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KKM Maps and Variational Inequalities (1).

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dedicated to Jean Leray

In 1929, Knaster-Kuratowski-Mazurkiewicz [5], using the Sperner Lemma as a tool, established the following geometrical result:

Let X be the set of vertices of a simplex in $E = R^n$ and let $G: X \to 2^E$ be a compact-valued map such that $\operatorname{conv}\{x_1, \ldots, x_s\} \subset \bigcup_{i=1}^{s} G(x_i)$ for each subset $\{x_1, \ldots, x_s\} \subset X$. Then $\bigcap \{G(x) | x \in X\} \neq \emptyset$.

The significance of this type of result (beyond that of being simply a convenient «Lemma» for proving the Brouwer fixed-point theorem) was established by Ky Fan many years later. In 1961, Ky Fan [2] proved that the assertion of the Knaster-Kuratowski-Mazurkiewicz Theorem remains valid when X is replaced by an arbitrary subset of any Hausdorff topological vector space E, and (what is more important) he gave numerous applications of this generalization; since then, many more applications have been found (cf. [1], [3], for example) and use of his methods is now a standard tool in some fields.

The requirement that G be compact-valued is not always met in practice [1] and prevents a direct application of Ky Fan's theorem. In this note, we present a slight modification of his result, and a technique that helps avoid the difficulty with compactness. We illustrate the method by giving a direct proof of a fairly general form of the Hartman-Stampacchia theorem on variational inequalities.

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1. – KKM maps.

Let E be a vector space. The set of all subsets of E is denoted by 2^E and $\operatorname{conv}(A)$ will denote the convex hull of any $A \in 2^E$. An $A \in 2^E$ is called finitely closed if its intersection with each finite-dimensional flat $L \subset E$ is closed in the Euclidean topology of L; note that a set closed in any topology making E a topological vector space is necessarily finitely closed. A family $\{A_\lambda | \lambda \in \mathfrak{L}\}$ of sets is said to have the finite intersection property if the intersection of each finite subfamily is not empty.

1.1. Definition. Let E be a vector space and $X \subset E$ an arbitrary subset. A function $G: X \to 2^E$ is called KKM map if $\operatorname{conv}\{x_1, \ldots, x_s\} \subset \bigcup_{i=1}^{s} G(x_i)$ for each finite subset $\{x_1, \ldots, x_s\} \subset X$.

The following result differs slightly from Ky Fan's generalization of the Knaster-Kuratowski-Mazurkiewicz theorem, in that we require only that the sets $G(x)$ be finitely closed; the topology in E plays no role.

1.2. Theorem. *Let E be a vector space, X an arbitrary subset of E, and $G: X \to 2^E$ a* KKM *map, such that each $G(x)$ is finitely closed. Then the family $\{G(x) | x \in X\}$ of sets has the finite intersection property.*

Proof. We argue by contradiction, so assume that $\bigcap_1^n G(x_i) = \emptyset$. Working in the finite-dimensional flat L spanned by $\{x_1, \ldots, x_n\}$, let d be the Euclidean metric in L and $C = \operatorname{conv}\{x_1, \ldots, x_n\} \subset L$; note that because each $L \cap G(x_i)$ is closed in L, we have $d(x, L \cap G(x_i)) = 0$ if and only if $x \in \in L \cap G(x_i)$. Since $\bigcap_1^n L \cap G(x_i) = \emptyset$ by assumption, the function $\lambda: C \to R$ given by $c \mapsto \sum_1^n d(c, L \cap G(x_i))$ would not be zero for any $c \in C$ and we would then have a continuous $f: C \to C$ by setting

$$f(c) = \frac{1}{\lambda(c)} \sum_1^n d(c, L \cap G(x_i)) \cdot x_i .$$

By Brouwer's theorem, f would have a fixed point $c_0 \in C$. Letting

$$I = \{i | d(c_0, L \cap G(x_i)) \neq 0\} ,$$

the fixed point c_0 cannot belong to $\bigcup\{G(x_i)|i\in I\}$; however

$$c_0 = f(c_0) \in \operatorname{conv}\{x_i|i\in I\} \subset \bigcup\{G(x_i)\,|i\in I\}$$

and, with this contradiction, the proof is complete.

As an immediate consequence,

1.3. Corollary (Ky Fan). *Let E be a topological vector space, $X\subset E$ an arbitrary subset, and $G\colon X\to 2^E$ a* KKM *map. If all the sets $G(x)$ are closed in E, and if one is compact, then $\bigcap\{G(x)|x\in X\}\neq\emptyset$.*

We now observe that the conclusion $\bigcap G(x)\neq\emptyset$ can be reached in another way, which avoids placing any compactness restriction on the sets $G(x)$; it involves using an auxiliary family of sets and a suitable topology on E.

1.4. Corollary. *Let E be a vector space, X an arbitrary subset of E, and $G\colon X\to 2^E$ a* KKM *map. Assume there is a set-valued map $\Gamma\colon X\to 2^E$ such that $G(x)\subset\Gamma(x)$ for each $x\in X$, and for which*

$$\bigcap\{\Gamma(x)|x\in X\} = \bigcap\{G(x)|x\in X\}\,.$$

If there is some topology on E such that each $\Gamma(x)$ is compact, then $\bigcap\limits_{x\in X} G(x)\neq\emptyset$.

Because of 1.2 the proof is obvious.

2. – Application to variational inequalities.

Let E be a Banach space, $E^* = \mathcal{L}(E, R)$ its dual space, and $e\colon E^*\times E\to R$ the natural pairing map $(A, x)\mapsto A(x)$; we denote $A(x)$ by $\langle A, x\rangle$. Let C be any subset of E; a map $f\colon C\to E^*$ is called monotone on C if $\langle f(x) - f(y), x-y\rangle\geqslant 0$ for all $x, y\in C$. The following theorem is a fairly general version of one of the basic facts in the theory of variational inequalities [4].

2.1. Theorem (Hartman-Stampacchia). *Let E be a reflexive Banach space, C a closed bounded convex subset of E, and $f\colon C\to E^*$ monotone. Assume that $f|L\cap C$ is continuous for each one-dimensional flat $L\subset E$. Then there exists a $y_0\in C$ such that $\langle f(y_0), y_0 - x\rangle\leqslant 0$ for all $x\in C$.*

Proof. For each $x\in C$, let

$$G(x) = \{y\in C|\langle f(y), y-x\rangle\leqslant 0\}\ ;$$

the theorem will be proved by showing $\bigcap\{G(x)|x\in C\}\neq\emptyset$.

First, $G\colon C \to 2^E$ is a KKM map. Indeed, let $y_0 \in \operatorname{conv}\{x_1, \ldots, x_n\}$.

If $y_0 \notin \bigcup_{i=1}^{n} G(x_i)$, we would have $\langle f(y_0), y_0 - x_i \rangle > 0$ for each $i = 1, \ldots, n$; since all the x_i would therefore lie in the half-space $\{x \in E | \langle f(y_0), y_0 \rangle > \langle f(y_0), x \rangle\}$, so also would $\operatorname{conv}\{x_1, \ldots, x_n\}$, and we have the contradiction $\langle f(y_0), y_0 \rangle > > \langle f(y_0)\, y_0 \rangle$. Thus, G is a KKM map.

Consider now the map $\Gamma\colon C \to 2^E$ given by

$$\Gamma(x) = \{y \in C | \langle f(x), y - x \rangle \leqslant 0\};$$

we show that Γ satisfies the requirements of 1.4.

(i) $G(x) \subset \Gamma(x)$ for each $x \in C$. For, let $y \in G(x)$, so that $0 \geqslant \langle f(y), y - x \rangle$. By monotonicity of $f\colon C \to E^*$ we have $\langle f(y) - f(x), y - x \rangle \geqslant 0$ so $0 \geqslant \langle f(y), y - x \rangle \geqslant \langle f(x), y - x \rangle$ and $y \in \Gamma(x)$.

(ii) Because of (i), it is enough to show $\bigcap \{\Gamma(x) | x \in C\} \subset \bigcap \{G(x) | x \in C\}$. Assume $y_0 \in \bigcap \Gamma(x)$. Choose any $x \in C$ and let $z_t = tx + (1-t)y_0 \equiv y_0 - t \cdot \cdot (y_0 - x)$; because C is convex, we have $z_t \in C$ for each $0 \leqslant t \leqslant 1$. Since $y_0 \in \Gamma(z_t)$ for each $t \in [0, 1]$, we find that $\langle f(z_t), y_0 - z_t \rangle \leqslant 0$ for all $t \in [0, 1]$. This says that $t\langle f(z_t), y_0 - x \rangle \leqslant 0$ for all $t \in [0, 1]$ and, in particular, that $\langle f(z_t), y_0 - x \rangle \leqslant 0$ for $0 < t \leqslant 1$. Now let $t \to 0$; the continuity of f on the ray joining y_0 and x gives $f(z_t) \to f(y_0)$ and therefore that $\langle f(y_0), y_0 - x \rangle \leqslant 0$. Thus, $y_0 \in G(x)$ for each $x \in C$ and $\bigcap \Gamma(x) = \bigcap G(x)$.

(iii) We now equip E with the weak topology. Then C, as a closed bounded convex set in a reflexive space, is weakly compact; therefore each $\Gamma(x)$, being the intersection of the closed half-space $\{y \in E | \langle f(x), y \rangle \leqslant \langle f(x), x \rangle\}$ with C is, for the same reason, also weakly compact.

Thus, all the requirements in 1.4 are satisfied; therefore $\bigcap \{G(x) | x \in C\} \neq \emptyset$ and, as we have observed, the proof is complete.

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