ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

JEFF E. LEWIS
CESARE PARENTI

Pseudodifferential operators and Hardy kernels on $L^p(\mathbf{R}^+)$

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 7, nº 3 (1980), p. 481-503
<http://www.numdam.org/item?id=ASNSP_1980_4_7_3_481_0>

Pseudodifferential Operators and Hardy Kernels on $L^p(\boldsymbol{R}^+)$ (*).

JEFF E. LEWIS - CESARE PARENTI

Introduction.

We develop an algebra of pseudodifferential operators on $L^p(\boldsymbol{R}^+)$, $1<p<\infty$, which includes H, the Hilbert transform restricted to $\boldsymbol{R}^+$, and some classical Hardy kernels. Such operators arise in the study of singular integral equations in $L^p(\boldsymbol{R}^+)$ since $H^2 = -I + K$, where K is a Hardy kernel operator. The algebra of operators described here, called Mellin operators in $OP\Sigma_{1/p}$, are defined via the Mellin transform. As in the case of the Hilbert transform [Sh 1] or a Hardy kernel [FJL 1], the spectrum of the operator depends upon the L^p space on which it acts. As described in the remarks of Section 5, there are operators in $OP\Sigma_{1/p}$ for all p, $1<p<\infty$, which admit a parametrix in $OP\Sigma_{1/p}$ for some values of p, but do not have a parametrix in $OP\Sigma_{1/p}$ for other values of p; the parametrices in $OP\Sigma_{1/p}$, for different values of p, do not necessarily agree on $C_0^\infty(\boldsymbol{R}^+)$.

E. Shamir [Sh 1, Sh 2] studied the spectrum of the Hilbert transform on $L^p(\boldsymbol{R}^+)$. G. I. Ěskin [E 1, E 2] has made an extensive study of operators defined via the Mellin transform and given applications to weighted L^2 spaces and boundary value problems. In [N] J. Nourrigat has defined a class of pseudodifferential operators on $\boldsymbol{R}^+$ defined by the Mellin transform and studied their properties on weighted L^2 spaces. B. A. Plamenevskii in [P] has studied an algebra of pseudodifferential operators in $\boldsymbol{R}^+\times S^{n-1}$ defined using the Mellin transform. H. O. Cordes and E. A. Herman [CH] studied singular integrals on $L^2(\boldsymbol{R}^+)$.

In Section 1 we state the properties of the Mellin transform and Mellin multipliers to be used in the sequel. A representation for variable symbol Mellin operators is studied in Section 2. The space of symbols, $\Sigma_{1/p}$, and

(*) Research partially supported by the National Science Foundation and by C.N.R., Gruppo G.N.A.F.A.

Pervenuto alla Redazione il 20 Febbraio 1979 ed in forma definitiva il 15 Marzo 1979.

the space of Mellin operators, $OP\Sigma_{1/p}$, are defined in Section 3; the principal symbol σ_p is defined. The symbolic calculus is developed in Section 4. The operators in $OP\Sigma_{1/p}$ which admit a parametrix in $OP\Sigma_{1/p}$ are characterized by their symbols in Theorem 5; the remarks following Theorem 5 describe a typical situation. The index of an elliptic operator in $OP\Sigma_{1/p}$ is studied in Section 6. An application to an oblique derivative problem for Laplace's equation in a plane sector is given in Section 7.

1. – Preliminaries on the Mellin transform.

We shall deal with functions in $L^p = L^p(\boldsymbol{R}^+)$, $1 < p < \infty$, with the norm

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p dx\right)^{1/p}.$$

It will be convenient to consider functions $g(x) \in L^p_*(\boldsymbol{R}^+)$ where

$$\|g\|_{p,*} = \left(\int_0^\infty |g(x)|^p \frac{dx}{x}\right)^{1/p}.$$

If $f(x) \in L^p$, we define $f_p(x) = x^{1/p} f(x) \in L^p_*(\boldsymbol{R}^+)$ and the functions $F(u) = f(\exp[-u])$, and $F_p(u) = f_p(\exp[-u])$, $u \in \boldsymbol{R}$. Note that

$$\|f\|_p = \|f_p\|_{p,*} = \|F_p\|_{L^p(\boldsymbol{R})} = \left(\int_{-\infty}^{+\infty} \exp[-u]|F(u)|^p du\right)^{1/p}.$$

If $f(x) \in C_0^\infty(\boldsymbol{R}^+)$ we define the Fourier transform of F_p as the function

$$\hat{F}_p(\xi) = \int_{-\infty}^{+\infty} \exp[-iu\xi] F_p(u)\, du = \int_0^\infty x^{1/p + i\xi - 1} f(x)\, dx\,, \qquad \xi \in \boldsymbol{R}\,. \tag{1.1}$$

The Mellin transform of a function $f \in C_0^\infty(\boldsymbol{R}^+)$ is defined as

$$\tilde{f}(z) = \int_0^\infty x^{z-1} f(x)\, dx\,, \qquad z \in \boldsymbol{C}\,. \tag{1.2}$$

It follows that for $f \in C_0^\infty(\boldsymbol{R}^+)$, $\tilde{f}(z)$ is an entire function and we have the inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} \tilde{f}(z)\, dz\,,$$

where the notation $1/2\pi i \int_{a-i\infty}^{a+i\infty} \ldots dz$ denotes contour integration along the path $z = a + i\xi$, $-\infty < \xi < \infty$. Integration by parts shows that

$$\left(-x \frac{d}{dx} f\right)^{\sim}(z) = z\tilde{f}(z) \quad \text{and} \quad ((\log x) f)^{\sim}(z) = \frac{d}{dz}\tilde{f}(z)\,.$$

For τ real, $\delta > 0$, we use the notation

$$S_{\tau,\delta} = \{z \in \boldsymbol{C}\colon\ \tau - \delta < \operatorname{Re} z < \tau + \delta\}\,.$$

If f is measurable on $\boldsymbol{R}^+$ and the integral (1.2) is absolutely convergent for all z in some strip $S_{\tau,\delta}$ we shall call the integral $\tilde{f}(z)$ the *Mellin transform* of f; under these conditions $\tilde{f}(z)$ is a holomorphic function in $S_{\tau,\delta}$. We make the following definition.

DEFINITION 1. *Let $b(z)$ be a bounded measurable function on the line $\operatorname{Re} z = 1/p$. Then we say b is a Mellin multiplier on L^p iff the map*

$$Bf(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} b(z) \tilde{f}(z)\, df\,, \quad x > 0, \quad f \in C_0^\infty(\boldsymbol{R}^+),$$

is extendable as a bounded linear operator on L^p.

By (1.1), $\hat{F}_p(\xi) = \tilde{f}(1/p + i\xi)$, so that b is a Mellin multipler on L^p iff the function $\xi \to b(1/p + i\xi)$ is a Fourier multiplier on $L^p(\boldsymbol{R})$ [H].

We give the following examples which will be essential ingredients in the algebra of operators to be constructed in Section 3.

1) *The Hilbert transform on $L^p(\boldsymbol{R}^+)$.* The Hilbert transform of a function $f \in L^p$ is defined as

$$Hf(x) = p.v.\ \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x - y}\, dy\,. \tag{1.3}$$

Following Shamir [Sh 1] and Eskin [E 1] we can represent H as

$$(1.4)\qquad Hf(x)=\frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-z}\left\{i\,\frac{1+\exp[2\pi iz]}{1-\exp[2\pi iz]}\right\}\tilde{f}(z)\,dz\,,\qquad f\in C_0^\infty(\boldsymbol{R}^+)\,.$$

It is well known that H is a bounded operator on L^p. Define the function

$$\theta(z)=\frac{1}{1-\exp[2\pi iz]}=\frac{1}{2}\left(1+\frac{1+\exp[2\pi iz]}{1-\exp[2\pi iz]}\right).$$

Then $\theta(z)$ is a Mellin multiplier on L^p, $1<p<\infty$.

2) *Hardy kernels on L^p.* Let $k(x)$ be a measurable function on $\boldsymbol{R}^+$ such that for some a, b with $0<a<b<1$,

$$\int_0^\infty x^{a-1}|k(x)|\,dx+\int_0^\infty x^{b-1}|k(x)|\,dx<\infty\,.$$

Then for all p, $a<1/p<b$, the Hardy operator with kernel k is defined on L^p by

$$(1.5)\qquad Kf(x)=\int_0^\infty k\left(\frac{x}{y}\right)f(y)\,\frac{dy}{y}\,.$$

Following [FJL 1], for $f\in C_0^\infty(\boldsymbol{R}^+)$,

$$(1.6)\qquad Kf(x)=\frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-z}\tilde{k}(z)\tilde{f}(z)\,dz\,,$$

where $\tilde{k}(z)$ is the Mellin transform of the kernel k which is defined and holomorphic for $a<\operatorname{Re} z<b$.

3) *The operator $T_{\zeta,p}$* (a particular Hardy kernel). Let $1<p<\infty$ and $\zeta\in\boldsymbol{C}$, $\operatorname{Re}\zeta\neq 1/p$. For $f\in C_0^\infty(\boldsymbol{R}^+)$, define

$$(1.7)\qquad T_{\zeta,p}f(x)=\frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{-z}\,\frac{1}{\zeta-z}\,\tilde{f}(z)\,dz\,.$$

The function $b\,(z)=1/(\zeta-z)$ is a Mellin multiplier on L^p, $1/p\neq\operatorname{Re}\zeta$, and the L^p norm of the operator $T_{\zeta,p}$ is bounded by $C|\mathrm{Re}\,\zeta - 1/p|^{-1}$ (C is independent of p). If $1/p < \mathrm{Re}\,\zeta$, the kernel for $T_{\zeta,p}$ on L^p is given by

$$k_{\zeta,p}(x) = 0\,, \quad 0 < x < 1\,, \quad k_{\zeta,p}(x) = x^{-\zeta}\,, \quad x > 1\,.$$

If $\mathrm{Re}\,\zeta < 1/p$, the kernel for $T_{\zeta,p}$ on L^p is given by

$$k_{\zeta,p}(x) = -\,x^{-\zeta}\,, \quad 0 < x < 1\,, \quad k_{\zeta,p}(x) = 0\,, \quad x > 1\,.$$

The bound for the L^p norm of $T_{\zeta,p}$ is a consequence of Young's inequality.

2. – A class of bounded operators on $L^p(\boldsymbol{R}^+)$.

We now introduce a class of Mellin integral operators on $L^p(\boldsymbol{R}^+)$ with variable kernels.

THEOREM 1. *Let $a(x, z)$ be a function defined for $x > 0$ and z in some strip $S_{1/p,\delta}$, $1 < p < \infty$. Suppose that for all x, $a(x, z)$ is holomorphic in $S_{1/p,\delta}$ and that there is an $\varepsilon > 0$ and a constant C such that*

$$\sup_{x>0} |a(x, z)| \leqslant C(1 + |z|)^{-1-\varepsilon}\,, \qquad z \in S_{1/p,\delta}\,.$$

Then the operator defined by

$$(2.6) \qquad Af(x) = \frac{1}{2\pi i} \int\limits_{1/p-i\infty}^{1/p+i\infty} x^{-z} a(x, z) \tilde{f}(z)\, dz\,, \qquad f \in C_0^\infty(\boldsymbol{R}^+)\,,$$

is extendable as a bounded operator on L^p.

PROOF. Let $0 < \delta_1 < \delta$ and let $\Gamma_{1/p,\delta_1}$ denote the contour

$$\left\{z = \frac{1}{p} + \delta_1 + i\xi,\ -\infty < \xi < \infty\right\} \cup \left\{z = \frac{1}{p} - \delta_1 - i\xi,\ -\infty < \xi < \infty\right\}.$$

If $\mathrm{Re}\, z = 1/p$, by the Cauchy integral formula

$$a(x, z) = \frac{1}{2\pi i} \int\limits_{\Gamma_{1/p,\delta_1}} \frac{a(x, \zeta)}{\zeta - z}\, d\zeta\,.$$

Using this representation for $a(x, z)$ in (2.6) and applying Fubini's Theorem, we obtain

$$Af(x) = \frac{1}{2\pi i} \int_{\Gamma_{1/p,\delta_1}} a(x, \zeta) \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} \frac{1}{\zeta - z} \tilde{f}(z)\, dz = \frac{1}{2\pi i} \int_{\Gamma_{1/p,\delta_1}} a(x, \zeta)\, T_{\zeta,p} f(x)\, d\zeta .$$

An application of Minkowski's Integral Inequality gives

$$\|Af\|_p \leqslant \frac{1}{2\pi} \int_{\Gamma_{1/p,\delta_1}} \sup_{x>0} |a(x, \zeta)|\, \|T_{\zeta,p} f\|_p\, d|\zeta| \leqslant C_1 \delta_1^{-1} \|f\|_p . \qquad q.e.d.$$

THEOREM 2. *Let $a(x, z)$ be a function defined for $x > 0$ and for z in some strip $S_{1/p,\delta}$. Suppose that*

1) *$a(x, z)$ is continuously differentiable in $\mathbf{R}^+ \times S_{1/p,\delta}$ and holomorphic in z,*

2) *For some $\varepsilon > 0$ there is a constant C such that for all x and z*

$$(2.7) \qquad |a(x, z)| \leqslant C \left(\frac{x}{1 + x^2}\right)^{\varepsilon} \left(\frac{1}{(1 + |z|)^{2+\varepsilon}}\right),$$

$$(2.8) \qquad \left| x \frac{\partial}{\partial x} a(x, z) \right| \leqslant \frac{C}{(1 + |z|)^{1+\varepsilon}} .$$

Then the operator A defined by (2.6) *is compact on L^p.*

PROOF. By the proof of Theorem 1 and (2.7) it follows that the operators $f \to \chi_{(0,\lambda)}(x) Af(x)$ and $f \to \chi_{(\lambda^{-1},\infty)}(x) Af(x)$ have small L^p norm if λ is small. The map $f \to -x(d/dx) Af(x) = Tf(x)$ is represented by

$$Tf(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} \left\{ a(x, z) z - x \frac{\partial}{\partial x} a(x, z) \right\} \tilde{f}(z)\, dz .$$

By (2.8) and Theorem 1, T is bounded on L^p. From these observations it follows that the family $\{Af\colon \|f\|_p \leqslant 1\}$ is equicontinuous in $L^p(0, N)$ for every N and that

$$\lim_{N\to\infty} \sup_{\|f\|_p \leqslant 1} \int_{|x|>N} |Af(x)|^p dx = 0 .$$

This establishes the compactness of A on L^p. $\qquad$ *q.e.d.*

3. – Spaces of symbols and Mellin operators.

As a preliminary step we introduce some spaces of functions and their Mellin transforms.

DEFINITION 2. *Let τ be real. If $\delta > 0$, by $\mathcal{F}_{\tau,\delta}$ we denote the class of functions $a(x) \in C^\infty(\boldsymbol{R}^+)$ such that the following property holds: for every δ_1, $0 < \delta_1 < \delta$, and every j there is a constant $C = C(\delta_1, j, a)$ such that*

$$\left|\left(x\frac{d}{dx}\right)^j a(x)\right| \leqslant Cx^{-\tau}\left(\frac{x}{1+x^2}\right)^{\delta_1}.$$

By $\mathcal{F}_\tau$ we denote the space of functions $a(x)$ such that $a \in \mathcal{F}_{\tau,\delta}$ for some δ.

DEFINITION 3. *Let τ be real. If $\delta > 0$, by $\tilde{\mathcal{F}}_{\tau,\delta}$ we denote the class of functions $b(z)$ which are defined and holomorphic in the strip $S_{\tau,\delta}$ and such that the following property holds: for every δ_1, $0 < \delta_1 < \delta$, and every j and k there is a constant $C = C(\delta_1, j, k, b)$ such that*

$$\left|z^j \frac{d^k}{dz^k} b(z)\right| \leqslant C$$

for all $z \in S_{\tau,\delta_1}$. By $\tilde{\mathcal{F}}_\tau$ we denote the space of functions $b(z)$ such that $b \in \tilde{\mathcal{F}}_{\tau,\delta}$ for some δ.

The fact that the functions in $\tilde{\mathcal{F}}_\tau$ are precisely the Mellin transforms of the functions in $\mathcal{F}_\tau$ is consequence of the following result whose proof is contained in the article of A. Avantaggiati [A, Sec. 2].

LEMMA 1. *If $a \in \mathcal{F}_{\tau,\delta}$, then its Mellin transform $\tilde{a} \in \tilde{\mathcal{F}}_{\tau,\delta}$. Conversely, given $b \in \tilde{\mathcal{F}}_{\tau,\delta}$, define the function*

$$a(x) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} x^{-z} b(z)\,dz, \qquad x > 0.$$

Then $a \in \mathcal{F}_{\tau,\delta}$ and $\tilde{a} = b$.

We define the symbols of the class of smoothing operators to be constructed.

DEFINITION 4. *Let τ be real. If $\varepsilon, \delta > 0$, by $\Phi_{\tau,\delta,\varepsilon}$ we denote the class of functions such that:*

1) $a(x, z) \in C^\infty(\boldsymbol{R}^+ \times S_{\tau,\delta})$,

2) *for all* x, $a(x, z)$ *defines a holomorphic function on* $S_{\tau,\delta}$,

3) *for each* ε_1, $0 < \varepsilon_1 < \varepsilon$, *and each* δ_1, $0 < \delta_1 < \delta$, *and each* M, j, k *there is a constant* $C = C(\varepsilon_1, \delta_1, M, j, k, a)$ *such that*

$$\left| z^M \left(x \frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial z}\right)^k a(x, z)\right| \leqslant C \left(\frac{x}{1 + x^2}\right)^{\varepsilon_1}$$

for $z \in S_{\tau,\delta_1}$.

By Φ_τ *we denote the class of functions* $a(x, z)$ *which belong to* $\Phi_{\tau,\varepsilon,\delta}$ *for some* ε, δ.

We recall that the function $\theta(z) = 1/(1 - \exp[2\pi i z])$ is holomorphic in the strip $0 < \operatorname{Re} z < 1$ and that for all M and j, uniformly in the strip $0 < \delta < \operatorname{Re} z < 1 - \delta < 1$,

$$\lim_{\operatorname{Im} z \to +\infty} \left| z^M \frac{d^j}{dz^j} (\theta(z) - 1) \right| = 0 \tag{3.1}$$

and

$$\lim_{\operatorname{Im} z \to -\infty} \left| z^M \frac{d^j}{dz^j} \theta(z) \right| = 0 . \tag{3.2}$$

It follows that $\theta(z)(1 - \theta(z)) \in \tilde{\mathcal{F}}_{1/p}$ for $1 < p < \infty$.

Finally we are ready for the definition of the space of symbols of an algebra of Mellin operators on $L^p(\boldsymbol{R}^+)$.

DEFINITION 5. *Let* $1 < p < \infty$. *Denote by* $\Sigma_{1/p}$ *the space of functions* $a(x, z) \in C^\infty(\boldsymbol{R}^+ \times S_{1/p,\delta})$ *for some* $\delta = \delta(a) > 0$ *and for which there is a representation of the following form in* $\boldsymbol{R}^+ \times S_{1/p,\delta}$:

$$a(x, z) = a_+(x)\theta(z) + a(x)(1 - \theta(z)) + a(z) + \alpha(x, z) \tag{3.3}$$

where

1) $a_+(x)$ *and* $a_-(x)$ *are extendable as continuous functions on* $\overline{\boldsymbol{R}^+}$ *in such a way that* $a_\pm(x) - a_\pm(0) \in \mathcal{F}_0$,

2) $a(z) \in \tilde{\mathcal{F}}_{1/p}$,

3) $\alpha(x, z) \in \Phi_{1/p}$.

DEFINITION 6. *For each symbol* $a \in \Sigma_{1/p}$ *we define the Mellin operator*

$$Af(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} a(x, z) \tilde{f}(z)\, dz , \qquad f \in C_0^\infty(\boldsymbol{R}^+) . \tag{3.10}$$

The space of all such operators will be denoted by $OP\Sigma_{1/p}$. *The function* $a(x, z) \in \Sigma_{1/p}$ *will be called the symbol of the Mellin operator* A. *If the symbol of the operator* A *is also in the class* $\Phi_{1/p}$, *we shall write* $A \in OP\Phi_{1/p}$ *and shall call* A *a smoothing operator.*

DEFINITION 7. *If* A *is a Mellin operator with symbol*

$$a(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z) + \alpha(x, z) ,$$

the function $a(x, z) - \alpha(x, z)$ *will be called the principal symbol of* A *and be denoted by* $\sigma_p(A)(x, z)$.

From Theorem 1 it follows that if $A \in OP\Sigma_{1/p}$, then A can be extended as a bounded operator on L^p; moreover, if $\sigma_p(A)(x, z) \equiv 0$, A is a compact operator on L^p.

4. – The symbolic calculus for $OP\Sigma_{1/p}$.

We study the compositions and adjoints of operators in $OP\Sigma_{1/p}$.

THEOREM 3. *Let* $A, B \in OP\Sigma_{1/p}$. *Then* $AB \in OP\Sigma_{1/p}$. *Moreover, if*

$$\sigma_p(A)(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z)$$

and

$$\sigma_p(B)(x, z) = b_+(x)\theta(z) + b_-(x)\big(1 - \theta(z)\big) + b(z) ,$$

then

$$\sigma_p(AB)(x, z) = c_+(x)\theta(z) + c_-(x)\big(1 - \theta(z)\big) + c(z) ,$$

where

(4.1) 1) $c_+(x) = a_+(x)b_+(x)$

(4.2) 2) $c_-(x) = a_-(x)b_-(x)$

(4.3) 3) $c(z) = \sigma_p(A)(0, z)\cdot\sigma_p(B)(0, z) - a_+(0)b_+(0)\theta(z) - a_-(0)b_-(0)\big(1 - \theta(z)\big)$.

PROOF. We shall first show that the composition of two smoothing operators is a smoothing operator. Let $a(x, z)$, $b(x, z) \in \Phi_{1/p}$ and let A and B

be the corresponding Mellin operators. For $f \in C_0^\infty(\boldsymbol{R}^+)$,

$$Bf(z) = \int_0^\infty x^{z-1} \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-w} b(x, w) \tilde{f}(w)\, dw\, dx$$
$$= \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} \tilde{b}(z - w, w) \tilde{f}(w)\, dw\,,$$

where $\tilde{b}(z, w)$ denotes the Mellin transform of $b(x, w)$ in the x-variable. Thus

$$ABf(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-z} a(x, z) \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} \tilde{b}(z - w, w) \tilde{f}(w)\, dw\, dz$$
$$= \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-w} c(x, w) \tilde{f}(w)\, dw\,,$$

where

$$c(x, w) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-(z-w)} a(x, z) \tilde{b}(z - w, w)\, dz$$
$$= \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} a(x, v + w) \tilde{b}(v, w) dv\,.$$

In the above calculations, the absolute convergence of the integrals justifies the use of Fubini's Theorem.

To verify that $c(x, w) \in \Phi_{1/p}$ we apply repeatedly the observation that if $a(x, z) \in \Phi_{1/p}$, then

$$z^M (\log x)^N \left(x \frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial z}\right)^j a(x, z) \in \Phi_{1/p}\,.$$

We next show that if $A \in OP\Sigma_{1/p}$, $B \in OP\Phi_{1/p}$, then $AB \in OP\Phi_{1/p}$. Let the symbol of A be

$$a(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z) \quad \text{and let } b(x, z) \in \Phi_{1/p}\,.$$

In this case we argue as above and again use Fubini's Theorem to obtain that

$$ABf(x) = \frac{1}{2\pi i} \int_{1/p-i\infty}^{1/p+i\infty} x^{-w} c(x, w) \tilde{f}(w)\, dw\,,$$

where

$$c(x, w) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} a(x, v + w) \tilde{b}(v, w)\, dv .$$

Consider, e.g., the contribution of

$$\frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} \theta(v + w) \tilde{b}(v, w)\, dv .$$

Modulo $\Phi_{1/p}$, this integral is $c_+(x, w)$ where

$$c_+(x, w) = \frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} [\theta(v + w) - \theta(w)] \tilde{b}(v, w)\, dv , \tag{4.4}$$

Now

$$c_+(x, w) = \frac{a_+(0)}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} \int_0^1 \frac{\partial \theta}{\partial w}(w + tv)\, v \tilde{b}(v, w)\, dt\, dv .$$

To show that $c_+(x, w) \in \Phi_{1/p}$, we observe that for some $\delta > 0$, it is possible to shift the integral $(1/2\pi i) \int_{0-i\infty}^{0+i\infty} \ldots dv$ to either of the integrals

$$\frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \ldots dv \qquad \text{or} \qquad \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \ldots dv .$$

Hence $x^{-\delta} w^N (x(\partial/\partial x))^j (\partial/\partial w)^k c_+(x, w)$ is a linear combination of integrals of the form

$$I(x, w) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta-i\infty} x^{-(v+\delta)} (\log x)^s \int_0^1 t^m (w + tv)^M \theta^{(1+q)}(w + tv) \tilde{b}_r(v, w)\, dt\, dv , \tag{4.5}$$

where $b_r(x, w) \in \Phi_{1/p}$. The integrals $I(x, w)$ are absolutely convergent and for w in some strip around $\operatorname{Re} w = 1/p$ are bounded by $C|\log x|^s$. Repeating the argument with δ replaced by $-\delta$ shows that $c_+(x, w) \in \Phi_{1/p}$. In the same manner one shows that

$$\frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} x^{-v} \{a_-(x)(1 - \theta(v + w)) + a(v + w)\} \tilde{b}(v, w)\, dv$$

gives a function in $\Phi_{1/p}$.

The next step is to show that if $A \in OP\Phi_{1/p}$ and $B \in OP\Sigma_{1/p}$ then $AB \in OP\Phi_{1/p}$. Let $a(x, z)$ be the symbol of A and suppose that $b(x, z) = b_+(x)\theta(z) + b_-(x)(1-\theta(z)) + b(z)$ is the symbol of B. Denote by B_+ the operator

$$B_+ f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} b_+(x)\theta(z)\tilde{f}(z)\, dz .$$

We have that

$$AB_+ f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-w} c(x, w)\tilde{f}(w)\, dw + C_1 f(x)$$

where

$$c(x, w) = \frac{\theta(w)}{2\pi i} \int_{0 - i\infty}^{0 + i\infty} x^{-v} a(x, v + w)\big(b_+(x) - b_+(0)\big)^{\sim}(v)\, dv$$

and C_1 is a smoothing operator. Since $\big(b_+(x) - b_+(0)\big)^{\sim}(v) \in \tilde{\mathcal{F}}_0$, it may be shown that $c(x, w) \in \Phi_{1/p}$. The other terms in the composition can be handled similarly.

As the next step we consider two operators A_+, B_+ with symbols $a_+(x)\theta(z)$ and $b_+(x)\theta(z)$. Let Θ be the operator with symbol $\theta(z)$. Then

$$A_+ B_+ = a_+(x) b_+(0) \Theta^2 + a_+(x)\big(\Theta b'_+(x) \Theta\big)$$

where

$$b'_+(x) = b_+(x) - b_+(0) \in \mathcal{F}_0 .$$

The second part can be written as

$$Cf(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-w} c(x, w)\tilde{f}(w)\, dw ,$$

where

$$c(x, w) = a_+(x)\theta(w) \frac{1}{2\pi i} \int_{0 - i\infty}^{0 + i\infty} x^{-v} \theta(v + w)\tilde{b}'_+(v)\, dv .$$

Using the argument for the integral (4.4), we have that modulo a symbol in $\Phi_{1/p}$, $c(x, w) = a_+(x) b'_+(x) \theta^2(w)$.

At this point we observe that $\theta^2 = \theta(\theta - 1) + \theta$ so that the operator θ^2 contains the Hardy kernel operator with symbol $\theta(z)\big(\theta(z) - 1\big) \in \tilde{\mathcal{F}}_{1/p}$. Thus the principal symbol of A_+B_+ is given by

$$\sigma_p(A_+B_+)(x, z) = a_+(x)b_+(x)\theta(z) + a_+(0)b_+(0)\theta(z)\big(\theta(z)-1\big)\,.$$

Since the composition of two Hardy kernel operators with symbols in $\tilde{\mathcal{F}}_{1/p}$ is a Hardy kernel operator with symbol in $\tilde{\mathcal{F}}_{1/p}$, we leave to the reader the proof of the cases not considered explicitly and the calculation of the principal symbol. *q.e.d.*

COROLLARY. *If* $A, B \in OP\Sigma_{1/p}$ *the commutator*

$$[A, B] = AB - BA \in OP\Phi_{1/p}\,.$$

We consider the adjoint of an operator $A \in OP\Sigma_{1/p}$. If $1/p + 1/q = 1$ we define $A^*\colon L^q \to L^q$ to be the operator such that

$$\int_0^\infty Af(x)\overline{g(x)}\,dx = \int_0^\infty f(x)\overline{A^*g(x)}\,dx\,, \qquad f, g \in C_0^\infty(\boldsymbol{R}^+)\,.$$

THEOREM 4. *Let* $A \in OP\Sigma_{1/p}$ *and* $1/p + 1/q = 1$. *Then* $A^* \in OP\Sigma_{1/q}$; *moreover, the principal symbol of* A^* *is*

$$\sigma_q(A^*)(x, z) = \overline{\sigma_p(A)(x, 1-\bar{z})}\,. \tag{4.6}$$

In particular if

$$\sigma_p(A)(x, z) = a_+(x)\theta(z) + a_-(x)\big(1-\theta(z)\big) + a(z)\,, \tag{4.7}$$

then

$$\sigma_q(A^*)(x, z) = \bar{a}_+(x)\theta(z) + \bar{a}_-(x)\big(1-\theta(z)\big) + \bar{a}(1-\bar{z})\,, \tag{4.8}$$

$\operatorname{Re} z$ *near* $1/q$.

PROOF. We recall that the Hilbert transform H is representable as $H = i(2\Theta - 1) \in OP\Sigma_{1/p}$. Using the kernel representation (1.3) of H, we have that $H^* = -H \in OP\Sigma_{1/q}$. A calculation shows that

$$\sigma_q(H^*)(z) = \overline{\sigma_p(H)(1-\bar{z})}\,.$$

Next consider the operator $Af(x) = a(x)f(x)$ where $a(x) - a(0) \in \mathcal{F}_0$. Then $A^*g(x) = \bar{a}(x)g(x)$. Representing Θ in terms of H and applying Theorem 3 proves the theorem for operators of the form

$$A = a_+(x)\Theta + a_-(x)(1-\Theta)\,.$$

We now consider an operator A which is a Hardy operator with symbol $a(z) \in \tilde{\mathcal{F}}_{1/p}$. There is a kernel $k(x) \in \mathcal{F}_{1/p}$ with $\tilde{k} = a$ such that $Af(x) = \int_0^\infty k(x/y) f(y)(dy/y)$. The adjoint A^* is representable with a kernel $k^*(x) = (1/x)\bar{k}(1/x)$. For $\operatorname{Re} w$ near $1/q$ we have that $\tilde{k}^*(w) = \overline{a(1-\overline{w})} \in \tilde{\mathcal{F}}_{1/q}$.

Finally we show that if $A \in OP\Phi_{1/p}$ then $A^* \in OP\Phi_{1/p}$. If the symbol of A is $a(x, z) \in \Phi_{1/p}$ and $f, g \in C_0^\infty(\mathbf{R}^+)$, represent

$$f(y) = \frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} y^{-z}\tilde{f}(z)\,dz \qquad \text{and} \qquad g(y) = \frac{1}{2\pi i}\int_{1/q-i\infty}^{1/q+i\infty} y^{-w}\tilde{g}(w)\,dw\,.$$

Using Fubini's Theorem, we obtain that

$$\int_0^\infty Af(y)\overline{g(y)}\,dy = \int_0^\infty f(x)\overline{\frac{1}{2\pi i}\int_{1/q-i\infty}^{1/q+i\infty} x^{-w}c(x, w)\tilde{g}(w)\,dw}\,dx\,,$$

where

$$\overline{c(x, w)} = \frac{1}{2\pi i}\int_{1/p-i\infty}^{1/p+i\infty} x^{\overline{w}+z-1}\int_0^\infty y^{-(z+\overline{w})}a(y, z)\,dy\,dz\,.$$

Performing the change of variables $z \to 1 - \overline{w} + v$, $\operatorname{Re} v = 0$, we have that

$$\overline{c(x, w)} = \frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} x^{-v}\tilde{a}(v, (1-\overline{w}) - v)\,dv\,.$$

Using arguments similar to those for (4.4) and (4.5), we can show that $c(x, w) \in \Phi_{1/q}$. *q.e.d.*

REMARK. If $A \in OP\Sigma_{1/p}$ the transposed operator ${}^tA$ is defined so that

$$\int_0^\infty Af(x)g(x)\,dx = \int_0^\infty f(x)\,{}^tAg(x)\,dx,\ f, g \in C_0^\infty(\mathbf{R}^+)\,.$$

Then if $1/p + 1/q = 1$, ${}^tA \in OP\Sigma_{1/q}$. If $\sigma_p(A)$ is given by (4.6) then

$$\sigma_q({}^tA)(x, z) = a_+(x)\big(1 - \theta(z)\big) + a_-(x)\theta(z) + a(1 - z),$$

$\operatorname{Re} z$ near $1/q$.

REMARK. We observe that smoothing operators map L^p into $\mathcal{F}_{1/p}$.

LEMMA 2. *Let $A \in OP\Phi_{1/p}$. Then if $f \in L^p$, $Af \in \mathcal{F}_{1/p}$.*

PROOF. Let the symbol of A be $a(x, z) \in \Phi_{1/p}$. Define the function $k(x, t)$ by

$$k(x, t) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} t^{-z} a(x, z)\, dz.$$

Then for some $\delta > 0$ and each i, j there is a $C = C(\delta, i, j, k)$ such that

$$\left|\left(x \frac{\partial}{\partial x}\right)^i \left(t \frac{\partial}{\partial t}\right)^j k(x, t)\right| \leqslant C t^{-1/p} \left(\frac{x}{1 + x^2}\right)^\delta \left(\frac{t}{1 + t^2}\right)^\delta. \tag{4.9}$$

Fix $\xi > 0$ and for $f \in C_0^\infty(\boldsymbol{R}^+)$ let

$$A_\xi f(x) = \int_0^\infty k\left(\xi, \frac{x}{y}\right) f(y) \frac{dy}{y}.$$

The Mellin transform of $A_\xi f$ is $a(\xi, z)\tilde{f}(z) \in \tilde{\mathcal{F}}_{1/p}$. Hence

$$A_\xi f(x) = \frac{1}{2\pi i} \int_{1/p - i\infty}^{1/p + i\infty} x^{-z} a(\xi, z) \tilde{f}(z)\, dz.$$

Putting $\xi = x$ we then have the representation

$$Af(x) = \int_0^\infty k\left(x, \frac{x}{y}\right) f(y) \frac{dy}{y}.$$

$$= \int_0^\infty y^{-1/p} k\left(x, \frac{x}{y}\right) [y^{1/p} f(y)] \frac{dy}{y}.$$

It follows that $x^{1/p}(x(d/dx))^j Af(x)$ is a linear combination of integrals of the form

$$I(x)=\int_0^\infty\left(\frac{x}{y}\right)^{1/p}k_r\left(x,\frac{x}{y}\right)[y^{1/p}f(y)]\frac{dy}{y},$$

where $k_r(x, t)$ satisfies estimates of the form (4.9). By Hölder's inequality,

$$|I(x)|\leqslant C\left(\frac{x}{1+x^2}\right)^\delta\left(\int_0^\infty\left(\frac{t}{1+t^2}\right)^{\delta_q}\frac{dt}{t}\right)^{1/q}\|f\|_p,$$

where $1/p+1/q=1$. *q.e.d.*

5. – Elliptic operators in $OP\Sigma_{1/p}$.

We characterize the operators $A\in OP\Sigma_{1/p}$ which are « elliptic », *i.e.*, for which there exists a parametrix $B\in OP\Sigma_{1/p}$ such that $AB-I$ and $BA-I$ are smoothing operators.

THEOREM 5. *Let $A\in OP\Sigma_{1/p}$ with principal symbol*

$$\sigma_p(A)(x,z)=a_+(x)\theta(z)+a_-(x)(1-\theta(z))+a(z).$$

The following two conditions are equivalent:

1) *There is an operator $B\in OP\Sigma_{1/p}$ such that $AB-I\in OP\Phi_{1/p}$.*

2) *The following three conditions are satisfied by $\sigma_p(A)(x,z)$:*

$$\text{(5.1)}\qquad\left\{\begin{aligned}&\inf_{\xi\in R}\left|\sigma_p(A)\left(0,\frac{1}{p}+i\xi\right)\right|>0,\\&\inf_{x>0}|a_+(x)|>0,\\&\inf_{x>0}|a_-(x)|>0.\end{aligned}\right.$$

PROOF. Suppose that 1. is satisfied and let

$$\sigma_p(B)(x,z)=b_+(x)\theta(z)+b_-(x)(1-\theta(z))+b(z).$$

By Theorem 3

$$\sigma_p(AB)(x,z)=1=1\cdot\theta(z)+1(1-\theta(z)).$$

By (4.1) and (4.2) and the observation that

$$\sigma_p(AB)(0, z) = \sigma_p(A)(0, z)\cdot\sigma_p(B)(0, z)\,,$$

we have the identities

$$1 = a_+(x)b_+(x)\,,$$

$$1 = a_-(x)b_-(x)\,,$$

$$1 = \sigma_p(A)(0, z)\sigma_p(B)(0, z)\,.$$

Condition 2 follows.

Conversely, suppose that 2. is satisfied. Note that for some $\delta > 0$,

$$\inf_{z\in S_{1/p,\delta}} |\sigma_p(A)(0, z)| > 0\,.$$

We define an operator B with symbol

$$b(x, z) = \frac{1}{a_+(x)}\theta(z) + \frac{1}{a_-(x)}\big(1-\theta(z)\big) + b(z)\,, \tag{5.2}$$

where

$$b(z) = \frac{1}{\sigma_p(A)(0, z)} - \frac{1}{a_+(0)}\,\theta(z) - \frac{1}{a_-(0)}\big(1-\theta(z)\big)\,, \tag{5.3}$$

$z\in S_{1/p,\delta}$. It may be shown, using the properties of $a_+(x)$, $a_-(x)$ and (3.1), (3.2), that $b(x, z)\in \Sigma_{1/p}$. A direct calculation using (4.1), (4.2), and (4.3) shows that B is a parametrix for A. *q.e.d.*

DEFINITION 8. *If $A\in OP\Sigma_{1/p}$ and A satisfies condition* 1. *or* 2. *of Theorem* 5, *we shall say that A is an elliptic operator in $OP\Sigma_{1/p}$.*

REMARK. We emphasize that the definition of ellipticity in $OP\Sigma_{1/p}$ depends on p. The following situation is typical.

Consider an operator A with principal symbol

$$\sigma_p(A)(x, z) = a(x, z) = a_+(x)\theta(z) + a_-(x)\big(1-\theta(z)\big) + a(z)\,.$$

Suppose that $\inf|a_+(x)|>0$, $\inf|a_-(x)|>0$, and that $a(z)\in\tilde{\mathcal{F}}_{1/p}$ for all p, $1<p<\infty$. Then the function $\psi(z) = \big(a(0, z)\big)^{-1}$ is meromorphic in the strip $0<\operatorname{Re} z<1$; moreover, in any strip $0<\delta<\operatorname{Re} z<1-\delta<1$, $\psi(z)$ has only a finite number of poles. Hence for all p outside a discrete set, $N(A)$, A is elliptic in $OP\Sigma_{1/p}$. Define $b(x, z)$ by (5.2) where $b(z)$ is defined by (5.3). Then $b(z)$ has the same poles and residues as $\psi(z)$. Then if $p \notin N(A)$, $b(x, z)$ is the principal symbol of a parametrix for A in $OP\Sigma_{1/p}$. If p_1 and $p_2 \notin N(A)$, let $B_{p_i} \in OP\Sigma_{1/p_i}$ be parametrices for A in $OP\Sigma_{1/p_i}$, $i = 1, 2$. The calculation in [FJL 1] shows that for $f \in C_0^\infty(\mathbf{R}^+)$,

$$B_{p_1}f(x) - B_{p_2}f(x) = \int_0^\infty k\left(\frac{x}{y}\right) f(y)\, \frac{dy}{y}\,,$$

where the kernel k is given by

$$k(x) = \frac{1}{2\pi i} \int_{1/p_1 - i\infty}^{1/p_1 + i\infty} x^{-z} b(z)\, dz - \frac{1}{2\pi i} \int_{1/p_2 - i\infty}^{1/p_2 + i\infty} x^{-z} b(z)\, dz\,.$$

If $a(0, \zeta) = 0$ for some ζ, $\operatorname{Re} \zeta$ between $1/p_1$ and $1/p_2$, then $k(x) \not\equiv 0$ (see [FJL 1]).

Remark. If A is elliptic in $OP\Sigma_{1/p}$ and $Af = 0$, $f \in L^p$, then $f \in \mathcal{F}_{1/p}$. This follows from Lemma 2.

6. – The index of an elliptic operator in $OP\Sigma_{1/p}$.

We will relate the index of an elliptic operator in $OP\Sigma_{1/p}$ to the winding numbers of the coefficients $a_\pm(x)$.

Lemma 3. *For ν an integer define*

$$\varphi_\nu(x) = \exp\left([2\pi i\nu]\frac{x}{x+1}\right).$$

Then

1) $\varphi_\nu\colon \overline{\mathbf{R}^+} \to S^1 = \{|z| = 1\}$ *and the winding number of* φ_ν *is* ν.

2) $\varphi_\nu(x) - 1 \in \mathcal{F}_0$.

3) *If $a(x)$ is a function mapping $\overline{\mathbf{R}_+} \to \mathbf{C}\setminus\{0\}$ such that $a(x) - a(0) \in \mathcal{F}_0$, and the winding number of a is ν, then there is a continuous homotopy*

   $F\colon [0, 1]\times \overline{\mathbf{R}^+} \to \mathbf{C}\setminus\{0\}$ *such that*

   (i) $F(0, x) = a(0)\varphi_\nu(x)$,

   (ii) $F(1, x) = a(x)$,

   (iii) $F(t, 0) = a(0)$, $0 \leqslant t \leqslant 1$,

   (iv) $F(t, x) - a(0) \in \mathcal{F}_0$, $0 \leqslant t \leqslant 1$.

PROOF. Parts 1 and 2 follow by a calculation. To prove 3, we remark that if we replace $a(x)$ by $a(x)\varphi_{-\nu}(x)$ it is sufficient to construct the homotopy when $\nu = 0$. In this case it is well known that there is a homotopy $G(t, x)$ which satisfies (i)-(iii) and such that $G(t, \cdot) \in C^\infty(\boldsymbol{R}^+)$ for every t. Let $\varepsilon = \frac{1}{2}|a(0)| > 0$ and choose $\delta > 0$ such that

$$|G(t, x) - a(0)| < \varepsilon \quad \text{for } 0 \leqslant t \leqslant 1,\ 0 < x < 4\delta \text{ or } (4\delta)^{-1} < x < \infty .$$

Construct a nonnegative partition of unity on $\overline{\boldsymbol{R}^+}$ as $1 = \alpha_1(x) + \alpha_2(x) + \alpha_3(x)$ where $\alpha_1(x) = 1$ if $0 \leqslant x \leqslant \delta$, $\alpha_1(x) = 0$ if $x > 2\delta$, $\alpha_3(x) = 1$ for $x > \delta^{-1}$, $\alpha_3(x) = 0$ for $x < \frac{1}{2}\delta^{-1}$ and $\alpha_2(x) = 1 - \alpha_1(x) - \alpha_3(x)$. Define the homotopy F as

$$F(t, x) = \big(\alpha_1(x) + \alpha_3(x)\big)\Big(a(0) + t\big(a(x) - a(0)\big)\Big) + \alpha_2(x)G(t, x) . \tag{6.1}$$

The verification of (i)-(iv) is left to the reader. *q.e.d.*

As an application of the previous lemma we have the following theorem.

THEOREM 6. *Let A be elliptic in $OP\Sigma_{1/p}$ and let*

$$\sigma_p(A)(x, z) = a_+(x)\theta(z) + a_-(x)\big(1 - \theta(z)\big) + a(z) . \tag{6.2}$$

Suppose that ν_+ and ν_- are the winding numbers of a_+ and a_-. Then A as an operator on L^p has index $\nu = \nu_+ - \nu_-$.

The proof of Theorem 6 is accomplished by a sequence of lemmas.

LEMMA 4. *With the notation of Theorem 6, A has the same index as the operator $A^\#$ with symbol*

$$a^\#(x, z) = a_+(0)\varphi_{\nu_+}(x)\theta(z) + a_-(0)\varphi_{\nu_-}(x)\big(1 - \theta(z)\big) + a(z) . \tag{6.3}$$

PROOF. By Lemma 3 there are homotopies $F_\pm(t, x)$ which connect $a_\pm(x)$ to the functions $a_\pm(0)\varphi_{\nu_\pm}(x)$ in such a way that the operators A_t with symbols

$$a_t(x, z) = F_+(t, z)\theta(z) + F_-(t, x)\big(1 - \theta(z)\big) + a(z)$$

are elliptic in $OP\Sigma_{1/p}$. Then $A_0 = A$ and $A_1 = A$ mod $OP\Phi_{1/p}$. Hence index $A^\# =$ index A in L^p. *q.e.d.*

LEMMA 5. *With the notation of Theorem 6 and Lemma 4, the operator A has the same index as the operator A_ν with symbol*

$$a_\nu(x, z) = \varphi_\nu(x)\theta(z) + \big(1 - \theta(z)\big) . \tag{6.4}$$

PROOF. Let A_0 be the operator with symbol $a_0^{\#}(z) \equiv a^{\#}(0, z)$ and B_0 be the operator with symbol $b_0(z) = (a_0^{\#}(z))^{-1}$. Then $B_0 A_0^{\#} = I = A_0^{\#} B_0$ so that $A^{\#}$ has the same index on L^p as the operator $B_0 A^{\#} = I + B_0(A^{\#} - A_0^{\#})$. Since $\sigma_p(A^{\#} - A_0^{\#})(0, z) = 0$, Theorem 3 yields that

$$\sigma_p(B_0 A^{\#}) = \varphi_{\nu_+}(x)\theta(z) + \varphi_{\nu_-}(x)(1 - \theta(z)) .$$

Then index (A_ν) = index $(\varphi_{-\nu_-} B_0 A)$ = index $(A^{\#})$. *q.e.d.*

PROOF OF THEOREM 6. It remains to calculate the index of A_ν on L^p. Since $\sigma_p(A_\nu)(0, z) \equiv 1$, A_ν is elliptic in $OP\Sigma_{1/r}$ for all r, $1 < r < \infty$. If $f \in L^p$, $A_\nu f = 0$, then $f \in \mathcal{F}_{1/r}$, $1 < r < \infty$. If $g \in L^q$, $1/p + 1/q = 1$, $A_\nu^* g = 0$, then $g \in \mathcal{F}_{1/r}$, $1 < r < \infty$. Thus the index of A_ν on L^p is the index of A_ν on L^2. As an operator on L^2, A_ν is in the algebra considered by Cordes and Herman [CH], and its symbol σA_ν, as defined in [CH], has winding number ν and hence index ν. (The particular operator considered in [CH] was $K_0 = \Theta + [(\log x - 2i)/(\log x + 2i)](I - \Theta))$. *q.e.d.*

7. – Application to an oblique derivative problem in a plane sector.

Operators in $OP\Sigma_{1/p}$ arise naturally in the oblique derivative problem in a plane sector.

Let $\Omega = \{(x, y) \in \boldsymbol{R}^2 \colon x > 0,\ y > 0\}$. We seek a solution of the following problem:

$$(7.1)\qquad \begin{cases} \Delta u = 0 \quad \text{in } \Omega \\[2mm] \lim\limits_{y \to 0^+} \left(\alpha_1(x) \dfrac{\partial u}{\partial y}(x, y) + \beta_1(x) \dfrac{\partial u}{\partial x}(x, y)\right) = \varphi_1(x) \in L^p(\boldsymbol{R}^+) , \\[2mm] \lim\limits_{x \to 0^+} \left(\alpha_2(y) \dfrac{\partial u}{\partial x}(x, y) + \beta_2(y) \dfrac{\partial u}{\partial y}(x, y)\right) = \varphi_2(y) \in L^p(\boldsymbol{R}^+) , \end{cases}$$

where α_j and β_j are real functions such that $\alpha_j^2(t) + \beta_j^2(t) \equiv 1$.

If $\Phi_1(t)$, $\Phi_2(t) \in C_0^\infty(\boldsymbol{R}^+)$ we study the single layer potential with density Φ_1 along the x-axis and density Φ_2 along the positive y-axis, namely,

$$\begin{aligned} u(x, y) &= \frac{1}{2\pi} \int_0^\infty \log\left((x - t)^2 + y^2\right) \varphi_1(t)\, dt \\ &\quad + \frac{1}{2\pi} \int_0^\infty \log\left(x^2 + (y - t)^2\right) \varphi_2(t)\, dt \\ &= u_1(x, y) + u_2(x, y) . \end{aligned}$$

Then it is known [St, FJL2] that in $L^p(\boldsymbol{R}^+)$,

$$\lim_{y\to 0^+} \frac{\partial u_1}{\partial y}(x, y) = \varphi_1(x)$$

$$\lim_{y\to 0^+} \frac{\partial u_1}{\partial x}(x, y) = p.v.\frac{1}{\pi}\int_0^\infty \frac{1}{x-t}\varphi_1(t)\,dt = H\varphi_1(x),$$

$$\lim_{x\to 0^+} \frac{\partial u_1}{\partial x}(x, y) = -\frac{1}{\pi}\int_0^\infty \frac{t}{y^2+t^2}\varphi_1(t)\,dt = -K_n\varphi_1(y),$$

$$\lim_{x\to 0^+} \frac{\partial u_1}{\partial y}(x, y) = \frac{1}{\pi}\int_0^\infty \frac{y}{y^2+t^2}\varphi_1(t)\,dt = K_\tau\varphi_1(y).$$

Note that K_n and K_τ are Hardy kernel operators with kernels

$$k_n(t) = \frac{1}{\pi}\frac{1}{1+t^2} \qquad \text{and} \qquad k_\tau(t) = \frac{1}{\pi}\frac{t}{1+t^2}.$$

Since $k_n(t)$ and $k_\tau(t) \in \mathcal{F}_{1/p}$, $1 < p < \infty$, we have that K_n and K_τ are operators in $OP\Sigma_{1/p}$ and that

$$\sigma_p(K_\tau)(z) = \frac{-\cos((\pi/2)z)}{\sin(\pi z)} \qquad \text{and} \qquad \sigma_p(K_n)(z) = \frac{\sin((\pi/2)z)}{\sin(\pi z)}.$$

Recall that the symbol of the operator H may be written as

$$\sigma_p(H)(z) = -\frac{\cos(\pi z)}{\sin(\pi z)}.$$

Similar formulas hold for the boundary values of the gradient of u_2.

We make the following assumptions on the coefficients of the boundary operators in (7.1):

$$\alpha_j(t)-\alpha_j(0), \quad \beta_j(t)-\beta_j(0) \in \mathcal{F}_0, \qquad j=1,2.$$

Then the boundary operators applied to u give functions $\psi_1(t)$, $\psi_2(t) \in L^p(\boldsymbol{R}^+)$ where

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\alpha_1(t)I + \beta_1(t)H & -\alpha_1(t)K_n + \beta_1(t)K_\tau\\ -\alpha_2(t)K_n + \beta_2(t)K_\tau & \alpha_2(t)I + \beta_2(t)H\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}. \tag{7.2}$$

We write the system (7.2) as $\vec{\psi} = A\vec{\varphi}$ where A is a matrix of operators in $OP\Sigma_{1/p}$. The matrix of principal symbols is given by

$$\sigma_p(A)(t,z) = \begin{pmatrix} v_1(t)\theta(z) + \overline{v_1(t)}(1-\theta(z)) & \dfrac{\alpha_1(0)\cos\big((\pi/2)z\big) + \beta_1(0)\sin\big((\pi/2)z\big)}{\sin(\pi z)} \\ -\dfrac{\alpha_2(0)\cos\big((\pi/2)z\big) + \beta_2(0)\big(\sin(\pi/2)z\big)}{\sin(\pi z)} & v_2(t)\theta(z) + \overline{v_2(t)}(1-\theta(z)) \end{pmatrix}$$

where $v_j(t) = \alpha_j(t) + i\beta_j(t)$.

THEOREM 7. *For $j = 1, 2$, let ν_j be the winding numbers of $v_j(t)$. Suppose that*

$$\inf_{\operatorname{Re} z = 1/p} |\det \sigma_p(A)(0,z)| > 0 .$$

Then

1) *There is a matrix B of operators in $OP\Sigma_{1/p}$ such that $AB - I$ and $BA - I$ are matrices of smoothing operators.*
2) *As an operator on $L^p \times L^p$, the index of A is $2\nu_1 + 2\nu_2$.*

PROOF. Since $\alpha_j^2 + \beta_j^2 = 1$, the function $\det \sigma_p(A)(t,z)$ is the symbol of an elliptic operator D in $OP\Sigma_{1/p}$. Denote by E a parametrix for D and let B be the matrix of operators

$$B = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \alpha_2(t)I + \beta_2(t)H & \alpha_1(0)K_n - \beta_1(0)K_\tau \\ \alpha_2(0)K_n - \beta_2(0)K_\tau & \alpha_1(t)I + \beta_1(t)H \end{pmatrix}.$$

The symbolic calculus establishes that, modulo a matrix of functions in $\Phi_{1/p}$, $\sigma_p(A)(t,z)\cdot\sigma_p(B)(t,z) = I$ and the first conclusion is established.

Let A_0 be the matrix of operators whose symbol is $\sigma_p(A)(0,z)$ and let B_0 be the matrix of operators whose symbol is $\sigma_p(B)(0,z) = [\sigma_p(A_0)(z)]^{-1}$. Then $B_0A_0 = A_0B_0 = I$ on $L^p \times L^p$ so that the index of A is the index of $B_0A = I + B_0(A - A_0)$. The matrix of principal symbols of B_0A is

$$\sigma_p(B_0A)(t,z) = \begin{pmatrix} \dfrac{v_1(t)}{v_1(0)}\theta(z) + \overline{\dfrac{v_1(t)}{v_1(0)}}(1-\theta(z)) & 0 \\ 0 & \dfrac{v_2(t)}{v_2(0)}\theta(z) + \overline{\dfrac{v_2(t)}{v_2(0)}}(1-\theta(z)) \end{pmatrix}. \tag{7.3}$$

By Theorem 6, the index of B_0A is $2\nu_1 + 2\nu_2$.

REMARK. For the operator $\boldsymbol{A}_0$ there are always values of p for which $\det \sigma_p(\boldsymbol{A}_0)(z) = 0$ for some z, $\operatorname{Re} z = 1/p$.

Suppose that $\alpha_j(0) + i\beta_j(0) = \cos\gamma_j + i \sin\gamma_j$, $j = 1, 2$. Another representation of $\sigma_p(\boldsymbol{A}_0)(z)$ is

$$\sigma_p(\boldsymbol{A}_0)(z) = \frac{1}{\sin(\pi z)} \begin{pmatrix} \sin(\pi z - \gamma_1) & \sin\left(\frac{\pi}{2} z - \left(\frac{\pi}{2} - \gamma_1\right)\right) \\ \sin\left(\frac{\pi}{2} z - \left(\frac{\pi}{2} - \gamma_2\right)\right) & \sin(\pi z - \gamma_2) \end{pmatrix}.$$

Then $\det \sigma_p(\boldsymbol{A}_0)(z) = 0$ when $z = (2k+1)/3$ or $z = (2/\pi)(\gamma_1 + \gamma_2) + (2k+1)$. In particular $\det \sigma_p(\boldsymbol{A}_0)(\frac{1}{3}) = 0$, and $\boldsymbol{A}_0$ is not a Fredholm operator on $L^3 \times L^3$. This is in accordance with the results of [FJL 2] for double layer potentials for the Dirichlet problem for which the operators were not Fredholm for $p = \frac{3}{2}$.

REFERENCES

[A] A. AVANTAGGIATI, *Sistemi di equazioni a derivate parziali a coefficienti costanti in due variabili e problemi ellittici*, Ricerche Mat., **24** (1975), pp. 203-279.

[CH] H. O. CORDES - E. A. HERMAN, *Singular integral operators on a half-line*, Proc. Nat. Acad. Sci. U.S.A., **56** (1966), pp. 1668-1673.

[E 1] G. I. ĚSKIN, *Boundary value problems for elliptic pseudodifferential operators* (in Russian), Nauka, Moscow (1973).

[E 2] G. I. ĚSKIN, *General mixed boundary problems for elliptic differential equations*, C.I.M.E., Course on Pseudodifferential Operators with Applications, II Ciclo (1977), Liguori, Napoli.

[FJL 1] E. B. FABES - M. JODEIT, Jr. - J. E. LEWIS, *On the spectra of a Hardy kernel*, J. Functional Analysis, **21** (1976), pp. 187-194.

[FJL 2] E. B. FABES - M. JODEIT, Jr. - J. E. LEWIS, *Double layer potentials for domains with corners and edges*, Indiana U. Math. J., **26** (1977), pp. 95-114.

[H] L. HÖRMANDER, *Translation invariant operators on L^p spaces*, Acta Math., **104** (1960), pp. 93-139.

[N] J. NOURRIGAT, *Une classe d'opérateurs pseudo-différentiels à une variable*, C. R. Acad. Sci. Sér. A, tome **283** (1976), pp. 457-460.

[P] B. A. PLAMENEVSKII, *An algebra of pseudodifferential operators in spaces with weight* (in Russian), Mat. Sb., **106** (1978), pp. 297-320.

[Sh 1] E. SHAMIR, *Reduced Hilbert transforms and singular integral equations*, J. Analyse Math., **12** (1964), pp. 277-305.

[Sh 2] E. SHAMIR, *Elliptic systems of singular integral operators*. I: *The half-space case*, Trans. Amer. Math. Soc., **127** (1967), pp. 107-123.

[St] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

University of Illinois at Chicago Circle
P. O. Box 4348
Chicago, Illinois, 60680

Istituto Matematico dell'Università
Piazza di Porta S. Donato, 5 - 40127 Bologna