Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

Frederick J. Almgren
Leon Simon

Existence of embedded solutions of Plateau's problem

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 6, nº 3 (1979), p. 447-495
<http://www.numdam.org/item?id=ASNSP_1979_4_6_3_447_0>

Existence of Embedded Solutions of Plateau's Problem (*).

FREDERICK J. ALMGREN Jr. (**) - LEON SIMON (***) (‡)

Consider a uniformly convex open set $A \subset \boldsymbol{R}^3$ such that ∂A is C^2, and let Γ be a C^3 Jordan curve contained in ∂A. The classical approach to Plateau's problem, as formulated by Douglas [D], Rado [R], and Courant [C], together with the work of Osserman [0], Alt [A1, 2] and Gulliver [G] concerning branch points (see also [GOR]), demonstrates that there is a minimal immersion X from the disc $D_1 = \{x \in \boldsymbol{R}^2\colon |x| \leqslant 1\}$ into $\boldsymbol{R}^3$ such that $X|\partial D_1$ is a homeomorphism onto Γ. Furthermore X is area minimizing relative to all immersions Y from D_1 into $\boldsymbol{R}^3$ such that $Y|\partial D_1$ is a homeomorphism onto Γ.

Osserman conjectured that (for convex A) the immersion X might be an embedding; that is, the solution surface might have no self intersections. This was proved subject to the (rather stringent) condition that Γ has total curvature $\leqslant 4\pi$ by Gulliver and Spruck [GS1] (1). Also, there is recent work of Tomi and Tromba [TT] which, based on an intriguing analysis of the pathwise connectivity of spaces of suitable minimal surfaces, asserts that Γ bounds a (perhaps unstable) embedded minimal surface.

We here wish to demonstrate by a geometric measure theory argument (quite different than these other approaches) that one can always find an embedded minimal surface $M \subset \bar{A}$ which is diffeomorphic to D_1, satisfies $\partial M = \Gamma$, and minimizes area relative to all diffeomorphs N of D_1 having $\partial N = \Gamma$. (See Theorem 6 in § 8 below.)

(*) This research was partly supported by NSF grants at Princeton and Stanford universities. The research of the second author was also partly supported by a Sloan Foundation grant.

(**) Princeton University.

(***) Stanford University, University of Adelaide.

(‡) Results announced in part in Notices A.M.S. (1977) A-13, Abstract No. 77T - B15.

(1) See also the correction [GS2] relating to Lemma 4.2 of [GS1].

Pervenuto alla Redazione il 13 Giugno 1978.

We here also discuss (in § 9) extensions of the result to the case when Γ consists of two or more contours, and (in § 10) we consider the case when the surfaces under consideration are allowed to have higher genus; Appendix B illustrates the fact that surfaces of higher genus will often have smaller area than the area minimizing discs obtained in § 8.

Concerning uniqueness, even in the single contour case, it is clear, by a slight modification of an argument of Nitsche [N, Section 2], that the embedded discs of § 8 are *not* necessarily unique. However, by using the methods of Morgan [M], it can be shown in a rather straightforward manner that there is a non-trivial (geometrically natural) measure μ defined on $C^{2,\alpha}$ curves Γ contained in the boundaries of uniformly convex regions in $\boldsymbol{R}^3$, such that the set of curves which do not bound a unique area minimizing disc has μ-measure zero.

We have also very recently received an announcement by Meeks and Yau [MY] dealing with similiar questions on three dimensional manifolds from a somewhat different perspective and by different methods (viz. showing « Douglas-Morrey » immersed solutions to Plateau's problem are in fact embedded). The techniques of our paper extend in a straightforward manner to yield embedded minimal surfaces in three dimensional manifolds (provided boundary regularity is guaranteed by convexity hypotheses, etc.). In this regard, see also [P].

1. – Terminology.

$\boldsymbol{B}(x_0, \varrho)$, $\boldsymbol{U}(x_0, \varrho)$ will denote respectively the closed and open balls, with radius ϱ and centre x_0, in $\boldsymbol{R}^3$.

$\mathcal{H}^k$, $k \geqslant 0$, denotes k-dimensional Hausdorff measure in $\boldsymbol{R}^3$.

$D_1 = \{x \in \boldsymbol{R}^2 \colon |x| \leqslant 1\}$.

$\mathcal{M}$ denotes the collection of all surfaces-with-boundary in $\boldsymbol{R}^3$ which are C^2 diffeomorphs of D_1.

If γ is a Jordan curve in $\boldsymbol{R}^2$, int γ denotes the bounded component of $\boldsymbol{R}^2 \sim \gamma$. More generally, given $M \in \mathcal{M}$ and a Jordan curve $\Delta \subset M$, we let $\operatorname{int}_M \Delta$ denote the (unique) component M' of $M \sim \Delta$ such that $M' \cap \partial M = \emptyset$.

$\tilde{\mathcal{M}}$ will denote the collection of surfaces of the form $\overline{\operatorname{int}_M \Delta}$, where $M \in \mathcal{M}$ and Δ is a piecewise C^2 Jordan curve in M.

A denotes a uniformly convex open subset of $\boldsymbol{R}^3$ with C^2 boundary ∂A.

Γ denotes a C^3 Jordan curve contained in ∂A.

The remaining notation concerning varifolds and currents is the standard notation of [AW1, 2] and [FH].

Now let $\{M_k\}$ be a sequence in $\mathcal{M}$ with $\partial M_k = \Gamma$, $k = 1, 2, \ldots$, and $\lim\limits_{k \to \infty} \mathcal{H}^2(M_k) = \inf\{\mathcal{H}^2(M) \colon M \in \mathcal{M}, \partial M = \Gamma\}$.

We can assume, by taking a subsequence if necessary, that there is a varifold $V \in V_2(\boldsymbol{R}^3)$ such that $V = \lim_{k\to\infty} \boldsymbol{v}(M_k)$. V is of course stationary in $\boldsymbol{R}^3 \sim \Gamma$ (in the sense of [AW1, 4.2]) because

$$\|h_\# V\|(U) \geq \|V\|(U)$$

whenever U is a bounded open subset of $\boldsymbol{R}^3$ and h is a diffeomorphism of U leaving a neighbourhood of $\partial U \cup \Gamma$ fixed. Also, by the convex hull property for stationary varifolds (see the appendix), we have

$$\operatorname{spt} \|V\| \subset A \cup \Gamma . \tag{1.1}$$

Our ultimate aim is to prove that there is an $M \in \mathcal{M}$ such that $\partial M = \Gamma$ and $V = \boldsymbol{v}(M)$. (Of course it will then follow that $\mathcal{H}^2(M) = \inf\{\mathcal{H}^2(N): N \in \mathcal{M}$ and $\partial N = \Gamma\}$ because of the manner in which V was constructed.) An outline of the remainder of the paper is as follows:

In §§ 2-4 we prove some preliminary results (of a rather elementary nature). In §§ 5-6 these are used to prove interior regularity of V: it is proved that for each $x_0 \in \operatorname{spt}\|V\| \sim \Gamma$ there is a positive integer n_{x_0}, a positive ϱ_{x_0}, and an analytic hypersurface M_{x_0} such that

$$V \llcorner \boldsymbol{U}(x_0, \varrho) \times G(3, 2) = n_{x_0} \boldsymbol{v}(M_{x_0}) .$$

In § 7 the corresponding boundary regularity results are given: we prove that for each $x_0 \in \Gamma$ there is a $\varrho = \varrho(x_0) > 0$ and a C^2 surface-with-boundary M_{x_0} such that $\partial M_{x_0} \cap \boldsymbol{U}(x_0, \varrho) = \Gamma \cap \boldsymbol{U}(x_0, \varrho)$ and $V \llcorner \big((\boldsymbol{R}^3 \sim \Gamma) \cap \cap \boldsymbol{U}(x_0, \varrho)\big) \times G(3, 2) = \boldsymbol{v}\big(M_{x_0} \cap \boldsymbol{U}(x_0, \varrho)\big)$.

In § 8 it is shown that these regularity results imply the existence of the required minimizing $M \in \mathcal{M}$. In § 9 an analogous existence result is proved for the case when Γ is a union of $k \geq 2$ smooth Jordan curves, and in § 10 we consider the possibility of finding area minimizing surfaces of higher genus.

2. – Some preliminary lemmas.

LEMMA 1. *Suppose F_1, F_2 are oriented compact connected surfaces-with-boundary, and suppose $\partial F_1 = \partial F_2$. Suppose further that for each $j = 1, 2$, there is a diffeomorphism μ_j of F_j onto a subset E_j of $\boldsymbol{S}^2$.*

Then there is a diffeomorphism χ of F_1 onto F_2 such that χ coincides with the identity map on ∂F_1.

PROOF. By replacing μ_j by $\theta\circ\mu_j$ if necessary, where θ is any orientation reversing diffeomorphism of $\boldsymbol{S}^2$, we can assume that μ_j is an orientation preserving map from F_j onto E_j, $j=1,2$. (We give E_1, E_2 the orientations induced by the inclusions $E_j\subset\boldsymbol{S}^2$, $j=1,2$.) Since $\mu_2\circ\mu_1^{-1}|\partial E_1$ is then an orientation preserving diffeomorphism of ∂E_1 onto ∂E_2, it is a standard fact that we can find an orientation preserving diffeomorphism μ of E_1 onto E_2 such that $\mu|\partial E_1=\mu_2\circ\mu_1^{-1}|\partial E_1$

We then define χ by $\chi=\mu_2^{-1}\circ\mu\circ\mu_1$.

Before stating the next lemma, we need the following additional terminology.

We let $\Gamma_1,\dots,\Gamma_n$ be pairwise disjoint C^2 Jordan curves in $\boldsymbol{R}^3$ and $D\subset\boldsymbol{R}^2$ we let be the union of a pairwise disjoint collection $D^{(1)},\dots,D^{(R)}$ of diffeomorphs of D_1. $\mathcal{Y}=\mathcal{Y}(\Gamma_1,\dots,\Gamma_n)$ denotes the (possibly empty) collection of Lipschitz maps $Y\colon D\to\boldsymbol{R}^3$ satisfying the following conditions

(2.1) $\bigcup_{j=1}^{n}Y^{-1}(\Gamma_j)=\mu_1\cup\dots\cup\mu_m$, where $\mu_1,\dots,\mu_m$ are pairwise disjoint C^2 Jordan curves contained in $D\sim\partial D$;

(2.2) for any component U of $D\sim\bigcup_{i=1}^{m}\mu_i$, $Y|\overline{U}$ is a C^2 embedding of $\overline{U}$ into $\boldsymbol{R}^3$;

(2.3) for any pair of components U, W of $D\sim\bigcup_{i=1}^{m}\mu_i$, either $Y(U)\cap Y(W)=\emptyset$, or else $Y(U)=Y(W)$.

Notice that the above conditions imply that $Y(D)\sim\Big(\big(\bigcup_{j=1}^{n}\Gamma_j\big)\cup Y(\partial D)\Big)$ is a 2-dimensional submanifold of $\boldsymbol{R}^3$, each component of which has closure of the form $Y(\overline{U})$, where U is a component of $D\sim\bigcup_{j=1}^{m}\mu_i$.

Subsequently, for $Y\in\mathcal{Y}$, we use the notation

$$
(2.4)\qquad \begin{cases} U(Y)=\bigcup_{i=1}^{m}\operatorname{int}(\mu_i) & (\mu_i \text{ as in } (2.1))\\ U_\eta(Y)=\{x\in D\colon \operatorname{dist}(x,U(Y))<\eta\}\,. \end{cases}
$$

If B is a Borel subset of D, then we let $A(Y|B)$ denote the area associated with $Y|B$; that is

$$
(2.5)\qquad A(Y|B)=\int_B\|\Lambda_2(DY(x))\|\,dx\,.
$$

(In case Y is 1-1 on B, $A(Y|B)$ is just $\mathcal{H}^2(Y(B))$.)

LEMMA 2. *Suppose $X \in \mathcal{Y}$. Then there is $\tilde{X} \in \mathcal{Y}$ such that*

$$(2.6)\quad \begin{cases} U(X) = U(\tilde{X});\, \tilde{X}(x) = X(x), & x \in D \sim U(X);\, \tilde{X}(D) \subset X(D) \\ A(\tilde{X}|U') \leqslant A(X|U') & \text{for each component } U' \text{ of } U(X)\,, \end{cases}$$

and such that for each $\eta > 0$ there is a C^2 embedding $\hat{X}$ of D into $\boldsymbol{R}^3$ with

$$(2.6)'\quad \begin{cases} \hat{X}(x) = X(x)\,, & x \in D \sim U_\eta(X);\ \sup |\hat{X}(x) - \tilde{X}(x)| < \eta \\ A(\hat{X}|U') \leqslant A(X|U') + \eta & \text{for each component } U' \text{ of } U_\eta(X)\,. \end{cases}$$

Before giving a proof of this lemma, we note the following corollary.

COROLLARY 1. *Suppose $M_1, \ldots, M_R \in \mathcal{M}$ with $\partial M_1, \ldots, \partial M_R$ pairwise disjoint and*

$$(2.7)\qquad M_i \sim \partial M_i \subset B \sim \partial B\,, \quad \partial M_i \subset \partial B\,, \qquad i = 1, \ldots, R\,,$$

where B is a diffeomorph of the closed unit ball in $\boldsymbol{R}^3$.

Then for each $\eta > 0$ there exist pairwise disjoint $\tilde{M}_1, \ldots, \tilde{M}_R \in \mathcal{M}$ with

$$(2.8)\qquad \partial\tilde{M}_i = \partial M_i\,, \quad \tilde{M}_i \sim \partial M_i \subset B \sim \partial B\,, \quad \mathcal{H}^2(\tilde{M}_i) \leqslant \mathcal{H}^2(M_i) + \eta$$

for each $i = 1, \ldots, R$.

PROOF OF COROLLARY 1. The proof is by induction on R. The result is trivial if $R = 1$. Hence take $R \geqslant 2$ and assume that there exist pairwise disjoint $\hat{M}_1, \ldots, \hat{M}_{R-1}$ such that (2.8) holds with $\hat{M}_i$ in place of $\tilde{M}_i$, $i = 1, \ldots, R-1$. Next, let $\hat{M}_R \in \mathcal{M}$ be such that $\partial\hat{M}_R = \partial M_R$, $M_R \sim \partial\hat{M}_R \subset \subset B \sim \partial B$, $\mathcal{H}^2(\hat{M}_R) < \mathcal{H}^2(M_R) + \eta$, and such that $\hat{M}_R$ intersects each of $\hat{M}_1, \ldots, \hat{M}_{R-1}$ transversally. (Such $\hat{M}_R$ can of course be obtained by making a slight perturbation of M_R.) We can then write $\hat{M}_R \cap \left(\bigcup_{i=1}^{R-1} \hat{M}_i\right) = \bigcup_{j=1}^{n} \Gamma_i$, where $\Gamma_1, \ldots, \Gamma_n$ are pairwise disjoint C^2 Jordan curves.

We now let X be a C^2 immersion of D into $\boldsymbol{R}^3$ such that the restriction of X to each $D^{(i)}$ is a C^2 diffeomorphism of $D^{(i)}$ onto $\hat{M}^{(i)}$. $\Big($Here $D = \bigcup_{j=1}^{R} D^{(j)}$ is as defined above.$\Big)$

It is now easy to see that $X \in \mathcal{Y}$ and hence by the lemma there is an embedding $\hat{X}$ of D onto $\boldsymbol{R}^3$ satisfying $\hat{X}(\partial D^{(i)}) = \partial M_i$, $\hat{X}(D \sim \partial D) \subset B \sim \partial B$ and $\mathcal{H}^2\big(\hat{X}(D^{(i)})\big) \leqslant \mathcal{H}^2(\hat{M}_i) + \eta$, $i = 1, \ldots, R$. Then, defining $\tilde{M}_i = \hat{X}(D^{(i)})$, $i = 1, \ldots, R$, we see that (2.8) holds with 2η in place of η. This completes the proof.

PROOF OF LEMMA 2. For $Y \in \mathcal{Y}$ we define $Y^{(k)}: D \to \boldsymbol{R}^3$ as follows:

(i) If $Y^{-1}(\Gamma_k) = \emptyset$, define $Y^{(k)} = Y$;

(ii) if $Y^{-1}(\Gamma_k) \neq \emptyset$, then (since $Y \in \mathcal{Y}$) we have $Y^{-1}(\Gamma_k) = \bigcup_{i=1}^{m} \mu_i$ for some pairwise disjoint C^2 Jordan curves $\mu_1, \dots, \mu_m$ in $D \sim \partial D$. We let $l \in \{1, \dots, m\}$ be such that

$$A(Y|\operatorname{int}\mu_l) \leqslant A(Y|\operatorname{int}\mu_j), \quad j = 1, \dots, m, \tag{2.9}$$

(notice that then $(\operatorname{int}\mu_l) \cap \mu_j = \emptyset$ for $j \neq l$), and let $U_1, \dots, U_p$ be the components of $\bigcup_{i=1}^{m} \operatorname{int}\mu_j$. For $j = 1, \dots, p$ let θ_j be any C^2 diffeomorphism of $\overline{U}_j$ onto $\overline{\operatorname{int}\mu_l}$ such that

$$\theta_j|\partial U_j = (Y|\mu_l)^{-1}\circ(Y|\partial U_j).$$

We then define $Y^{(k)}$ by

$$Y^{(k)} = \begin{cases} Y \text{ on } D \sim \bigcup_{j=1}^{p} U_j \\ Y\circ\theta_j \text{ on } U_j, \quad j = 1, \dots, p. \end{cases}$$

Of course $Y^{(k)}$, so defined, depends on the choice of $\theta_1, \dots, \theta_p$ (and on the choice of l in case there is more than one integer l satisfying (2.9)).

For each $j = 1, \dots, n$ define

$$\begin{cases} \mathcal{Y}^{(j)} = \{Y^{(j)}: Y \in \mathcal{Y}\} \\ \mathcal{Y}_j \;= \bigcap_{r=1}^{j} \mathcal{Y}^{(r)}. \end{cases} \tag{2.10}$$

We note that $\mathcal{Y}^{(j)} \subset \mathcal{Y}$, and $Y \in \mathcal{Y}^{(j)}$ if and only if $Y \in \mathcal{Y}$ and $Y^{-1}(\Gamma_j) = \bigcup_{i=1}^{m} \mu_i$, where $\operatorname{int}\mu_1, \dots, \operatorname{int}\mu_m$ are pairwise disjoint and such that there are C^2 diffeomorphisms θ_j of $\overline{\operatorname{int}\mu_j}$ onto $\overline{\operatorname{int}\mu_1}$ with $Y|\overline{\operatorname{int}\mu_j} = (Y|\overline{\operatorname{int}\mu_1})\circ\theta_j$, $j = 1, \dots, m$.

The following properties, except possibly the first, are straightforward consequences of the definition of $Y^{(k)}$:

(i) $Y \in \mathcal{Y}^{(j)} \Rightarrow Y^{(k)} \in \mathcal{Y}^{(j)}$, $k, j = 1, \dots, n$;

(ii) $U(Y) = U(Y^{(k)})$ (notation as in (2.4));

(iii) $Y^{(k)}(D) \subset Y(D)$, $Y^{(k)}|(D \sim U(Y)) = Y|(D \sim U(Y))$;

(iv) $A(U^{(k)}|U') \leqslant A(Y|U')$ whenever U' is a component of $U(Y)$.

To check (i), we first let $\tilde{\mu}_1, \ldots, \tilde{\mu}_r$ be pairwise disjoint Jordan curves in $D \sim \partial D$ such that $(Y^{(k)})^{-1}(\Gamma_k) = \bigcup_{i=1}^{r} \tilde{\mu}_i$ (2). (Thus $\bigcup_{i=1}^{r} \operatorname{int} \tilde{\mu}_i = \bigcup_{i=1}^{m} \operatorname{int} \mu_i$, where $\mu_1, \ldots, \mu_m$ are as in the definition of $Y^{(k)}$.) Since $Y \in \mathcal{Y}^{(j)}$, one can now check that if a component γ of $(Y^{(k)})^{-1}(\Gamma_j)$ is contained in $\bigcup_{i=1}^{r} \operatorname{int} \tilde{\mu}_i$, then any other component γ' is either likewise contained in $\bigcup_{i=1}^{r} \operatorname{int} \tilde{\mu}_i$, or else $\operatorname{int} \gamma' \cap \cap \left(\bigcup_{i=1}^{r} \operatorname{int} \tilde{\mu}_i\right) = \emptyset$. (Otherwise it is quite a straightforward matter to deduce that there is an $i \in \{1, \ldots, r\}$ with $\mu_i \subset \operatorname{int} \mu_l$, thus contradicting (2.9).) Using the above fact, together with the definition of $Y^{(k)}$, it then quite easily follows that $Y^{(k)} \in \mathcal{Y}^{(j)}$ as required.

We are now ready to define $\tilde{X}$; we define $\tilde{X} = X_n$, where X_n is the final element in the sequence $X_0, X_1, \ldots, X_n$ defined inductively as follows:

(*a*) $X_0 = X$.

(*b*) Assume $k \in \{1, \ldots, n\}$ and suppose $X_0, \ldots, X_{k-1} \in \mathcal{Y}$ have already been defined. Then define $X_k = (X_{k-1})^{(k)}$.

By properties (ii)-(iv) above, it is clear that $\tilde{X}$ has the required properties (2.6). It remains to show that there exists a suitable embedding $\hat{X}$ for each $\eta > 0$.

First, note that by property (i) above we have the implication

$$Y \in \mathcal{Y}_{k-1} \Rightarrow Y^{(k)} \in \mathcal{Y}_k\,, \qquad k = 1, \ldots, n\,.$$

(Here we adopt the convention that $\mathcal{Y}_0 = \mathcal{Y}$ in case $k = 1$.) Hence from definition of $\tilde{X}$ we have $\tilde{X} \in \mathcal{Y}_n$.

The proof will now be completed by showing that for *each* $Y \in \mathcal{Y}_n$ and each $\eta > 0$, there is a C^2 embedding $\hat{Y}$ of D_1 into $\boldsymbol{R}^3$ with

$$Y(x) = \hat{Y}(x) \quad \text{for } x \in D \sim U_\eta(Y)\,, \qquad \sup_D |Y - \hat{Y}| < \eta$$

and

$$A(\hat{Y}|U') \leqslant A(Y|U') + \eta \quad \text{for each component } U' \text{ of } U_\eta(Y)\,.$$

From now on we thus take Y to be an arbitrary fixed element of $\mathcal{Y}_n$.

(2) In case $(Y^{(k)})^{-1}(\Gamma_j) = \emptyset$, then $Y^{(k)} = Y$ and hence $Y^{(k)} \in \mathcal{Y}^{(j)}$ trivially in this case.

We let

$$k = \max\{\operatorname{card}\big(Y^{-1}(p) : p \in \boldsymbol{R}^3\}\,,$$
$$K = \{p \in \boldsymbol{R}^3 : \operatorname{card}\big(Y^{-1}(p)\big) = k\}\,.$$

We want to show that K is closed. Let γ be any component of $\bigcup_{j=1}^{n} Y^{-1}(\Gamma_j)$, and let U_γ^+, U_γ^- be the (unique) components of $D \sim \bigcup_{j=1}^{n} Y^{-1}(\Gamma_j)$ such that $\gamma \subset \bar{U}_\gamma^+ \cap \bar{U}_\gamma^-$, $\operatorname{int}\gamma \cap U_\gamma^- \neq \emptyset$, $(D \sim \operatorname{int}\gamma) \cap U_\gamma^+ \neq \emptyset$. By (2.3) we have either $Y(U_\gamma^+) \cap Y(U_\gamma^-) = \emptyset$ or $Y(U_\gamma^+) = Y(U_\gamma^-)$. We claim that since $Y \in \mathcal{Y}_n$ the latter possibility cannot occur. Indeed suppose $Y(U_\gamma^+) = Y(U_\gamma^-)$, and let μ_1 denote the outermost component of ∂U_γ^+. Note that then $\mu_1 \cap \partial D = \emptyset$ by (2.1), hence μ_1 is a component of $\bigcup_{j=1}^{n} Y^{-1}(\Gamma_j)$. Since $Y(U_\gamma^+) = Y(U_\gamma^-)$, it then follows that there is a $\mu_2 \subset \partial U_\gamma^-$ with $Y(\mu_2) = Y(\mu_1) = \Gamma_j$ for some $j \in \{1, \ldots, n\}$. On the other hand, because of our selection of μ_1, we have $\operatorname{int}\mu_2 \subset \operatorname{int}\mu_1$. However, since $Y(\mu_2) = Y(\mu_1) = \Gamma_j$, this contradicts the fact that $Y \in \mathcal{Y}_n \subset \mathcal{Y}^{(j)}$ (in particular, see the characterization of $\mathcal{Y}^{(j)}$ following (2.10)). Thus we deduce that $Y(U_\gamma^+) \cap Y(U_\gamma^-) = \emptyset$ for any component γ of $\bigcup_{j=1}^{n} Y^{-1}(\Gamma_j)$. Because of this, it is now quite a straightforward matter to show that K is closed. (One now needs only to use the fact that $Y \in \mathcal{Y}$; no further use need be made of the fact that $Y \in \mathcal{Y}_n$.)

We let $K_1, \ldots, K_R$ denote the (compact) components of K, so that

$$\operatorname{card}\big(Y^{-1}(p)\big) = k\,, \qquad p \in K_r\,, \tag{2.11}$$

and we define $L_r = Y^{-1}(K_r)$, $r = 1, \ldots, R$.

If $k = 1$, we simply set $\hat{Y} = Y$ and we are finished. Hence we assume $k > 1$. Using (2.1)-(2.3) we can see that $\operatorname{card}\big(Y^{-1}(p)\big)$ is constant on each component of $Y(D) \sim \bigcup_{j=1}^{n} \Gamma_j$, and

$$\partial L_r \subset \bigcup_{j=1}^{n} Y^{-1}(\Gamma_j)\,. \tag{2.12}$$

Also, if $\Gamma_j \cap K_r \neq \emptyset$, then, by (2.11) and the fact that $Y \in \mathcal{Y}_n \subset \mathcal{Y}^{(j)} \subset \mathcal{Y}$, we have

$$Y^{-1}(\Gamma_j) = \bigcup_{i=1}^{k} \gamma_i^j$$

for some pairwise disjoint smooth Jordan curves $\gamma_1^j, \ldots, \gamma_k^j$ with

$$\operatorname{int}\gamma_i^j \cap \operatorname{int}\gamma_l^j = \emptyset\,, \qquad i \neq l\,, \qquad \bigcup_{i=1}^{k} \operatorname{int}\gamma_i^j \subset L_r\,. \tag{2.13}$$

By combining (2.11), (2.12), (2.13) with the fact that K_r is closed, we deduce that

$$L_r = \bigcup_{i=1}^{k} E_i^r, \quad r = 1, \ldots, R, \tag{2.14}$$

where $E_1^r, \ldots, E_k^r$ are pairwise disjoint diffeomorphs of D_1 with $\partial E_i^r \subset \bigcup_{j=1}^{n} Y^{-1}(\Gamma_j)$ and

$$Y(E_1^r) = \ldots = Y(E_k^r) = K_r. \tag{2.15}$$

By (2.11) and (2.15) it is clear that $Y|E_i^r$ is 1-1 for $i = 1, \ldots, k$. Notice also that $Y(B \sim E_i^r) \cap Y(E_i^r) = \emptyset$ whenever $B \cap \bigcup_{l=1, l\neq i}^{k} E_l^r = \emptyset$, because otherwise by (2.15) we would have a contradiction to the definition of k. Using this fact together with the definition of $\mathcal{Y}$, it then follows that we can find pairwise disjoint subsets $F_1^r, \ldots, F_k^r$ of $D \sim \partial D$ having the properties

$$F_i^r \in \mathcal{M}, \quad F_i^r \sim \partial F_i^r \supset E_i^r, \quad Y|F_i^r \quad \text{is 1-1, } \operatorname{dist}\{E_i^r, \partial F_i^r\} < \eta. \tag{2.16}$$

Since $Y(\partial E_i^r)$ is one of the Γ_j, we can also clearly arrange

$$\Big(\bigcup_{i=1}^{k} Y(F_i^r \sim E_i^r)\Big) \cap \Big(\bigcup_{j=1}^{n} \Gamma_j\Big) = \emptyset. \tag{2.17}$$

Furthermore by (2.3) we can arrange to choose the F_i^r so that, for each $i, j = 1, \ldots, k$, *either*

$$Y(F_i^r \sim E_i^r) \cap Y(F_j^r \sim E_j^r) = \emptyset \tag{2.18}$$

or

$$Y(F_i^r) = Y(F_j^r). \tag{2.18$'$}$$

Now let $\tilde{Y}$ be a Lipschitz map of D into $\boldsymbol{R}^3$ such that

(i) $\tilde{Y}|F_i^r$ is a C^2 embedding into $\boldsymbol{R}^3$, $i = 1, \ldots, k$;

(ii) $\begin{cases} \tilde{Y}(F_i^r) = \tilde{Y}(F_j^r) \text{ for each } i, j \text{ such that (2.18)}' \text{ holds} \\ \tilde{Y}(F_i^r) \cap \tilde{Y}(F_j^r) = \emptyset \text{ for each } i,\ j \text{ such that (2.18) holds;} \end{cases}$

(iii) $\tilde{Y}\Big(\bigcup_{r=1}^{R} \bigcup_{i=1}^{k} F_i^r\Big) \cap \Big(\bigcup_{j=1}^{n} \Gamma_j\Big) = \emptyset$, $\tilde{Y}\Big(\bigcup_{r=1}^{R} \bigcup_{i=1}^{k} F_i^r\Big) \cap Y\Big(D \sim \bigcup_{r=1}^{R} \bigcup_{i=1}^{k} F_i^r\Big) = \emptyset$;

(iv) $\tilde{Y}(x) = Y(x)$ for x in some neighbourhood of $D \sim \bigcup_{r=1}^{R} \bigcup_{i=1}^{k} \operatorname{interior}(F_i^r)$.

(v) $\sup |\tilde{Y} - Y| < \eta$;

(vi) $A(\tilde{Y}|F_i^r) \leqslant A(Y|F_i^r) + \eta,\ i = 1, \ldots, k.$

Because of (2.16), (2.17), (2.18), (2.18)′, it is clear that such a $\tilde{Y}$ can be constructed by a perturbation and smoothing of Y inside the sets F_i^r $\left(\text{holding } Y \text{ fixed in some neighbourhood of } D \sim \bigcup_{r=1}^{R} \bigcup_{i=1}^{k} \text{interior } (F_i^r)\right)$.

We now claim $\tilde{Y} \in \mathcal{Y}_n$. One can readily check this on the basis of (2.16), (2.17) and (i)-(iv) above. It of course follows from the construction of $\tilde{Y}$ that $\max\{\text{card}\left(\tilde{Y}^{-1}(p)\right): p \in \boldsymbol{R}^3\} < k$.

Then in view of (iv)-(vi) and the arbitraryness of η, the existence of the desired embedding $\hat{Y}$ now follows by induction on k.

3. – A replacement theorem.

The following theorem guarantees that if U is an open convex subset of $\boldsymbol{R}^3$ and if $M \in \mathcal{M}$ intersects ∂U in a sufficiently nice manner (to be made precise below), then we can always replace M by $\tilde{M} \in \mathcal{M}$ in such a way that $\partial\tilde{M} = \partial M$, $\tilde{M} \sim \{x \in U: \text{dist}\,(x, \partial U) < \theta\} \subset M \sim \{x \in U: \text{dist}\,(x, \partial U) < \theta\}$ (θ any preassigned positive constant), $\mathcal{H}^2(\tilde{M}) \leqslant \mathcal{H}^2(M)$, and such that $\tilde{M} \cap \overline{U}$ is a disjoint union of elements of $\mathcal{M}$. The replacement procedure also has other nice properties. The precise result is given in the following theorem. In this theorem we use the notation that

$$U_\theta = \{x \in U: \text{dist}\,(x, \partial U) > \theta\}\,.$$

THEOREM 1 (Replacement Theorem). *Suppose $\theta > 0$ is given and*

(i) *U is a C^2 convex open set in $\boldsymbol{R}^3$;*

(ii) *$M \in \mathcal{M}$, $\partial M \subset \boldsymbol{R}^3 \sim U$, and $\partial M \sim \partial U$ is contained in the union of the unbounded components of $\boldsymbol{R}^3 \sim \left(\overline{U} \cup (M \sim \partial M)\right)$;*

(iii) *M intersects ∂U transversally; in case $\partial M \cap \partial U \neq \emptyset$, we will always take this to mean that there is a C^2 (open) surface N with $M \subset N$, with $(N \sim M) \cap \partial U = \emptyset$ and with N intersecting ∂U transversally.*

Then there is $\tilde{M} \in \mathcal{M}$ such that

(iv) $\partial M = \partial\tilde{M}$, $\tilde{M} \sim U \subset M \sim U$, $\tilde{M} \cap U_\theta \subset M \cap U_\theta$;

(v) *$\tilde{M}$ intersects ∂U transversally (in the same sense as in* (iii)*);*

(vi) $\mathcal{H}^2(\tilde{M}) + \mathcal{H}^2\left((M \sim \tilde{M}) \cap U_\theta\right) \leqslant \mathcal{H}^2(M)$;

(vii) *$\tilde{M} \cap \overline{U}$ is a disjoint union $\bigcup_{j=1}^{k} N_j$ of elements $N_j \in \mathcal{M}$ (and consequently $\tilde{M} \subset \overline{U}$ in case $\partial M \subset \partial U$).*

If in addition to the hypotheses (i)-(iii) *we have*

(viii) $\mathcal{H}^2(M) \leqslant \mathcal{H}^2(P) + \theta$ *for every* $P \in \mathcal{M}$ *with* $\partial P = \partial M$, *then there are non-negative numbers* $\theta_1, \ldots, \theta_k$ *with* $\sum_{j=1}^{k} \theta_j \leqslant \theta$ *and*

(ix) $\mathcal{H}^2(N_j) \leqslant \mathcal{H}^2(P) + \theta_j$ *for every* $P \in \mathcal{M}$ *with* $\partial P = \partial N_j$, $j = 1, \ldots, k$.

Proof. In the proof we will need the following area minimizing property of subsets F of ∂U:

Suppose $F \subset \partial U$ and suppose each component of F is a C^2 surface-with-boundary; suppose that $E \subset \boldsymbol{R}^3 \sim U$ is a C^2 surface-with-boundary satisfying $\partial E = E \cap \partial U = \partial F$ and $E \cup F = \partial G$ for some open $G \subset \boldsymbol{R}^3 \sim \overline{U}$. Then

$$\mathcal{H}^2(F) < \mathcal{H}^2(E)\,. \tag{3.1}$$

The proof of (3.1) consists of an application of the divergence theorem on $\overline{G}$. We note first that since U is convex, then the outward unit normal ν of ∂U extends, by defining it to be constant on rays normal to ∂U, to a $C^1(\boldsymbol{R}^3 \sim U)$ function ν^* with $|\nu^*| = 1$ and $\operatorname{div} \nu^* \geqslant 0$. Applying the divergence theorem on $\overline{G}$ we then deduce that

$$\int_F \nu \cdot \nu^* \, d\mathcal{H}^2 \leqslant \int_E \eta \cdot \nu^* \, d\mathcal{H}^2$$

where η denotes the outward unit normal of G at points of E. Since $\nu \equiv \nu^*$ on F and $\eta \cdot \nu^* \leqslant 1$ on E, with equality if and only if $\eta = \nu^*$, the required result (3.1) then easily follows.

We now proceed with the proof of the theorem. Let E be any component of $M \sim \overline{U}$ such that $E \cap \partial M = \emptyset$ (3). Since E is connected and $\overline{E} \sim E$ is a union of smooth Jordan curves contained in ∂U, a straightforward modification of a standard topological result enables us to assert that there is a unique bounded component U_E of $\boldsymbol{R}^3 \sim (\overline{U} \cup E)$. Let F be defined by

$$F = \partial U_E \cap \partial U\,. \tag{3.2}$$

Then $\overline{E}$, F are C^2 surfaces-with-boundary (although of course F is not necessarily connected) with $\partial \overline{E} = \partial F$. Furthermore, given any component E' of $M \sim \overline{U}$ with $E' \subset U_E$, it is clear that $\partial M \cap E' = \emptyset$ (by condition (ii) of the theorem) and $U_{E'} \subset U_E$. From this last inclusion it follows

(3) Notice of course that $(\overline{E} \sim E) \cap \partial M = \emptyset$, by (iii).

that we can select a component E of $M \sim \overline{U}$ such that $\partial M \cap E = \emptyset$ and

$$U_E \cap M = \emptyset\,. \tag{3.3}$$

Henceforth it is assumed that E has been so selected, and F is defined by (3.2).

Now let χ be a C^2 diffeomorphism of D_1 onto M, and consider the set $\chi^{-1}(\overline{E})$; this is a compact connected subset H of D_1 bounded by a finite collection $\gamma_1, \dots, \gamma_k$ of pairwise disjoint C^2 Jordan curves. Suppose without loss of generality that γ_1 is the outermost of these curves. That is, suppose $\gamma_2, \dots, \gamma_k$ are all contained in $\operatorname{int}\gamma_1$. Notice that, by the connectedness of H, we can then write

$$H = \overline{\operatorname{int}\gamma_1} \sim \bigcup_{j=2}^{k} \operatorname{int}\gamma_j\,.$$

We now define $F^* \subset F$ to be the component of F which contains the Jordan curve $\chi(\gamma_1)$, and we define $I = \cup \operatorname{int}\gamma_j$, where the union is over those j such that $\chi(\gamma_j) \subset \partial F \sim \partial F^*$. Also, define $E^* = \chi(I)$. Then $E \cup E^*$ is diffeomorphic to the connected subset $H \cup I$ of D_1, and $\partial(E \cup E^*) = \partial F^*$. Hence by Lemma 1 we can construct a diffeomorphism μ of $E \cup E^*$ onto F^* such that μ coincides with the identity on $\partial(E \cup E^*)$ $(= \partial F^*)$. We now define a Lipschitz mapping $\hat{\chi}\colon D_1 \to \boldsymbol{R}^3$ by

$$\hat{\chi}(x) = \begin{cases} (\mu \circ \chi)(x) & \text{if } x \in H \cup I \\ \chi(x) & \text{otherwise}\,, \end{cases}$$

and we consider the following two cases:

Case I. $\partial M \cap (F^* \sim \partial F^*) = \emptyset$. In this case $\big($by (3.3)$\big)$ it is clear that $\hat{\chi}$ is 1-1 on D_1, and we define

$$\hat{M} = \hat{\chi}(D_1) \qquad \big(= (M \sim (E \cup E^*)) \cup F^*\big)\,.$$

Case II. $\partial M \cap (F^* \sim \partial F^*) \neq \emptyset$. In this case we know by (3.3) and the hypotheses (ii) and (iii) of the theorem that $\partial M \subset F^* \sim \partial F^*$, and hence $\hat{\chi}^{-1}(\partial M) = \partial D_1 \cup \gamma$, where γ is a C^2 Jordan curve in $D_1 \sim \partial D_1$. In this case we define $\hat{M} = \hat{\chi}(\operatorname{int}\gamma)$ and we note that

$$\hat{M} \subset \hat{\chi}(D_1) = \big(M \sim (E \cup E^*)\big) \cup F^*\,.$$

In either of the above two cases we thus obtain $\hat{M}$ which is homeomorphic to D_1 (via a bilipschitz homeomorphism). Also, writing $\hat{E}^* = E^*$

in Case I and $\hat{E}^* = E^* \cup \hat{\chi}(D_1 \sim \operatorname{int} \gamma)$ in Case II, we have

$$\mathcal{H}^2(\hat{M}) + \mathcal{H}^2(\hat{E}^*) < \mathcal{H}^2(M) . \tag{3.4}$$

To check this we first note that, by (3.1),

$$\mathcal{H}^2(F^*) \leqslant \mathcal{H}^2(F) < \mathcal{H}^2(E) . \tag{3.5}$$

Next we note that, by definition of $\hat{M}$,

$$\mathcal{H}^2(\hat{M}) = \mathcal{H}^2(M) - \mathcal{H}^2(E) - \mathcal{H}^2(\hat{E}^*) + \mathcal{H}^2(F^*) . \tag{3.6}$$

By adding (3.5) and (3.6) we then have (3.4) as required.

The next step in the argument involves a smoothing and slight perturbation of $\hat{M}$ near $F^* \cap \hat{M}$, holding the sets $\partial M \cap \partial U$, $\hat{M} \sim \overline{U}$ and $\hat{M} \cap U_\theta$ fixed, and taking points of $(F^* \cap \hat{M}) \sim \partial M$ into $U \sim \overline{U}_\theta$. In this way we can obtain an $M_1 \in \mathcal{M}$ with M_1 intersecting ∂U transversally, with

$$\begin{cases} \partial M_1 = \partial M , \quad M_1 \sim \overline{U} \subset \hat{M} \sim \overline{U} \subset M \sim \overline{U} , \\ M_1 \cap U_\theta = \hat{M} \cap U_\theta = (M \sim \hat{E}^*) \cap U_\theta , \end{cases} \tag{3.7}$$

and with

$$\mathcal{H}^2(M_1) \leqslant \mathcal{H}^2(\hat{M}) + \delta \tag{3.8}$$

for any preassigned $\delta > 0$. Then, if δ is taken small enough, (3.8) together with (3.4) implies

$$\mathcal{H}^2(M_1) + \mathcal{H}^2(\hat{E}^*) \leqslant \mathcal{H}^2(M) .$$

By (3.7) this of course gives

$$\mathcal{H}^2(M_1) + \mathcal{H}^2\big((M \sim M_1) \cap U_\theta\big) \leqslant \mathcal{H}^2(M) . \tag{3.9}$$

Next we note that $M_1 \sim \overline{U}$ has fewer components than $M \sim \overline{U}$ (by construction). Thus, by induction on the number of components of $M \sim \overline{U}$, we obtain a sequence $M_0 = M$, M_1, $M_2, \ldots, M_k = \tilde{M}$, where $\partial M_j = \partial M$, where $\tilde{M} \sim \overline{U}$ has no components E such that $E \cap \partial M = \emptyset$ (notice this gives conclusion (vii)), and where, for $j = 1, \ldots, k$, M_j intersects ∂U transversally, $\partial M_j = \partial M$, $M_j \sim U \subset M_{j-1} \sim U$, $M_j \cap U_\theta \subset M_{j-1} \cap U_\theta$, and

$$\mathcal{H}^2(M_j) + \mathcal{H}^2\big((M_{j-1} \sim M_j) \cap U_\theta\big) \leqslant \mathcal{H}^2(M_{j-1}) .$$

Summing over j in this last inequality, we then obtain (vi). The remaining conclusions are now evident.

It thus remains to prove the last part of the theorem subject to the additional hypothesis (viii). Let $N_1, \ldots, N_k$ be as in (vii) and suppose that (ix) fails; then we must have $\inf\{\mathcal{H}^2(P_j)\colon P_j \in \mathcal{M},\ \partial P_j = \partial N_j\} = \mathcal{H}^2(N_j) - \alpha_j$, $j = 1, \ldots, k$, where $\sum_{j=1}^{k} \alpha_j > \theta$. Then choose $\tilde{P}_1, \ldots, \tilde{P}_k \in \mathcal{M}$ with $\partial\tilde{P}_j = \partial N_j$ and

$$\mathcal{H}^2(\tilde{P}_j) + \beta_j \leqslant \mathcal{H}^2(N_j)\,, \tag{3.10}$$

$$\sum_{j=1}^{k} \beta_j > \theta\,. \tag{3.11}$$

By applying the first part of the theorem with $\tilde{P}_j$ in place of M, we see by conclusion (vii) that we may take $\tilde{P}_j$ such that $\tilde{P}_j \sim \partial\tilde{P}_j \subset U$. By Corollary 1, we can also choose the $\tilde{P}_j$ to be pairwise disjoint. We can then define $M^* = \left(\tilde{M} \sim \bigcup_{j=1}^{k} N_j\right) \cup \left(\bigcup_{j=1}^{k} \tilde{P}_j\right)$. Clearly M^* is homeomorphic to D_1 and $\partial M^* = \partial\tilde{M} = \partial M$. Also by (3.10) and (3.11) we have

$$\mathcal{H}^2(M^*) = \mathcal{H}^2(\tilde{M}) - \sum_{j=1}^{k} \mathcal{H}^2(N_j) + \sum_{j=1}^{k} \mathcal{H}^2(\tilde{P}_j) < \mathcal{H}^2(\tilde{M}) - \theta \leqslant \mathcal{H}^2(M) - \theta\,.$$

That is

$$\mathcal{H}^2(M^*) + \theta < \mathcal{H}^2(M)\,;$$

in view of the fact that M^* may be smoothed to give $M' \in \mathcal{M}$ having the same boundary as M^* and area arbitrarily close to M^*, this last inequality contradicts the assumption (viii).

(3.12) REMARK. By applying the above theorem with A in place of U and with M_k (as in § 1) in place of M [4], and noting again that $\operatorname{spt}\|V\| \subset A \cup \Gamma$, it is not difficult to see that we can replace the sequence $\{M_k\}$ by a sequence $\{\tilde{M}_k\} \subset \mathcal{M}$ such that $\partial\tilde{M}_k = \Gamma$, $\tilde{M}_k \subset A \cup \Gamma$, $\lim_{k\to\infty} \mathcal{H}^2(\tilde{M}_k) = \lim_{k\to\infty} \mathcal{H}^2(M_k)$ $\left(= \inf\{\mathcal{H}^2(M)\colon M \in \mathcal{M},\ \partial M = \Gamma\}\right)$ and $\left(\lim_{k\to\infty} \boldsymbol{v}(\tilde{M}_k)\right) \llcorner (\mathbf{R}^3 \sim \Gamma) \times G(3, 2) = V \llcorner (\mathbf{R}^3 \sim \Gamma) \times G(3, 2)$. *That is, we may assume that the sequence* $\{M_k\}$ *of* § 1 *is such that* $M_k \subset A \cup \Gamma$, $k = 1, 2, \ldots$, *without changing the varifold limit* V *in* $\mathbf{R}^3 \sim \Gamma$. *We will therefore henceforth make this assumption.*

(4) If M_k does not intersect ∂A transversally, it is necessary to apply Theorem 1 to a slightly perturbed version of M_k. That a transversally intersection surface can be obtained by arbitrarily slight perturbations of M_k holding Γ fixed follows from Sard's Theorem together with the fact that if $\Gamma' \subset M_k \sim \Gamma$ is a C^2 Jordan curve, then Γ, Γ' are not linked. (This last fact is a consequence or the orientability of M_k, and does not depend on the fact that M_k has genus zero).

4. – Filigree lemma.

Here the word « filigree » means (very roughly) « a collection of thread-like protrusions from a surface ». For example, if M is a surface in $\boldsymbol{R}^3$ with $(\bar{M} \sim M) \cap \boldsymbol{U}(x_0, \varrho) = \emptyset$ and $\mathcal{H}^2(M \cap \boldsymbol{U}(x_0, \varrho)) = \varepsilon\varrho^2$, where ε is small, then $M \cap U(x_0, \varrho/2)$ would be classed as filigree.

The following lemma will enable us to « prune off » such sets under appropriate circumstances.

LEMMA 3 (Filigree Lemma). *Suppose $\{Y_t\}_{t\in[0,1]}$ is an increasing family of convex sets with $Y_t = \{x \in \boldsymbol{R}^3 : f(x) < t\}$, $t > 0$, where f is a non-negative function on $\boldsymbol{R}^3$ which is C^2 on $\boldsymbol{R}^3 \sim \bar{Y}_0$, $Df \neq 0$ on $Y_1 \sim \bar{Y}_0$ and*

$$\sup_{\bar{Y}_1 \sim \bar{Y}_0} |Df| \leqslant c_1$$

for some constant $c_1 > 0$. Suppose also that there is a constant $c_2 < \infty$ such that, whenever Γ_1 is a C^2 Jordan curve which is nullhomotopic in ∂Y_t, then there is a region $E \subset \partial Y_t$ with $\partial E = \Gamma_1$ and

$$\mathcal{H}^2(E) \leqslant c_2(\mathcal{H}^1(\Gamma_1))^2 . \tag{4.1}$$ [5]

Finally, suppose $M \in \mathcal{M}$ and $\varepsilon > 0$ are such that ∂M is contained in the union of the unbounded components of $\boldsymbol{R}^3 \sim (\bar{Y}_t \cup (M \sim \partial M))$ for all $t \in (0, 1)$, and

$$\mathcal{H}^2(M) \leqslant \mathcal{H}^2(N) + \varepsilon, \quad \forall N \in \mathcal{M} \text{ with } \partial N = \partial M . \tag{4.2}$$

Then

$$\mathcal{H}^2(M \cap Y_t) \leqslant 2\varepsilon \text{ whenever } t \leqslant 1 - 2c_1\sqrt{c_2}\sqrt{\mathcal{H}^2(M \cap Y_1)} . \tag{4.3}$$

PROOF. By Sard's Theorem, we know M itersects Y_t transversally for almost all $t \in (0, 1)$. Then for almost all $t \in (0, 1)$ we can apply Theorem 1, with Y_t in place of U, to give $\tilde{M}$ such that $\mathcal{H}^2(\tilde{M}) \leqslant \mathcal{H}^2(M)$, $\tilde{M} \cap \partial Y_t \subset$

([5]) There exists such a constant c_2 in the cases when all the ∂Y_t are spheres, or planes, or cylinders with circular cross-section; these are the only cases considered subsequently.

$\subset M \cap \partial Y_t$, and $\tilde{M} \cap \overline{Y}_t = \bigcup_{j=1}^{k} N_j$ for some $N_j \in \mathcal{M}$ satisfying

$$\mathcal{H}^2(N_j) \leqslant \mathcal{H}^2(N) + \varepsilon_j\,, \qquad \forall N \in \mathcal{M} \text{ with } \partial N = \partial N_j\,, \tag{4.4}$$

where $\sum_{j=1}^{k} \varepsilon_j < \varepsilon$.

Let $j \in \{1, \ldots, k\}$ and let $\Gamma_1 = \partial N_j$. By hypothesis we have an $E \subset \partial Y_t$ with $\partial E = \Gamma_1$ and with (4.1) holding. However, by (4.4) we have

$$\mathcal{H}^2(N_j) < \mathcal{H}^2(E) + \varepsilon_j\,,$$

hence

$$\mathcal{H}^2(N_j) \leqslant c_2\big(\mathcal{H}^1(N_j \cap \partial Y_t)\big)^2 + \varepsilon_j\,.$$

Summing over j, we then have

$$\mathcal{H}^2(\tilde{M} \cap Y_t) \leqslant c_2\big(\mathcal{H}^1(\tilde{M} \cap \partial Y_t)\big)^2 + \varepsilon\,.$$

Since $\tilde{M} \cap \partial Y_t \subset M \cap \partial Y_t$, this gives

$$\mathcal{H}^2(\tilde{M} \cap Y_t) \leqslant c_2\big(\mathcal{H}^1(M \cap \partial Y_t)\big)^2 + \varepsilon\,. \tag{4.5}$$

However $\mathcal{H}^2(M \cap Y_t) \leqslant \mathcal{H}^2(\tilde{M} \cap Y_t) + \varepsilon$ (by (4.2) with $N = \tilde{M}$ together with the fact that $\tilde{M} \sim Y_t \subset M \sim Y_t$). Hence (4.5) gives

$$\mathcal{H}^2(M \cap Y_t) \leqslant c_2\big(\mathcal{H}^1(M \cap \partial Y_t)\big)^2 + 2\varepsilon\,. \tag{4.6}$$

We can now suppose $\mathcal{H}^2(M \cap Y_1) > 2\varepsilon$, otherwise the required conclusion is trivial. Then let $t_0 = \inf\{t\colon \mathcal{H}^2(M \cap Y_t) > 2\varepsilon\}$ and define $f(t) = \mathcal{H}^2(M \cap Y_t) - 2\varepsilon$, $t \in [t_0, 1]$. By the co-area formula and the definition of c_1 we see that (4.6) implies

$$f(t) \leqslant c_1^2\, c_2\big(f'(t)\big)^2\,, \quad \text{a.e. } t \in [t_0, 1]\,.$$

Integrating this inequality (using the fact that $f(t)$ is an increasing function of t), we obtain

$$\sqrt{f(t_0)} \leqslant \sqrt{f(1)} - \frac{1 - t_0}{2\sqrt{c_2}\,c_1}\,.$$

However $f(1) = \mathcal{H}^2(M \cap Y_1) - 2\varepsilon < \mathcal{H}^2(M \cap Y_1)$; hence we deduce $1 - t_0 < 2\sqrt{c_2}\,c_1\sqrt{\mathcal{H}^2(M \cap Y_1)}$. That is, $t_0 > 1 - 2\sqrt{c_2}\,c_1\sqrt{\mathcal{H}^2(M \cap Y_1)}$, and the required result is proved.

5. – Interior regularity.

THEOREM 2. *Suppose $\{A_k\}$ is an increasing sequence of open sets such that $\boldsymbol{R}^3 \sim A_k$ has no bounded components, $\{N_k\}$ is a sequence in $\mathcal{M}$ with $\partial N_k \subset \partial A_k$, $N_k \sim \partial N_k \subset A_k$, and*

$$\mathcal{H}^2(N_k) \leqslant \mathcal{H}^2(N) + \varepsilon_k\,, \qquad \forall N \in \mathcal{M} \text{ with } \partial N = \partial N_k\,,$$

where $\varepsilon_k \to 0$ as $k \to \infty$. Suppose further that $W = \lim_{k\to\infty} \boldsymbol{v}(N_k)$ exists in $\boldsymbol{V}_2(\boldsymbol{R}^3)$.

Then W is a stationary integral varifold in $\bigcup_{k=1}^{\infty} A_k$, with the property that if $x_0 \in \operatorname{spt}\|W\| \cap \left(\bigcup_{k=1}^{\infty} A_k\right)$ and if W has a varifold tangent C at x_0 with $\operatorname{spt}\|C\| \subset H$, where H is a plane, then there is a $\varrho > 0$ such that

$$W \llcorner U(x_0, \varrho) \times G(3, 2) = n\boldsymbol{v}(M)\,, \tag{5.1}$$

where n is a positive integer and M is an analytic oriented connected minimal surface containing x_0.

Notice that (in view of (3.12) and the fact that $\operatorname{spt}\|V\| \subset A \cup \Gamma$) the above theorem (in case $A_k = A$, $N_k = M_k$) implies that the varifold V of § 1 is a stationary integral varifold in $\boldsymbol{R}^3 \sim \Gamma$, and V is regular in a neighbourhood of any point of $\boldsymbol{R}^3 \sim \Gamma$ where there is a varifold tangent with support contained in a plane. (By rectifiability there is such a tangent plane at $\|V\|$-almost all points of $\boldsymbol{R}^3$.)

PROOF OF THEOREM 2. $W \llcorner \bigcup_{k=1}^{\infty} A_k \times G(3, 2)$ is stationary in $\bigcup_{k=1}^{\infty} A_k$ because $\|h_{\#} W\|(U) \geqslant \|W\|(U)$ whenever U is a bounded open subset of $\bigcup_{k=1}^{\infty} A_k$ and h is a diffeomorphism of U leaving a neighbourhood of ∂U fixed.

We first want to show that there is a constant $c > 0$ such that

$$\Theta^2_*(\|W\|, x_1) \geqslant c \tag{5.2}$$

whenever $x_1 \in \operatorname{spt}\|W\| \cap \left(\bigcup_{k=1}^{\infty} A_k\right)$. (It follows from this that $W \llcorner \bigcup_{k=1}^{\infty} A_k \times \times G(3, 2)$ is rectifiable by virtue of [AW1, 5.5].) Suppose $x_1 \in \operatorname{spt}\|W\|$ and $\boldsymbol{U}(x_1, \varrho) \subset \bigcup_{k=1}^{\infty} A_k$, and let c_2 be a constant such that if Γ_1 is a Jordan curve in the unit sphere $\boldsymbol{S}^2$, and if E_1, E_2 are the two components of $\boldsymbol{S}^2 \sim \Gamma_1$, then

$$\min\{\mathcal{H}^2(E_1), \mathcal{H}^2(E_2)\} \leqslant c_2(\mathcal{H}^1(\Gamma_1))^2 .$$

By the filigree lemma (which can be applied with $f(x) \equiv |x|/\varrho$, $c_1 = 1/\varrho$, $Y_t = \boldsymbol{U}(x_1, \varrho t)$), we know that if $\mathcal{H}^2(N_k \cap \boldsymbol{U}(x_1, \varrho)) \leqslant (1/4c_2)\varrho^2$, then $\mathcal{H}^2(N_k \cap \cap \boldsymbol{U}(x_1, \varrho/2)) < 2\varepsilon_k$. Hence if there is a subsequence $\{k'\} \subset \{k\}$ with $\mathcal{H}^2(N_{k'} \cap \boldsymbol{U}(x_1, \varrho)) \leqslant (1/4c_2)\varrho^2$, then we would have $(\operatorname{spt}\|W\|) \cap \boldsymbol{U}(x_1, \varrho/2) = \emptyset$, thus contradicting the fact that $x_1 \in \operatorname{spt}\|W\|$. Hence for all sufficiently large k we have

$$\mathcal{H}^2(N_k \cap \boldsymbol{U}(x_1, \varrho)) \geqslant \frac{1}{4c_2}\varrho^2 , \tag{5.3}$$

from which we deduce

$$\|W\|(\boldsymbol{U}(x_1, \varrho)) \geqslant \frac{1}{4c_2}\varrho^2 . \tag{5.4}$$

Thus we obtain (5.2) with $c = 1/(4c_2)$. In particular, $W \llcorner \bigcup_{k=1}^{\infty} A_k \times G(3, 2)$ is rectifiable by [AW1, 5.5].

Now let x_0, C, H be as in the statement of the theorem. For convenience of notation we will suppose that $x_0 = 0$ and that $(0, 0, 1)$ is normal to the plane H. We write

$$D_\varrho = \{x \in \boldsymbol{R}^2 \colon |x| \leqslant \varrho\} , \qquad K_{\varrho,\sigma} = (D_\varrho \sim \partial D_\varrho) \times (-\sigma, \sigma) ,$$

and we let $\boldsymbol{\mu}_t$ denote the transformation of $\boldsymbol{R}^3$ defined by $\boldsymbol{\mu}_t(x) = tx$. By definition of C, we know there is a sequence $\{r_k\} \to \infty$ such that

$$\boldsymbol{\mu}_{r_k\#} W \to C \quad \text{as } k \to \infty . \tag{5.5}$$

By (5.4) it is then clear that for any $\sigma_0 \in (0, 1)$ we can find r such that

$$K_{1,1} \cap \operatorname{spt}\|\mu_{r\#} W\| \subset K_{1,\sigma_0/2} . \tag{5.6}$$

We can of course also choose r such that

$$\mathcal{H}^2(\operatorname{spt}\|\mu_{r\#} W\| \cap (\partial D_{\frac12} \times \boldsymbol{R})) = 0 . \tag{5.7}$$

Henceforth we will suppose that r has been thus chosen.

Now by hypothesis

$$\mathcal{H}^2(\boldsymbol{\mu}_r(N_k)) \leqslant \mathcal{H}^2(N) + r^2\varepsilon. \tag{5.8}$$

for every $N \in \mathcal{M}$ with $\partial N = \partial \boldsymbol{\mu}_r(N_k)$. From (5.6) and the co-area formula we know that for almost all $\sigma \in (\sigma_0/2, 1)$

$$\mathcal{H}^1\big(\boldsymbol{\mu}_k(N) \cap (D_1 \times (\{-\sigma\} \cup \{\sigma\}))\big) \to 0 \qquad \text{as } k \to \infty .$$

Thus for any given $\eta > 0$ we can assert that, for sufficiently large k, there is a $\sigma_k \in \big((3/4)\sigma_0, \sigma_0\big)$ such that

$$\mathcal{H}^1\big(\boldsymbol{\mu}_r(N_k) \cap (D_1 \times (\{-\sigma_k\} \cup \{\sigma_k\}))\big) < \eta . \tag{5.9}$$

Hence, assuming η is taken small enough, we can deduce

$$\boldsymbol{\mu}_r(N_k) \cap \big(\partial D_{\varrho_k} \times (\{-\sigma_k\} \cup \{\sigma_k\})\big) = \emptyset , \tag{5.10}$$

for some $\varrho_k \in (\frac{3}{4}, 1)$. We can also arrange, by Sard's theorem, that $\boldsymbol{\mu}_r(N_k)$ intersects both $D_{\varrho_k} \times \big(\{-\sigma_k\} \cup \{\sigma_k\}\big)$ and $\partial D_{\varrho_k} \times [-1, 1]$ transversally.

We now apply Theorem 1 with $\boldsymbol{\mu}_r(N_k)$ in place of N and with K_{ϱ_k, σ_k} in place of U [6]. Then we find $P_k^1, \ldots, P_k^{R_k'}, P_k^{R_k'+1}, \ldots, P_k^{R_k} \in \mathcal{M}$, where R_k', R_k are integers depending on k, such that

$$\begin{gathered} \partial P_k^1, \ldots, \partial P_k^{R_k'} \subset D_{\varrho_k} \times \big(\{-\sigma_k\} \cup \{\sigma_k\}\big) \\ \partial P_k^{R_k'+1}, \ldots, \partial P_k^{R_k} \subset \partial D_{\varrho_k} \times (-\sigma_k, \sigma_k) \end{gathered}$$

and such that (by virtue of Theorem 1)

$$\mathcal{H}^2(P_k^i) \leqslant \mathcal{H}^2(P) + \varepsilon_{k,i} , \qquad \forall P \in \mathcal{M} \text{ with } \partial P = P_k^i , \tag{5.11}$$

$i = 1, \ldots, R_k$, where $\sum_{i=1}^{R_k} \varepsilon_{k,i} \leqslant r^2 \varepsilon_k$, and (by (5.7) and [AW1, 2.6(2) (*d*)]

$$(\boldsymbol{\mu}_{r\#} W) \llcorner K_{\frac12,1} \times G(3, 2) = \lim_{k \to \infty} \sum_{i=1}^{R_k} \boldsymbol{v}(P_k^i \cap K_{\frac12,1}) . \tag{5.12}$$

Next we claim that P_k^i, $i = R_k' + 1, \ldots, R_k$ can be discarded without changing the varifold limit in (5.12); that is, we claim

$$(\mu_{r\#} W) \llcorner K_{\frac12,1} \times G(3, 2) = \lim_{k \to \infty} \sum_{i=1}^{R_k'} \boldsymbol{v}(P_k^i \cap K_{\frac12,1}) . \tag{5.13}$$

[6] To be strictly precise, we first apply Theorem 1 with the « edges » $\partial D_{\varrho_k} \times \times (\{-\sigma_k\} \cup \{\sigma_k\})$ smoothed out; because of (5.10), no modifications are needed in subsequent parts of the argument.

To justify this we notice that, by (5.11) and the isoperimetric inequality, each of the curves ∂P_k^i, $i = R_k' + 1, \ldots, R_k$ encloses an area $A_k \subset D_{\varrho_k} \times \times(\{-\sigma_k\} \cup \{\sigma_k\})$ which is such that $\mathcal{H}^2(A_k^i) \leqslant (4\pi)^{-1}\eta^2$; hence by (5.11) $\mathcal{H}^2(P_k^i) \leqslant \eta^2$ for all sufficiently large k. Thus, applying the filigree lemma with $f(x) = |x_3/\sigma_k|$, $Y_t = \{x: x_3/\sigma_k| < t\}$, $c_1 = 2\sigma_0^{-1}$ and $c_2 = (4\pi)^{-1}$, we deduce that, provided η is sufficiently small ($\eta < \sigma_0/4$ will suffice),

$$\mathcal{H}^2(P_k^i \cap K_{\frac{1}{2},\sigma_0/2}) \leqslant 2\varepsilon_{i,k}\,.$$

(5.13) now follows from this, (5.6) and (5.12).

Suppose now that $i \in \{1, \ldots, R_k'\}$ and $\Gamma_k^i = \partial P_k^i$ is nullhomotopic in $\partial D_{\varrho_k} \times [-\sigma_0, \sigma_0]$. Then letting A_k^i be that part of the cylinder $\partial D_{\varrho_k} \times \boldsymbol{R}$ interior to Γ_k^i, we clearly have

$$\mathcal{H}^2(A_k^i) < \mathcal{H}^2\big(\partial D_{\varrho_k} \times [-\sigma_0, \sigma_0]\big) = 4\pi\varrho_k\sigma_0\,.$$

By using (5.11) with $N = A_k^i$, we deduce

$$\mathcal{H}^2(P_k^i) < 4\pi\sigma_0 + \varepsilon_{k,i} < 5\pi\sigma_0$$

for sufficiently large k. Hence we can again use the filigree lemma, this time with $Y_t = \{x = (x_1, x_2, x_3): \sqrt{x_1^2 + x_2^2} < t\varrho_k\}$, $f(x) = \varrho_k^{-1}\sqrt{x_1^2 + x_2^2}$, $c_1 = \varrho_k^{-1}$, and a suitable c_2. (By homogeneity we can take c_2 to be such that (4.1) holds for any null-homotopic Γ_1 in Y_1.) We thus obtain

$$\mathcal{H}^2(P_k^i \cap K_{\frac{1}{2},1}) \leqslant 2\varepsilon_{k,i}\,,$$

provided σ_0 is sufficiently small. Hence by (5.13) we have

$$\sum_{i \in \mathcal{T}_k} \boldsymbol{v}(P_k^i \cap K_{\frac{1}{2},1}) \to (\boldsymbol{\mu}_{r\#} W) \llcorner K_{\frac{1}{2},1} \times G(3, 2)\,, \tag{5.14}$$

where $\mathcal{T}_k$ denotes the collection of those $P_k^i\big(i \in \{1, \ldots, R_k'\}\big)$ which are such that ∂P_k^i is *not* null-homotopic in $\partial D_{\varrho_k} \times [-\sigma_0, \sigma_0]$. For each $i \in \mathcal{T}_k$ we clearly have

$$\pi\varrho^2 \leqslant \mathcal{H}^2(P_k^i \cap K_{\varrho,1})\,, \qquad \varrho \in (0, \varrho_k]\,. \tag{5.15}$$

Also, by letting A_k^i denote the component of $\big(\partial D_{\varrho_k} \times [-\sigma_0, \sigma_0]\big) \sim P_k^i$ which contains the circle $\partial D_{\varrho_k} \times \{\sigma_0\}$, we define $N = A_k^i \cup \big(D_{\varrho_k} \times \{\sigma_0\}\big)$. Clearly N is homeomorphic to D_1 (via a bilipschitz homeomorphism) and $\mathcal{H}^2(N) \leqslant \leqslant \pi\varrho_k^2 + 4\pi\varrho_k\sigma_0$. However we can now use (5.11) (with N is place of P)

to deduce

$$\mathcal{H}^2(P_k^i) \leqslant \pi\varrho_k^2 + 5\pi\sigma_0 \tag{5.16}$$

for all sufficiently large k.

Using (5.15) together with the fact that $\sum_{i\in\mathcal{I}_k} \boldsymbol{v}(P_k^i \cap K_{\frac{1}{2},1})$ is bounded (e.g. by (5.14)), we then deduce that card $\mathcal{I}_k$ is bounded independent of k. Thus we can find a positive integer n and a subsequence $\{k'\} \subset \{k\}$ such that, after a suitable relabelling of P_k^i,

(i) $\varrho_{k'} \to \varrho_0 \in [\frac{3}{4}, 1]$ as $k' \to \infty$;

(ii) for $i = 1, \ldots, n$, $\boldsymbol{v}(\boldsymbol{\mu}_{\varrho_{k'}^{-1}}(P_{k'}^i))$ converges to a varifold W_i, where (by (5.15), (5.16) and (5.6)), for each $\varrho \in (0, 1]$,

$$\pi\varrho^2 \leqslant \|W_i\|(K_{\varrho,1}), \qquad \|W_i\|(K_{1,1}) \leqslant \pi + 5\pi\sigma_0\varrho_0^{-2} \leqslant \pi + 32\sigma_0, \tag{5.17}$$

$$\operatorname{spt}\|W_i\| \subset K_{1,\sigma_0}, \tag{5.18}$$

and

$$(\boldsymbol{\mu}_{r\varrho_0^{-1}\#} W) \llcorner K_{\frac{1}{2},1} \times G(3,2) = \sum_{i=1}^{n} W_i \llcorner K_{\frac{1}{2},1} \times G(3,2)\,. \tag{5.19}$$

Now by (5.17), (5.18) and the fact that $K_{1-\sigma_0,\sigma_0} \subset \boldsymbol{U}(0,1) \subset K_{1,1}$ we have

$$\|W_i\|(\boldsymbol{U}(0,1)) \geqslant \|W_i\|(K_{1-\sigma_0,\sigma_0}) = \|W_i\|(K_{1-\sigma_0,1}) \geqslant \pi(1-\sigma_0)^2$$

and

$$\|W_i\|(\boldsymbol{U}(0,1)) \leqslant \|W_i\|(K_{1,1}) \leqslant \pi + 32\sigma_0\,.$$

Then by using the arbitraryness of σ_0, together with (5.18), (5.19) and the monotonicity of $\varrho^{-2}\|W\|(\boldsymbol{U}(x_0,\varrho))$ ([AW1, 5.1(3)]), we conclude that $\Theta(\|W\|, x_0) = n$ for some positive integer n. Thus we have proved that $\Theta(\|W\|, x_0)$ is a positive integer whenever $x \in \operatorname{spt}\|W\| \cap \bigcup_{k=1}^{\infty} A_k$ and W has a varifold tangent at x_0 supported in a plane. Since $W \llcorner \bigcup_{k=1}^{\infty} A_k \times G(3,2)$ is rectifiable, we know that such a varifold tangent exists for $\mathcal{H}^2$-almost all $x_0 \in \operatorname{spt}\|W\| \cap \left(\bigcup_{k=1}^{\infty} A_k\right)$. Hence we finally deduce that W is a stationary integral varifold in $\bigcup_{k=1}^{\infty} A_k$, thus completing the proof of the first part of the theorem.

We can now apply this first part of the theorem to the sequences $\{\mu_{\varrho_{k'}^{-1}}(P^i_{k'})\}$ (in place of $\{N_k\}$), and hence deduce that each W_i in (5.17)-(5.19) is a stationary integral varifold in $K_{1,1}$. But then by (5.17) we can apply the regularity theorem of W. Allard [AW1, § 8]; for small enough σ_0 we then deduce that

$$W_i \llcorner K_{\frac{1}{2},1} \times G(3, 2) = \boldsymbol{v}(M_i)\,, \qquad i = 1, \dots, n\,,$$

where M_i is an analytic minimal surface which can be represented in the form

$$M_i = \{(x, z) : z = u_i(x),\, x \in D_{\frac{1}{2}}\}$$

where u_i is a solution of the minimal surface equation on $D_{\frac{1}{2}}$ satisfying $\sup_{D_{\frac{1}{2}}} |Du| \leqslant 1$. Further, since $\Theta(\|W\|, x_0) = n$, we easily see from (5.19) that $u_i(0) = 0$, $i = 1, \dots, n$ (7). On the other hand, from the construction of the W_i, it is clear that for each $i, j = 1, \dots, n$ we have *either* $u_i(x) \geqslant u_j(x)$ for all $x \in D_{\frac{1}{2}}$ *or* $u_i(x) \leqslant u_j(x)$ for all $x \in D_{\frac{1}{2}}$. But $u_i(0) = u_j(0) = 0$, and hence, since the difference of two solutions of the minimal surface equation satisfies the strong maximum principle, we deduce $u_i = u_j$ on $D_{\frac{1}{2}}$. Thus by (5.19) $(\mu_{r\#} W) \llcorner K_{\frac{1}{2},1} \times G(3, 2) = n\boldsymbol{v}(M_1)$. This completes the proof of Theorem 1.

(5.20) Remark. For later reference we wish to make a note concerning another consequence of the above theorem and its proof. Suppose instead of the sequence $\{\boldsymbol{v}(N_k)\}$ of varifolds we were to consider the sequence $\{\tau(N_k)\}$ of *currents*, where $\tau(N)$ denotes the rectifiable current associated (in the usual way) with an oriented surface N. With respect to the flat norm topology ([FH: 4.1.12]) we get convergence of some subsequence $\{\tau(N_{k'})\}$ to a rectifiable current T, with $\partial T \llcorner \bigcup_{k=1}^{\infty} A_k = 0$. Now if x_0 is as in the statement of the above theorem, then an examination of the proof shows clearly that we must have

$$T \llcorner \boldsymbol{U}(x_0, \varrho) = n'\,\tau(M)\,,$$

where M is as in (5.1) and n' is an integer such that $|n'| \leqslant n$ and $n - n'$ is *even*. (n also as in (5.1).) In particular, if $n = 1$ then $n' = \pm 1$.

(7) Of course for all sufficiently large r it follows from (5.17), (5.18) that the integer n in (5.19) is precisely $\Theta(\|W\|, x_0)$.

6. – Interior regularity (continued).

THEOREM 3. *Suppose* $\{A_k\}$, $\{N_k\}$, W *are as in Theorem 2, suppose* $B_1, \ldots, B_R$ *are distinct open discs of radius* ϱ *which intersect in a common diameter* L *of a ball* $\boldsymbol{U}(x_0, \varrho)$, *where* $\boldsymbol{B}(x_0, \varrho) \subset \bigcup_{k=1}^{\infty} A_k$. *Suppose also that* $\operatorname{spt}\|W\| \cap \cap \boldsymbol{U}(x_0, \varrho) \subset \bigcup_{j=1}^{R} B_j$ *and that* $\operatorname{spt}\|W\| \cap (B_1 \sim L) \neq \emptyset$. *Then*

$$W \llcorner \boldsymbol{U}(x_0, \varrho) \times G(3, 2) = n\boldsymbol{v}(B_1)$$

for some positive integer n.

PROOF. For each $i = 1, \ldots, R$, let B_i^+, B_i^- be $\frac{1}{2}$-discs with common diameter L such that $B_i^+ \cup B_i^- = B_i$. According to Theorem 2

$$W \llcorner U(x_0, \varrho) \times G(3, 2) = \sum_{j=1}^{R} \big(n_j^+ \boldsymbol{v}(B_j^+) + n_j^- \boldsymbol{v}(B_j^-)\big)\,, \tag{6.1}$$

where n_j^+, n_j^- are non-negative integers.

We now consider 3 cases:

Case I. $\sum_{j=1}^{R} (n_j^+ + n_j^-) \leqslant 2$.

Using the fact that W has first variation zero in $\boldsymbol{U}(x_0, \varrho)$, the case $\sum_{j=1}^{R} (n_j^+ + n_j^-) = 1$ is easily shown to be impossible, hence we need only consider the possibility $\sum_{j=1}^{R} (n_j^+ + n_j^-) = 2$. However, using the facts that W is stationary in $\boldsymbol{U}(x_0, \varrho)$ and that $\operatorname{spt}\|W\| \cap (B_1 \sim L) \neq \emptyset$, it is easy to show in this case that we must have $n_1^+ = n_1^- = 1$ (and $n_j^+ = n_j^- = 0$, $j \neq 1$). Hence the required result, with $n = 1$, is established in *Case I*.

Case II. $\sum_{j=1}^{R} (n_j^+ + n_j^-) = 3$.

In this case we can find 3 half-discs H_1, H_2, H_3 with common diameter L such that $W \llcorner \boldsymbol{U}(x_0, \varrho) \times G(3, 2) = \sum_{i=1}^{3} \boldsymbol{v}(H_i)$; furthermore, using the fact that W has first variation zero in $\boldsymbol{U}(x_0, \varrho)$, one easily shows that H_1, H_2, H_3 are distinct. (In fact they must meet along L at angles of 120°.) Now by the remark (5.20) we know that (provided the H_i are appropriately oriented and provided we use the notation of (5.20))

$$T \llcorner \boldsymbol{U}(x_0, \varrho) = \sum_{i=1}^{3} \tau(H_i)\,.$$

However we then clearly have $\partial T \llcorner \boldsymbol{U}(x_0, \varrho) \neq 0$, thus contradicting the fact that $\partial T \llcorner \bigcup_{k=1}^{\infty} A_k = 0$.

Case III. $\sum_{j=1}^{R} (n_j^+ + n_j^-) \geqslant 4$.

We will prove in this case that for each sufficiently small $\theta \in (0, \varrho)$, there is a positive integer n such that

$$W \llcorner U_\theta \times G(3, 2) = n\boldsymbol{v}(B_1 \cap U_\theta)\,, \tag{6.2}$$

where, here and subsequently, $U_\theta = \boldsymbol{U}(x_0, \varrho - \theta)$. This clearly suffices to establish the desired result.

Choose a C^2 convex open set G such that $\bigcup_{j=1}^{R} B_j \subset G \subset \boldsymbol{U}(x_0, \varrho)$, and

$$\mathcal{H}^2(\partial G) = 4\pi\varrho^2 - \alpha \quad \text{for some } \alpha > 0\,. \tag{6.3}$$

We can suppose N_k intersects ∂G transversally (otherwise modify N_k slightly without upsetting the hypotheses concerning N_k), $k = 1, 2, \ldots$. By Theorem 1 (with G in place of U and N_k in place of M) we can then find the pairwise disjoint $P_k^1, \ldots, P_k^{T_k} \in \mathcal{M}$ (T_k a positive integer) such that $P_k^j \sim \partial P_k^j \subset G$, $\partial P_k^j \subset \partial G$,

$$\mathcal{H}^2(P_k^j) \leqslant \mathcal{H}^2(P) + \varepsilon_{k,j}\,, \qquad \forall P \in \mathcal{M} \text{ with } \partial P = \partial P_k^j\,, \tag{6.4}$$

where $\sum_{j=1}^{T_k} \varepsilon_{k,j} \leqslant \varepsilon_k$, and such that (by conclusions (iv) and (vi) of Theorem 1 together with [AW1, 2.6(2) (*d*)])

$$W \llcorner U_\theta \times G(3, 2) = \lim_{k\to\infty} \sum_{j=1}^{T_k} \boldsymbol{v}(P_k^j \cap U_\theta) \tag{6.5}$$

for any preasigned $\theta > 0$. (Notice that $\operatorname{spt} \|W\| \cap \overline{U}_\theta$ is a compact subset of G because $\operatorname{spt} \|W\| \cap \overline{U}_\theta \subset \left(\bigcup_{j=1}^{R} B_j\right) \cap \overline{U}_\theta$, hence (6.5) *does* follow from Theorem 1.)

By (6.4) and the filigree lemma, there is an $\alpha > 0$ (independent of k, but depending on θ) such that if $\mathcal{H}^2(P_k^j \cap U_\theta) < \alpha$, then $\mathcal{H}^2(P_k \cap U_{2\theta}) \leqslant 2\varepsilon_{k,j}$; hence we have by (6.5) and [AW1, 2.6(2) (*d*)] that

$$\lim_{k\to\infty} \sum_{j\in\mathcal{J}_k} \boldsymbol{v}(P_k^j \cap U_{2\theta}) = W \llcorner U_{2\theta} \times G(3, 2)\,,$$

where $\mathcal{T}_k$ is the set of j such that $\mathcal{H}^2(P_k \cap U_\theta) > \alpha$. In view of the fact that $\lim\limits_{k\to\infty}\sup \sum\limits_{j\in\mathcal{T}_k} \mathcal{H}^2(P_k^j \cap U_\theta) < \infty$ (by (6.5)) we then deduce that card $(\mathcal{T}_k)$ is bounded independent of k. Thus we can select a subsequence $\{k'\} \subset \{k\}$ and assert that (after a suitable relabelling of the P_k^j)

$$\lim_{k'\to\infty} \sum_{j=1}^{n} \boldsymbol{v}(P_{k'}^j \cap U_{2\theta}) = W \llcorner U_{2\theta} \times G(3, 2)\,.$$

We can choose the subsequence $\{k'\}$ such that each individual sequence $\{\boldsymbol{v}(P_{k'}^j)\}$, $j = 1, \ldots, n$, has a varifold limit W_j; thus

$$(6.5)' \qquad \begin{aligned} W_j &= \lim_{k'\to\infty} \boldsymbol{v}(P_{k'}^j)\,, \qquad j = 1, \ldots, n\,,\\ W \llcorner U_{2\theta} \times G(3, 2) &= \sum_{j=1}^{n} W_j \llcorner U_{2\theta} \times G(3, 2)\,. \end{aligned}$$

By (6.4) and Theorem 2 (with $\{P_{k'}^j\}$ in place of $\{N_k\}$), we know that W_i is a stationary integral varifold in G, and since spt $\|W_j\| \cap U_\theta \subset$ spt $\|W\| \cap$ $\cap\, U_\theta \subset \bigcup\limits_{j=1}^{R} B_j$, Theorem 2 also implies that

$$(6.6) \qquad W_i \llcorner U_\theta \times G(3, 2) = \sum_{i=1}^{R} \left(m_{ij}^+ \boldsymbol{v}(B_i^+ \cap U_\theta) + m_{ij}^- \boldsymbol{v}(B_i^- \cap U_\theta)\right),$$

$j = 1, \ldots, n$, where m_{ij}^+, m_{ij}^- are non-negative integers.

Now by (6.4)

$$\mathcal{H}^2(P_k^j) \leqslant \min\{\mathcal{H}^2(S^+), \mathcal{H}^2(S^-)\} + \varepsilon_{k,j}\,,$$

where S^+, S^- are the components of $\partial G \sim \partial P_k^j$. In view of (6.3), this last inequality gives

$$\mathcal{H}^2(P_k^j) \leqslant 2\pi\varrho^2 - \frac{\alpha}{2} + \varepsilon_{k,j}\,,$$

whereupon we must have

$$\|W\|(U_\theta) < 2\pi(\varrho - \theta)^2$$

for sufficiently small $\theta \in (0, \varrho)$. But then the m_{ij}^+, m_{ij}^- in (6.6) must satisfy $\sum\limits_{i=1}^{R} (m_{ij}^+ + m_{ij}^-) \leqslant 3$, and hence we deduce by *Cases I* and *II* above that

$$(6.7) \qquad W_j \llcorner U_\theta \times G(3, 2) = \boldsymbol{v}(B_k \cap U_\theta)\,,$$

for any $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, R\}$ with $\operatorname{spt} \|W_j\| \cap B_k \cap U_\theta \neq \emptyset$. Because $\operatorname{spt} \|W\| \cap (B_1 \sim L) \neq \emptyset$, we see that (for any $\theta \in (0, \varrho)$ such that $\operatorname{spt} \|W\| \cap B_1 \cap U_\theta \neq \emptyset$) there is a j, say j_1, such that (6.7) holds with $j = j_1$ and $k = 1$. If on the other hand there is an $i \neq 1$ such that $\operatorname{spt} \|W\| \cap \cap (B_i \sim L) \neq \emptyset$ then (for $\theta \in (0, \varrho)$ sufficiently small) there is a j_2 such that (6.7) holds with $j = j_2$ and $k = i$.

Thus we would have

$$(6.8) \qquad \boldsymbol{v}(P^{j_1}_{k'} \cap U_\theta) \to \boldsymbol{v}(B_1 \cap U_\theta)\,, \qquad \boldsymbol{v}(P^{j_2}_{k'} \cap U_\theta) \to \boldsymbol{v}(B_i \cap U_\theta)\,, \qquad i \neq 1\,.$$

This is clearly impossible in view of the fact that $P^{j_1}_{k'}$, $P^{j_2}_{k'}$ are *disjoint* elements of $\mathcal{M}$ for each k'. (For example we can argue as follows: since $P^{j_1}_{k'} \cap P^{j_2}_{k'} = \emptyset$, we have *disjoint* open sets $U^{j_1}_{k'}$, $U^{j_2}_{k'}$ in $\boldsymbol{U}(x_0, \varrho)$ with $\partial U^{j_r}_{k'} = P^{j_r}_{k'}$, $r = 1, 2$. But by (6.8) and (5.20) we see that the characteristic functions of $U^{j_1}_{k'}$, $U^{j_2}_{k'}$ converge in the L_1 sense to the characteristic functions of hemispheres U^1, U^2 of $\boldsymbol{U}(x_0, \varrho)$ with $\partial U^1 \cap \boldsymbol{U}(x_0, \varrho) = B_1$ and $\partial U^2 \cap \cap \boldsymbol{U}(x_0, \varrho) = B_i$. Such convergence is clearly impossible in view of the disjointness of $U^{j_1}_{k'}$ and $U^{j_2}_{k'}$.) Thus we deduce that $W_j \llcorner U_\theta \times G(3, 2) = = \boldsymbol{v}(B_1 \cap U_\theta)$ whenever $W_j \llcorner U_\theta \times G(3, 2) \neq 0$, $j = 1, \dots, n$. The desired conclusion (6.2) now follows from (6.5)$'$.

COROLLARY 2. *Let V be as in § 1. Then at each point $x_0 \in \operatorname{spt} \|V\| \sim \Gamma$ there is a varifold tangent C of V having the form $C = n\boldsymbol{v}(H)$, where H is a plane in $\boldsymbol{R}^3$ and n is a positive integer.*

PROOF. For convenience of notation we suppose $x_0 = 0$. Let $\{t_k\} \to \infty$ be a sequence such that $C = \lim_{k\to\infty} \boldsymbol{\mu}_{t_k\#} V$. Clearly we can find a sequence $\{N_k\} \subset \mathcal{M}$ such that

$$N_k = \boldsymbol{\mu}_{r_k}(M_k) \qquad (r_k \to \infty \text{ as } k \to \infty)$$
$$\boldsymbol{v}(N_k) \to C \qquad \text{as } k \to \infty\,,$$

and

$$\mathcal{H}^2(N_k) \leqslant \mathcal{H}^2(N) + \varepsilon'_k\,, \qquad \forall N \in \mathcal{M} \text{ with } \partial N = \partial N_k\,,$$

where $\varepsilon'_k \to 0$ as $k \to \infty$.

If $\operatorname{spt} \|C\|$ is contained in a plane, Theorem 2 applies and the required result is immediately obtained. If $\operatorname{spt} \|C\|$ is not contained in a plane, then it follows (see [AW1, 6.5], [AA]) that we can find $x_1 \neq 0$ and $\varrho > 0$ such that

$$\operatorname{spt} \|C\| \cap \boldsymbol{B}(x_1, \varrho) = \bigcup_{j=1}^{R} \boldsymbol{H}_j\,,$$

where $R \geqslant 2$ and $H_1, \ldots, H_R$ are distinct closed half-discs having common diameter L, L also being a diameter of $\boldsymbol{B}(x_1, \varrho)$. But then we can apply Theorem 3 with C in place of W, whereupon we deduce

$$C \llcorner \boldsymbol{U}(x_1, \varrho) \times G(3, 2) = n\,,\boldsymbol{v}(B)$$

for some disc B, where n is a positive integer. This completes the proof.

By combining Theorems 2 and 3 the following interior regularity theorem is now evident.

THEOREM 4 (Interior regularity of V). *Let the varifold V be as in* § 1. *At each point $x_0 \in \operatorname{spt} \|V\| \sim \Gamma$ there is a positive integer n_{x_0}, a $\varrho_{x_0} > 0$, and an analytic minimal surface M_{x_0} such that*

$$V \llcorner \boldsymbol{U}(x_0, \varrho) \times G(3, 2) = n_{x_0} \boldsymbol{v}(M_{x_0}) \, .$$

7. – Boundary regularity.

In this section x_0 denotes a fixed point of the boundary curve $\Gamma \subset \partial A$. Since Γ is C^2 and A is uniformly convex, we know there are planes

$$\pi_{x_0}^+ = \{x \in \boldsymbol{R}^3 \colon (x - x_0)\cdot\nu^+ = 0\}\,, \qquad \pi_{x_0}^- = \{x \in \boldsymbol{R}^3 \colon (x - x_0)\cdot\nu^- = 0\}\,,$$

where $\nu^+, \nu^- \in \boldsymbol{S}^2$ satisfy $|\nu^+\cdot\nu^-| < 1$, such that $\pi_{x_0}^+ \cap \Gamma = \pi_{x_0}^- \cap \Gamma = \{x_0\}$ and

$$(7.1) \qquad \Gamma \sim \{x_0\} \subset W = \{x \in \boldsymbol{R}^3 \colon (x - x_0)\cdot\nu^+ > 0\,, \quad (x - x_0)\cdot\nu^- > 0\}\,.$$

We assume that the minimizing sequence $\{M_k\}$ of § 1 has been chosen so that $M_k \sim \Gamma \subset A$, $k = 1, 2, \ldots$. (By Remark (3.12) this can always be arranged without any change in $V \llcorner (\boldsymbol{R}^3 \sim \Gamma) \times G(3, 2)$.)

We let $V' = V \llcorner (\boldsymbol{R}^3 \sim \Gamma) \times G(3, 2)$, and we claim that

$$(7.2) \qquad \Gamma \subset \operatorname{spt} \|V'\|\,.$$

In fact some subsequence of $\tau(M_k)$ (notation as in (5.20)) converges with respect to the flat norm to an integral current T such that $\partial T = \tau(\Gamma)$, where $\tau(\Gamma)$ denotes the one-dimensional current associated with a suitably oriented version of Γ. Thus $\Gamma \subset \operatorname{spt} T$. On the other hand (by (5.20)) $\operatorname{spt} T (= \operatorname{spt} T \llcorner (\boldsymbol{R}^3 \sim \Gamma)) \subset \operatorname{spt} \|V'\|$, and hence we deduce (7.2) as required.

Now define $L = \pi_{x_0}^+ \cap \pi_{x_0}^-$, and henceforth (for convenience of notation only) assume $x_0 = 0 \in \Gamma$. Letting $C_+ = \lim_{k\to\infty} \boldsymbol{\mu}_{r_k\#} V'$ ($r_k \to \infty$ as $k \to \infty$) be any varifold tangent of V' at 0 ([AW1, 3.4(1)]) we then deduce that the line L defined above is contained in spt $\|C_+\|$ (by applying [AW2, 3.4(1)] to each $\boldsymbol{\mu}_{r_k\#} V'$). Furthermore (by applying [AW2, 3.4(2)] to each $\boldsymbol{\mu}_{r\#} V'$) we deduce

$$\|C_+\|(L) = 0 \tag{7.3}$$

$$r^{-2}\|C_+\|\big(U(x_1, r)\big) \quad \text{is non-decreasing in } r \text{ for } x_1 \in L\,. \tag{7.4}$$

By using (7.4) with $x_1 = 0$, together with the definition C_+, we then see that

$$r^{-2}\|C_+\|\big(\boldsymbol{U}(0, r)\big) \equiv \Theta\big(\|V'\|, 0\big)\,, \qquad \forall r > 0\,. \tag{7.5}$$

Also, by definition of C_+, we can use the sequence $\{M_k\}$ to construct a sequence $\{N_k\} \subset \mathcal{M}$ with the following properties:

$$N_k \subset W \cup L\big(W \text{ as in } (7.1)\big)\,, \qquad \partial N_k \cap \boldsymbol{U}(0,1) \subset L \cap \boldsymbol{U}(0,1)\,; \tag{7.6}$$

$$\mathcal{H}^2(N_k) \leqslant \mathcal{H}^2(N) + 1/k\,, \qquad \forall N \in \mathcal{M} \text{ with } \partial N = \partial N_k\,; \tag{7.7}$$

$$\lim_{k\to\infty} \boldsymbol{v}(N_k) \llcorner (\overline{W} \sim L) \times G(3,2) = C_+\big(\equiv C_+ \llcorner (\boldsymbol{R}^3 \sim L) \times G(3,2)\big)\,. \tag{7.8}$$

Thus by Theorem 2 we deduce that C_+ is a stationary integral varifold in $\boldsymbol{R}^3 \sim L$, and

$$\text{spt } \|C_+\| \subset \overline{W}\,. \tag{7.9}$$

Next, define $C = C_+ + \varphi_\# C_+$, where φ is the reflection of $\boldsymbol{R}^3$ through the line L. By (7.3) and the reflection principle [AW2, 3.2], C is a stationary integral varifold in $\boldsymbol{R}^3$. Then by (7.5) and [AW1, 5.1(2)] it is clear that $\boldsymbol{\mu}_{r\#} C = C$, $\forall r > 0$, hence

$$\boldsymbol{\mu}_{r\#} C_+ = C_+\,, \qquad \forall r > 0\,. \tag{7.10}$$

By Theorem 3 together with the classification of 1-dimensional stationary varifold tangents, given in [AA], we then know that (since C_+ is integral and $L \subset \text{spt } \|C_+\| \subset \overline{W}$)

$$C_+ = \sum_{i=1}^{R} \boldsymbol{v}(H_i)\,, \tag{7.11}$$

where $H_1, \ldots, H_R$ are half-planes with common boundary L and $\bigcup_{i=1}^{R} H_i \subset \overline{W}$.

Our aim now is to show that $R = 1$. We let

$$U^+(0, \varrho) = \{x = (x_1, x_2, x_3) \in \boldsymbol{R}^3\colon |x| < \varrho,\ x_3 > 0\} .$$

and suppose (for notational convenience) that $W \subset \{x \in \boldsymbol{R}^3\colon x_3 > 0\}$.

By Sard's theorem we can find $\varrho_k \in (\frac{3}{4}, 1)$ such that N_k intersects $\partial \boldsymbol{U}^+(0, \varrho_k)$ transversally. (Since $\partial \boldsymbol{U}^+(0, \varrho_k)$ is not smooth at the edge $E = \{x = (x_1, x_2, x_3)\colon x_3 = 0,\ |x| = \varrho_k\}$ this last statement has to be appropriately interpreted; at points in E we require the tangent plane of N_k to differ from both the tangent plane of $\partial \boldsymbol{U}(0, \varrho_k)$ and the plane $x_3 = 0$.) Then $N_k \cap \partial \boldsymbol{U}^+(0, \varrho_k) = \bigcup_{j=1}^{n_k} \Gamma_k^j$, where $\Gamma_k^1, \ldots, \Gamma_k^{n_k}$ are pairwise disjoint Jordan curves such that Γ_k^1 is a union of $L \cap \boldsymbol{U}(0, \varrho_k)$ and a C^2 Jordan arc $\gamma_k \subset \partial \boldsymbol{U}(0, \varrho_k) \cap \overline{W}$, and where $\Gamma_k^2, \ldots, \Gamma_k^{n_k}$ are C^2 Jordan curves contained in $\partial \boldsymbol{U}(0, \varrho_k) \cap W$. Let $\varepsilon > 0$ be given. Because of (7.8), (7.11), it is not difficult to show (e.g. by using the co-area formula) that we can also arrange for ϱ_k to be such that

$$\sum_{j=2}^{n_k} \mathcal{H}^1\Big(\Gamma_k^j \sim \Big\{x\colon \operatorname{dist}\Big(x, \bigcup_{i=1}^{R} H_i\Big) > \varepsilon\Big\}\Big) < \varepsilon, \tag{7.12}$$

provided k is sufficiently large (depending on ε). From this it clearly follows that (for k sufficiently large) each of the curves Γ_k^j, $j = 2, \ldots, n_k$, encloses an area $A_k^j \subset \partial U(0, \varrho_k)$ with

$$\mathcal{H}^2(A_k^j) \leqslant c\varepsilon\,, \tag{7.13}$$

where c is an absolute constant.

Now recall that Theorem 1 was stated for the case when U was a C^2 convex open set. The theorem is easily seen to be valid (with only minor modifications in the proof) in case U is the *intersection* of two C^2 convex open sets (like $\boldsymbol{U}^+(0, \varrho)$ for example), although in this case we can only require the N_j of (vii) to be elements of $\widetilde{\mathcal{M}}$ (rather than $\mathcal{M}$) and (ix) holds for all $P \in \widetilde{\mathcal{M}}$ with $\partial P = \partial N_j$. (Here $\widetilde{\mathcal{M}}$ is as in § 1.) Using such a modified version of Theorem 1, with $\boldsymbol{U}^+(0, \varrho_k)$ in place of U, we deduce that there is an $\tilde{N}_k \in \mathcal{M}$ with $\partial \tilde{N}_k = \partial N_k$, $\tilde{N}_k \cap \boldsymbol{U}^+(0, \varrho_k) = \bigcup_{j=1}^{m_k} \boldsymbol{P}_k^j$, $P_k^1 \in \widetilde{\mathcal{M}}$, $\partial P_k^1 = \Gamma_k^1$, $P_k^2, \ldots, P_k^{m_k} \in \mathcal{M}$, $\partial P_k^j \subset \partial \boldsymbol{U}(0, \varrho_k) \cap W$

$$C_+ \llcorner \boldsymbol{U}(0, \tfrac{1}{2}) \times G(3, 2) = \lim_{k \to \infty} \sum_{j=1}^{m_k} \boldsymbol{v}(P_k^j \cap \boldsymbol{U}(0, \tfrac{1}{2})) \tag{7.15}$$

(using conclusions (iv) and (vi) of Theorem 1 together with [AW1, 2.6(2) (d)]), and where

$$\mathcal{H}^2(P_k^j) \leqslant \mathcal{H}^2(P) + \varepsilon_{k,j}, \qquad \forall P \in \tilde{\mathcal{M}} \text{ with } \partial P = \partial P_k^j, \tag{7.16}$$

$j = 1, \ldots, m_k$. Here $\varepsilon_{k,j}$ are constants such that $\sum_{j=1}^{m_k} \varepsilon_{kj} \leqslant 1/k$ (which Theorem 1 guarantees by virtue of (7.7)).

By (7.13) and (7.16) we then have $\mathcal{H}^2(P_j^k) \leqslant (c+1)\varepsilon$, $j = 2, \ldots, m_k$, for all sufficiently large k. Hence by the filigree lemma (for ε small enough) we can discard $P_2^k, \ldots, P_{m_k}^k$ without changing the limit in (7.18). That is,

$$C^+ \llcorner \boldsymbol{U}(0, \tfrac{1}{2}) \times G(3, 2) = \lim_{k\to\infty} \boldsymbol{v}(P_{k'}^1 \cap \boldsymbol{U}(0, \tfrac{1}{2})) . \tag{7.15$'$}$$

Also, by (7.16) and Theorem 2 we have a subsequence $\{k'\} \subset \{k\}$ with

$$\varrho_{k'} \to \varrho \in [\tfrac{3}{4}, 1], \qquad \boldsymbol{v}(P_{k'}^1) \to Y, \tag{7.17}$$

where Y is a stationary integral varifold in $\boldsymbol{U}^+(0, \varrho)$ with $\operatorname{spt} \|Y\| \subset \overline{W}$. Clearly by (7.15)$'$ and (7.17)

$$Y \llcorner \boldsymbol{U}^+(0, \tfrac{1}{2}) \times G(3, 2) = C_+ \llcorner \boldsymbol{U}^+(0, \tfrac{1}{2}) \times G(3, 2) . \tag{7.18}$$

We next let S^+, S^- be the two components of $(\partial \boldsymbol{U}(0, \varrho_k) \cap W) \sim \Gamma_k^1$ and let D^+, D^- be the half discs having common diameter $L \cap \boldsymbol{U}(0, \varrho_k)$ such that $\partial(\boldsymbol{U}(0, \varrho_k) \cap W) = \bar{S}^+ \cup \bar{S}^- \cup \bar{D}^+ \cup \bar{D}^-$, and suppose the labelling is such that $(\bar{D}^+ \cap \bar{S}^+) \sim L \neq \emptyset$. Then $E^+ = \bar{D}^+ \cup \bar{S}^+$ and $\bar{E}^- = \bar{D}^- \cup \bar{S}^-$ are both bilipschitz homeomorphs of D_1, and $\partial E^+ = \partial E^- = \Gamma_k^1$. Furthermore

$$\begin{aligned}\min\{\mathcal{H}^2(E^+), \mathcal{H}^2(E^-)\} &= \mathcal{H}^2(D^+) + \min\{\mathcal{H}^2(S^+), \mathcal{H}^2(S^-)\} \\ &\leqslant \left(\frac{\pi}{2} + \frac{2\pi - 2\alpha}{2}\right)\varrho_k^2 = \left(\frac{3\pi}{2} - \alpha\right)\varrho_k^2\end{aligned}$$

for some $\alpha > 0$. $\left(2\alpha = \frac{1}{2}\mathcal{H}^2(\partial \boldsymbol{U}(0, \varrho_k)) - \mathcal{H}^2(\partial U(0, \varrho_k) \cap W)\right)$. Hence by (7.16) and (7.17) we conclude

$$\|Y\|(\boldsymbol{U}(0, \varrho)) < \frac{3\pi}{2}\varrho^2 . \tag{7.19}$$

Also, writing $Y' = Y \llcorner \boldsymbol{U}^+(0, \varrho) \times G(3, 2)$, we then deduce from (7.19) and the monotonicity of $r^{-2}\|Y'\|(\boldsymbol{U}(0, r))$ ([AW2, 3.2; AW1, 5.1 (3)]) that

$$\Theta(\|C_+\|, 0) = \Theta(\|Y'\|, 0) < \tfrac{3}{2} . \tag{7.20}$$

Thus we can deduce that the integer R in (7.11) satisfies $R \leqslant 2$. On the other hand we know that by (5.20) there is a rectifiable current T with

$$\text{(7.21)} \quad \partial T \llcorner \boldsymbol{U}(0, \varrho) = \tau\big(L \cap \boldsymbol{U}(0, \varrho)\big), \quad T \llcorner \boldsymbol{U}(0, \varrho) = \sum_{i=1}^{R} \chi_i \tau\big(H_i \cap \boldsymbol{U}(0, \varrho)\big),$$

where each $\chi_i = \pm 1$, $i = 1, \ldots, R$. (Here we take account of the fact that the half-spaces $H_1, \ldots, H_R$ are not necessarily distinct.) In (7.21) $\tau\big(L \cap \boldsymbol{U}(0, \varrho)\big)$ denotes the one-dimensional current associated with a suitably oriented version of $L \cap \boldsymbol{U}(0, \varrho)$, and the H_i are also assumed to have been given some definite orientation. Now (7.21) clearly implies that R is *odd*, hence we deduce finally that $R = 1$. That is we have

$$\Theta(\|V'\|, 0) = \Theta(\|C_+\|, 0) = \tfrac{1}{2},$$

and by the regularity theorem of Allard [AW2, § 4] we then deduce the following result concerning V' in a neighbourhood of $x_0 \in \Gamma$.

THEOREM 5 (Boundary regularity of V). *Suppose V is as in § 1, $x_0 \in \Gamma$ and $V' = V \llcorner (\boldsymbol{R}^3 \sim \Gamma) \times G(3, 2)$. Then there is a $\varrho > 0$ and a C^2 surface-with-boundary M such that $\partial M \cap \boldsymbol{U}(x_0, \varrho) = \Gamma \cap \boldsymbol{U}(x_0, \varrho)$ and $V' \llcorner \boldsymbol{U}(x_0, \varrho) \times \times G(3,2) = \boldsymbol{v}\big(M \cap \boldsymbol{U}(x_0, \varrho)\big)$.*

REMARK. The regularity theorem of [AW2, § 4] guarantees M is $C^{1,\gamma}$ for each $\gamma \in (0, 1)$; the fact that M is C^2 then follows from the fact that Γ is C^3 together with the fact that M can be locally represented as the graph of a solution of the minimal surface equation. (The Schauder theory for elliptic equations with Hölder continuous coefficients is thus applicable.)

8. – Main result.

From Theorems 4 and 5 (together with the fact that there exist no compact minimal surfaces without boundary) we now immediately deduce that $V' = \boldsymbol{v}(M)$, where M is a compact connected C^2 surface-with-boundary such that $\partial M = \Gamma$. It thus only remains to show $M \in \mathcal{M}$.

In order to do this we first show that there is a C^2 map χ of D_1 into M such that $\chi | \partial D_1$ is a diffeomorphism of ∂D_1 onto Γ. Notice that the existence of such a map χ immediately implies orientability of M (because, for example, we can use the map χ to demonstrate that the 2-dimensional singular integral homology group $H_2(M, \partial M)$ is non-zero). It then follows from standard classification theory for 2-dimensional surfaces that either $M \in \mathcal{M}$ or else M is a disc with $h \geqslant 1$ handles [AS 41G, 46]. However the latter case cannot occur because the existence of the map χ clearly implies that the homeomorphism $\phi\colon [0,1] \to \Gamma$ defined by $\phi(t) = \chi(e^{it})$, is null-homotopic in M. (A homotopy is given by $F(s,t) = \chi(se^{it})$, $(s,t) \in [0,1]\times[0,1]$.) However, if M has $h \geqslant 1$, then one can express the homotopy group of ϕ explicitly in terms of generators of the free group $\pi(M)$ ([AS, 43B]); certainly one can conclude in this case that ϕ is not null-homotopic.

Thus to prove $M \in \mathcal{M}$ it merely remains to prove the existence of a map χ as described above. To do this, first note that there is a $\delta > 0$ such that there is a C^2 retract μ of $S_\delta = \{x \in \boldsymbol{R}^3\colon \operatorname{dist}(x, M) < \delta\}$ onto M. We can write $\boldsymbol{R}^3 = \bigcup_{j=1}^{\infty} K_j$, where $K_1, K_2, \ldots$ are closed cubes of edge length $\delta/4$, with pairwise disjoint interiors and with faces contained in the planes $x_i = k\delta/4$, $i = 1, 2, 3$, $k = 0, \pm 1, \pm 2, \ldots$. Suppose the labelling is such that for some R

$$(8.1) \qquad (\boldsymbol{R}^3 \sim S_\delta) \cap \bar{A} \subset \bigcup_{i=1}^{R} K_i$$

and

$$(8.2) \qquad M \cap \bigcup_{i=1}^{R} K_i = \emptyset .$$

Now let $e(K_i)$ denote the union of the edges of K_i (thus $e(K_i)$ is a union of 12 closed line segments, each of length $\delta/4$). Since $\mathcal{H}^2\Big(M_k \cap \Big(\bigcup_{i=1}^{R} K_i\Big)\Big) \to 0$ as $k \to \infty$ (by (8.2)), it follows that, for k sufficiently large, we can slightly perturb M_k to give $N \in \mathcal{M}$ with $N \sim \partial N \subset A$, $\partial N = \Gamma$, $N \cap \Big(\bigcup_{i=1}^{R} e(K_i)\Big) = \emptyset$, and such that N meets $\bigcup_{i=1}^{R} (\partial K_i \sim e(K_i))$ transversally.

Having thus chosen N, apply Theorem 1 with N in place of M and interior $(A \cap K_R)$ in place of U (8). Then we obtain $\tilde{N} \in \mathcal{M}$ with $\partial\tilde{N} = \Gamma$, $\tilde{N} \sim \partial\tilde{N} \subset A$, $\tilde{N} \cap K_R = \bigcup_{i=1}^{p} P_r$, where $P_r \in \mathcal{M}$, and with $\tilde{N} \cap \Big(\bigcup_{i=1}^{R} \partial K_i\Big) \subset$ $\subset N \cap \Big(\bigcup_{i=1}^{R} \partial K_i\Big)$. It is then clear, since $\partial P_r \subset (\partial K_R \sim e(K_R)) \cap N$, that we can modify $\tilde{N}$ to give $N^* \in \mathcal{M}$ with $\partial N^* = \Gamma$, $N^* \sim \partial N^* \subset A$, $N^* \cap \Big(\bigcup_{i=1}^{R} e(K_i)\Big) = \emptyset$, N^* intersecting $\bigcup_{i=1}^{R} \partial K_i$ transversally, $N^* \cap \Big(\bigcup_{i=1}^{R} \partial K_i\Big) \subset N \cap \Big(\bigcup_{i=1}^{R} \partial K_i\Big)$, and

(8) To be strictly precise, we should here replace $A \cap K_R$ by a C^2 convex set U with $U \subset A \cap K_R$ and $\bar{U} \cap N = K_R \cap N$. The subsequent conclusions are then still valid.

$N^* \cap \partial K_R = \emptyset$. Thus, by induction on R, we can obtain $N \in \mathcal{M}$ with $\partial N = \Gamma$, $N \sim \partial N \subset A$, and $N \cap \left(\bigcup_{i=1}^{R} \partial K_i\right) = \emptyset$ $\left(\text{and hence } N \cap \left(\bigcup_{i=1}^{R} K_i\right) = \emptyset\right)$. Thus by (8.1) we deduce that $N \subset S_\delta$. We now let χ_1 be a C^2 diffeomorphism of D_1 onto N and let μ be the retract of S_δ onto M described above. We finally define $\chi = \mu \circ \chi_1$; this gives the desired mapping χ.

Thus we have finally proved the following.

THEOREM 6. *If A, Γ are as in § 1, then there is an $M \in \mathcal{M}$ such that $\partial M = \Gamma$ and $\mathcal{H}^2(M) = \inf\{\mathcal{H}^2(N): N \in \mathcal{M}, \partial N = \Gamma\}$.*

9. – The k-contour case, $k \geq 2$.

In this section we wish to point out that the techniques of the previous sections imply analogous results for the case when Γ is a disjoint union $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$ of $k \geq 2$ smooth Jordan curves contained in ∂A. For simplicity we will here only explicitly consider the case $k = 2$. We let $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are disjoint C^3 Jordan curves in ∂A, $\mathcal{M}_2$ denotes the set of surfaces $M \subset \boldsymbol{R}^3$ which are C^2 diffeomorphs of the annulus $\{x: \frac{1}{2} \leq |x| \leq 1\}$, and we let

$$\begin{aligned} \alpha_1 &= \inf\{\mathcal{H}^2(M):\ M \in \mathcal{M},\ \partial M = \Gamma_1\} \\ \alpha_2 &= \inf\{\mathcal{H}^2(M):\ M \in \mathcal{M},\ \partial M = \Gamma_2\} \\ \alpha &= \inf\{\mathcal{H}^2(M):\ M \in \mathcal{M}_2,\ \partial M = \Gamma\}\,. \end{aligned}$$

THEOREM. 7. *Suppose $\alpha < \alpha_1 + \alpha_2$.*

Then there is an $M \in \mathcal{M}_2$ with $\partial M = \Gamma$ and $\mathcal{H}^2(M) = \alpha$.

The general approach to the proof is as for the single contour case. We take a sequence $\{M_k\} \subset \mathcal{M}_2$ with $\partial M_k = \Gamma$, $\mathcal{H}^2(M_k) \to \alpha$, and such that $V = \lim_{k\to\infty} \boldsymbol{v}(M_k) \in \boldsymbol{V}_2(\boldsymbol{R}^3)$ exists. We need to show that there is a compact C^2 surface-with-boundary $M \in \mathcal{M}_2$ with $\partial M = \Gamma$ and $\boldsymbol{v}(M) = V \llcorner (\boldsymbol{R}^3 \sim \Gamma) \times \times G(3, 2)$. (The fact that $\mathcal{H}^2(M) = \alpha$ then of course easily follows from the construction of V.) In order to show the existence of such an M we need the following variant of Theorem 1.

THEOREM 1'. *Suppose the hypotheses are as in Theorem 1, except that $M \in \mathcal{M}_2$.*

Then either

(*a*) *there exist $M_1, M_2 \in \mathcal{M}$ with $\partial M = \partial M_1 \cup \partial M_2$ and $\mathcal{H}^2(M_1) + + \mathcal{H}^2(M_2) \leq \mathcal{H}^2(M)$, or* (*b*) *there is an* $\tilde{M} \in \mathcal{M}_2$ *such that the conclusions* (iv), (v), (vi) *of Theorem* 1 *hold, and in place of conclusion* (vii) *we have:*

(vii)′ $\tilde{M} \sim \bar{U}$ *has no components* E *with* $\partial M \cap E = \emptyset$; *in case* $\partial M \subset \partial U$ *we deduce* $\tilde{M} \subset \bar{U}$ *as in* (vii).

If in addition neither of the two components Γ_1, Γ_2 *of* ∂M *is contained entirely in* ∂U, *and if there is a single component* F *of* $M \sim \bar{U}$ *containing* $\partial M \sim \partial U$, *then conclusion* (vii) *of Theorem* 1 *holds as before; if hypothesis* (viii) *holds for every* $P \in \mathcal{M}_2$ *with* $\partial P = \partial M$, *then conclusion* (ix) *of Theorem* 1 *also holds.*

PROOF. Let χ denote a diffeomorphism of $\bar{G}$ onto M, where $G = S^2 \sim \sim (D^{(1)} \cup D^{(2)})$, with $D^{(1)}$, $D^{(2)} \subset S^2$ disjoint diffeomorphs of D_1, and let $\Gamma_i = \chi(\partial D^{(i)})$, $i = 1, 2$.

Now let U_E, E, F be constructed as in the proof of Theorem 1 (with $\partial M \cap E = \emptyset$ and $U_E \cap M = \emptyset$). For $i = 1, 2$ let $\delta_i \subset G$ be the component of $\chi^{-1}(\partial E)$ such that some component K of $\bar{G} \sim \chi^{-1}(E)$ contains both δ_i and $\partial D^{(i)}$. Let F_i^* denote the component of F which contains $\chi(\delta_i)$, let I_i denote the union of those components W of $G \sim \chi^{-1}(\bar{E})$ such that $\chi(\partial W \sim \partial G) \cap \partial F_i^* = \emptyset$, and let $E_i^* = \chi(I_i)$, $i = 1, 2$. (cf. the definition of F^*, I, E^* in the proof of Theorem 1.)

We now consider two cases:

Case I. $\Gamma_2 \not\subset \bar{E}_1^*$ or $\Gamma_1 \not\subset \bar{E}_2^*$.

Case II. $\Gamma_2 \subset \bar{E}_1^*$ and $\Gamma_1 \subset \bar{E}_2^*$.

In *Case I* we only explicitly consider the alternative $\Gamma_2 \not\subset \bar{E}_1^*$; the procedure in case of the other alternative is almost identical.

We define a Lipschitz mapping $\hat{\chi}$ by setting $\hat{\chi}(x) = \chi(x)$ for $x \in \bar{G} \sim \sim \chi^{-1}(\bar{E} \cup \bar{E}_1^*)$ and $\hat{\chi}(x) = (\mu_1 \circ \chi)(x)$ for $x \in \chi^{-1}(\bar{E} \cup \bar{E}_1^*)$, where μ_1 denotes a diffeomorphism of $\bar{E} \cup \bar{E}_1^*$ onto F_1^* with $\mu_1 =$ identity on ∂F_1^*. (Notice that such a μ_1 exists by Lemma 1, because $\Gamma_2 \not\subset \bar{E}_1^*$.)

If $(\Gamma_1 \cup \Gamma_2) \cap F_1^* \sim \partial F_1^* = \emptyset$, then $\hat{\chi}$ is 1-1 on the whole of $\bar{G}$ and we can define a Lipschitz surface $\hat{M} = \hat{\chi}(\bar{G})$. Then by (3.1) $\mathcal{H}^2(\hat{M}) + \mathcal{H}^2(E_1^*) < < \mathcal{H}^2(M)$ and, by making a slight perturbation of $\hat{M}$ in a neighbourhood of F_1^* (cf. the procedure of Theorem 1), we obtain $\tilde{M}_1$ satisfying

$$(9.1) \quad \begin{cases} \tilde{M}_1 \in \mathcal{M}_2\,, \quad \partial \tilde{M}_1 = \partial M\,, \quad \tilde{M}_1 \sim \bar{U} \subset M \sim \bar{U}\,, \quad \tilde{M}_1 \cap U_\theta \subset M \cap U_\theta \\ \mathcal{H}^2(\tilde{M}_1) + \mathcal{H}^2((M \sim \tilde{M}_1) \cap U_\theta) \leqslant \mathcal{H}^2(M)\,. \end{cases}$$

If $(\Gamma_1 \cup \Gamma_2) \cap F_1^* \sim \partial F_1^* \neq \emptyset$ (cf. *Case II* of Theorem 1), then we have a curve $\gamma_i \subset G$ with $\hat{\chi}(\gamma_i) = \Gamma_i$ for either $i = 1$ or 2 (or both). Suppose for example that such a curve γ_1 exists. We consider the possibilities:

(i) γ_1 is not null-homotopic in G;

(ii) γ_1 is null-homotopic in G.

In case (i) holds, we define $\tilde{\chi} = \hat{\chi}|G'$, where G' denotes the component of $G \sim \gamma_1$ with closure containing $\partial D^{(2)}$. In case (ii) holds, we first let μ be a diffeomorphism of $D^{(1)}$ onto $\overline{\operatorname{int}\gamma_1}$ ($\subset G$) with $\mu|\partial D^{(1)} = (\hat{\chi}|\gamma_1)^{-1}\circ(\hat{\chi}|\partial D^{(1)})$, and we define G' and $\tilde{\chi}$ by $G' = (G \sim \overline{\operatorname{int}\gamma_1}) \cup D^{(1)}$, $\tilde{\chi} = \hat{\chi}$ on $\bar{G} \cap \bar{G}'$, $\tilde{\chi} = \chi\circ\mu$ on $D^{(1)}$. Then (regardless of whether (i) or (ii) holds), if there is no curve $\gamma_2 \subset G'$ with $\tilde{\chi}(\gamma_2) = \Gamma_2$, we can slightly perturb $\tilde{\chi}$ to give a smooth embedding $\tilde{\chi}^*\colon \bar{G}' \to \boldsymbol{R}^3$ such that $\tilde{M}_1 = \tilde{\chi}^*(\bar{G}')$ satisfies (9.1). If on the other hand there is a curve $\gamma_2 \subset G'$ with $\tilde{\chi}(\gamma_2) = \Gamma_2$, then a repetition of the above argument, starting with $\tilde{\chi}$, G', γ_2 in place of $\hat{\chi}$, G, γ_1 respectively, again yields $\tilde{M}_1$ as in (9.1). Thus *Case I* always leads to a surface $\tilde{M}_1$ as in (9.1).

We now turn to *Case II*. In this case, since $\partial D^{(i)} \not\subset \bar{I}_i$, we must have

$$F_1^* \neq F_2^*. \tag{9.2}$$

Defining $\hat{M}_i = \big(M \sim (E \cup \bar{E}_i^*)\big) \cup F_i^*$, we then see by (9.2) and (3.1) that

$$\mathcal{H}^2(\hat{M}_1) + \mathcal{H}^2(\hat{M}_2) < \mathcal{H}^2(M),$$

and, with the aid of Lemma 1, we also see that $\hat{M}_i = Y_i(D_1)$, where $Y_i\colon D_1 \to \boldsymbol{R}^3$ and $Y_i|\overline{\operatorname{int}\gamma_i}$ is a bilipschitz homeomorphism for some Jordan curve $\gamma_i \subset D_1$ with $Y_i(\gamma_i) = \Gamma_i$. It follows that we can find $M_1, M_2 \in \mathcal{M}$ such that alternative (*a*) of Theorem 1′ holds. Thus *Case II* implies that alternative (*a*) holds.

We can now complete the proof that there is $\tilde{M}$ as in (vii)′ by using induction on the number of components E of $M \sim \bar{U}$ such that $\partial M \cap E = \emptyset$. (cf. the proof of (vii) in Theorem 1; the only essential difference here is that at the inductive step of the argument the occurrence of *Case II* will always yield alternative (*a*) and the argument can therefore be immediately terminated.)

To prove the second part of Theorem 1′, we simply note that (vii)′ implies (vii) in case neither component of ∂M is contained entirely in ∂U and there is a single component F of $M \sim \bar{U}$ containing $\partial M \sim \partial U$. (The remaining part is then also proved in an almost identical fashion to the corresponding part of Theorem 1.)

To prove (vii)′ implies (vii) under the above hypotheses, first note that we can take N in condition (iii) of Theorem 1 to be such that $\bar{N} \in \mathcal{M}_2$ and

$(\bar{N} \sim M) \cap \bar{U} = \emptyset$. It then easily follows that there is a $\tilde{N} \in \mathcal{M}_2$ with $\tilde{M} \subset \tilde{N} \sim \partial\tilde{N}$ and $(\tilde{N} \sim \tilde{M}) \cap \bar{U} = \emptyset$. Furthermore (by conclusion (iv)) $F \subset \tilde{M}$; F is thus a component of $\tilde{M} \sim \bar{U}$ containing all of $\partial\tilde{M} \sim \partial U$. It then follows that $\tilde{N} \sim \bar{U}$ has a component containing all of $\partial\tilde{N}$, and hence (by (vii)′) $\tilde{N} \sim \bar{U}$ is connected. Thus if we let χ be a C^2 diffeomorphism of $\{x: \frac{1}{2} \leqslant |x| \leqslant 1\}$ onto $\tilde{N}$, so that $\chi^{-1}(\tilde{N} \cap \partial U)$ $\left(= \chi^{-1}(\tilde{M} \cap \partial U)\right) = \bigcup_{i=1}^{m} \gamma_i$, where $\gamma_1, \ldots, \gamma_m$ are pairwise disjoint C^2 Jordan curves in $\{x: \frac{1}{2} < |x| < 1\}$, then we must have $\operatorname{int}\gamma_1, \ldots, \operatorname{int}\gamma_m$ pairwise disjoint and contained in $\{x: \frac{1}{2} < |x| < 1\}$. (Otherwise we contradict connectedness of $\tilde{N} \sim \bar{U}$.) It is then clear that $\tilde{N} \cap \bar{U}$ $(= \tilde{M} \cap \bar{U})$ is a union $\bigcup_{j=1}^{m} \overline{\chi(\operatorname{int}\gamma_j)}$ of elements of $\mathcal{M}$ as required.

In conjunction with the above modification of Theorem 1, we point out the following property of the minimizing sequence $\{M_k\}$.

LEMMA 4. *Suppose $\alpha < \alpha_1 + \alpha_2$ as in Theorem 7, and suppose $\{M_k\} \subset \mathcal{M}_2$ satisfies $\partial M_k = \Gamma$, $k = 1, 2, \ldots$, and $\lim_{k\to\infty} \mathcal{H}^2(M_k) = \alpha$.*

Then we have the following conclusion: there is a constant $\varrho > 0$ such that if U is any C^2 open convex subset of A with diameter $U \leqslant \varrho$, if M_k intersects ∂U transversally for $k = 1, 2, \ldots$, and if $\Gamma_1 \sim \partial U$, $\Gamma_2 \sim \partial U$ are both non-empty and connected, then for all sufficiently large k both $\Gamma_1 \sim \partial U$ and $\Gamma_2 \sim \partial U$ are contained in the same component of $M_k \sim \bar{U}$.

PROOF. Suppose no such ϱ exists. Then for each $\varepsilon > 0$ we can find a convex set $U \subset A$ with diameter $U \leqslant \varepsilon$, with the sets $\Gamma_1 \sim \partial U$, $\Gamma_2 \sim \partial U$ non-empty, connected, and contained in different components of $M_k \sim \bar{U}$ for all k in some subsequence $\{k'\} \subset \{k\}$, and with M_k intersecting ∂U transversally. We can suppose $\Gamma_1 \cap \partial U = \emptyset$ and $\Gamma_2 \cap \partial U = \emptyset$; otherwise this can be achieved, without upsetting the above hypotheses, by replacing U by a slightly smaller set. We then let $X_{k'}$ be a C^2 diffeomorphism of $\{x \in \mathbf{R}^2: \frac{1}{2} \leqslant |x| \leqslant 1\}$ onto $M_{k'}$, such that $X_{k'}(\partial D_1) = \Gamma_1$ and $X_{k'}(\partial D_{\frac{1}{2}}) = \Gamma_2$. Since Γ_1, Γ_2 are in different components of $M_{k'} \sim \bar{U}$, we can find a Jordan curve $\gamma \subset \{x \in \mathbf{R}^2: \frac{1}{2} < |x| < 1\}$ such that $D_{\frac{1}{2}} \subset \operatorname{int}\gamma$ and $X_{k'}(\gamma) \subset \partial U$. Now let $P_{k'}$ be one of the two components of $\partial U \sim X_{k'}(\gamma)$ and let $Y_{k'}$ be a bilipschitz immersion of D_1 into $\mathbf{R}^3$ such that $Y_{k'}(D_1) = P_{k'} \cup X_{k'}(D_1 \sim \operatorname{int}\gamma)$. Then by Lemma 2 it follows that there is an $M^1_{k'} \in \mathcal{M}$ such that

$$\text{(i)}\quad \partial M^1_{k'} = \Gamma_1, \quad \mathcal{H}^2(M^1_{k'}) \leqslant \mathcal{H}^2(P_{k'}) + \mathcal{H}^2\left(X_{k'}(D_1 \sim \operatorname{int}\gamma)\right) + \varepsilon^2 .$$

Similary we can find $M^2_{k'} \in \mathcal{M}$ with

$$\text{(ii)}\quad \partial M^2_{k'} = \Gamma_2, \quad \mathcal{H}^2(M^2_{k'}) \leqslant \mathcal{H}^2(P_{k'}) + \mathcal{H}^2\left(X_{k'}(\operatorname{int}\gamma \sim D_{\frac{1}{2}})\right) + \varepsilon^2 .$$

Now $\mathcal{H}^2(M^1_{k'}) + \mathcal{H}^2(M^2_{k'}) \geqslant \alpha_1 + \alpha_2$, hence adding (i) and (ii), we see that

$$\alpha_1 + \alpha_2 \leqslant 2\mathcal{H}^2(P_{k'}) + \mathcal{H}^2(M_{k'}) + 2\varepsilon^2 .$$

Since $\mathcal{H}^2(P_{k'}) \leqslant 4\pi\varepsilon^2$ and $\mathcal{H}^2(M_{k'}) \to \alpha$, we then deduce, because $\varepsilon > 0$ is arbitrary, that $\alpha_1 + \alpha_2 \leqslant \alpha$, contrary to the hypothesis of the lemma. This completes the proof of Lemma 4.

Because of Theorem 1′ and Lemma 4 it should now be clear that the techniques of § 5-8 can be modified in a very straightforward manner to yield a proof of Theorem 7.

10. – Minimizing surfaces of higher genus.

In this section we wish to show that there is an analogue of Theorem 6 for surfaces of higher genus. The discussion of Appendix B illustrates the wide range of applicability of the results obtained here.

We let A, Γ, $\mathcal{M}$ be as in §§ 1-8, and we also introduce the following further notation:

$\mathcal{M}(g, \Gamma)$ ($g \geqslant 0$ an integer) denotes the collection of connected oriented C^2 surfaces-with-boundary M with $\partial M = \Gamma$, and genus $M = g$;

$\alpha_g = \inf\{\mathcal{H}^2(M) \colon M \in \mathcal{M}(g, \Gamma)\}\big(= \inf\{\mathcal{H}^2(M) \colon M \in \mathcal{M}(h, \Gamma) \text{ for some } h \leqslant g\}\big)$;

$\{M_k\}$ will denote a sequence in $\mathcal{M}(g, \Gamma)$ such that

$$\lim_{k\to\infty} \mathcal{H}^2(M_k) = \alpha_g \tag{10.1}$$

and such that there is a varifold V_g with

$$V_g = \lim_{k\to\infty} \boldsymbol{v}(M_k) . \tag{10.2}$$

Of course by the convex hull property of Appendix A we have

$$\operatorname{spt} V_g \subset A \cup \Gamma \tag{10.3}$$

(cf. (1.1)).

The main result we want to prove is the following.

THEOREM 8. *Suppose* $\alpha_g < \alpha_{g-1}$.
Then there is an $M \in \mathcal{M}(g, \Gamma)$ *with* $\mathcal{H}^2(M) = \alpha_g$.

We will show that, like the main result of § 9, this theorem can be proved by rather straightforward modifications of the techniques of §§ 5-8.

The key point is to prove interior and boundary regularity of V_g in the sense (cf. Theorems 2, 3) that there is a smooth surface-with-boundary M satisfying $\partial M = \Gamma$ and

$$v(M) = V_g \llcorner (\boldsymbol{R}^3 \sim \Gamma)\,; \tag{10.4}$$

it then remains to prove genus $M = g$ (cf. § 8).

We first need the following generalization of Theorem 1.

THEOREM 9. *Suppose the hypotheses are as in Theorem* 1, *except that* $M \in \mathcal{M}(g, \Gamma)$ *rather than* $M \in \mathcal{M}$.

Then there is an integer $0 \leqslant h \leqslant g$ *and* $\tilde{M} \in \mathcal{M}(h, \Gamma)$ *such that the conclusions* (iv), (v), (vi) *of Theorem* 1 *hold and such that* $\boldsymbol{R}^3 \sim (\tilde{M} \cup \overline{U})$ *is connected.* (*Consequently* $\tilde{M} \subset \overline{U}$ *in case* $\Gamma \subset \partial U$.)

PROOF. Let E, U_E, F (satisfying (3.2), (3.3)) be as in the proof of Theorem 1 and consider the cases $\Gamma \subset F \sim \partial F$, $\Gamma \not\subset F \sim \partial F$.

Case I. $\Gamma \not\subset F \sim \partial F$. In this case we let M^* be the compact (not necessarily connected) oriented Lipschitz surface with boundary Γ defined by $M^* = (M \sim E) \cup F$, and we let $M^{(1)}$ be the component of M^* containing Γ. We want to show genus $M^{(1)} \leqslant$ genus M.

To show this we are going to use the *Euler characteristic* $\chi(Y)$ of a compact (not necessarily connected) oriented surface Y; $\chi(Y)$ has the properties that it is an integer, and if Y is connected then

$$\chi(Y) = 2 - R - 2g\,. \tag{10.5}$$

Here $R = 0$ if $\partial Y = \emptyset$ and R is the number of components of ∂Y otherwise; g denotes the genus (number of handles) of Y and is zero if and only if Y is homeomorphic to a compact surface-with-boundary in $\boldsymbol{S}^2$. Thus, in case Y is connected we always have

$$\chi(Y) \leqslant 2\,, \tag{10.6}$$

with equality if and only if Y is homeomorphic to $\boldsymbol{S}^2$. The Euler characteristic has the additional property that if $Z \subset Y$ with Z also a compact surface with boundary, then

$$\chi(Y) = \chi\big(Y \sim (Z \sim \partial Z)\big) + \chi(Z)\,. \tag{10.7}$$

Now let k be the number of components of F and let $M^{(2)}, \ldots, M^{(l)}$ be the components of $M^* \sim M^{(1)}$. Using the fact that $l \leqslant k$ together with (10.5), (10.6) and (10.7), we have

$$\begin{aligned} 2(k-1) + \chi(M^{(1)}) \geqslant \sum_{j=1}^{l} \chi(M^{(j)}) &= \chi(M^*) \\ &= \chi(M) + \chi(F) - \chi(E) \\ &= \chi(M) + 2k - 2 + 2g_E , \end{aligned}$$

where g_E denotes the genus of E. Thus we have $\chi(M^{(1)}) \geqslant \chi(M)$, and hence genus $M^{(1)} \leqslant$ genus M as required. Of course by (3.1) and the construction of $M^{(1)}$ we have

$$M^1 \sim \partial U \subset M , \quad \mathcal{H}^2\big((M \sim M^{(1)}) \cap U\big) + \mathcal{H}^2(M^{(1)}) < \mathcal{H}^2(M) . \tag{10.8}$$

Case II. $\Gamma \subset F \sim \partial F$. Let $\Gamma' \subset M \sim \Gamma$ be a C^2 Jordan curve homotopic to Γ in M, and let $N \subset M$ be the compact surface with boundary Γ'. Since $M \cap U_E = \emptyset$, we can arrange to choose Γ' sufficiently close to Γ to ensure that $\Gamma' \subset U$. We now write $N^* = (N \sim E) \cup F$ and let $N^{(1)}$ be the component of N^* containing Γ'. Exactly as in *Case I* we show that genus $N^{(1)} \leqslant$ genus N ($=$ genus M). If $\Gamma \not\subset N^{(1)}$, we write $M^{(1)} = N^{(1)} \cup (M \sim N)$ and we note that genus $M^{(1)} \leqslant$ genus M and (10.8) again holds. If $\Gamma \subset N^{(1)}$ we first want to argue that $N^{(1)} \sim \Gamma$ is *not* connected. Indeed $U \sim M$ consists of open components, each lying *on one side* of the oriented surface M. Let W be any component of $U \sim M$ with $\Gamma \subset \overline{W}$. If $N^{(1)} \sim \Gamma$ is connected we could, after letting F^* be the component of F which contains Γ, construct a curve in $N^{(1)} \sim \Gamma$ joining any two given points on $F^* \sim \Gamma$. Since $N^{(1)}$ is oriented we could then construct a curve in $(U \cup F^*) \sim M$ joining any two given points on $F^* \sim \Gamma$. Thus it would follow that $F^* \subset \overline{W}$, and this in turn clearly implies that there are points of $M \sim \Gamma$ (close to Γ) which lie in the *interior* of $\overline{W}$, thus contradicting the fact that W lies on one side of M. Thus $N^{(1)} \sim \Gamma$ is not connected. We now let $M^{(1)}$ be the closure of the component of $N^{(1)} \sim \Gamma$ which does *not* contain Γ'. We then note (by using (10.5), (10.6), (10.7), together with the established fact that genus $N^{(1)} \leqslant$ genus M) that genus $M^{(1)} \leqslant$ genus M and that (10.8) again holds.

Thus in either *Case I* or *Case II* we obtain a Lipschitz surface $M^{(1)}$ with genus $M^{(1)} \leqslant$ genus M and with (10.8) holding. By making an arbitrarily slight perturbation of $M^{(1)}$ in a neighborhood of F (and leaving the rest of $M^{(1)}$ fixed) we obtain a smooth surface $\tilde{M}^{(1)}$ with $(\tilde{M}^{(1)} \sim U) \cup (\tilde{M}^{(1)} \cap U_\theta) \subset M$ and with $\mathcal{H}^2\big((M \sim \tilde{M}^{(1)}) \cap U_\theta\big) + \mathcal{H}^2(\tilde{M}^{(1)}) < \mathcal{H}^2(M)$. (Cf. the relevant part of the proof of Theorem 1.) By induction on the number of components of $M \sim \overline{U}$ (as in Theorem 1), we thus obtain the desired surface $\tilde{M}$.

We next define $\mathcal{M}^{(\varrho)}(g, \Gamma)$ (for $\varrho > 0$) to be the collection of surfaces $M \in \mathcal{M}(g, \Gamma)$ with the following property:

If U is an open C^2 convex set with diameter $\leqslant \varrho$, if $\partial M \subset \boldsymbol{R}^3 \sim U$, and if M and ∂U intersect transversally (in the sense of Theorem 1 (iii)), then for each component Λ of $M \cap \partial U$ there is an $N \in \mathcal{M}$ with $N \subset M$ and $\partial N = \Lambda$.

We have the following lemma.

LEMMA 5. *Suppose $g \geqslant 1$ and $M \in \mathcal{M}(g, \Gamma)$ is such that $\mathcal{H}^2(M) < \alpha_{g-1}$. Then $M \in \mathcal{M}^{(\varrho)}(g, \Gamma)$ for any $\varrho < \{(\alpha_{g-1} - \mathcal{H}^2(M))/(8\pi + 3)\}^{\frac{1}{2}}$.*

Before proving this, we note that the following corollary can be obtained with the aid of Theorem 9 and Corollary 1.

COROLLARY 3. *If the hypotheses are as in Theorem 9, with $\mathcal{H}^2(M) < \alpha_{g-1} - \beta$ and diameter $U < \{\beta/(8\pi + 3)\}^{\frac{1}{2}}$ for some $\beta > 0$, then the surface $\tilde{M}$ of Theorem 9 is such that*

$$\tilde{M} \cap \overline{U} = \bigcup_{j=1}^{k} N_j ,$$

where $N_1, \ldots, N_k$ are pairwise disjoint elements of $\mathcal{M}$.

If it is in addition hypothesized that $\mathcal{H}^2(M) \leqslant \mathcal{H}^2(P) + \theta$ for each $P \in \mathcal{M}(g, \Gamma)$, then there are positive $\theta_1, \ldots, \theta_k$ so that $\sum_{j=1}^{k} \theta_j \leqslant \theta$ and

$$\mathcal{H}^2(N_j) \leqslant \mathcal{H}^2(P) + \theta_j , \quad \forall P \in \mathcal{M} \text{ with } \partial P = \partial N_j,\ j = 1, \ldots, k .$$

PROOF OF LEMMA 5. Let U have diameter $\leqslant \varrho$, where for the moment $\varrho > 0$ is arbitrary, and suppose that M intersects ∂U transversally. We can also suppose $\partial M \cap \overline{U} = \emptyset$, otherwise initially replace U by a slightly smaller set.

Let Γ_1 be a component of $M \cap \partial U$ which is not null-homotopic in M such that one of the two components (let us call it P) of $\partial U \sim \Gamma_1$ satisfies either $M \cap P = \emptyset$ or each component of $M \cap P$ is null-homotopic in M. (Of course such a component Γ_1 of $M \cap \partial U$ exists unless every component of $M \cap \partial U$ is null-homotopic in M.) Then (even if $M \cap P$ is empty) we can find pairwise disjoint diffeomorphs $M_1, \ldots, M_R$ of D_1, with $\bigcup_{i=1}^{R} M_i \subset M$ and $\left(\bigcup_{i=1}^{R} (M_i \sim \partial M_i)\right) \cap P = M \cap P$. By Lemma 2 we then have pairwise disjoint $\tilde{M}_1, \ldots, \tilde{M}_R$, $\tilde{P} \in \mathcal{M}$ with $\partial \tilde{M}_i = \partial M_i$, $\partial \tilde{P} = \partial P$

$$\text{(10.9)} \qquad \mathcal{H}^2(\tilde{M}_i) \leqslant \mathcal{H}^2(M_i) + \varrho^2/R , \quad \mathcal{H}^2(\tilde{P}) \leqslant \mathcal{H}^2(P) + \varrho^2$$

and

$$\tilde{M}_i \cap \Big(M \sim \bigcup_{i=1}^{R} M_j\Big) = \emptyset\,, \quad i = 1, \ldots, R\,. \tag{10.10}$$

(Notice that this last conclusion follows from the first two assertions in (2.6)$'$ of Lemma 2.) Then, defining $\tilde{M} = \Big(M \sim \big(\bigcup_{i=1}^{R} M_i\big)\Big) \cup \big(\bigcup_{i=1}^{R} \tilde{M}_i\big)$, we see that $\tilde{M}$ is homeomorphic (via a bilipschitz homeomorphism) to M, and also $\tilde{M} \cap \tilde{P} = \emptyset$, $\tilde{M} \cap (\partial U \sim P) = M \cap (\partial U \sim P)$.

Since U is convex with diameter $\leqslant \varrho$, we have $\mathcal{H}^2(P) \leqslant 4\pi\varrho^2$, and hence by (10.9) we deduce

$$\mathcal{H}^2(\tilde{P}) < (4\pi + 1)\varrho^2\,, \quad \mathcal{H}^2(\tilde{M}) \leqslant \mathcal{H}^2(M) + \varrho^2\,. \tag{10.11}$$

We now construct a surface N as follows (the construction depends on consideration of two cases):

Case I. If $M \sim \Gamma_1$ is not connected, we let Q be the component of $\tilde{M} \sim \Gamma_1$ which does not contain Γ, and we define $N = (\tilde{M} \sim Q) \cup \tilde{P}$.

Case II. If $M \sim \Gamma_1$ is connected, we let $S = \{x \in \tilde{M}: \operatorname{dist}(x, \Gamma_1) < \varepsilon\}$, where ε is chosen small enough to ensure that $\bar{S}$ is diffeomorphic via a diffeomorphism X, to the annulus $\{x: \frac{1}{2} \leqslant |x| \leqslant 1\}$. For ε sufficiently small, we can construct $E_1, E_2 \in \mathcal{M}$ (each being constructed by a slight perturbation of $\tilde{P}$) such that

$$E_1 \cap E_2 = \emptyset\,, \quad \partial E_1 \cup \partial E_2 = \partial S\,, \quad (E_1 \cup E_2) \cap \tilde{M} = \partial S\,, \tag{10.12}$$
$$\mathcal{H}^2(E_1) + \mathcal{H}^2(E_2) \leqslant 2(4\pi + 1)\varrho^2\,.$$

We then define $N = (\tilde{M} \sim S) \cup E_1 \cup E_2$.

(That is, N is constructed in this case by cutting out the annulus S and replacing it by two discs.)

By (10.11), (10.12) we now have (in either *Case I* or *Case II*)

$$\begin{aligned} \mathcal{H}^2(N) &\leqslant \mathcal{H}^2(M) + \big(2(4\pi + 1) + 1\big)\varrho^2 \\ &= \mathcal{H}^2(M) + (8\pi + 3)\,\varrho^2\,. \end{aligned} \tag{10.13}$$

We now claim that (again in either *Case I* or *Case II*)

$$\text{genus } N < g (= \text{genus } \tilde{M} = \text{genus } M)\,. \tag{10.14}$$

Of course once (10.14) is established we immediately conclude

$$\varrho^2 \geqslant \frac{\alpha_{g-1} - \mathcal{H}^2(M)}{8\pi + 3}$$

from (10.13) because otherwise (10.13) would imply $\mathcal{H}^2(N) < \alpha_{g-1}$.

However (10.14) is an immediate consequence of (10.5)-(10.7) and hence the proof is complete.

Using Theorem 9 and Corollary 3, we can now very directly modify the method of §§ 5-7 in order to prove that there is a C^2 surface M with boundary Γ as in (10.4). (Also, analogously to Remark 3.12, we note that Theorem 9 and (10.3) can be used to construct a minimizing sequence $\{\tilde{M}_k\} \subset \mathcal{M}(g, \Gamma)$ with $\tilde{M}_k \sim \Gamma \subset A$ and $\lim\limits_{k\to\infty} \boldsymbol{v}(\tilde{M}_k) \llcorner (\boldsymbol{R}^3 \sim \Gamma) \times G(3, 2) = \lim\limits_{k\to\infty} \boldsymbol{v}(M_k) \llcorner (\boldsymbol{R}^3 \sim \Gamma) \times G(3, 2)$ $\big(= V_g \llcorner (\boldsymbol{R}^3 \sim \Gamma) \times G(3, 2)\big)$; that is, we may suppose without loss of generality that the minimizing sequence $\{M_k\}$ is such that $M_k \sim \Gamma \subset A$.)

It thus remains only to show that M is orientable and genus $M = g$. By virtue of Theorem 9 and Corollary 3, for each $\delta > 0$ we can use the argument of § 8 in order to construct a surface $N^\delta \in \mathcal{M}(g, \Gamma)$ with $N^\delta \sim \Gamma \subset A$ and such that there is a smooth mapping $\varphi^\delta \colon N^\delta \to M$ with $\varphi^\delta | \Gamma$ a diffeomorphism of Γ, and with dist $\big(x, \varphi^\delta(x)\big) < \delta$ for each $x \in N^\delta$. As in § 8 this immediately implies the orientability of M. Letting K_1, K_2 denote the closures of the two components of $A \sim M$, we note that the exactness of the Mayer-Vietoris sequence for the couple K_1, K_2 (together with the facts that $H_1(\bar{A}) = 0$, $H_2(\bar{A}) = 0\big)$ implies that there is an isomorphism

$$H_1(M)\big(= H_1(K_1 \cap K_2)\big) \approx H_1(K_1) \oplus H_1(K_2)\,. \tag{10.15}$$

Likewise

$$H_1(N^\delta) \approx H_1(K_1^\delta) \oplus H_1(K_2^\delta)\,, \tag{10.16}$$

where K_1^δ, K_2^δ are the closures of the two components of $A \sim N$, labelled so that dist $(x, K_i^\delta) \leqslant \delta$ for each $x \in K_i$, $i = 1, 2$. By smoothness of M there exists a compact $C_i \subset$ interior K_i and a retract $r_i \colon K_i \to C_i$ which induces an isomorphism $H_1(K_i) \approx H_1(C_i)$. Taking δ small enough to ensure $K_i^\delta \supset C_i$ it then follows that the inclusion map $C_i \subset K_i^\delta$ induces a monomorphism of $H_1(C_i)$ into $H_1(K_i^\delta)$. Thus we conclude that rank $H_1(K_i) \leqslant$ rank $H_1(K_i^\delta)$, and hence that rank $H_1(M) \leqslant$ rank $H_1(N^\delta)$ by (10.15) and (10.16). This of course gives genus $M \leqslant$ genus N^δ. Since $\mathcal{H}^2(M) = \alpha_g < \alpha_{g-1}$, we then have genus $M = g$ as required, and hence Theorem 8 is completely proved.

Appendix A. The convex hull property.

THEOREM. *Suppose Γ is any compact subset of $\boldsymbol{R}^n$, suppose $V \in \boldsymbol{V}_k(\boldsymbol{R}^n)$ $(k \leqslant n)$ is stationary in $\boldsymbol{R}^n \sim \Gamma$, and suppose $\|V\|(\boldsymbol{R}^n) < \infty$. Then* spt $\|V\|$ *is contained in the convex hull of Γ.*

PROOF. It suffices to prove spt $\|V\| \subset \bar{H}_+$ for any half-space H_+ such that $\Gamma \subset \bar{H}_+$. For convenience of notation we will prove this only for the case when $H_+ = \{x = (x_1, \ldots, x_n) \colon x_n < 0\}$ and $\Gamma \subset \{x = (x_1, \ldots, x_n) = x_n \leqslant 0\}$. (The general case of course follows by considering $\phi_{\#} V$, where ϕ is a suitable isometry of $\boldsymbol{R}^n$.)

Since V is stationary in $\boldsymbol{R}^3 \sim \Gamma$ we can write (in the notation of [AW1])

$$\text{(A.1)} \qquad \int_{G_k(\boldsymbol{R}^n)} S \cdot Dg(x)\, dV(x, S) = 0$$

whenever g is smooth and spt g is a compact subset of $\boldsymbol{R}^n \sim \bar{H}_+$. Since $\|V\|(\boldsymbol{R}^n) < \infty$, one can then easily verify that (A.1) holds whenever g is smooth and spt $g \subset \boldsymbol{R}^n \sim \bar{H}_+$ (even if spt g is not compact.)

Then we may choose $g(x)$ of the form $(0, \ldots, 0, \gamma(x_n))$ in (A.1), where γ is any $C^1(\boldsymbol{R})$ function with spt γ contained in the positive real numbers. This choice of g immediately yields

$$\int_{G_k(\boldsymbol{R}^n)} \gamma'(x_n) s^{nn}\, dV(x, S) = 0\,,$$

where (s^{ij}) denotes the matrix of the orthogonal projection of $\boldsymbol{R}^n$ onto S, so that $s^{nn} \geqslant 0$. In view of the arbitraryness of γ (for any $\varepsilon > 0$ we can choose γ so that $\gamma'(t) \geqslant 0$ for all $t \in \boldsymbol{R}$ and $\gamma'(t) > 0$ for $t > \varepsilon$), this of course implies that $V[(\boldsymbol{R}^n \sim \bar{H}_+) \times \{S \in G(n, k) \colon s^{nn} \neq 0\}] = 0$. However, since (s^{ij}) is the matrix of an orthogonal projection $\left(\text{so that } \sum_{j=1}^{n} (s^{nj})^2 = s^{nn}\right)$, $s^{nn} = 0$ implies $s^{nj} = s^{jn} = 0$, $j = 1, \ldots, n$. Then choosing $g(x) = \gamma(x_n) x$ in (A.1), and noting that $S \cdot Dx \equiv k$ for every $S \in G(n, k)$, we obtain

$$\int_{G_k(\boldsymbol{R}^n)} \gamma(x_n)\, dV(x, S) = 0\,.$$

Again using the arbitraryness of γ, this gives spt $\|V\| \subset \{x \colon x_n \leqslant 0\}$ as required.

Appendix B. Comparisons with surfaces of higher topological type.

Our reason for including the discussion of § 10 above is that the discs obtained (in § 8) bounding simple closed curves Γ on the boundaries of uniformly convex sets A frequently have area which is larger than that of manifolds of higher topological type such as arise as two dimensional mass minimizing integral currents having Γ as boundary. For example, for suitable $1 < m < n < p < \infty$ let

$$K = \{(x, y, z): x^2 + y^2 \leqslant 1\},$$

$$A = \{(x, y, z): x^2 + y^2 + (nz)^2 \leqslant m^2\},$$

and $\Gamma \subset \partial A$ be the simple closed curve

$$[\partial A \cap \partial K \sim \{(x, y, z): x > 0, (py)^2 < 1\}] \cup [\partial A \cap \{(x, y, z): (py)^2 = 1\}]$$

with smoothing near the four corners

$$\partial A \cap \partial K \cap \{(x, y, z): (py)^2 = 1\}$$

and with the orientation illustrated in Figure 1. For proper choices of m, n, p it is intuitively clear and readily checkable with the use [FH1, 5.4.3., 5.4.5., 5.4.15.] and [AW2] that the support of any two-dimensional mass minimizing integral current T having Γ as boundary must be an embedded minimal submanifold of $\boldsymbol{R}^3$ having Γ as boundary and having genus at least 1. In particular, when n is large one checks by area comparison that

$$\operatorname{spt} T \cap \{(x, y, z): x = y = 0\} = \emptyset$$

and, also, since for large values of p, Γ is close in the flat metric topology to $\partial A \cap \partial K$, it follows that T very nearly must lie within K and hence for most $1 < r < m$,

$$\operatorname{spt} T \cap \{(x, y, z): x = r,\ z = 0\} = \emptyset .$$

More generally the references cited above imply that if $T, T_1, T_2, T_3, \ldots \in$ $\in \boldsymbol{I}_2(R^3)$ are mass minimizing, if $\lim_i T_i = T$ in the flat topology, if L is a compact subset of $\boldsymbol{R}^3$, and if

$$\lim_i \text{Hausdorff distance } (\operatorname{spt} \partial T_i, L) = 0,$$

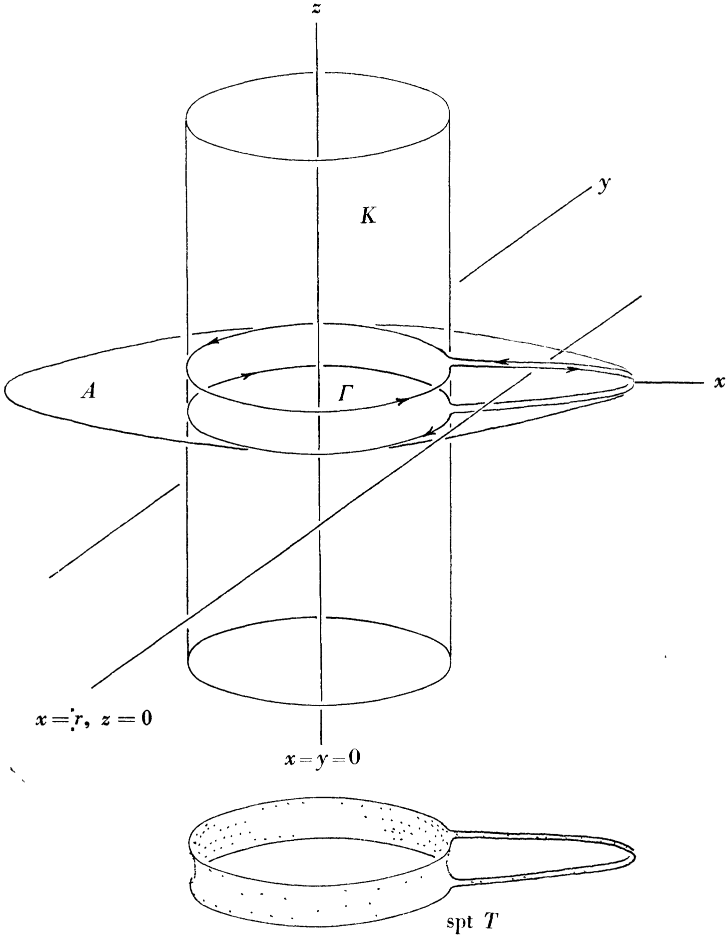

Figure 1

then

$$\lim_i \text{ Hausdorff distance } (\text{spt } T_i, \text{ spt } T \cup L) = 0 .$$

For such r as above one notes that the homotopy group

$$\pi_1(\boldsymbol{R}^3 \sim [\{(x, y, z) : x = y = 0\} \cup \{(x, y, z) : x = r, z = 0\}] ; (0, 0, 0))$$

is the free group $\boldsymbol{Z} * \boldsymbol{Z}$ on two generators and that the homotopy class of Γ is the commutator of two such generators, which commutator is of course not equal to the identity. In particular then, spt T cannot topologically be a disc. The support of T presumably resembles the surface indicated in Figure 1.

For each $g = 1, 2, 3, \ldots$ our construction readily generalizes to produce smooth simple closed curves Γ on the boundary of any uniformly convex subset of $\boldsymbol{R}^3$ such that the genus of the support of any mass minimizing integral current T having such Γ as boundary is at least g. Such a curve Γ

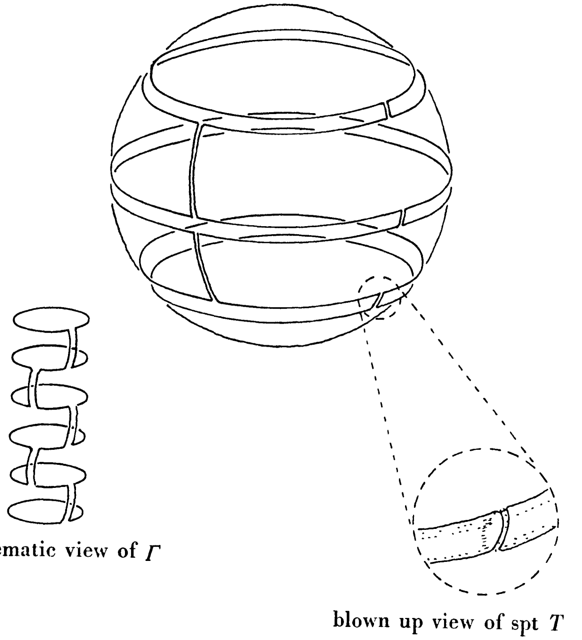

Figure 2

on $\boldsymbol{S}^2$ corresponding to $g = 3$ is illustrated in Figure 2. If one whishes, such curves Γ, corresponding to any g, can be required to have arbitrarily short prescribed length; the curve if Figure 2 for example can be slid on $\boldsymbol{S}^2$ to a small neighbourhood of any point.

Curves like that illustrated in Figure 2 have increasingly large curvature as g increases. Indeed, it follows from [AW1] [AW2] [FH1, 4.2.17, 5.4.5, 5.4.15] that if $\boldsymbol{\Gamma}$ is any collection of twice continuously differentiable oriented *simple* closed curves Γ lying in, say, $\boldsymbol{S}^2$ which is *compact* in the C^2 topology, then there is a finite number $g_0 = g_0(\boldsymbol{\Gamma})$ such that whenever T is a mass minimizing integral current with $\partial T \in \boldsymbol{\Gamma}$ then the genus of spt T does not exceed g_0.

On the other hand William P. Thurston has pointed out that for each $\varepsilon > 0$ and $g_0 < \infty$ there is a C^∞ simple closed oriented curve Γ lying on $\boldsymbol{S}^2$

such that the length of Γ does not exceed ε, the total absolute curvature of Γ does not exceed $4\pi + \varepsilon$, and the genus of the support of any mass minimizing integral current having Γ as boundary is not less than g_0. Such a curve Γ

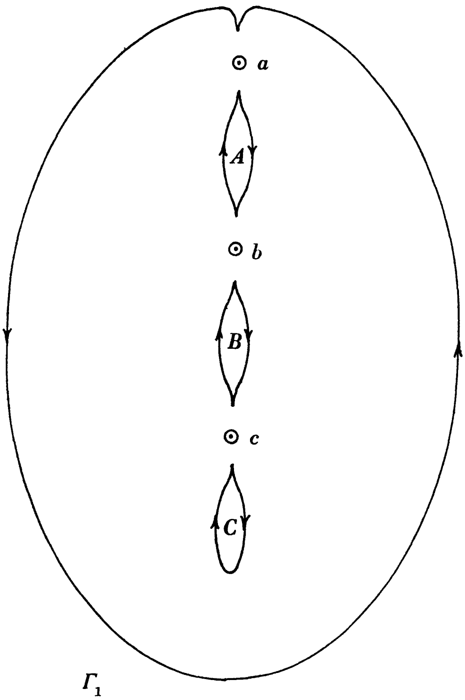

Figure 3*a*

can be obtained as follows for $g_0 = 3$. We consider one dimensional integral cycles shaped like Γ_1 in Figure 3*a* and note that in case Γ_1 lies in a plane and T_1 is the unique mass minimizing integral current with $\partial T = \Gamma_1$, then spt T_1 does not intersect the open region $A \cup B \cup C$. It readily follows that if an integral cycle like Γ_1 lies on a sphere of sufficiently large radius, then any mass minimizing integral current T_1 having Γ_1 as boundary largely will not cover the region $A \cup B \cup C$. We now fix such Γ_1 on such a sphere of large radius and observe that the points *on* Σ at positions a, b, c as indicated in Figure 3*a* lie somewhat outside the convex hull of Γ_1. If Γ_1 is now modified to become the C^∞ simple closed curve Γ_2 lying on Σ of Figure 3*b* by adding extremely thin bridges as indicated (so that Γ_2 is very close in the flat topology to Γ_1) one checks, by arguments similar to those above, that, whenever T_2 is a mass minimizing integral current having Γ_2 as

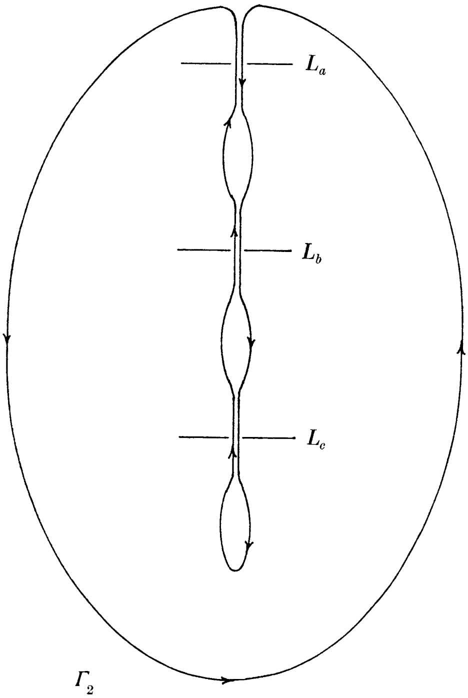

Figure 3*b*

boundary, the region $A \cup B \cup C$ still will largely be uncovered and there will exist straight lines L_a, L_b, L_c as indicated entirely missing spt T_2 while passing just beneath the three bridges and above the main part of spt T_2. One checks, for example by computing the rank of the intersection matrix of appropriately chosen elements of the homology group $H_1(\text{spt}\ T_2; \boldsymbol{Z}_2)$ as in [AT], that the genus of spt T_2 is at least 3. It is clear from the illustration that such Γ_2 can be constructed with total curvature as close to 4π as desired, and, since the radius of Σ is large, the curve Γ on our standard two dimensional sphere $\boldsymbol{S}^2$ (which is our desired curve) corresponding to Γ_2 on Σ will be short.

REFERENCES

[AA] W. K. Allard - F. J. Almgren, Jr., *The structure of stationary one dimensional manifolds with positive density*, preprint.

[Al] H. W. Alt, *Verzweigungspunkte von H-Flächen I*, Math. Z., **127** (1972), pp. 333-362.

[A2] H. W. ALT, *Verzweigungspunkte von H-Flächen II*, Math. Ann., **201** (1973), pp. 33-55.

[AS] L. V. AHLFORS - L. SARIO, *Riemann Surfaces*, Princeton University Press, 1960.

[AT] F. J. ALMGREN, Jr. - W. P. THURSTON, *Examples of unknotted curves which bound only surfaces of high genus with their convex hulls*, Ann. of Math., **105** (1977), pp. 527-538.

[AW1] W. K. ALLARD, *On the first variation of a varifold*, Ann. of Math., **95** (1972), pp. 417-491.

[AW2] W. K. ALLARD, *On the first variation of a varifold: boundary behaviour*, Ann. of Math., **101** (1975), pp. 418-446.

[C] R. COURANT, *Dirichlet's Principle*, Interscience, 1950.

[D] J. DOUGLAS, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc., **33** (1931), pp. 263-321.

[FH] H. FEDERER, *Geometric measure theory*, Springer-Verlag, 1969.

[G] R. GULLIVER, *Regularity of minimizing surfaces of prescribed mean curvature*, Ann. of Math. (1973), pp. 275-305.

[GOR] R. GULLIVER - R. OSSERMAN - H. ROYDEN, *A theory of branched immersions of surfaces*, Amer. J. Math., **95** (1973), pp. 750-812.

[GS1] R. GULLIVER - J. SPRUCK, *On embedded minimal surfaces*, Ann. of Math., **103** (1976), pp. 331-347.

[GS2] R. GULLIVER - J. SPRUCK, *Correction to « On embedded minimal surfaces »*, Ann. of Math., **109** (1979), pp. 407-412.

[M] F. MORGAN, *Almost every curve in $\boldsymbol{R}^3$ bounds a unique area minimizing surface*, Invent. Math., **45** (1978), pp. 253-297.

[MY] W. H. MEEKS III - S. T. YAU, *The classical Plateau problem and the topology of 3-manifolds*, Arch. Rational Mech. Anal.

[N] J. C. C. NITSCHE, *Contours bounding at least three solutions of Plateau's problem*, Arch. Rational Mech. Anal., **30** (1968), pp. 1-11.

[O] R. OSSERMAN, *A proof of the regularity everywhere of the classical solution of Plateau's problem*, Ann. of Math., **91** (1970), pp. 550-569.

[P] J. PITTS, *Existence and regularity of minimal surfaces in Riemannian manifolds*, Bull. Amer. Math. Soc., **82** (1976), pp. 503-504.

[R] T. RADO, *The problem of least area and the problem of Plateau*, Math. Z., **32** (1930), pp. 763-796.

[TT] F. TOMI - A. J. TROMBA, *Extreme curves bound embedded minimal surfaces of the type of disc*, Math. Z., **158** (1978), pp. 137-145.