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An Embedding Theorem for Real Analytic Spaces (*).

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Introduction.

Let V be a real analytic, paracompact connected manifold of dimension n.

H. Grauert has proved (see [4]) that V is isomorphic to a closed submanifold of $\boldsymbol{R}^{2n+1}$.

If $(X, \mathcal{O}_X)$ is a paracompact connected coherent real analytic space and $N = \sup_{x\in X} \dim \tau_x$, $N > n$, where τ_x is the Zariski tangent space, then $(X, \mathcal{O}_X)$ can be embedded in $\boldsymbol{R}^{n+N}$ (i.e. is isomorphic to a closed real analytic subspace of $\boldsymbol{R}^{n+N}$) (see [16]).

The purpose of this paper is to prove that if $(X, \mathcal{O}_X)$ is a (reduced) real analytic space, paracompact and connected and if $N = \sup_{x\in X} \dim \tau_x < +\infty$ then $(X, \mathcal{O}_X)$ can be embedded in an euclidean space $\boldsymbol{R}^q$.

Using the above result one can prove that the embeddings $X \to \boldsymbol{R}^{n+N}$ are dense in the space of the C^∞ maps of X into $\boldsymbol{R}^{n+N}$.

1. – Definitions and preliminary remarks.

In this paragraph we shall recall some definitions and well known facts that we shall use in the following.

DEFINITION 1. Let $(X, \mathcal{O}_X)$ be a ringed space, $(X, \mathcal{O}_X)$ is called a real (complex) coherent analytic space iff locally $(X, \mathcal{O}_X)$ is isomorphic to a local model $(S, \mathcal{O}_U/\mathcal{I})$ where U is open in $\boldsymbol{R}^n(\boldsymbol{C}^n)$, $\mathcal{O}_U =$ sheaf of germs of analytic functions (holomorphic functions), $\mathcal{I}$ is a coherent ideal sheaf of $\mathcal{O}_U$ such that $S =$ support of $\mathcal{O}_U/\mathcal{I}$.

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DEFINITION 2. Let $(X, \mathcal{O}_X)$ be a ringed space, $(X, \mathcal{O}_X)$ is called a reduced real (complex) analytic space iff locally $(X, \mathcal{O}_X)$ is isomorphic to a local model $(S, \mathcal{O}_U/\mathcal{I})$ where U, $\mathcal{O}_U$ are as above, S is an analytic closed subset of U and $\mathcal{I}$ is the subsheaf of $\mathcal{O}_U$ of the germs of all the analytic functions that are zero on S.

In the following the analytic spaces are considered as ringed spaces, the morphisms of ringed spaces are often called analytic (or holomorphic) maps.

It is well known that any reduced complex analytic space is a coherent complex analytic space, but there exist reduced real analytic spaces that are not coherent.

Let $(X, \mathcal{O}_X)$ be a reduced real analytic space, we shall say that X is coherent in the point x if there exists an open set $U \ni x$ such that $(U, \mathcal{O}_{X|U})$ is a coherent real analytic space.

Let $(X, \mathcal{O}_X)$ be a real analytic reduced space and let us suppose there exists a coherent real analytic space $(X', \mathcal{O}_{X'})$ such that $(X, \mathcal{O}_X)$ is the reduced space associated to $(X', \mathcal{O}_{X'})$. In this hypothesis we have (see [11]):

i) the set S_X of the singular points of X is contained in a proper real analytic subspace of X,

ii) the set of the points where X is not coherent is a semianalytic subset (contained in S_X) of codimension at least two in X.

DEFINITION 3. Let $(X, \mathcal{O}_X)$ be a coherent real analytic space and $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ a complex analytic space.

We shall say that $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is a complexification of $(X, \mathcal{O}_X)$ if $(X, \mathcal{O}_X)$ is a closed real analytic subspace of the real analytic space associated to $\tilde{X}$ and for any $x \in X$ we have: $\mathcal{O}_{\tilde{X},x} \simeq \mathcal{O}_{X,x} \otimes_{\boldsymbol{R}} \boldsymbol{C}$ (if $\mathcal{F}$ is a sheaf $\mathcal{F}_x$ means the stalk of $\mathcal{F}$ at x).

In the following all the real or complex analytic spaces we shall consider are paracompact and hence metric spaces.

We remember that for a connected reduced real analytic space $(X, \mathcal{O}_X)$ such that $\sup_{x \in X} \dim \tau_x < +\infty$, where τ_x is the Zariski tangent space, the following statements are equivalent (see [15]):

a) $(X, \mathcal{O}_X)$ is the reduced analytic space associated to a coherent real analytic space $(X', \mathcal{O}_{X'})$ such that $\sup_{x \in X'} \dim \tau_x < +\infty$.

b) $(X, \mathcal{O}_X)$ is isomorphic to a closed real analytic subspace Y of some $\boldsymbol{R}^n$ and Y has global equations in $\boldsymbol{R}^n$.

c) there exists a Stein space $\tilde{Y} \subset \boldsymbol{C}^n$ such that $\tilde{Y} \cap \boldsymbol{R}^n$ is isomorphic to X.

H. Cartan has proved (see [12]) that not any real reduced compact analytic space satisfies one of the equivalent conditions *a*), *b*), *c*). The aim of this paper is to prove that also these « patological » real reduced analytic spaces can be embedded in the euclidean space.

Let $(X, \mathcal{O}_X)$ be a complex reduced analytic space and $\sigma: \boldsymbol{C} \to \boldsymbol{C}$ the usual *conjugation*. A function $f: X \to \boldsymbol{C}$ is called *antiholomorphic* if $\sigma \circ f$ is holomorphic.

If $(X_i, \mathcal{O}_{X_i})$ $i = 1, 2$ are two complex, reduced, analytic spaces and $\varphi: X_1 \to X_2$ is a continuous map then φ is said to be antiholomorphic if locally φ is described by antiholomorphic functions (for the details see [15]).

Let $(X, \mathcal{O}_X)$ be a complex analytic space. Then a map $\sigma: X \to X$ is called an *antiinvolution* if:

1) $\sigma \circ \sigma = \mathrm{id}$.

2) σ is antiholomorphic.

DEFINITION 4. Let $(X, \mathcal{O}_X)$ be a complex, reduced, analytic space and $\sigma: X \to X$ an antiinvolution.

The couple $\{(X, \mathcal{O}_X), \sigma\}$ is called a complex, reduced, analytic space *defined on the real numbers* (briefly defined on $\boldsymbol{R}$).

Given two complex, reduced, analytic spaces defined on the real numbers $\{(X_i, \mathcal{O}_{X_i}), \sigma_i\}$ and a morphism $\varphi: (X_1, \mathcal{O}_{X_1}) \to (X_2, \mathcal{O}_{X_2})$ we shall say that φ *is defined on* $\boldsymbol{R}$ iff: $\sigma_2 \circ \varphi = \varphi \circ \sigma_1$.

If $\{(X, \mathcal{O}_X), \sigma\}$ is as above the set $X_\sigma = \{x \in X | \sigma(x) = x\}$ has a natural structure of reduced real analytic space. X_σ is called the *real part* of X.

Clearly if $\varphi: (X_1, \mathcal{O}_{X_2}) \to (X_2, \mathcal{O}_{X_2})$ is defined on $\boldsymbol{R}$ then φ induces a morphism $\varphi_\sigma: X_{1\sigma} \to X_{2\sigma}$.

2. – Preliminary results.

In this paragraph we shall expose some facts that are necessary to prove the main result.

These lemmata can be found, in a similar form, in [1], [2].

LEMMA 1. *Let V_i, $i = 1, 2$ be two real analytic, closed submanifolds of the open sets A_i of $\boldsymbol{R}^n$.*

Let $\varphi: V_1 \to V_2$ be a real analytic isomorphism and let us suppose $\boldsymbol{R}^n$ canonically embedded in $\boldsymbol{R}^{n+q}$ for any q.

In these hypotheses there exist two open neighbourhoods A_i' of V_i in $\boldsymbol{R}^{n+n}$ such that φ can be extended to an analytic isomorphism $\varphi': A_1' \to A_2'$.

PROOF. Let $F \xrightarrow{\pi_1} V_2$ be a vector bundle, then we shall denote by $\varphi^*(F) \xrightarrow{\pi_1} V_1$ the inverse image of F.

It is well known that if F is an analytic vector bundle the same is true for $\varphi^*(F)$ and we have the following commutative diagram:

$$\begin{array}{ccc} \varphi^*(F) & \xrightarrow{\varphi^*} & F \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ V_1 & \xrightarrow{\varphi} & V_2 \end{array}$$

where φ^* is an analytic isomorphism of vector bundles.

In the following we shall denote by T_{V_i}, N_{V_i} the tangent and the normal bundle of $V_i \subset A_i$, $i = 1, 2$.

It is known (see [9]) that the zero section of N_{V_i} has a neighbourhood analytitically isomorphic to a neighbourhood U_{V_i} of V_i in A_i.

In the following U_{V_i} is called a tubular neighbourhood.

We have the relations (see [9]):

$$T_{V_i} \oplus N_{V_i} \simeq T_{\mathbf{R}^n|_{V_i}} \simeq V_i \times \mathbf{R}^n \tag{1}$$

(we remember that if two analytic vector bundles are C^∞ isomorphic then they are also analytically isomorphic, see [10]). In the following $\simeq$ shall mean analytically isomorphic (as vector bundles, as manifolds, as analytic spaces ...).

Let us consider the commutative diagram:

$$\begin{array}{ccc} T_{V_1} \oplus N_{V_1} \simeq \varphi^*(T_{V_2} \oplus N_{V_2}) & \xrightarrow{\varphi^*} & T_{V_2} \oplus N_{V_2} \\ \downarrow & & \downarrow \\ V_1 & \xrightarrow{\varphi} & V_2 \end{array} \quad . \tag{2}$$

From (2) we deduce the commutative diagram:

$$\begin{array}{ccc} \varphi^*(N_{V_2}) \oplus T_{V_1} \oplus N_{V_1} & \xrightarrow{\psi} & N_{V_2} \oplus T_{V_2} \oplus N_2 \\ \downarrow & & \downarrow \\ V_1 & \xrightarrow{\varphi} & V_2 \end{array} \quad . \tag{3}$$

Now we remark that φ is an analytic isomorphism hence $T_{V_1} = \varphi^*(T_{V_2}) \simeq T_{V_1}$ and $N_{V_2} \simeq \varphi^*(N_{V_2})$ and the isomorphism ψ defined in (3) gives:

$$(V_1 \times \mathbf{R}^n) \oplus N_{V_1} \overset{\psi}{\simeq} (V_1 \times \mathbf{R}^n) \oplus N_{V_2} . \tag{4}$$

The relation (4) can be written

$$N_{V_1}\times \boldsymbol{R}^n \simeq N_{V_2}\times \boldsymbol{R}^n . \tag{5}$$

Relation (5) and the existence of the tubular neighbourhood prove the assertion of the lemma.

Let X be a real analytic space and $X = X_1 \cup X_2$ an open covering. Suppose $\varrho_i\colon X_i \hookrightarrow A_i$, $i = 1, 2$ are two proper embeddings of X_i into open subsets A_i of $\boldsymbol{R}^n$.

Then we have the diagram:

$$X_1 \cap X_2 \begin{array}{l} \xhookrightarrow{\varrho_1} \varrho_1(X_1 \cap X_2) \overset{\text{def}}{=} Y_1 \subset A_1 \subset \boldsymbol{R}^n \\ \xhookrightarrow{\varrho_2} \varrho_2(X_1 \cap X_2) \underset{\text{def}}{=} Y_2 \subset A_2 \subset \boldsymbol{R}^n \end{array} \tag{6}$$

that defines the analytic isomorphism $\varphi = \varrho_2 \circ \varrho_1^{-1}\colon Y_1 \to Y_2$.

LEMMA 2. *Let us suppose that $\varphi\colon Y_1 \to Y_2$ can be extended to an analytic isomorphism $\tilde{\varphi}\colon \Omega_1 \to \Omega_2$ of a neighborhood of Y_1 in A_1 onto a neighbourhood Ω_2 of Y_2 in A_2. Then the real analytic space $X = X_1 \bigcup X_2$ can be embedded in a euclidean space $\boldsymbol{R}^q$.*

PROOF. It is enough to prove that X can be embedded in a paracompact real analytic manifold (any such manifold is isomorphic to a closed submanifold of some $\boldsymbol{R}^q$ (see [4])).

The space X is paracompact, hence there exists an open covering $X = X_1' \cup X_2'$ such that $\bar{X}_i' \subset X_i$ and therefore $\overline{X_1' \cap X_2'} \subset X_1 \cap X_2$.

Let $A_i' \subset A_i$, $i = 1, 2$ be open sets of $\boldsymbol{R}^n$ such that: $X_i' = A_i' \cap X_i$, $\bar{X}_i' = \bar{A}_i' \cap X_i$ and let us denote $Y_i' = \varrho_i(X_1' \cap X_2')$.

By hypothesis φ can be extended to $\tilde{\varphi}\colon \Omega_1 \to \Omega_2$ where Ω_1 is open and contains $\bar{Y}_1'$.

Let $\Omega_1' \subset \Omega_1$ be an open set such that: $\Omega_1' \cap Y_1 = Y_1'$, $\bar{\Omega}_1' \cap Y_1 = \bar{Y}_1'$, $\bar{\Omega}_1' \subset \Omega_1$. We have the analytic isomorphism $\tilde{\varphi}\colon \bar{\Omega}_1' \to \tilde{\varphi}(\bar{\Omega}_1') \overset{\text{def}}{=} \bar{\Omega}_2'$.

By the fact that $\tilde{\varphi}$ is an omeomorphism we obtain: $\Omega_2' \cap Y_2 = Y_2'$.

Let now $\bar{A}_1' \coprod \bar{A}_2'$ be the disjoint union of $\bar{A}_1'$ and $\bar{A}_2'$ and $\mathscr{R}\tilde{\varphi}$ the equivalence relation $x \sim y \Leftrightarrow x \in \bar{A}_1'$, $y \in A_2'$ and $y = \tilde{\varphi}(x)$.

We claim that the quotient space $\bar{A}_1' \coprod \bar{A}_2'/\mathscr{R}_{\tilde{\varphi}} = \tilde{X}'$ is Hausdorff because the gluing map $\tilde{\varphi}$ is defined on closed sets of $\bar{A}_i'$. Let x, y be points of $\tilde{X}'$ and let us denote by $p\colon \bar{A}_1' \coprod \bar{A}_2' \to \tilde{X}'$ the canonical projection.

If there don't exist open, disjoint neighbourhoods $U_x \ni x$, $U_y \ni y$ it means that we may construct $x_n \to x'$, $y_n \to y'$, $x_n, y_n, x', y' \in \bar{A}_1' \coprod \bar{A}_2'$ such that: $p(x') = x$, $p(y') = y$, $p(x_n) = p(y_n)$, $x_n \neq y_n$.

From the fact that the homeomorphism $\tilde{\varphi}$ is defined on a closed set we deduce that $\tilde{\varphi}(x') = y'$ and hence $x = y$. So we have proved that $\tilde{X}'$ is T_2.

Now we remark that if we take a subset $A \subset \bar{A}_1' \coprod \bar{A}_2$ then the topology of $A/\mathscr{R}_{\tilde{\varphi}|A} = \hat{A}$ is finer than the topology induced by $\tilde{X}'$ on $p(A)$. Hence $\hat{A}$ is a Hausdorff space.

By the above remark it follows that $A_1' \coprod A_2'/\mathscr{R}_{\tilde{\varphi}} = \tilde{X}$ is a Hausdorff space.

From the fact that $\tilde{\varphi}$ is an homeomorphism and the A_i' are open it follows that the equivalence relation $\mathscr{R}_{\tilde{\varphi}|A_1' \coprod A_2'}$ is open and hence $\tilde{X}$ is an analytic Hausdorff manifold containing canonically X.

Clearly the A_i have a countable base of open sets, hence $\tilde{X}$ has the same property and therefore is paracompact. The lemma is now proved.

LEMMA 3. *Let X be a real analytic space and $X = X_1 \cup X_2$ be an open covering.*

Let us suppose there exist two analytic embeddings $\varrho_i\colon X_i \hookrightarrow A_i$, A_i open set of $\mathbf{R}^{n_i}$, $\varrho(X_i)$ closed in A_i.

Let us suppose $\varrho_2 \circ \varrho_1^{-1} \overset{\text{def}}{=} \varphi\colon \varrho_1(X_1 \cap X_2) \to A_2$ can be extended to an embedding $\tilde{\varphi}\colon \Omega_1 \to A_2$ of an open neighbourhood Ω_1 of $\varrho_1(X_1 \cap X_2)$ in A_1 into the open set A_2.

In these hypotheses the space X can be embedded in an euclidean space.

PROOF. Using lemma 1 we prove that the conditions of lemma 2 are satisfied and hence the thesis follows.

DEFINITION 5. Let $X = X_1 \cup X_2$ be an open covering of a reduced real analytic space. We shall say that the covering has *the extension property* if there exist two embeddings $\varrho_i\colon X_i \to A_i \subset \boldsymbol{R}^{n_i}$ satisfing the condition of the lemma 3.

Lemma 3 can now be written: if $X = X_1 \cup X_2$ has the extension property then the real analytic reduced space X can be embedded in an euclidean space $\boldsymbol{R}^N$.

3. – The embedding theorem.

To prove the embedding theorem we need some criteria to ensure that a covering $X = X_1 \cup X_2$ of a reduced real analytic space has the extension property (see definition 5).

PROPOSITION 1. *Let* $(X, \mathcal{O}_X)$ *be a real, reduced, analytic space and* $X = X_1 \cup X_2$ *an open covering of* X.

Everyone of the following conditions is sufficient to ensure that the covering $X = X_1 \cup X_2$ *has the extension property*:

(I) *there exist two embeddings* $\varrho_i\colon X_i \to A_i$, A_i *open set of* $\boldsymbol{R}^{n_i}$, $i = 1, 2$ *and* $(X_1 \cap X_2, \mathcal{O}_{X|X_1\cap X_2})$ *is a reduced coherent real analytic space.*

(II) *there exist two embeddings* $\varrho_i\colon X_i \to A_i$, A_i *open set of* $\boldsymbol{R}^{n_i} \subset \boldsymbol{C}^{n_i}$, *two Stein spaces* $\tilde{Y}_i \subset \tilde{A}_i$ *contained in two open sets* $\tilde{A}_i \subset \boldsymbol{C}^{n_i}$ *and defined on* $\boldsymbol{R}$ *with respect to the antiinvolutions* σ_i *given by the usual conjugation of* $\boldsymbol{C}^{n_i}$, *such that*:

i) $\tilde{Y}_i \cap \boldsymbol{R}^{n_i} = \varrho_i(X_1 \cup X_2)$;

ii) *the isomorphism* $\varphi = \varrho_2 \circ \varrho_1^{-1}\colon \varrho_1(X_1 \cup X_2) \to \varrho_2(X_1 \cup X_2)$ *can be extended to an isomorphism* $\varphi\colon \tilde{Y}_1' \to \tilde{Y}_2'$ *defined on* $\boldsymbol{R}$ *of two open neighbourhoods* $\tilde{Y}_i'$ *of* $\varrho_i(X_1 \cup X_2)$ *in* $\tilde{Y}_i$.

PROOF:

(I) Let $\varrho_i\colon X_1 \to A_i \subset \boldsymbol{R}^{n_i}$ be the embeddings given by the hypotheses and let us denote $\varphi = \varrho_2 \varrho_1^{-1}\colon \varrho_1(X_1 \cap X_2) \to \varrho_2(X_1 \cap X_2)$. We may suppose $n_2 \geqslant 2n_1 + 1$ (otherwise we take $\boldsymbol{R}^{n_1'} = \boldsymbol{R}^{n_2} \times \boldsymbol{R}^{q}$). From a result of H. Whitney (see [13]) we have that φ can be extended to a C^∞ embedding $\hat{\varphi}\colon \varrho_1(\widehat{X_1 \cap X_2}) \to A_2 \subset \boldsymbol{R}^{n_2}$ where $\varrho_1(\widehat{X_1 \cap X_2})$ is an open neighbourhood of $\varrho_1(X_1 \cap X_2)$ in A_1.

From the fact that $X_1 \cap X_2$ is a coherent real analytic space we deduce, using the results of [14], that $\hat{\varphi}$ can be approached by an analytic embedding $\tilde{\varphi}\colon \varrho_1(\widehat{X_1 \cap X_2}) \to A_2 \subset \boldsymbol{R}^{n_2}$ and we may suppose $\tilde{\varphi}$ extends φ.

Part (I) is now proved.

(II) In [3] the following result is proved:

Let $(X, \mathcal{O}_X)$ be a reduced Stein space, Y a closed analytic subspace and $\varphi\colon Y \to \boldsymbol{C}^l$ an embedding. Let us suppose $l \geqslant 2n + 1$, $n = \dim X$. Then the set of all analytic maps $\psi\colon X \to \boldsymbol{C}^l$ that are proper, one to one, regular on the regular points of X and extend φ are dense in the space of the analytic maps that extend φ.

If we suppose that $((X, \mathcal{O}_X), \sigma)$ is defined on $\boldsymbol{R}$, $\sigma(Y) = Y$ and φ is a morphism defined on $\boldsymbol{R}$ then we have: the proper, one to one, regular in the regular points, extension of φ defined on $\boldsymbol{R}$ are dense in the space of the extensions of φ defined on $\boldsymbol{R}$.

Going back to the proof of this result one checks that if φ is an embedding, not necessarily proper, then the one to one, regular in the regular points, analytic extensions $\psi\colon X \to \boldsymbol{C}^l$ are dense in the extensions of φ.

The same result is true for mapping defined on $\boldsymbol{R}$.

Now we can apply these results to our case.

We can suppose $n_2 \geqslant 2n_1 + 1$, then there exists an embedding $\psi\colon \tilde{A}_1' \to \tilde{A}_2$ of a neighbourhood $\tilde{A}_1'$ of $\tilde{Y}_1$ in $\boldsymbol{C}^{n_1}$ into $\boldsymbol{C}^{n_2}$ and hence into $\tilde{A}_2$ that extends φ (if $\psi'\colon \tilde{A}_1' \to \boldsymbol{C}^{n_2}$ is an extension of φ we take $\psi = \psi'|_{\psi'^{-1}(\tilde{A}_2')}$).

We may suppose that ψ is defined on $\boldsymbol{R}$ and hence ψ defines an embedding of a neighbourhood of $\varrho_1(X_1 \cap X_2)$ in A_1 into $\boldsymbol{R}^{n_2}$. The proposition is now proved.

Now we can prove the main results:

THEOREM 1. *Let $(X, \mathcal{O}_X)$ be a real, reduced, connected analytic space and let us suppose* $\sup\limits_{x \in X} \dim \tau_x < +\infty$, $\tau_x =$ *Zariski tangent space at* x.

In these hypotheses $(X, \mathcal{O}_X)$ is isomorphic to a closed real analytic subspace of some euclidean space $\boldsymbol{R}^q$.

PROOF:

i) We wish to prove the following topological fact: let us suppose $n =$ topological dimension of X, $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. In this hypothesis there exists an open refinement $\mathcal{V} = \{V_j\}_{j \in J}$ of $\mathcal{U}$ such that:

a) $\mathcal{V}$ is locally finite;

b) J is the disjoint union of $n+1$ subset $J_1, \ldots, J_{n+1}$ such that any connected component of $V^k = \bigcup\limits_{i \in J_k} V_i$ is contained in some U_i, $k = 1, \ldots, n+1$.

The space X is metric and has a countable base of open sets hence the usual definitions of topological dimension coincide on X (see [6]).

The space X is paracompact of dimension n hence we can suppose, eventually refining $\mathcal{U}$, that $\mathcal{U} = \{U_i\}_{i \in I}$ is locally finite and that any $x \in X$ is contained, at most, in $n+1$ open sets U_i. From the fact that X has a countable base of open sets we may also suppose $I = \boldsymbol{N}$.

Let $\{\alpha_i\colon X \to \boldsymbol{R}\}_{i \in \boldsymbol{N}}$ be a continuous partition of the unity associated to $\mathcal{U} = \{U_i\}_{i \in \boldsymbol{N}}$.

Let us state:

$$\hat{U}_{k_1,\ldots,k_j} = \{x \in X | \alpha^k(x) \leqslant \min(\alpha_{k_1}(x), \ldots, \alpha_{k_j}(x)),\ k \neq k_1, \ldots, k\}\,.$$

We have from the above construction:

1) any $\hat{U}_{k_1,\dots,k_j}$ is open because $\mathscr{U}$ is locally finite;

2) $\hat{U}_{k_1,\dots,k_j} \subset U_{k_1} \cap U_{k_2} \dots \cap U_{k_j}$;

3) $\hat{U}_{k_1,\dots,k_j} \cap \hat{U}_{k_1,\dots,k_j} \neq \phi$ only if $h_1 = k_1, \dots, h_j = k_j$;

4) if $j > n + 1$ we have $\hat{U}_{h_1,\dots,h_j} = \phi$;

5) $\bigcup\limits_{j=1}^{n+1} \bigcup\limits_{k_i \in N} \hat{U}_{k_1,\dots,k_j} = X$.

The families $\mathscr{V}^1 = \{\hat{U}_i\}_{i \in N}$, $\mathscr{V}^2 = \{\hat{U}_{ik}\}_{i,k \in N} \dots \mathscr{V}^{n+1} = \{\hat{U}_{i_1,\dots,i_{n+1}}\}_{i_1,\dots,i_{n+1} \in N}$ give the required decomposition of the refinement given by

$$\mathscr{V} = \mathscr{V}^1 \cup \dots \cup \mathscr{V}^{n+1} .$$

The assertion is now proved.

ii) From the above result there exists an open covering $X = \bigcup\limits_{i=1}^{n+1} X_i$ such that:

1) for any $i = 1, \dots, n + 1$ there exists an embedding $\varrho_i \colon X_i \to A_i$ where A_i is an open set of $\boldsymbol{R}^{n_i}$ and $\varrho_i(X_i)$ is a closed analytic subset of A_i;

2) there exist open sets $\tilde{A}_i$ of $\boldsymbol{C}^{n_i}$ and Stein closed subspace $\tilde{X}_i \subset \tilde{A}_i$ such that: $\tilde{A}_i \cap \boldsymbol{R}^{n_i} = A_i$, $\tilde{X}_i \cap A_i = X_i$ and the $\tilde{X}_i$ are defined on $\boldsymbol{R}$.

It is sufficient to choose a covering $\mathscr{U}$ by local models, to construct $\mathscr{V}$ and then to take $X_i = \bigcup\limits_{V_j \in \mathscr{V}^i} V_j$.

Let us define:

$$T = \{x \in X_1 \cap X_2 | X \text{ is not coherent in the point } x\} .$$

It is known (see § 1) that T is contained in a proper analytic subspace of X_1 (and of X_2).

If we consider $X_1 \cup X_2 - \overline{T} = X^1$ then this analytic reduced space is covered by X_1 and $X_2 - \overline{T}$ and the hypothesis (I) of proposition 1 is satisfied.

Following the construction of the previous lemmata we obtain a real analytic manifold

$$V_1 = A_1' \coprod A_2'/\mathscr{R}_1 \quad \text{that contains } X^1 .$$

From the construction we see that we may suppose that if A_2' is open in $\boldsymbol{R}^{n_2'}$ then there exists, in an open set of $\boldsymbol{C}^{n_2'}$, a Stein space defined on $\boldsymbol{R}$ isomorphic to a neighbourhood of $X_2 - \overline{T}$ in $\tilde{X}_2$.

In fact A_2' contains an embedding of an open subset of A_2 and this embedding can be extended to an open set of $\boldsymbol{C}^{n_2'}$.

Let us consider the real reduced analytic space $X_1 \cup X_2$ covered by V_1 and X_2.

The charts A_2' and $X_2 \subset A_2$ are now in the hypothesis (II) of proposition 1 and hence we can construct an analytic manifold W that contains $X_1 \cup X_2$.

We can now repeat the arguments taking $X_1 \cup X_2$ and X_3 and after p_{n+1} steps we have embedded X into a real analytic manifold.

The theorem is now proved.

Let $(X, \mathcal{O}_X)$ be a real reduced analytic space, we shall denote by $C^\infty(X, \boldsymbol{R}^q)$, $(C^\omega(X, \boldsymbol{R}^q))$ the spaces of the C^∞, (analytic) maps of X into $\boldsymbol{R}^q$ endowed with the usual C^∞ topology.

We have

THEOREM 2. *Let $(X, \mathcal{O}_X)$ be a real reduced, paracompact, connected, analytic space of dimension n such that: $N = \sup_{x \in X} \dim \tau_x < +\infty$, $\tau_x =$ Zariski tangent space, $N \neq n$. In these hypotheses we have:*

(I) *The set of proper, analytic, one to one maps $\varphi\colon X \to \boldsymbol{R}^{2n+1}$ that are regular in the regular points of X is dense in $C^\infty(X, \boldsymbol{R}^{2n+1})$.*

(II) *The set of the proper embeddings $\varphi\colon X \hookrightarrow \boldsymbol{R}^{n+N}$ is dense in $C^\infty(X, \boldsymbol{R}^{n+N})$.*

PROOF. Let us suppose that X is a real analytic subspace of $\boldsymbol{R}^p$.

We remark that any C^∞ map $\varphi\colon X \to \boldsymbol{R}^s$ can be extended to a C^∞ map $\hat{\varphi}\colon \boldsymbol{R}^p \to \boldsymbol{R}^s$ and $\hat{\varphi}$ can be approached by analytic maps $\psi\colon \boldsymbol{R}^p \to \boldsymbol{R}^s$.

It follows that it is enough to prove the theorem for the space of analytic maps $\varphi\colon X \to \boldsymbol{R}^{2n+1}$ (or $\varphi\colon X \to \boldsymbol{R}^{n+N}$) that are restrictions of analytic maps defined on $\boldsymbol{R}^p$.

From theorem 1, the above remark and the fact that the cubes are Runge's sets in $\boldsymbol{C}^p$ it follows that the theorem is proved by

PROPOSITION 2. *Let $(X, \mathcal{O}_X)$ be a reduced, real analytic subspace of $\boldsymbol{R}^p \subset \boldsymbol{C}^p$ satisfying the hypotheses of theorem 2.*

Then the holomorphic maps $\tilde{\varphi}\colon \boldsymbol{C}^p \to \boldsymbol{C}^{2n+1}$ ($\tilde{\varphi}\colon \boldsymbol{C}^p \to \boldsymbol{C}^{n+N}$) *defined on* $\boldsymbol{R}$ *and such that*:

$\tilde{\varphi}|_X$ *is proper, one to one, regular in the regular points of* X (*an embedding*) *are dense in the space of the holomorphic maps* $\psi\colon \boldsymbol{C}^p \to \boldsymbol{C}^{2n+1}$ ($\psi\colon \boldsymbol{C}^p \to \boldsymbol{C}^{n+N}$) *defined on* $\boldsymbol{R}$.

PROOF. The proof is the same as in [7].

We write here some remarks to simplify the adaptation of Narasimhan's proof to our case.

1) The admissible systems defined in § 2 of [7] can be constructed taking the complement of a locally finite family of hyperplanes of $\boldsymbol{C}^p$.

2) $n+1$ admissible systems of $\boldsymbol{C}^p$ are enough, if well chosen, to cover X.

3) X, in general, has no good decomposition into irreducible components. Hence the argument of taking a point on any irreducible component of X (or of an analytic subspace) must be replaced in the following way: take a stratification of X and a point on any connected component of the strata of maximal dimension.

4) All the constructions of [7] can be adapted to holomorphic maps defined on $\boldsymbol{R}$ (for the details see [16]).

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