C})$. Therefore, G is a Lie group which acts holomorphically on X. Note that G also stabilizes the real projective subspace $V_{\boldsymbol{R}} = \{[ix_0:x_1:\ldots:x_n] | x_j \in \boldsymbol{R}\} \subset \boldsymbol{P}^{n+1}$, so that G can be explicitly described in matrix form as

$$\begin{aligned} G &= SO(n+2, \boldsymbol{C}) \cap SL(V_{\boldsymbol{R}}) \\ &= \left\{ \begin{pmatrix} a & ib \\ ic & D \end{pmatrix} \middle| \begin{pmatrix} a & -b \\ c & D \end{pmatrix} \in U^1(n+2) \cap SL(n+2, \boldsymbol{R}) \right\} \\ &\cong SO^1(n+2)\,, \end{aligned}$$

the indefinite orthogonal group in $SL(n+2, \boldsymbol{R})$. This group is easily seen to be a real form of $SO(n+2, \boldsymbol{C})$.

Now let $[z]$ and $[w]$ be two distinct points in X. Since X is not contained in a hyperplane, and since $X \cap \boldsymbol{P}_\infty^{n-1} \cong Q^{n-1}$, it is clear that we can choose bases for $\boldsymbol{C}^{n+2}$, $\{z_0, \ldots, z_{n+1}\}$, and $\{w_0, \ldots, w_{n+1}\}$ such that $[z_i]$, $[w_i] \in X$ for $i = 0, \ldots, n+1$, and such that $[z_0] = [z]$ and $[w_0] = [w]$. Also, since any $[x] \in X$ satisfies ${}^t\bar{x} E x \neq 0$, we can choose the z_i and w_i such that ${}^t\bar{z}_i E z_i = 1$ and ${}^t\bar{w}_i E w_i = 1$. If A is the change of basis matrix such that $A(z_i) = w_i$, then we have that $A \in G$, and $[A(z)] = [w]$. Thus, G acts transitively on X.

(3) Our next example is $X = \boldsymbol{P}^n \setminus \boldsymbol{R}\boldsymbol{P}^n$, where $\boldsymbol{R}\boldsymbol{P}^n = \{[t_0 : \ldots : t_n] | t_i \in R\}$ is clearly a totally real submanifold of $\boldsymbol{P}^n$. Note that $\boldsymbol{R}\boldsymbol{P}^n$ is the set of fixed points of the involution $[z] \mapsto [\bar{z}]$ of $\boldsymbol{P}^n$. Here again, X is strictly pseudoconcave and homogeneous under a Lie group. Before we show this, let us first discuss some connections that exist between this example and the previous one. Define $p: Q^n \to \boldsymbol{P}_n^\infty$ to be the restriction of the projection map $\boldsymbol{P}^{n+1} \setminus \{0\} \to \boldsymbol{P}_n^\infty$. Then, p is surjective onto $\boldsymbol{P}_n^\infty$, for it $[z] \in \boldsymbol{P}_n^\infty$, ${}^t zz = \lambda^2$, then $[\pm i\lambda : z] \in Q^n$. Thus, p is a 2-to-1 ramified covering map with ramification set $Q^n \cap P_n^\infty = Q^{n-1}$. In addition, the n-sphere, $S^n \subset Q^n \setminus \boldsymbol{P}_n^\infty$, is mapped 2-to-1 onto $\boldsymbol{R}\boldsymbol{P}^n \subset \boldsymbol{P}_n^\infty \setminus Q^{n-1}$ under p. We can now utilize p to construct a strictly pseudoconcave exhaustion for X. Recall, we defined $\tilde{\varphi}$: $Y = Q^n \setminus \boldsymbol{P}_n^\infty \to \boldsymbol{R}$ by $\tilde{\varphi}([1:z]) = {}^t\bar{z}z - 1$. Since $\tilde{\varphi}$ is constant on the p-fibers, we obtain a strictly plurisubharmonic function $\varphi = \tilde{\varphi} \circ p^{-1}: \boldsymbol{P}_n^\infty \setminus Q^{n-1} \to \boldsymbol{R}$. Note that $\{z | \varphi(z) = 0\} = p(S^n) = \boldsymbol{R}\boldsymbol{P}^n \subset \boldsymbol{P}_n^\infty \setminus Q^{n-1}$, and $\varphi(z) > 0$ for $z \in \boldsymbol{P}_n^\infty \setminus (Q^{n-1} \cup \boldsymbol{R}\boldsymbol{P}^n)$. If we smooth φ to a constant function in some neighborhood of $Q^{n-1} \subset \boldsymbol{P}_n^\infty$, then φ becomes a strictly pseudoconcave exhaustion for X.

To show that X is a homogeneous manifold of a Lie group, we define G to be the stabilizer of X in $SL(n+1, \boldsymbol{C})$. Then we obtain:

$$\begin{aligned} G &= \{A \in SL(n+1, \boldsymbol{C}) | [Az] \in X \text{ for all } [z] \in X\} \\ &= \{A \in SL(n+1, \boldsymbol{C}) | [Az] \in \boldsymbol{R}\boldsymbol{P}^n \subset \boldsymbol{P}^n \text{ for all } [z] \in \boldsymbol{R}\boldsymbol{P}^n \subset \boldsymbol{P}^n\} \\ &= \{A \in SL(n+1, \boldsymbol{C}) | Az \in \boldsymbol{R}^{n+1} \text{ for all } z \in \boldsymbol{R}^{n+1} \subset \boldsymbol{C}^{n+1}\} \\ &= SL(n+1, \boldsymbol{R}) . \end{aligned}$$

Thus, G is a Lie group acting holomorphically on X, and is obviously a real form of $SL(n+1, \boldsymbol{C})$. To see that G acts transitively on X, let $[z]$ and $[w]$ be any two distinct points in $X = \boldsymbol{P}^n \setminus \boldsymbol{R}\boldsymbol{P}^n$. Choose representatives $z_0 = x + iy$ and $w_0 = u + iv$ in $\boldsymbol{C}^{n+1}$ such that $[z_0] = [z]$ and $[w_0] = [w]$. Note that both $\{x, y\}$ and $\{u, v\}$ are linearly independent sets of vectors in $\boldsymbol{R}^{n+1}$ (e.g. if $x = \lambda y$, then $[z_0] = [(\lambda + i)y] = [y] \in \boldsymbol{RP}^n$). Expand these sets to bases for $\boldsymbol{R}^{n+1}$, and let $\tilde{A} \in GL(n+1, \boldsymbol{R})$ be the change of basis matrix such that $\tilde{A}(x) = u$ and $\tilde{A}(y) = v$. Define $A = (\det(\tilde{A}))^{-1}\tilde{A} \in SL(n+1, \boldsymbol{C})$. Then we have that $[Az] = \left[(\det(\tilde{A}))^{-1} w_0\right] = [w]$, showing that $SL(n+1, \boldsymbol{R})$ acts transitively on X.

(4) The example is $X = \boldsymbol{P}^n \times \boldsymbol{P}^n \setminus \boldsymbol{P}^n_{\boldsymbol{R}}$, where $\boldsymbol{P}^n_{\boldsymbol{R}} = \{([z], [w]) \in \boldsymbol{P}^n \times \boldsymbol{P}^n | [z] = [\bar{w}]\}$ is a totally real submanifold of $\boldsymbol{P}^n \times \boldsymbol{P}^n$ which is real analytically isomorphic to complex projective space, $\boldsymbol{P}^n$. Note that $\boldsymbol{P}^n_{\boldsymbol{R}}$ is the set of fixed points of the involution $([z], [w]) \mapsto ([\bar{w}], [\bar{z}])$. We first show that X is strictly pseudoconcave by inspecting the Segre imbedding $s\colon \boldsymbol{P}^n \times \boldsymbol{P}^n \to \boldsymbol{P}^N$ $(N = (n+1)^2 - 1)$ given by

$$([z_0 : \ldots : z_n], [w_0 : \ldots : w_n]) \mapsto [z_0 w_0 : \ldots : z_j w_j : \ldots : z_n w_n] .$$

For convenience, we will denote the homogeneous coordinates of $\boldsymbol{P}^N$ by $[x_{00} : \ldots : x_{ij} : \ldots : x_{nn}]$ so that $x_{ij} = z_i w_j$ on $s(\boldsymbol{P}^n \times \boldsymbol{P}^n)$. Define $\boldsymbol{P}^{N-1}_\infty := \{x \in \boldsymbol{P}^N | x_{00} + x_{11} + \ldots + x_{nn} = 0\}$. Note that

$$s^{-1}(s(\boldsymbol{P}^n \times \boldsymbol{P}^n) \cap \boldsymbol{P}^{N-1}_\infty) = \{([z], [w]) \in \boldsymbol{P}^n \times \boldsymbol{P}^n | z_0 w_0 + \ldots + z_n w_n = 0\} ,$$

so that

$$s^{-1}(s(\boldsymbol{P}^n_{\boldsymbol{R}}) \cap \boldsymbol{P}^{N-1}_\infty) = \{([z], [\bar{z}]) \in \boldsymbol{P}^n_{\boldsymbol{R}} | z_0 \bar{z}_0 + \ldots + z_n \bar{z}_n = 0\} = \emptyset .$$

Therefore, $s(\boldsymbol{P}^n_{\boldsymbol{R}}) \subset s(\boldsymbol{P}^n \times \boldsymbol{P}^n) \setminus \boldsymbol{P}^{N-1}_\infty \subset \boldsymbol{C}^N = \boldsymbol{P}^N \setminus \boldsymbol{P}^{N-1}_\infty$. Let $v = x_{00} + x_{11} + \ldots + x_{nn}$ and define coordinates in $\boldsymbol{C}^N = \boldsymbol{P}^N \setminus \boldsymbol{P}^{N-1}_\infty$ by $y_{ij} = x_{ij}/v$. As in the example of the quadrics, we define $\tilde{\varphi}\colon \boldsymbol{C}^N \to \boldsymbol{R}$ by $\tilde{\varphi}(y) = {}^t y y - 1$ and $\varphi\colon Y := \boldsymbol{P}^n \times \boldsymbol{P}^n \setminus s^{-1}(s(\boldsymbol{P}^n \times \boldsymbol{P}^n) \cap \boldsymbol{P}^{N-1}_\infty) \to \boldsymbol{R}$ by $\varphi = \varphi \circ s$. It is clear that φ is strictly plurisubharmonic on Y, since $\tilde{\varphi}$ is strictly plurisubharmonic on $\boldsymbol{C}^N$. Note that

$$\begin{aligned}
\{([z], [w]) &\in \boldsymbol{P}^n \times \boldsymbol{P}^n | \varphi(z, w) = 0\} \\
&= s^{-1}\{y \in s(\boldsymbol{P}^n \times \boldsymbol{P}^n) \cap \boldsymbol{C}^N | \tilde{\varphi}(y) = 0\} \\
&= s^{-1}\Big\{x \in s(\boldsymbol{P}^n \times \boldsymbol{P}^n) \setminus \boldsymbol{P}^{N-1}_\infty \Big| \sum_{i,j} \bar{x}_{ij} x_{ij} = \sum_{i,j} \bar{x}_{ii} x_{jj}\Big\} \\
&= \{([z], [w]) \in \boldsymbol{P}^n \times \boldsymbol{P}^n | ({}^t\bar{z}z)({}^t\bar{w}w) = ({}^t z w)({}^t z w)\} \\
&= \{([z], [w]) \in \boldsymbol{P}^n \times \boldsymbol{P}^n | [z] = [\bar{w}]\} = \boldsymbol{P}^n_{\boldsymbol{R}} .
\end{aligned}$$

If we smooth φ to a constant function outside a relatively compact neighborhood of $\boldsymbol{P}^n_{\boldsymbol{R}} \subset \boldsymbol{P}^n \times \boldsymbol{P}^n$ we obtain a strictly pseudoconcave exhaustion for X. It is interesting to note that the above construction provides a more or less canonical way to imbed $\boldsymbol{P}^n$ in $S^{2N-1}\subset \boldsymbol{C}^N$, $N=n^2+2n$, using only a real analytic map of degree 2.

Let G be the stabilizer of X in $S := SL(n+1, \boldsymbol{C})\times SL(n+1, \boldsymbol{C})$. Then,

$$\begin{aligned} G &= \{(A, B)\in S|(A, B)([z], [w])\in \boldsymbol{P}^n_{\boldsymbol{R}} \text{ for all } ([z], [w])\in \boldsymbol{P}^n_{\boldsymbol{R}}\} \\ &= \{(A, B)\in S|[Az]=[\bar{B}z] \text{ for all } [z]\in \boldsymbol{P}^n\} \\ &= \{A, B)\in S|A=\bar{B}\}\,. \end{aligned}$$

We observe that G is a real form of S and that it is real analytically isomorphic to the complex Lie group $SL(n+1, \boldsymbol{C})$. (Thus, G is a simple Lie group while its complexification is not.) Although G could be given the structure of a complex Lie group via this isomorphism, the map $G\times X\to X$ would then not be holomorphic in both variables.

To show that G acts transitively on X, let $([z_0], [w_0])$ and $([z_1], [w_1])$ be two points of X. Then, the sets of vectors in $\boldsymbol{C}^{n+1}$, $\{z_0, \bar{w}_0\}$ and $\{z_1, \bar{w}_1\}$, are linearly independent over $\boldsymbol{C}$. As in the previous example, it follows that there exists an $A\in SL(n+1, \boldsymbol{C})$ such that

$$(A, \bar{A})([z_0], [w_0]) = ([Az_0], [Aw_0]) = ([z_1], [w_1])\,.$$

(5) The next example is a little more difficult to describe. Recall that the quaternions, which we denote by $\boldsymbol{Q}$, are defined as the 4-dimensional $\boldsymbol{R}$-module over the finite group $\{\pm 1, \pm i, \pm j, \pm k|ij=k, jk=i, ki=j, i^2=j^2=k^2=-1\}$. The quaternions form a non-commutative field, the inverse of a typical element $a+ib+jc+kd$ being $(a-ib-jc-kd)/(a^2+ + b^2+c^2+d^2)$. Since $\boldsymbol{Q}$ contains a subfield, $\{a+ib|a, b\in\boldsymbol{R}\}$, isomorphic to $\boldsymbol{C}$, it can also be realized as a 2-dimensional vector space over $\boldsymbol{C}$. A typical element $a+ib+jc+kd$ has the form $(a+ib)+(c+id)j$, so that $\{1, j\}$ is a basis for $\boldsymbol{Q}$ over $\boldsymbol{C}$. Let $\boldsymbol{Q}^n$ denote the left vector space of dimension n over $\boldsymbol{Q}$. Then $\boldsymbol{Q}^n$ is isomorphic to $\boldsymbol{C}^{2n}$ over $\boldsymbol{R}$ and a $\boldsymbol{Q}$-linear transformation from $\boldsymbol{Q}^n$ to $\boldsymbol{Q}^n$ is a $\boldsymbol{C}$-linear transformation from $\boldsymbol{C}^{2n}$ to $\boldsymbol{C}^{2n}$ which commutes with « multiplication by j ». Let $\{e_1, \ldots, e_n\}$ be the usual basis for $\boldsymbol{C}^n$. With respect to the basis $\{e_1, je_1, \ldots, e_n, je_n\}$, multiplication by j on $\boldsymbol{C}^{2n}$ takes $(a_1, b_1, \ldots, a_n, b_n)$ to $(-\bar{b}_1, \bar{a}_1, \ldots, -b_n, \bar{a}_n)$. Denote this real linear transformation of $\boldsymbol{C}^{2n}$ by τ. Note that τ always takes a complex subspace of $\boldsymbol{C}^{2n}$ to another complex subspace of $\boldsymbol{C}^{2n}$, since $i\circ\tau=-\tau\circ i$.

Now consider the set of all 2-dimensional complex linear subspaces of $\boldsymbol{C}^{2n}$ which are invariant under τ. These are precisely the set of quaternionic lines in $\boldsymbol{Q}^n$ (under the bove isomorphism), which is by definition the quaternionic projective space of dimension $n-1$, $\boldsymbol{Q}P^{n-1}$. In this way we obtain an imbedding of $\boldsymbol{Q}P^{n-1}$ into the Grassmannian manifold $G_{2,2n}$ of 2-dimensional complex subspaces of $\boldsymbol{C}^{2n}$. This $\boldsymbol{Q}P^{n-1}$ is totally real in $G_{2,2n}$ because it is the set of points $[P]\in G_{2,2n}$ such that $[P]=[\tau(P)]=:\tilde{\tau}[P]$, where $\tilde{\tau}$ is the anti-holomorphic involution of $G_{2,2n}$ induced by τ. One can see this explicitly by taking a basis $\{x_1, y_1, \dots, x_n, y_n\}$ for $\boldsymbol{C}^{2n}$ such that $\tau(x_i)=y_i$ and $\tau(y_i)=-x_i$, and representing a point $[P]\in G_{2,2n}$ by $[P]=\Big[\sum_{i,j} a_{ij}x_i\wedge x_j + b_{ij}x_i\wedge y_j + c_{ij}y_i\wedge y_j\Big]$. We find that $[\tau(P)]=[P]$ means

$$
\begin{aligned}
[P] &= \Big[\sum_{i,j} \bar{a}_{ij}\tau(x_i\wedge x_j) + \bar{b}_{ij}\tau(x_i\wedge y_j) + \bar{c}_{ij}\tau(y_i\wedge y_j)\Big] \\
&= \Big[\sum_{i,j} \bar{a}_{ij}y_i\wedge y_j + \bar{b}_{ij}x_j\wedge y_i + \bar{c}_{ij}x_i\wedge x_j\Big]\,,
\end{aligned}
$$

or, $a_{ij}=\lambda\bar{c}_{ij}$, $b_{ij}=\lambda\bar{b}_{ji}$, $c_{ij}=\lambda\bar{a}_{ij}$, for some λ, $|\lambda|=1$.

We define $X=G_{2,2n}\setminus\boldsymbol{Q}P^{n-1}$. Again, it turns out that X is strictly pseudoconcave and homogeneous under a Lie group. At this point, however, we appeal to the theoretical results in § 9 for the proof, since the geometry is not so clear as in the previous examples. We mention only that the Lie group which acts holomorphically and transitively on X is the subgroup of $SL(2n, \boldsymbol{C})$ (acting on $G_{2,2n}$) which stabilizes X (or equivalently, $\boldsymbol{Q}P^{n-1}$). This group is isomorphic to the special linear group of quaternionic transformations,

$$
\begin{aligned}
G &= \{A\in SL(2n,\boldsymbol{C})\,|\,[A(P)]\in \boldsymbol{Q}P^{n-1} \text{ for all } [P]\in\boldsymbol{Q}P^{n-1}\} \\
&= \{A\in SL(2n,\boldsymbol{C})\,|\,A\circ\tau=\tau\circ A\} \\
&\cong SL(n,\boldsymbol{Q})\,.
\end{aligned}
$$

The compact group $G\cap U(2n)=:Sp(n)$ is called the symplectic group and is a maximal compact subgroup of G which acts transitively on $\boldsymbol{Q}P^{n-1}$. The complexification of $Sp(n)$ in $SL(2n,\boldsymbol{C})$ is denoted by $Sp(n,\boldsymbol{C})$ and is usually denoted by $SU^*(n)$, a real form of $SL(2n,\boldsymbol{C})$.

(6) The final example is very difficult to describe geometrically, so we will only give its definition in terms of quotients of Lie groups. Let $(EIII)= E_6/(\text{Spin}\,(10)\times SO(2))$. The Cayley projective plane, $F_4/\text{Spin}\,(9)$, can be imbedded in $(EIII)$ in such a way that $X=(EIII)\setminus$(Cayley projective plane) is strictly pseudoconcave and homogeneous under the Lie group $G=E_7^3$ (see [6] for the definitions of these groups). Again, we appeal to the results in § 9 for the proof of this fact.

4. – Compactification to a homogeneous rational manifold.

Let X be a non-compact homogeneous strictly pseudoconcave manifold, and let V be a minimal normal compactification of X. Recall that this compactification always exists if $\dim X \geqslant 3$. Then, by the remarks in § 2, we may assume $X = G/H$, where $G := \mathrm{Aut}^0(X)$ is realized as a closed (not necessarily complex) Lie subgroup of the complex Lie group $\mathrm{Aut}^0(V)$.

Assuming that no complex Lie group acts transitively on X, we first show that V is a compact homogeneous rational manifold and then study the structure of the group G.

PROPOSITION 4.1. *Let X be a non-compact homogeneous strictly pseudoconcave manifold, and let V be a minimal normal compactification of X. Assume that no complex Lie group acts transitively on X. Then V is a compact homogeneous rational manifold.*

PROOF. The first step of the proof is to show that V is homogeneous. We define $S = G^{\boldsymbol{C}}$, the complexification of G in $\mathrm{Aut}^0(V)$. Then, for any $x_0 \in X$, we have $X = G(x_0)$ is open in $S(x_0) \subset V$, showing that V is almost homogeneous. Let E be the compact complex analytic set $V \setminus S(x_0)$. Since $E \subset V \setminus X \subset V \setminus \bar{X}_\nu$, and $V \setminus \bar{X}_\nu$ is Stein, it follows that E is a finite set of points. If $E = \{e_1, e_2, \ldots, e_k\}$ is not empty, then $S(x_0) = V \setminus E$ is non-compact, strictly pseudoconcave, and a homogeneous manifold of a complex Lie group. Therefore (see § 1) $S(x_0)$ is a homogeneous cone over a compact homogeneous rational manifold Q. Let $\pi\colon S(x_0) \to Q$ be the canonical projection with fiber $\boldsymbol{C}$. Note that now the set $V \setminus S(x_0)$ consists of a single point, say p. This bundle is equivariant under S and so it is equivariant under G. Thus, we can realize the restricted holomorphic map $\pi'\colon X \to Q$ as a homogeneous fibration $\pi'\colon G/H \to G/J$. By Lemma 2.3, π' is surjective, i.e. $Q = G/J$. It is important to note that this fibration may not be holomorphically locally trivial, even though it is real analytically locally trivial and the map π' itself is holomorphic. The π'-fiber is connected (by the homotopy sequence for this fibration) and is a homogeneous complex submanifold of X contained in the original π-fiber $\boldsymbol{C}$. That is, $F = J/H$ is a homogeneous connected subset of $\boldsymbol{C}$. It is well-known that the homogeneous connected subsets of $\boldsymbol{C}$ are biholomorphic to either $\boldsymbol{C}$ itself, $\boldsymbol{C}^*$, or the unit disk, D. We now show that each of these possibilities leads us to a contradiction.

If $F \cong \boldsymbol{C}$, then of course $X = S(x_0)$, contradicting our assumption that no complex Lie group acts transitively on X. If $F \cong \boldsymbol{C}^*$, then in each π-fiber, $\pi^{-1}(q)$, there is a point $\sigma(q)$ such that $\{\sigma(q)\} = \pi^{-1}(q) \setminus \pi'^{-1}(q)$ (corresponding to $\{0\} = \boldsymbol{C} \setminus \boldsymbol{C}^*$). Thus, we obtain a real analytic section $\sigma\colon Q \to$

$\to S(x_0)$ with $\sigma(Q) = S(x_0)\setminus X$. By the G-equivariance of the fibration, we have that $g(\sigma(q)) = \sigma(\hat{g}(q))$ for all $g \in G$ (here, $\hat{g}$ denotes the automorphism of Q induced by g), so that $\sigma(Q)$ is actually an orbit of G in $S(x_0)$. Since $\sigma(Q)$ is compact and simply connected, any maximal compact subgroup K_G of G acts transitively on $\sigma(Q)$. Let K_S be a maximal compact subgroup of S containing K_G. Since K_S fixes the vertex p of the cone $S(x_0)$, it stabilizes and acts transitively on a complementary hyperplane section $\mathfrak{D}$ (the representation of K_S is semisimple; see § 1). However, since the cone is a topologically non-trivial line bundle, $\sigma(Q)$ intersects $\mathfrak{D}$ non-trivially. Thus, we must have that $\sigma(Q) = \mathfrak{D}$. This implies that $X = S(x_0)\setminus\mathfrak{D} = V\setminus\mathfrak{D} \cup \{p\}$ which is contrary to the fact that no complex Lie group acts transitively on X.

The last possibility is $F = J/H \cong D$. In this case we realize a fixed π-fiber as the complex plane $\boldsymbol{C}$ and the π'-fiber F as a subdomain. Thus, we have a representation $\varrho\colon J \to \operatorname{Aut}(\boldsymbol{C})$ such that $\varrho(J)$ stabilizes and acts transitively on the domain F. Now, $\operatorname{Aut}(F)$ is a semisimple real Lie group (isomorphic to $PSL(2, \boldsymbol{R})$), and no proper Lie subgroup of $\operatorname{Aut}(F)$ acts transitively on F. Therefore, $\varrho(J) \cong \operatorname{Aut}(F)$. This contradicts the fact that $\operatorname{Aut}(\boldsymbol{C})$ contains no semisimple Lie subgroups, real or complex.

The above shows that $E = \emptyset$. Therefore, V is in fact a compact homogeneous manifold under the complex Lie group S, i.e. $V = S/P$, where P is chosen to be the isotropy of some fixed point $x_0 \in X \subset V$.

Our last step in the proof is to show that V is a compact *rational* manifold. We begin with the normalizer fibration $\mu\colon S/P \to S/N$, where $N = N_S(P^0)$. In the category of compact homogeneous manifolds, it is well-known (see e.g. [3]) that S/N is a compact *rational* manifold and the fiber $N/P = (N/P^0)/(P/P^0) = \hat{N}/\Gamma$ is connected and parallelizable (i.e. Γ is discrete). We want to show, of course, that $\dim N/P = 0$ so that $S/P = S/N$, finishing the proof.

We first observe that this fibration is equivariant under G. As before, it follows that the restricted holomorphic map $\mu'\colon X \to S/N$ is a homogeneous fibration, $G/H \to G/J$. If $\dim J/H > 0$, then, by Lemma 2.3, μ' is surjective. Consequently, $S/N = G/J$, and $\dim J/H = \dim N/P$. Note that $H = G \cap P$, $J = G \cap N$, and $H^0 = (G \cap P)^0 = (G \cap P^0)^0$. Thus, if $j \in J$ and $g \in H^0$, then $jgj^{-1} \in G \cap P^0$. Therefore, since jH^0j^{-1} is connected, contains the identity, and is contained in $G \cap P^0$, we have $jH^0j^{-1} \subset H^0$ and H^0 is a normal subgroup of J. This allows us to express the μ' fiber J/H as $(J/H^0)/(H/H^0) = \hat{J}/\Gamma' \subset \hat{N}/\Gamma$. Since $J \subset N$ and $H^0 \subset P^0$, we have the natural homomorphism $\alpha\colon \hat{J} \to \hat{N}$, whose kernel is discrete because $J \cap P^0/H^0 \subset$ $\subset G \cap P^0/(GP^0)^0$. Thus, we can identify $\hat{J}/\ker\alpha$ with a subgroup $\tilde{J} \subset \hat{N}$. It is clear that $J(x) = \hat{J}(x) = \tilde{J}(x) = \tilde{J}^0(x)$, and $N(x) = \hat{N}(x) = \hat{N}^0(x)$ for

$x \in X$. Now,

$$\dim \hat{N} = \dim N(x) = \dim J(x) = \dim \hat{J} = \dim \tilde{J}\,,$$

so that, as real Lie groups, $\hat{N}^0 = \tilde{J}^0$. This implies that $J(x) = N(x)$ so that $X = V$ is compact! This contradiction followed from the assumption that $\dim J/H > 0$. Therefore, $\dim \tilde{J}/H = 0$, showing that $\dim N/P = 0$ ($\dim G/H = \dim S/P = \dim S/N$). Since N/P is connected, we must have that $S/P = S/N$ and S/P is a compact homogeneous rational manifold as claimed. □

5. – The structure of the groups.

As in the previous section, let X be a non-compact strictly pseudoconcave homogeneous manifold and V a minimal normal compactification of X. We have seen that $X = G/H$, where $G := \mathrm{Aut}^0(X)$ is a Lie subgroup of $\mathrm{Aut}^0(V)$, and that if no complex Lie group acts transitively on X, then $V = S/P$ is a compact homogeneous manifold of the complex Lie group $S := G^{\boldsymbol{C}}$.

We gather here a few facts about the groups G and S.

PROPOSITION 5.1. *Under the above assumptions, G is a real form of S. (That is, if $\mathfrak{g}$ and $\mathfrak{s}$ denote the Lie algebras of G and S respectively, and if J is the complex structure on S, then $\mathfrak{s} = \mathfrak{g} \oplus J\mathfrak{g}$.) Moreover, S is a semisimple complex Lie group and G is a simple real Lie group* (*).

PROOF. By the definition of the complexification of G we have $\mathfrak{s} = \mathfrak{g}^{\boldsymbol{C}} := \mathfrak{g} + J\mathfrak{g}$. We show $\hat{\mathfrak{g}} := \mathfrak{g} \cap J\mathfrak{g}$ is $\{0\}$ as follows. Let $\hat{G} = \exp(\hat{\mathfrak{g}}) \subset G$ be the connected complex Lie group associated to $\hat{\mathfrak{g}}$. Then, for point p in the compact set $E = V \setminus X$, we must have $\hat{G}(p) \subset E \subset V \setminus \bar{X}_\nu$. This orbit is the image of a holomorphic map from $\mathfrak{g}$ (which is biholomorphic to $\boldsymbol{C}^k$) into the bounded Stein manifold $V \setminus \bar{X}_\nu$ showing that $\hat{G}(p) = \{p\}$. Let $F(\hat{G}) := \{q \in V | \hat{G}(q) = q\}$. Clearly, $F(\hat{G})$ is a compact complex analytic subset of V and $E \subset F(\hat{G})$. Consider the complex analytic set $F := \cap \{gF(\hat{G}) | g \in G\}$. Since $g(E) = E$ for all $g \in G$, we have that $E \subset F$. If $X \cap F = \emptyset$, then $F = E$, showing that E is a compact *complex analytic* set. Thus, there is a complex Lie group acting transitively on X, contrary to our assumption. Therefore, $X \cap F \neq \emptyset$. Now, for any $q \in X \cap F$ and for all $g \in G$ we know $g(q) \in F$, so that $X = G(q) \subset F$. Therefore, $F(\hat{G}) = X$, showing that $\hat{G} = \{1\}$ and $\mathfrak{g} = \{0\}$.

(*) By *simple* we mean that the Lie algebra of G is simple.

To see that S is semisimple, let $R(S)$ denote the radical of S. A standard flag argument [3] shows that $R(S)$ fixes every point of V. Now, since $R(S) \subset \subset S \subset \mathrm{Aut}^0(V)$, we must have that $R(S) = \{1\}$. Thus, S is a semisimple complex Lie group.

Since G is a real form of S, it follows that G is also semisimple [6]. To show that G is *simple*, we express $\mathfrak{g}$ as a direct sum of its simple ideals, $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_t$. Then, the Lie algebras of S and P have the form $\mathfrak{s} = = \mathfrak{s}_1 \oplus \ldots \oplus \mathfrak{s}_t$ and $\mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_t$, where $\mathfrak{s}_i = \mathfrak{g}_i^{\mathbf{C}}$ and $\mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{s}_i$. From these decompositions, it follows that $V = V_1 \times V_2 \times \ldots \times V_t$ where $V_i = S_i/P_i$, and S_i (resp. P_i) is the Lie subgroup of S whose Lie algebra is $\mathfrak{s}_i$ (resp. $\mathfrak{p}_i$) .Thus, for anypoint $x \in X \subset V$, we have $x = (x_1, \ldots, x_t)$ so that $G(x) = = G_1(x_1) \times \ldots \times G_t(x_t)$, where G_i is the Lie subgroup of G whose Lie algebra is $\mathfrak{g}_i$. Since $X = G(x)$ is open in V, we have that $G_i(x_i)$ is open in V_i. Since X is non-compact, we may assume that $G_1(x_1)$ is non-compact. Now, by Lemma 2.3, the fiber $G_2(x_2) \times \ldots \times G_t(x_t)$ of the holomorphic projection $X \to G_1(x_1)$ cannot have positive dimension, otherwise $G_1(x_1)$ would be compact. Therefore, since G acts effectively on X, $\mathfrak{g} = \mathfrak{g}_1$.

In the above proposition it should be noted that S need not be simple (see § 3, example (4)).

6. – The special case of $\boldsymbol{P}^n \setminus \overline{\boldsymbol{B}}^n$.

So far we have shown that, if X is a non-compact strictly pseudoconcave homogeneous manifold with no complex Lie group acting transitively on X, then the minimal normal compactification of X is a compact homogeneous rational manifold $V = S/P$ with S semisimple and $G = \mathrm{Aut}^0(X)$ a simple real form of S. We are now in a position to utilize the results of J. A. Wolf [25] on the structure of the orbits of such a real form G acting on a compact homogeneous rational manifold $V = S/P$.

This section is devoted to proving the following theorem.

THEOREM 6.1. *Let X and V be as above. If $V \setminus X$ contains an open set, then $V \cong \boldsymbol{P}^n$ and $X \cong \boldsymbol{P}^n \setminus \boldsymbol{B}^n$.*

PROOF. Easy computations show that G has only a finite number of orbits on V ([25] Thm. 2.6). Therefore, if $V \setminus X$ contains an open set, it must also contain an open orbit of G, say $G(y)$ for some $y \in V \setminus X$. Now, $G(y) \subset V \setminus X \subset V \setminus \overline{X}_\nu$, where $V \setminus \overline{X}_\nu$ is a Stein manifold, so that, in particular, the holomorphic functions on $G(y)$ separate the point of $G(y)$. Under this assumption, it follows ([25] Thm. 5.4, Cor. 5.8) that $G(y) = G/K_G$ is a hermitian symmetric space with K_G a maximal compact subgroup of G.

Let K_S be a maximal compact subgroup of S containing K_G. We now show that $V = S/P$ is the compact hermitian symmetric space dual to G/K_G, which is by definition K_S/K_G. Since V is simply-connected we know that K_S acts transitively on V. Thus, $V = K_S/(P \cap K_S)$. We may choose P to be the isotropy of the point y so that $K_G \subset P \cap K_S$. Now consider the homogeneous fibration $\alpha: K_S/K_G \to K_S/(P \cap K_S)$. By construction, $G(y)$ is open in K_S/K_G (see e.g. [6]), and we know $G(y)$ is open in $V = K_S/(P \cap K_S)$. Therefore, α is a finite map. But, V is simply-connected so that α is one-to-one. Therefore, $K_G = P \cap K_S$ and $V = K_S/K_G$ is the compact hermitian symmetric space dual of $G(y)$.

Again we quote Wolf ([25] Cor. 11.11) who shows that every open G-orbit, $G(x)$, on such a compact hermitian symmetric space $V = S/P$ has a holomorphic fibration $\beta: G(x_0) \to K_G(x_0)$ for some $x_0 \in G(x)$, where $K_G(x_0)$ is a compact homogeneous rational manifold contained in $G(x_0)$. In addition, $G(x_0)$ is biholomorphically equivalent to a relatively compact open subset of the holomorphic normal bundle, N, of $K_G(x_0)$ in $G(x_0)$, in such a way that the β fibration becomes a (not necessarily holomorphically locally trivial) subbundle of N. Furthermore, the β-fiber F is a non-compact hermitian symmetric space.

Now, let $G(x_0)$ be the open orbit X. If $\dim F > 1$, then F inherits the strict pseudoconcavity of X and so F cannot possess non-constant holomorphic functions. But, by the above remarks, we know that F is open and relatively compact in the fibers of N, i.e. in $\mathbf{C}^k$, $K > 1$. This contradiction implies that $\dim F = 1$, so that the fibration $\beta: G(x_0) \to K_G(x_0)$ is then a disk subbundle of the *line* bundle N. Since X is strictly pseudoconcave we have that N is a *positive* line bundle. If we compactify N by adding the point at infinity to each fiber, we obtain a new compactification for X. We can make this compactification a minimal normal compactification, $\hat{V}$, by blowing down the newly added infinity section to a point $p_0 \in V$ and then normalizing, if necessary. By Proposition 4.1 we again have that $\hat{V}$ is a compact homogeneous rational *manifold* with respect to the complex Lie group $S = G^{\mathbf{C}}$. By [1], such minimal compactifications of X to manifolds are unique, so that we must have $\hat{V} \simeq V$. We claim that $V \setminus \{p_0\} \simeq N$ is a homogeneous manifold with respect to some complex Lie group. This follows from the fact that a positive line bundle over a compact homogeneous rational manifold is always homogeneous with respect to a complex group of bundle preserving automorphisms (see e.g. [8]). Note that any automorphism of N trivially extends to V, so that a result of E. Oeljeklaus [16] now applies: If a complex Lie group acts almost transitively on a compact manifold, and if its fixed set contains an isolated point, then the manifold is projective n-space, $\mathbf{P}^n$! This implies that $V \simeq \mathbf{P}^n$ and that $G(y)$ is the non-compact symmetric space dual of $\boldsymbol{P}^n$, i.e. $G(y) \simeq \boldsymbol{B}^n$ and $G \simeq \mathrm{Aut}\,(\boldsymbol{B}^n)$. It follows that $X \simeq \boldsymbol{P}^n \setminus \overline{\boldsymbol{B}}^n$, as claimed. □

7. – The minimal orbit.

A real form G of S acting on a compact homogeneous rational manifold $V = S/P$ has exactly one closed orbit, $G(v)$ for some $v \in V$. This closed orbit can be characterized by the condition that $G(v)$ is the lowest dimensional G-orbit on V, or that some (equivalently every) maximal compact subgroup of G acts transitively on $G(v)$, or that $G(v)$ is contained in the closure of every G-orbit on V. Furthermore, for *any* orbit $G(v)$ we have $2 \dim_{\boldsymbol{R}} G(v) \geqslant \dim_{\boldsymbol{R}} V$ ([25], Thm. 3.6).

In this section we wish to continue our investigation of $X = G(x)$, a non-compact strictly pseudoconcave open subset of the compact homogeneous rational manifold $V = S/P$, $S = G^{\boldsymbol{C}}$. In § 6 we showed that if $V \setminus X$ contains an open set, then $V = \boldsymbol{P}^n$ and $X = \boldsymbol{P}^n \setminus \boldsymbol{B}^n$. (In this case it is clear that the unique closed orbit of G is the $(2n-1)$-sphere, $S^{2n-1} = \partial \overline{\boldsymbol{B}}^n$.) In the present section we assume that $V \setminus X$ does *not* contain an open set of V. Since the action of G is real analytic, we can say that $A := V \setminus X = \{v \in V \mid \dim_{\boldsymbol{R}} G(v) \leqslant \dim_{\boldsymbol{R}} V - 1\}$ is a closed, proper, real analytic subvariety of V. In fact, we prove the following:

PROPOSITION 7.1. *Let X be a non-compact, strictly pseudoconcave, open G-orbit on the compact homogeneous rational manifold $V = S/P$, $S = G^{\boldsymbol{C}}$, as above. If $A = V \setminus X$ does not contain an open set, then A is the unique closed orbit of G on V, and $2 \dim_{\boldsymbol{R}} A = \dim_{\boldsymbol{R}} V$.*

PROOF. For every orbit, $G(v)$, $v \in V$, there is an intrinsically defined closed complex Lie subgroup Q of S such that $Q_0 := (G \cap Q)^0$ is a real form of Q, and such that $Q_0(v)$ and $Q(v)$ are both (not necessarily closed) submanifolds of $V = S/P$, with $Q_0(v)$ open in $Q(v)$ ([25] Lemma 8.4 and Thm. 8.5). It can be shown that if $Q_0(v) = \{v\}$ then $G(v)$ is the unique closed orbit of G on V ([25] Cor. 8.20).

Consider $G(v) \subset A$ for any $v \in A$. By a real analytic identity principle, we must have that $Q(v) \subset A$, since $Q_0(v) \subset A$ and $Q_0(v)$ is open in $Q(v)$. But $A \subset V \setminus \overline{X}_v$ can be realized as a bounded Stein space in $\boldsymbol{C}^N$. Therefore, Q can never have a non-trivial orbit in A (otherwise we would obtain non-constant bounded holomorphic functions on the Lie algebra of Q), and so $Q_0(v) = Q(v) = \{v\}$. As mentioned above, $G(v)$ must then be the *unique* closed orbit of G on V. Since v was an arbitrary point of A, it follows that $A = G(v)$.

To show that $2 \dim_{\boldsymbol{R}} A = \dim_{\boldsymbol{R}} V$, we first note that A is « holomorphically convex » in the sense that the envelope of holomorphy of A, $E(A)$, is equal to A itself. This follows from [19] Thm. 2.12, by noting that A is the intersection of the countable number of Stein manifolds $U_\nu := V \setminus \bar{X}_\nu = \{x \in V | x \notin X, \text{ or } x \in X \text{ and } \varphi(x) \leqslant c_\nu\}$. However, if $2k = 2 \dim_{\boldsymbol{R}} A > \dim_{\boldsymbol{R}} V$, then $E(A)$ contains a differentiable submanifold of real dimension $k+1$ (see [24]). Therefore, $2 \dim_{\boldsymbol{R}} A = \dim_{\boldsymbol{R}} V$. □

From now on, we will refer to A as the *minimal orbit* of G. It should be noted that in this setting it is possible to realize V as a complex projective variety defined over $\boldsymbol{R}$ in such a way that A is the set of « real points » ([25] Thm. 3.6).

8. – An orbit decomposition of the compactification.

We now investigate the compactification $V = S/P$ of the non-compact strictly pseudoconcave homogeneous manifold X obtained in the previous sections, by studying the orbits of K_G and $K_G^{\boldsymbol{C}}$, where K_G is a maximal compact subgroup in G. As before, we assume $A = V \setminus X$ does not contain an open set so that by Proposition 7.1, A is the minimal orbit (necessarily compact) of G in V. Thus, we already have a decomposition of V into two G-orbits: X, a unique open orbit of G in V, and A, a unique closed orbit of G in V. With respect to the complex Lie group $K_G^{\boldsymbol{C}}$ we have the following:

PROPOSITION 8.1. *There are exactly two $K_G^{\boldsymbol{C}}$-orbits in $V = S/P$. For $v \in A$, $K_G^{\boldsymbol{C}}(v)$ is an open Stein submanifold of V; and for some $x_0 \in X$, $K_G^{\boldsymbol{C}}(x_0) = V \setminus K_G^{\boldsymbol{C}}(v)$ is a compact complex rational submanifold of V of complex codimension* 1.

PROOF. Our first step is to show that $M := K_G^{\boldsymbol{C}}(v)$, for $v \in A$, is an *open* submanifold of V. Note that $A = K_G(v)$ is contained in M. We define W_ν to be the connected component of $U_\nu \cap M$ containing A, where $U_\nu := V \setminus \bar{X}_\nu$ are Stein manifolds. Thus, we have that A is a countable intersection of Stein manifolds, W_ν. As in the proof of Proposition 7.1, using envelopes of holomorphy, it follows that $2 \dim_{\boldsymbol{R}} A = \dim_{\boldsymbol{R}} M$. But, from the same proposition, $2 \dim_{\boldsymbol{R}} A = \dim_{\boldsymbol{R}} V$. Therefore, M is open in V.

To show that M is Stein, we write $A = K_G/L \subset M = K_G^{\boldsymbol{C}}/I$, where L (resp. I) is the isotropy in K_G (resp. $K_G^{\boldsymbol{C}}$) of some point $v \in A$. Clearly, $L^{\boldsymbol{C}} \subset I$. Now, since $2 \dim_{\boldsymbol{R}} A = \dim_{\boldsymbol{R}} M$, we have $\dim_{\boldsymbol{R}} K_G^{\boldsymbol{C}}(v) = \dim_{\boldsymbol{R}} K_G^{\boldsymbol{C}} - \dim_{\boldsymbol{R}} I = 2(\dim_{\boldsymbol{R}} K_G - \dim_{\boldsymbol{R}} L)$, so that $\dim_{\boldsymbol{R}} I = 2 \dim_{\boldsymbol{R}} L = \dim_{\boldsymbol{R}} L^{\boldsymbol{C}}$. These facts imply that $(L^0)^{\boldsymbol{C}} = (L^{\boldsymbol{C}})^0 = I^0$. Furthermore, since K_G is compact, we can choose a representation for S such that $K_G^{\boldsymbol{C}}$ is an algebraic subgroup of S

acting algebraically on V (see [4]). Therefore, I is an algebraic subgroup of $K_G^{\mathbf{C}}$ and we obtain the homogeneous fibration $K_G^{\mathbf{C}}/L^{\mathbf{C}} \to K_G^{\mathbf{C}}/I$ with *finite* fiber $I/L^{\mathbf{C}}(I^0 \subset L^{\mathbf{C}})$. A theorem of Matsushima [12] implies that $K_G^{\mathbf{C}}/L^{\mathbf{C}}$ in a Stein manifold. Therefore, $K_G^{\mathbf{C}}(v) = K_G^{\mathbf{C}}/I$, which is the image of $K_G^{\mathbf{C}}/L^{\mathbf{C}}$ under the finite map above, is also Stein.

We now turn our attention to $C := V \setminus M$. Since C is the complement of an open orbit of a complex Lie group acting on V, we know that C is a proper complex analytic set. On the other hand, $M = V \setminus C$ is Stein, so that C is pure $(n-1)$-dimensional.

It is easy to show that every open orbit of G contains a K_G-orbit, $K_G(x_0)$, which is a compact complex rational submanifold of $G(x_0)$, i.e. $K_G(x_0) = K_G^{\mathbf{C}}(x_0)$, and that $G(x_0)$ has a deformation retraction onto $K_G(x_0)$ ([25] Lemma 5.1, Thm. 5.4). If we take the open orbit to be $X = G(x_0)$, then clearly $K_G(x_0) = K_G^{\mathbf{C}}(x_0) \subset C$. The retraction gives us an isomorphism $H_*(K_G^{\mathbf{C}}(x_0), \mathbf{Z}) \simeq H_*(X, \mathbf{Z})$. Now, since the homology class of C is non-trivial in $H_{2n-2}(V, \mathbf{Z})$, it is non-trivial in $H_{2n-2}(X, \mathbf{Z})$. Therefore, by the above isomorphism, there exists a non-trivial element in $H_{2n-2}(K_G^{\mathbf{C}}(x_0), \mathbf{Z})$, showing that $\dim_{\mathbf{R}} K_G^{\mathbf{C}}(x_0) \geqslant 2n-2$. But $K_G^{\mathbf{C}}(x_0) \subset C$, so $\dim_{\mathbf{R}} K_G^{\mathbf{C}}(x_0) = 2n-2$, and $K_G^{\mathbf{C}}(x_0)$ is a branch of C. Note that, since $K_G^{\mathbf{C}}$ must stabilize the branch locus B of C, we have $B \cap K_G^{\mathbf{C}}(x_0) = \emptyset$, and thus $K_G^{\mathbf{C}}(x_0)$ is a connectivity component of C. However, a homogeneous Stein manifold (other than $\mathbf{C}^*$) can only have one end (see [23]), that is, C must be connected and thus $C = K_G^{\mathbf{C}}(x_0)$ as claimed. □

For the maximal compact group K_G of G we have the following:

PROPOSITION 8.2. *There are exactly three orbit types of K_G in V. For $v \in A$, the minimal orbit of G in V, we have $K_G(v) = A$ and so $2 \dim_{\mathbf{R}} K_G(v) = \dim_{\mathbf{R}} V$. For $x_1 \in C = K_G^{\mathbf{C}}(x_0)$, the divisor given in Proposition 8.1, we have $K_G(x_1) = C$ and so $\dim_{\mathbf{R}} K_G(x_1) = \dim_{\mathbf{R}} V - 2$. For $y \in V \setminus (A \cup C)$, $K_G(y)$ is a real hypersurface in V, $\dim_{\mathbf{R}} K_G(y) = \dim_{\mathbf{R}} V - 1$.*

PROOF. We have already seen that $K_G(v) = A$, $v \in A$, and $K_G(x_1) = C$, $x_1 \in C$. To prove the last assertion, we use the deformation retraction r: $X \to K_G(x_0) = C$ mentioned in the proof of Proposition 8.1. We may assume that r is K_G-equivariant by taking a K_G-equivariant imbedding of X into some $\mathbf{R}^N$, averaging the coordinate functions over the compact group K_G, and pulling this new retraction back to X. Thus, we obtain a surjective map $r_y = r \circ i \colon K_G(y) \to K_G(x_0)$ (where $i \colon K_G(y) \to X$ is the inclusion), such that $r_y(k(x)) = k(r_y(x))$ for $x \in K_G(y)$ and $k \in K_G$.

Suppose, for some $y \in V \setminus (A \cup C)$, the orbit $K_G(y)$ is not a real hypersurface. Then, necessarily, $\dim_{\mathbf{R}} K_G(y) = \dim_{\mathbf{R}} V - 2$, and r_y is a covering map of $K_G(y)$ onto $K_G(x_0)$. But $K_G(x_0)$ is simply-connected, so, in fact, r_y is a diffeomorphism. Since r_y is the restriction of the deformation retraction r, we have that $r_*[C'] = [C] \in H_{2n-2}(C, \mathbf{Z}) = H_{2n-2}(X, \mathbf{Z})$, where $C' := K_G(y)$. Therefore, since C' is homologous to C (in V), the line bundles L_C and $L_{C'}$ over V defined by C and C', respectively, are topologically equivalent ([5] Prop. (A.1.9)). Let m and m' be smooth (not necessarily holomorphic) sections of $L_C \simeq L_{C'}$, such that $C = \{p \in V | m(p) = 0\}$ and $C' = \{p \in V | m'(p) = 0\}$. The section i^*m is now a smooth non-vanishing section of i^*L_C (where $i\colon C \to V$ is the inclusion) because $C \cap C' = \emptyset$. Therefore, i^*L_C is topologically trivial, and in fact holomorphically trivial, since C is a compact homogeneous rational manifold and $H^1(C, \mathcal{O}) = 0$. On the other hand, $X \setminus C$ is Stein so that $gC \cap C \neq \emptyset$ for all $g \in G$. Since i^*L_C is G-equivariant (the Picard variety of C is trivial), gC defines a holomorphic section σ_g of i^*L_C. Choosing $g \in G$ such that $gC \neq C$, we see that σ_g is a non-constant holomorphic section, which is a contradiction. Therefore, $K_G(y)$ is a real hypersurface in V for any $y \in V \setminus (A \cup C)$. $\square$

9. – A classification.

We retain the assumption and notation of § 8. The existence of hypersurface orbits of K_G in the Stein manifold $M = K_G^{\mathbf{C}}(x)$, $x \in A$, leads us to a final classification for the non-compact strictly pseudoconcave manifold $X \subset V = S/P$. Given such a hypersurface, $K(y)$ for $y \in M$, Morimoto and Nagano ([15] Thm. 2) show that M is differentiably and K-equivariantly the tangent bundle of $A = K_G(x)$ in M, and that A is a compact symmetric space of rank 1. (In [13] it is assumed that the hypersurface is simply-connected. However, to obtain the above conclusion, one need only require that the fundamental group of the hypersurface be *finite*. See [22] $A(2)$, p. 133.) The list of compact symmetric spaces of rank 1 is actually quite short. In fact, the only possibilities for A are the following:

1) the n-sphere, S^n, $n \geqslant 2$;
2) real projective space, $\mathbf{RP}^n$, $n \geqslant 2$;
3) complex projective space, $\mathbf{P}^n$, $n \geqslant 1$;
4) quaternionic projective space, $\mathbf{QP}^n$, $n \geqslant 1$;
5) the Cayley projective plane.

The question then arises, which of these compact symmetric spaces of rank 1 actually gives rise to an example of a strictly pseudoconcave homogeneous manifold of a Lie group, and can a given symmetric space of rank 1 produce more than one example. The answers to these questions are implicitly contained in the work of T. Nagano [15]. The following theorem is essentially due to him.

THEOREM 9.1. *Any compact symmetric space A of rank 1 can be imbedded in a compact complex homogeneous rational manifold V in such a way that $X = V \setminus A$ is a strictly pseudoconcave homogeneous manifold.*

PROOF. Let $A = K/L$ be a compact symmetric space of rank 1, with K the connected isometry group. Then, by [15], the Lie algebra $\mathfrak{k}$ of K is always contained in a simple Lie algebra $\mathfrak{g}$ of a Lie group G which has the following properties:

i) $\mathfrak{g}$ is generated by $\mathfrak{k}$ and a single element z of $\mathfrak{g}$;

ii) $\mathfrak{g}$ is precisely $\mathfrak{k} + [z, \mathfrak{m}] + [[z, \mathfrak{m}], \mathfrak{m}]$ as a vector space, where $\mathfrak{m}$ is the complementary space of $\mathfrak{l}$, the Lie algebra of L, in the Cartan decomposition, $\mathfrak{k} = \mathfrak{l} + \mathfrak{m}$; $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{l}$;

iii) the Lie algebra structure of g is completely determined by the following: the adjoint ad (z) has eigenvalues $+1$, -1, and 0 on $\mathfrak{g}$ with corresponding eigenspaces

$$\mathfrak{m}_+ := \{x + [z, x] | x \in \mathfrak{m}\}$$

$$\mathfrak{m}_- := \{x - [z, x] | x \in \mathfrak{m}\}$$

and

$$\mathfrak{f} := \mathfrak{k} + [[z, \mathfrak{m}], \mathfrak{m}] .$$

The isotropy subgroup H of G acting on $A = K/L = G/H$ has the Lie algebra $\mathfrak{h} = \mathfrak{f} + \mathfrak{m}_-$. The complexification $\mathfrak{g}^{\boldsymbol{C}} = \mathfrak{g} + i\mathfrak{g}$ contains the compact form

$$\mathfrak{g}_u := \mathfrak{f} + [iz, \mathfrak{m}] + [[iz, \mathfrak{m}], \mathfrak{m}] .$$

Also, $\mathfrak{f}_g := \mathfrak{k} + [[iz, \mathfrak{m}], \mathfrak{m}]$ is the Lie algebra of the isotropy group of G_u acting on $V := G^{\boldsymbol{C}}/H^{\boldsymbol{C}} = G_u/F_u$, where $H^{\boldsymbol{C}}$ of course has the Lie algebra

$$(\mathfrak{k} + [[z, \mathfrak{m}], \mathfrak{m}] + \mathfrak{m}_-) \otimes \boldsymbol{C} = (\mathfrak{f} + \mathfrak{m}_-) \otimes \boldsymbol{C} .$$

The imbedding $A \to V$ is induced by the monomorphism $K \to G_u$.

We now check that

(1) A is totally real in V;

(2) G is transitive on $V \setminus A$;

(3) $V \setminus A$ is strictly pseudoconcave.

To prove (1), we observe that the tangent spaces to A and V at the point fixed by F_u may be identified with $\mathfrak{m}$ and $\mathfrak{m} + i\mathfrak{m}$, respectively. This i corresponds to the complex structure of V. On the Lie algebra level we have the complex conjugation $x + iy \mapsto x - iy$ in $\mathfrak{g}_C$, for x and y in $\mathfrak{g}$. This automorphism extends to that of G^C (see e.g. [4] p. 49), leaving invariant F_u and H^C. Thus, « conjugation » is well-defined on V and fixes A pointwise. That no other points are fixed follows from (2). To show (2), we note that the Stein manifold $M := K^C/L^C \subset V$ is differentiably and K-equivariantly the tangent bundle of $A = K/L \subset M$ with the orbits of K in $M \setminus A$ being hypersurfaces, by Morimoto and Nagano ([13] Prop. 2). Therefore, the orbit of G of any point y on such a hypersurface must properly contain the hypersurface, for otherwise G would have more than one closed orbit on V (see § 7). Thus, $G(y)$ is open and the connected component of $V \setminus G(y)$ which contains A must be A itself. Similarly, if $V \setminus G(y)$ contained other components, then these components would contain other closed orbits of G. Therefore, G acts transitively on the complement of A in V. For (3), we observe that the hypersurface orbits of K are strongly pseudoconvex, since they lie in the Stein manifold $M = K^C/L^C$ and are the boundaries of open submanifolds, namely, the tube neighborhoods of the zero section in the above mentioned identification with the tangent bundle of A. This identification then yields a natural strictly pseudoconcave exhaustion for $X = V \setminus A$. □

In summary, we state the final classification.

THEOREM 9.2. *Let X be a non-compact strictly pseudoconcave homogeneous manifold. Assume that no complex Lie group acts holomorphically and transitively on X. (If $\dim_C X = 2$, we also assume that X has a compactification.) Then X is equivariantly biholomorphic to one of the following (see* § 3):

1) $\boldsymbol{P}^n \setminus \overline{\boldsymbol{B}}^n$; $G = SU^1(n+1, \boldsymbol{C})$,

2) $Q^n \setminus S^n$; $G = SO^1(n+2)$,

3) $\boldsymbol{P}^n \setminus \boldsymbol{R}P^n$; $G = SL(n+1, \boldsymbol{R})$,

4) $\boldsymbol{P}^n \times \boldsymbol{P}^n \setminus \boldsymbol{P}^n_{\boldsymbol{R}}$; $G = \{(A, \bar{A}) \in SL(n+1, \boldsymbol{C}) \times SL(n+1, \boldsymbol{C})\}$,

5) $G_{2,2n} \setminus \boldsymbol{Q}P^{n-1}$; $G = SU^*(n)$,

6) $(EIII) \setminus$(Cayley projective plane);
$(EIII) = E_6/(\mathrm{Spin}\,(10) \times_{4} SO(2))$,
Cayley projective plane $= F_4/\mathrm{Spin}\,(9)$; $G = E_7^3$.

In each case $\mathrm{Aut}^0(X) \cong G/\mathrm{Center}\,(G)$.

PROOF. Let V be the compactification of X given in Proposition 4.1. If $V\setminus X$ contains an open set, then Theorem 6.1 applies and $X \cong \boldsymbol{P}^n\setminus\overline{\boldsymbol{B}}^n$. If $V\setminus X$ does not contain an open set, then § 7, § 8, and the opening paragraph of this section imply that $X = V\setminus A$, where A is a compact symmetric space of rank 1. Theorem 9.1 shows that any compact symmetric space A of rank 1 determines V and therefore determines $X = V\setminus A$. The uniqueness of V follows from the fact that G is *simple* so that V must be contained in Nagano's list of indecomposable possibilities ([15], § 5). The construction of V in Theorem 9.1 leads to (2), (3), (4), (5) and (6). □

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