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Variational Theory of Set-Valued Hammerstein Operators in Banach Function Spaces. The Eigenvalue Problem (*).

CHARLES V. COFFMAN (**)

1. – Introduction.

This paper is concerned with the eigenvalue problem for monotone, variational, set-valued Hammerstein operators. The methods that were used in [6] to treat the single-valued operator are extended here so as to be applicable to the set-valued operator; we prove a generalization of the main result of [6].

Let (Ω, Σ, μ) be a measure space and let g be a real-valued function on $\Omega \times R$ such that $g(\cdot, x)$ is μ-measurable for each $x \in R$ and $g(t, \cdot)$ is monotone for each $t \in \Omega$. Let g_+, g_- be the real-valued functions on $\Omega \times R$ defined by

$$g_+(t, x) = \lim_{\xi \downarrow x} g(t, \xi)\,, \qquad g_-(t, x) = \lim_{\xi \uparrow x} g(t, x)\,. \tag{1.1}$$

Finally, let k be a real-valued kernel defined on $\Omega \times \Omega$. By the value of the Hammerstein operator, denoted

$$x \to \int_\Omega k(\cdot, s)\, g\big(s, x(s)\big)\, d\mu\,, \tag{1.2}$$

on the function x, we shall understand the set of all functions of the form

$$y = \int_\Omega k(\cdot, s)\, h(s)\, d\mu$$

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where h is a real-valued, μ-measurable function satisfying

$$g_+(t, x(t)) \geqslant h(t) \geqslant g_-(t, x(t)) \quad \text{a.e. on } \Omega. \tag{1.3}$$

In the particular case, for example, where Ω is a smooth region in R^n, and k is the Green's function for

$$-\Delta u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

this set consists of all functions which have generalized derivatives of first and second order, vanish in a generalized sense on $\partial\Omega$, and satisfy

$$g_+(t, x(t)) \geqslant -\Delta y \geqslant g_-(t, x(t)), \quad \text{a.e. on } \Omega.$$

An *eigenfunction* of (1.2) is then understood to be a function x such that, for some real number λ,

$$x \in \lambda \int_\Omega k(\cdot, s) g(s, x(s))\, d\mu$$

i.e.

$$x = \lambda \int_\Omega k(\cdot, s) h(s)\, d\mu$$

where h satisfies (1.3). Suppose that for almost all $t \in \Omega$ g satisfies

$$xg(t, x) = -xg(t, -x) \geqslant 0$$

for all $x \in R \setminus \{0\}$ and that k is a symmetric, non-degenerate, non-negative definite kernel. The main result obtained here states that, under some additional compactness and continuity conditions, the operator (1.2) has an infinity of eigenfunctions.

We restrict our attention here to Hammerstein operators of the form (1.2) acting in a Banach function space X. The motivation for choosing to consider the operator acting in a Banach function space is the following. Let φ be the convex Nemytskii potential that is defined by

$$\varphi(x) = \int_\Omega \int_0^{x(t)} g(t, \xi)$$

and which we suppose to be defined and finite-valued on the Banach space X in which the Hammerstein operator is to be assumed to act. In order that the eigenfunctions of the Hammerstein operator in X whose existence we prove be eigenfunctions in the sense described above, rather than merely in some more generlized sense, it is essential that φ have as its subgradient at any $x \in X$ precisely the set of measurable functions that satisfy (1.3). If we take X to be a Banach function space, and if φ is defined and finite on X then its subgradient indeed has just such a characterization. To appreciate the problem that is involved here consider the case where $X = L^{\infty}(0,1)$, then the subgradient of φ at x is a set in $(L^{\infty}(0,1))^{*}$ and it is by no means immediately obvious that this is a set of functions; (in this particular case, however, the desired result is known, see, e.g. Rockafellar [19].)

The construction of the space H, in § 6, is equivalent to a construction used by Amann in [2] and by Browder and Gupta in [4], see also Lemma 5.1 [1]. The variant of this construction that is used here and in [6] results in a Hilbert space of functions H in which the Hammerstein operator acts and is a subgradient. In the partial differential equations case, e.g. in the example mentioned earlier, this construction leads to the Sobolev space naturally associated with the problem.

The topological lemma of § 8, which is essential to the proof of the main result, was suggested by Day's generalization [8] to set-valued functions of the Borsuk-Ulam theorem and the proof thereof, see also Jaworowski, [12].

For further references on the eigenvalue problems for Hammerstein operators, see the bibliography of [6]. An additional reference is Amann [1], whose results include a generalization of the main result of [6]. In particular, it is shown in [1] that the assumption of positive definiteness of the kernel that is made in [6] is not necessary. Eigenvalue problems for multiple-valued operators are treated in [9], [10], however the problems discussed there, unlike the problem treated here, admit transformation to an eigenvalue problem for a single-valued operator.

I wish to acknowledge a number of very helpful conversations with Professor V. J. Mizel concerning Banach function spaces and Nemytskii functionals. I am indebted to the referee for supplying simpler proofs than my original ones of Lemmas 4.1 and 7.1 and for bringing to my attention several relevant references of which I was not aware.

2. – The function space X.

Let (Σ, Ω, μ) be a complete σ-finite non-atomic measure space. In what follows this space will be regarded as fixed and all measure-theoretic notions, such as almost everywhere, measurable, integrable, etc., except where explicitly stated otherwise, are to be understood as with respect to μ. M will denote the set of equivalence classes, modulo difference on a μ-null set, of real-valued measurable functions on Ω. We shall not distinguish between an element of M and a function which is a representative of it. The natural order on M will be denoted by « $\geqslant$ », i.e. for $x, y \in M$

$$x \geqslant y \quad \text{iff} \quad x(t) \geqslant y(t) \qquad \text{a.e. on } \Omega .$$

If $x \in M$, $|x|$ denotes the measurable function defined by

$$|x|(t) = |x(t)| \qquad \text{a.e. on } \Omega .$$

Finally

$$M^+ = \{x \in M : x \geqslant 0\} .$$

In what follows X will denote a Banach space with norm $\|...\|$, whose element are members of M, and which has the following properties.

(A) If $E \in \Sigma$ and $\mu(E) \neq 0$ then there exists $F \in \Sigma$ such that $F \subset E$, $\mu(F) \neq 0$ and $\chi_F \in X$, where χ_F is the characteristic function of F. (The assumption that X is a Banach space whose elements are members of M implies that $\|\chi_F\| \neq 0$.)

(B) If $x \in X$ then $y \in M$ and $-|x| \leqslant y \leqslant |x|$ implies $y \in X$ and $\|y\| \leqslant \|x\|$.

(C) The unit ball in X is closed under monotone convergence, i.e. if $\{x_n\}$ is a sequence in X with

$$\|x_n\| \leqslant 1 , \quad x_n \leqslant x_{n+1} , \qquad n = 1, 2, ...$$

then $x_0 \in X$ and $\|x_0\| \leqslant 1$ where

$$x_0(t) = \lim_{n\to\infty} x_n(t) \qquad \text{a.e. on } \Omega .$$

These assumptions concerning X are equivalent to the assumption that X is a real Banach function space with the strong Fatou property, see [13], [15]; more specifically, $x = L_\varrho$, with the underlying measure space (Ω, Σ, μ) and where ϱ is the function norm defined on M^+ by

$$\varrho(x) = \begin{cases} \|x\| , & x \in M^+ \cap X \\ \infty , & x \in M^+ \setminus X . \end{cases}$$

For standard notions in the theory of Banach function spaces, e.g. *associate space, absolutely continuous norm,* the reader is referred to [13], [15]. Here we shall identify the associate space X' of X with the subspace of X^* which is naturally isometrically isomorphic to it. The normal subspace of X consisting of elements of absolutely continuous norm is denoted X_a.

As indicated, $\|...\|$ will be used to denote the norm in X; for other normed spaces we shall in most cases use double-bars with the symbol for the space as subscript to denote the norm, *e.g.* the norm in X' will be denoted $\|...\|_{X'}$. For operator norms, however, when the domain and codomain are specified we shall use double bars without a subscript. When two distinct Banach spaces are in duality, $(\cdot,\cdot)$ will be used to denote the duality pairing, e.g., for $x \in X$, $\xi \in X'$

$$(x, \xi) = \int_\Omega x(t)\xi(t)\, d\mu\ ;$$

the lack of explicit distinction between such pairings for different pairs of spaces, even though several may occur in the same computation, should not cause confusion. For the inner product in a Hilbert space we shall use $\langle\cdot,\cdot\rangle$. For an arbitrary, i.e. not necessarily continuous linear functional ξ on the space X we will use $\xi(x)$ to denote the value of ξ at x.

Let Y be a Banach function space and T a bounded linear operator from Y to X. If the range of $T^*|X'$ lies in Y', the associate space of Y, we shall say that $T' = T^*|X'$ is the *associate* of T. We shall also say that $T^*|X'$ and T are associates when T is a bounded linear operator from H to X where H is a Hilbert space whose elements are equivalence classes of measurable functions.

We shall denote by (X', X) the space X' furnished with the weak X-topology; (X', X) is sequentially complete [13], [15], therefore, if X is separable then the unit ball in X' is sequentially compact in (X', X). The unit ball in X' is also sequentially compact in (X', X) when X has absolutely continuous norm [15].

The following lemmas contain the properties of X that are required in the sequel; these hold merely by virtue of the fact that X is a real Banach function space. (In particular we need not have assumed that X has the strong Fatou property, neither for our purposes, is there any loss of generality in doing so.)

LEMMA 2.1. *Let $\{x_n\}$ be a convergent sequence in X with limit x_0. Then there exists an element $y \in X$ and a subsequence $\{x_{n_k}\}$ of the original sequence such that*

$$|x_0|, \quad |x_{n_k}| \leqslant y, \qquad k = 1, 2, \ldots$$

and

$$x_0(t) = \lim_{k\to\infty} x_{n_k}(t)\,, \qquad \text{a.e. } on\ \Omega\,.$$

LEMMA 2.2. (Luxemburg) *Let* $\xi \in X^*$. *Then* $\xi \in X'$ *if and only if*

$$\lim_{n\to\infty} (x_n, \xi) = 0 \tag{2.1}$$

for every sequence $\{x_n\}$ *in* X *such that*

$$0 \leqslant x_{n+1} \leqslant x_n\,, \qquad n = 1, 2, \ldots \tag{2.2}$$

and

$$\lim_{n\to\infty} x_n(t) = 0 \text{ a.e. } on\ \Omega\,. \tag{2.3}$$

For the proof that a convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ that converges almost everywhere see the proof of Theorem 4.8 [14]. It can be assumed further that

$$\sum_{k=1}^{\infty} \|x_{n_k} - x_0\| < \infty$$

and then we can take

$$y = |x_0| + \sum_{k=1}^{\infty} |x_{n_k} - x_0|$$

which, because of (C), will belong to X.

The proof of Lemma 2.2 can be found in [13] or [15].

From Lemma 2.2 and the remarks above concerning absolutely continuous norm we immediately obtain the following result.

COROLLARY 2.1. *If* X *has absolutely continuous norm then* $X' = X^*$. *If both* X *and* X' *have absolutely continuous norm then* X *is reflexive.*

3. – Nemytskii functionals.

A real valued function f defined on $\Omega \times R$ is said to satisfy the *Carathéodory conditions* or to be a *Carathéodory function* on $\Omega \times R$ if $f(\cdot, x)$ is measurable for each $x \in R$ and $f(t, \cdot)$ is continuous for all $t \in \Omega$. (It is convenient and involves no real loss of generality to assume that $f(t, \cdot)$ is defined and continuous for *all* $t \in \Omega$ rather than only for almost all $t \in \Omega$.) In what follows here « Carathéodory function » will always be understood to mean « Carathéodory function on $\Omega \times R$ ».

If f is a Carathéodory function, $A \subset M$ and $f(\cdot, x(\cdot)) \in L^1(\Omega, \Sigma, \mu)$ whenever $x \in A$ then φ, defined on A by

$$\varphi(x) = \int_\Omega f(t, x(t))\, d\mu\,, \tag{3.1}$$

is said to be a *Nemytskii functional* on A.

LEMMA 3.1. *Let f be a convex Carathéodory function and suppose that $f(\cdot, x(\cdot)) \in L^1(\Omega, \Sigma, \mu)$ whenever $x \in X$. Then the Nemytskii functional φ on X defined by* (3.1) *is:*

(*a*) *continuous with respect to dominated convergence in X*, i.e.

$$\lim_{n\to\infty} \varphi(x_n) = \varphi(x_0) \tag{3.2}$$

if $\{x_n\}$ is a sequence in X, and there exist x_0, $y \in X$ such that

$$x_0(t) = \lim_{n\to\infty} x_n(t) \qquad \text{a.e. } on\ \Omega\,, \tag{3.3}$$

and

$$|x_0|\,, \quad |x_n| \leqslant y\,, \qquad n = 1, 2, \ldots,$$

(*b*) *continuous with respect to monotone convergence in X*, i.e.

$$x_{n+1} \leqslant x_n\,, \quad n = 1, 2, \ldots,$$

and (3.3) *for $x_0 \in X$ imply* (3.2),

(*c*) *continuous with respect to the X-topology.*

PROOF. If $|x| \leqslant y$, the convexity of $f(t, \cdot)$ implies that

$$|f(t, x)| \leqslant |f(t, y)| + |f(t, -y)| + 2|f(t, 0)|$$

and thus (*a*) follows from the Lebesgue dominated convergence theorem. Continuity with respect to monotone convergence is an immediate consequence of continuity with respect to dominated convergence.

To prove the last continuity assertion it suffices to show that if $\{x_n\}$ is any convergent sequence in X, say with limit x_0, then every subsequence of $\{x_n\}$ contains a subsequence such that the corresponding sequence of the values of φ converges to $\varphi(x_0)$. This however follows immediately from Lemma 2.1 and continuity with respect to dominated convergence.

REMARK. When X is an L^p space then one can also conclude from the hypothesis of Lemma 3.1 that φ is bounded on bounded subsets on X, see Krasnosel'skii [17]. In the general case, when φ is defined and finite on the space X, then a sufficient condition for φ to be bounded on bounded sets is that f satisfy the Δ_2'-condition of Portnov [18], i.e. that there exist a non-negative function $g \in L^1(\Omega, \Sigma, \mu)$ and a constant C such that for all $x \in R$,

$$|f(t, 2x)| \leqslant C|f(t, x)| + g(t) \qquad \text{a.e. on } \Omega .$$

LEMMA 3.2. *Suppose that X has absolutely continuous norm. If there exists an $a > 0$ such that*

$$f(\cdot, x(\cdot)) \in L^1(\Omega, \Sigma, p) \tag{3.4}$$

whenever $x \in X$ and $\|x\| \leqslant a$, then in fact [3.4) *holds for all $x \in X$.*

PROOF. Since X has absolutely continuous norm and (Ω, Σ, p) is non-atomic, then given any $x \in X$, Ω can be partitioned, say as follows,

$$\Omega = E_1 \cup E_2 \cup \ldots \cup E_n$$

where $E_i \in \Sigma$, $i = 1, \ldots, n$, $E_i \cup E_j = \emptyset$, $i \neq j$, so that

$$\|\chi_{E_i} x\| \leqslant a , \qquad i = 1, 2, \ldots, n .$$

The assertion then follows immediately.

REMARK. This is the only place where the non-atomicity of the measure space is used.

Suppose that $f(t, \cdot)$ is convex for all $t \in \Omega$ and satisfies

$$f(t, 0) = 0$$

$$f(t, x) > 0 \qquad \text{for } x \neq 0$$

$$f(t, x) = f(t, -x)$$

and finally that for any $E \in \Sigma$ with $\mu(E) \neq 0$ there exists an $F \in \Sigma$ and an $\varepsilon > 0$ such that $F \subset E$, $\mu(F) \neq 0$ and $f(\cdot, \varepsilon x_E) \in L^1(\Omega, \Sigma, \mu)$. Let $Z_1 = \{x \in M$: for some $\tau > 0$, $f(\cdot, \tau^{-1}x(\cdot)) \in L^2(\Omega, \Sigma, \mu)\}$, and for $x \in Z_1$ put

$$\|x\|_{Z_1} = \inf\left\{\tau : \int_\Omega f(t, \tau^{-1}x(t))\, dt \leqslant 1\right\} .$$

Then Z_1 with the norm $\|\ldots\|_{Z_1}$ is a real Banach function space with the strong

Fatou property, as follows readily from our assumptions concerning f. We now put

$$Z = Z_{1,a}\,,$$

i.e. Z is the normal subspace of Z_1 that consists of functions of absolutely continuous norm. Since Z is a Banach function space with absolutely continuous norm and (3.4) holds for all $x \in Z$ with $\|x\|_Z \leqslant 1$ it follows from Lemma 3.2 that (3.4) holds for all $x \in Z$.

LEMMA 3.3. *Suppose that X is as above and* (3.4) *holds for all $x \in X$, then X is stronger than Z,* i.e. *X is contained in Z algebraically and the inclusion mapping $X \subset Z$ is bounded.*

The above discussion shows how a Banach function space having absolutely continuous norm and satisfying (3.4) can be constructed when f is given. Lemma 3.3 shows moreover that this space is the weakest Banach function space among those that satisfy (3.4). Note that the last assumption above concerning f is necessarily satisfied if there exists any space X satisfying the hypothesis of Lemma 3.3. Since Lemma 3.3 is not used explicitely in what follows its proof will be omitted.

Finally, in connection with the remark following the proof of Lemma 3.1 we note that the following are equivalent:

a) φ is bounded on bounded sets in Z,

b) $Z = Z_1$,

c) f satisfies the Δ_2'-condition.

For the equivalence of *b*) and *c*) see Portnov [19].

4. – Subgradients of convex Nemytskii functionals.

Let g be a real-valued function on $\Omega \times R$ such that $g(\cdot, x)$ is measurable for each $x \in R$ and $g(t, \cdot)$ is monotone nondecreasing for all t. Let g_+, g_- be the functions on $\Omega \times R$ that are defined by (1.1).

Let

$$f(t, x) = \int_0^x g(t, \xi)\, d\xi = \int_0^x g_+(t, \xi)\, d\xi = \int_0^x g_-(t, \xi)\, d\xi\,, \tag{4.1}$$

so that f is a Carathéodory function on $\Omega \times R$ with

$$f(t, 0) = 0\,, \quad \text{all } t \in \Omega\,,$$

and $f(t, \cdot)$ is convex for all $t \in \Omega$.

Let f be as above, let φ be given by (3.1). We shall assume throughout this section that φ is defined and finite on X, i.e. that $f(\cdot, x(\cdot)) \in L^1(\Omega, \Sigma, \mu)$ for all $x \in X$. The functional φ is then a convex functional on X and satisfies

$$\varphi(0) = 0\,. \tag{4.2}$$

The *subgradient* $\partial\varphi(x)$ of φ at x is defined as follows:

$$\partial\varphi(x) = \{\xi \in X^*: \forall y \in X,\, \varphi(y) - \varphi(x) \geqslant (y - x, \xi)\}\,.$$

LEMMA 4.1. *For any* $x \in X$,

$$\partial\varphi(x) = \{\xi \in M: g_-(t, x(t)) \leqslant \xi(t) \leqslant g_+(t, x(t)) \text{ a.e. on } \Omega\} \subset X'\,.$$

PROOF. Let $\xi \in M$ satisfy

$$g_-(t, x(t)) \leqslant \xi(t) \leqslant g_+(t, x(t)) \quad \text{a.e. on } \Omega\,.$$

For $y \in M$ we have

$$f(t, y(t)) - f(t, x(t)) \geqslant \xi(t)(y(t) - x(t)) \quad \text{a.e. on } \Omega\,.$$

Since φ is continuous it is bounded on the ball $\{y: \|y - x\|_X \leqslant r\}$ for some $r > 0$. Upon taking $y = x \pm z$ we see that $\int_\Omega |\xi(t) z(t)| d\mu$ is uniformly bounded on the ball $\{z: \|z\|_X \leqslant r\}$. Thus $\xi \in X'$ and $\varphi(y) - \varphi(x) \geqslant (y - x, \xi)$ i.e. $\xi \in X' \cap \cap\, \partial\varphi(x)$.

Conversely, if $\xi \in \partial\varphi(x)$ then we conclude from the definition of $\partial\varphi(x)$ and Lemma 3.1 that a sequence $\{x_n\}$ satisfying (2.2) and (2.3) satisfies (2.1) and thus, by Lemma 2.2, $\xi \in X'$. Now let $E \in \Sigma$ with $\chi_E \in X$, then for $\lambda > 0$

$$\int_\Omega \lambda^{-1}[f(t, x(t) + \lambda\chi_E(t)) - f(t, x(t))]\, d\mu \geqslant \int_E \xi(t)\, d\mu\,.$$

As $\lambda \downarrow 0$ the integrand on the left decreases to the limit $g_+(t, x(t))\, \chi_E(t)$ and therefore by the monotone convergence theorem we have

$$\int_E g_+(t, x(t))\, d\mu \geqslant \int_E \xi(t)\, d\mu\,.$$

Similarly, we get

$$\int_E g_-(t, x(t))\, d\mu \leqslant \int_E \xi(t)\, d\mu .$$

From condition (A) of § 2 it follows that

$$g_-(t, x(t)) \leqslant \xi(t) \leqslant g_+(t, x(t)) \qquad \text{a.e. on } \Omega .$$

This completes the proof of the lemma.

LEMMA 4.2 (Moreau [17]). *For each $x \in X$, $\partial\varphi(x)$ is a closed, bounded, convex subset of X'. The set-valued mapping $x \to \partial\varphi(x)$ is upper semi-continuous from X to (X', X).*

We recall that a set-valued function f from a topological space S to a topological space T is *upper semi-continuous* [3], if

$$\{x : f(x) \subset U\}$$

is open in S for every open set U in T.

5. – Compact symmetric integral operators on X'.

By a symmetric integral operator on X' we shall understand an operator A from X' to X of the form

$$(5.1) \qquad Ax = \int_\Omega k(\cdot, s)x(s)\, ds , \qquad \xi \in X' ,$$

where k is a $(\mu\times\mu)$-measurable function on $\Omega\times\Omega$,

$$k(t, s) = k(s, t) ,$$

$k(t, s)\xi(t)\eta(s)$ is $(\mu\times\mu)$-integrable over $\Omega\times\Omega$ for each pair ξ, $\eta \in X'$ and

$$\sup\left\{ \int_{\Omega\times\Omega} k(t, s)\xi(t)\eta(s)\, d(\mu\times\mu) : \xi, \eta \in X',\ \|\xi\|_{X'},\ \|\eta\|_{X'} < 1\right\} < \infty .$$

The last two conditions are obviously necessary and sufficient in order that the integral operator A act from X' to X. Because of the symmetry of the kernel k, the operator $A : X' \to X$ satisfies

$$(A\xi, \eta) = (A\eta, \xi) \qquad \text{all } \xi, \eta \in X' ,$$

A is therefore *self-associate*; it is *self-adjoint* if and only if X is reflexive.

It is proved in [15] that if $AX' \subset X_a$ then a necessary and sufficient condition for A to be compact is that the image of the unit ball in X' under A be of uniformly absolutely continuous norm in X. A sufficient condition for A to map X' into X_a and satisfy the above condition is that k have absolutely continuous norm in the Banach function Γ of functions on $\Omega \times \Omega$ with norm

$$\|\gamma\|_\Gamma = \sup\left\{ \int_{\Omega\times\Omega} |\gamma(t, s)\xi(t)\eta(s)| d(\mu\times\mu) : \xi, \eta \in X',\ \|\xi\|_{X'},\ \|\eta\|_{X'} \leqslant 1 \right\}.$$

In the case $X = L^p(\Omega, \Sigma, \mu)$, $2 \leqslant p < \infty$, $\mu(\Omega) < \infty$, we have, as a special case of the result of [15] quoted above, that $A: X' \to X$ is compact if

$$\int_\Omega |k(\cdot, s)|^a d\mu \in L^\infty(\Omega, \Sigma, \mu)$$

for some $a > p/2$.

Finally we note that so far as the condition $AX' \subset X_a$ is concerned there is, in fact, in view of Lemma 3.3, no loss of generality in assuming that $X = X_a$.

6. – The intermediate space H.

Let A be, as in § 5, a symmetric integral operator from X' to X. Assume moreover that A is compact and non-negative definite, i.e. that

$$\int_{\Omega\times\Omega} k(t, s)\xi(t)\xi(s) d(\mu\times\mu) \geqslant 0$$

for all $\xi \in X'$. Let

$$\mathfrak{R} = \text{range } A \subset X,$$

and for $x, y \in \mathfrak{R}$, say

$$x = A\xi, \quad y = A\eta, \quad \xi, \eta \in X'$$

let

$$\begin{aligned} \langle x, y\rangle &= (A\xi, \eta) \\ &= (A\eta, \xi). \end{aligned}$$

Since A is non-negative definite

$$\langle x, x\rangle = (A\xi, \xi) = 0$$

for $x = A\xi \in \mathfrak{R}$ if and only if $A\xi = 0$. Thus $\langle \cdot, \cdot \rangle$ is a positive definite symmetric bilinear form on $\mathfrak{R}$. Let H denote the Hilbert space that is obtained by completing $\mathfrak{R}$ with respect to $\langle \cdot, \cdot \rangle$ and let $B\colon X' \to H$ be the operator that assigns to $\xi \in X'$ the element $A\xi$ considered as a member of H. For $\xi \in X'$ we have

$$\begin{aligned}\|B\xi\|_H^2 &= \langle B\xi, B\xi\rangle \\ &= (A\xi, \xi) \\ &\leqslant \|A\| \, \|\xi\|_{X'}^2 .\end{aligned}$$

Therefore

$$\|B\| \leqslant \|A\|^{\frac{1}{2}} .$$

Since B clearly has dense range in H, its adjoint

$$B^*\colon H \to (X')^*$$

is injective. Moreover, for $\xi, \eta \in X'$, $y = A\eta$,

$$(\xi, B^* y) = \langle B\xi, y\rangle = (\xi, A\eta) ,$$

so

$$B^* y = A\eta$$

i.e. B^* agrees on $\mathfrak{R}$ with the inclusion mapping $R \subset X$ (where X is regarded as a subspace of $(X')^*$). Since B^* is bounded and injective we can identify the elements of H with elements of X, i.e. we can regard H as being algebraically contained in the function space X and then the inclusion mapping $i\colon H \subset X$ is the associate of the mapping $B\colon X' \to H$ and $A = iB$.

LEMMA 6.1. *The operators B and i are compact.*

PROOF. Let $\{\xi_n\}$ be a bounded sequence in X'. Since A is compact we can choose a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that $\{A\xi_{n_i}\}$ is convergent. But then

$$\begin{aligned}\|B(\xi_{n_i} - \xi_{n_j})\|_H^2 &= \langle B(\xi_{n_i} - \xi_{n_j}), B(\xi_{n_i} - \xi_{n_j})\rangle \\ &= \big(A(\xi_{n_i} - \xi_{n_j}), (\xi_{n_i} - \xi_{n_j})\big)\end{aligned}$$

so that $\{B\xi_{n_i}\}$ is a Cauchy sequence in H. Since $\{\xi_n\}$ was an arbitrary bounded sequence in X' it follows that B is compact. By standard results it follows that B^* is compact and hence i is compact also.

REMARK. By using an alternate construction, it can be shown that the space H does not depend on the space X.

7. – The potential ψ.

Let φ be a Nemytskii functional which satisfies the hypothesis of Lemma 3.1, i.e. which is defined and finite on all of X. With H as in § 6, we define the potential ψ on H by

$$\psi(y) = \varphi(iy)\,, \qquad y \in H\,, \tag{7.1}$$

where, as before, i denotes the inclusion $H \subset X$.

LEMMA 7.1. *The potential ψ is weakly sequentially continuous on H.*

PROOF. Immediate from Lemma 6.1.

LEMMA 7.2. *For $y_0 \in H$,*

$$\partial\psi(y_0) = B\,\partial\varphi(iy_0)\,.$$

PROOF. By Lemma 4.1, $\partial\varphi(iy_0) \subset X'$ and by the results of section 6, the restriction to X' of i^* is B; the result then follows from [11, Prop. 5.7, p. 27].

8. – The genus, and a topological lemma.

Let N be a normed linear space and let $\mathcal{G} = \mathcal{G}(N)$ be the class of subsets of $N \setminus \{0\}$ which are closed in N and invariant under the involution

$$x \to -x \qquad N \to N\,.$$

The *genus* $\gamma(G) = \gamma_N(G)$ of an element $G \in \mathcal{G}$ is zero if G is empty; otherwise, $\gamma(G)$ is the supremum of the set of integers n such that every odd continuous map $f\colon G \to R^{n-1}$ has a zero on G. Following are the relevant properties of the genus; in these statements G, with or without subscript, will denote an element of $\mathcal{G}$.

1) If there exists an odd continuous map $f\colon G_1 \to G_2$, in particular if $G_1 \subset G_2$, then $\gamma(G_1) \leqslant \gamma(G_2)$.

2) $\gamma(G_1 \cup G_2) \leqslant \gamma(G_1) + \gamma(G_2)$.

3) If G is compact then $\gamma(G) < \infty$ and G has a neighborhood U with $\overline{U} \in \mathcal{G}$ and $\gamma(\overline{U}) = \gamma(G)$.

4) If $\{G_n\}$ is a decreasing sequence of compact sets then

$$G = \bigcap_{n=1}^{\infty} G_n \in \mathcal{G}$$

and

$$\gamma(G) = \lim_{n\to\infty} \gamma(G_n) .$$

5) If there exists an odd homeomorphism of the n-sphere onto G then

$$\gamma(G) = n + 1 .$$

For proofs of the above properties see [5].

The first result of this section is a partial generalization to set-valued mappings of property 1 of the genus.

LEMMA 8.1. *Let M and N be normed linear spaces and let f be a set-valued mapping from M to N such that*:

i) *f is odd*, i.e.

$$f(x) = - f(- x) ,$$

ii) *$f(x)$ is a compact convex subset of N for each $x \in M$,*

iii) *f is upper semi-continuous*

iv) *$0 \notin f(x)$ for $x \neq 0$.*

Then for any compact set G, with $G \in \mathcal{G}(M)$, $f(G) \in \mathcal{G}(N)$ and

$$\gamma_M(f(G)) \geqslant \gamma_N(G) .$$

PROOF. It follows readily from the hypothesis that $G_1 = f(G)$ is compact and belongs to $\mathcal{G}(N)$. Let G_2 be a closed neighborhood of G_1 which belongs to $\mathcal{G}(N)$ and is such that $\gamma(G_2) = \gamma(G_1)$. For each $x \in G$ let U_x be a convex open neighborhood of $f(x)$ with $U_x \subset G_1$. Let V_x be a convex open neighborhood of x such that

$$f(G \cap V_x) \subset U_x .$$

Finally, let ε_0 be the Lebesgue number of the covering $\{V_x : x \in G\}$ of G. Let $x_1, \ldots, x_{2m}$ be chosen in G so that

$$x_{i+m} = - x_i , \quad i = 1, 2, \ldots, m ,$$

$x_1, \ldots, x_m$ are in general position and

$$G \subset \bigcup_{i=1}^{2m} \left\{x\colon \|x - x_i\|_N < \frac{\varepsilon}{4}\right\} = W\,,$$

where ε is chosen so small that $0 < \varepsilon \leqslant \varepsilon_0$ and $0 \notin W$. Let S be the simplicial complex in M whose vertices are $x_1, \ldots, x_{2m}$ and such that $x \in M$ belongs to S if and only if

$$x = \sum_{i=1}^{2m} \alpha_i x_i$$

where

$$\alpha_i \geqslant 0\,, \qquad i = 1, \ldots, 2m\,, \qquad \sum_{i=}^{2m} \alpha_i = 1$$

and

$$\alpha_i \alpha_j > 0 \qquad \text{implies} \qquad \|x_i - x_j\|_N < \frac{\varepsilon}{2}\,.$$

Let $h(t)$ be a real valued function which is defined, continuous and non-negative on $[0, \infty)$ and satisfies

$$h(t) > 0 \qquad \text{on} \qquad \left[0, \frac{\varepsilon}{4}\right),\ h(t) = 0\,,\ t \geqslant \frac{\varepsilon}{4}\,.$$

Define an odd mapping $T\colon G \to M$ by

$$Tx = \sum_{i=1}^{2m} h(\operatorname{dist} x, x_i) x_i \Big/ \sum_{i=1}^{2m} h(\operatorname{dist} x, x_i)\,;$$

it can be verified immediately that $TG \subset S$. We next define an odd simplicial map

$$f_1\colon S \to N$$

by choosing $f_1(x_i)$ in $f(x_i)$ for $i = 1, \ldots, m$, putting $f_1(x_{m+i}) = -f_1(x_i)$, $i = 1, \ldots, m$, and extending f_1 to S by linearity. All of the vertices of a simplex σ of S lie in a ball of radius ε, hence in one of the V_x. Consequently, $f_1(\sigma)$ lies in the corresponding U_x. Thus $f_1\colon S \to G_2$ and therefore

$$\gamma_N(G_1) = \gamma_N(G_2) \geqslant \gamma_M(S) \geqslant \gamma_M(TG) \geqslant \gamma_M(G)\,,$$

and the lemma is proved.

9. – Variational problems.

We now turn to the central problem of this paper, namely the question of existence of eigenfunctions of the Hammerstein operator

$$x \to \int_{\Omega} k(\cdot, s) g(s, x(s))\, d\mu \,.$$

Following is a summary of the assumptions and notations introduced earlier and which will be implicit below. The function g is as in § 4, f is given by (4.1) and it is assumed that φ, defined by (3.1), is defined and finite on the Banach function space X. The kernel k is as in § 5, i.e. k determines a compact, non-negative definite, self-associate operator from X' to X. The space H and the operators B and i are as constructed in § 6 and, finally, ψ is defined by (7.1). We now make the following further assumptions concerning g, namely, for all $x \in R\setminus\{0\}$

$$xg(s, x) = -xg(s, -x) > 0 \qquad \text{a.e. on } \Omega \,.$$

We also exclude the case in which A has finite dimensional range and H is finite dimensional. The assumptions lead immediately to the conclusion that φ, hence also ψ, is even and vanishes only at 0, where we have used the injectivity of i; in particular, this implies

$$0 \notin \partial\varphi(x) \qquad \text{for } x \neq 0 \,.$$

We now consider the two dual variational problems of determining the critical values of ψ relative to the side condition $\|x\|_H = \alpha > 0$, and of determining the critical values of $\|\ldots\|_H$ subject to the side condition $\varphi(x) = \beta > 0$. We shall exhibit two variational principles, a maximum-minimum principle and a minimum-maximum principle, which yield, respectively, critical values of these two problems. These can be more conveniently formulated if we first introduce the following notations. We let $\mathcal{G}(\alpha)$, $\alpha > 0$, denote the collection of subsets G of $H\setminus\{0\}$ which are weakly compact, invariant under

$$x \to -x \,,$$

and such that

$$x \in G \qquad \text{implies } \|x\|_H \leqslant \alpha \,.$$

We let $\mathcal{G}'(\beta)$ denote the collection of subsets G of $H\diagdown\{0\}$ which are weakly compact, invariant under

$$x \to -x$$

and such that

$$x \in G \quad \text{implies } \psi(x) \geqslant \beta .$$

Next we put

$$\mathcal{G}_n(\alpha) = \{G \in \mathcal{G}(\alpha) \colon \gamma(iG) \geqslant n\}$$

$$\mathcal{G}'_n(\beta) = \{G \in \mathcal{G}'(\beta) \colon \gamma(iG) \geqslant n\} ;$$

here $\gamma(iG) = \gamma_X(iG)$ is the genus of iG in X, or less formally, the genus of G regarded as a subset of X; note that this is always finite. Finally, if G is a weakly compact subset of H,

$$\Psi(G) = \min\{\psi(x) \colon x \in G\} ,$$

$$\Delta(G) = \operatorname{diam} G .$$

We now define sequences $\{\mu_n(\alpha)\}$, $\{\mu'_n(\beta)\}$ as follows:

$$\mu_n(\alpha) = \sup\{\Psi(G) \colon G \in \mathcal{G}_n(\alpha)\} ,$$

$$\mu'_n(\beta) = \inf\{\tfrac{1}{2}\Delta(G) \colon G \in \mathcal{G}'_n(\beta)\} .$$

THEOREM 9.1. *The sequences $\{\mu_n(\alpha)\}$ and $\{\mu'_n(\beta)\}$ are, respectively, non-increasing and non-decreasing sequences of positive numbers with*

$$\lim_{n\to\infty} \mu_n(\alpha) = 0 , \qquad \lim_{n\to\infty} \mu'_n(\beta) = \infty .$$

Corresponding to each $\mu_n(\alpha)$ there is an $x \in X$ with

$$\|x\|_H = \alpha , \quad \psi(x) = \mu_n(\alpha) \tag{9.1}$$

such that

$$x \in \lambda\partial\psi(x) \tag{9.2}$$

for some real λ, and corresponding to each $\mu'_n(\beta)$ there is a $y \in H$ with

$$\|y\|_H = \mu'_n(\beta) , \quad \psi(y) = \beta \tag{9.3}$$

and such that

$$y \in \omega \partial \psi(y) \tag{9.4}$$

for some real ω. *If* $\mu_n(\alpha) = \mu_{n+k}(\alpha)$ *for some* $k > 0$, *then the set of* $x \in H$ *for which* (9.1), (9.2) *hold is a set of genus not less than* $k+1$. *Similarly, if* $\mu_n(\beta) = \mu_{n+k}(\beta)$ *for some* $k > 0$ *then the set of* $y \in H$ *for which* (9.3), (9.4) *hold is a set of genus not less than* $k+1$. (Since a set of elements satisfying (9.1), (9.2) or (9.3), (9.4) is compact in H its genus as a subset of H is the same as its genus when it is regarded as a subset of X).

PROOF. From (7.1) it follows that ψ is both continuous and, since i is compact, weakly sequentially continuous, thus bounded on bounded sets. From convexity and the fact that $\psi(x) > 0$ except for $x = 0$ it follows that $\psi(\tau x)$ varies from 0 to ∞ as τ varies from 0 to ∞ when $x \neq 0$. Using these facts and property 5 of the genus we easily conclude, since H is assumed to be infinite dimensional, that when $\alpha, \beta > 0$ and n is any positive integer then $\mathcal{G}_n(\alpha)$ and $\mathcal{G}'_n(\alpha)$ are non-empty and $\mu_n(\alpha)$, $\mu'_n(\beta)$ are finite positive numbers. The monotonicity of the two sequences $\{\mu_n(\alpha)\}$, $\{\mu'_n(\beta)\}$ is an immediate consequence of the definitions.

The remainder of the proof will be carried out only for the first of our two variational problems; the proof for the second is similar. We begin by showing that the suprema in the definition of the $\mu_n(\alpha)$ are attained. To this end let the natural number n be given and suppose

$$\mu_n(\alpha) > c > 0 .$$

Put

$$S(\alpha, c) = \{x \in H : \psi(x) \geqslant c, \ \|x\|_H \leqslant \alpha\} ,$$

and note that $S(\alpha, c)$ is weakly compact and moreover, $S(\alpha, c) \in \mathcal{G}(\alpha)$. In fact, if

$$m = \gamma\big(iS(\alpha, c)\big)$$

then

$$\mu_k(\alpha) < c \qquad \text{for } k > m ,$$

this leads immediately to a proof that

$$\lim_{n\to\infty} \mu_n(\alpha) = 0 .$$

Let

$$\mathcal{G}(\alpha, c) = \{G \in \mathcal{G}(\alpha) : G \subset S(\alpha, c)\} .$$

We furnish $\mathcal{G}(\alpha, c)$ with a metric topology by defining the distance $\varrho(G_1, G_2)$ between two sets $G_1, G_2 \in \mathcal{G}(\alpha, c)$ to be the Hausdorff distance between iG_1 and iG_2 induced by the X-norm, i.e.

$$\varrho(G_1, G_2) = \inf \{r: G_2 \subset \{x + y: x \in G_1, \|iy\| \leqslant r\} \text{ and } \{x + y: x \in G_2, \|iy\| \leqslant r\}\} .$$

Henceforth $\mathcal{G}(\alpha, c)$ is to be regarded as a topological space with the topology induced by the metric ϱ. Since $iS(\alpha c)$ is compact in X it follows readily, using general results on the Hausdorff metric, that $\mathcal{G}(\alpha, c)$ is compact. Next we observe that the functions $\frac{1}{2}\operatorname{diam} G$ and $\gamma(iG)$ are, respectively, lower and upper semi-continuous on $\mathcal{G}(\alpha, c)$. The first of these assertions is obvious, the second almost equally so. To verify the second note that if $\{G_n\}$ is a sequence in $\mathcal{G}(\alpha, c)$ which tends to G_0, then, for any neighborhood U of iG_0 in X, $iG_n \subset U$ for all but finitely many values of n. Using properties 1 and 3 of the genus we conclude that

$$\gamma(iG_0) \geqslant \lim_{n\to\infty} \gamma(iG_n) .$$

We conclude from the compactness of $\mathcal{G}(\alpha, c)$ and the semi-continuity assertions above that there exists a $G_n \in \mathcal{G}(\alpha, c) \cap \mathcal{G}_n(\alpha)$ such that

$$\mu_n(\alpha) = \max \{\psi(x): x \in G_n\} .$$

The next step in the proof consists in showing that the set G_n, whose existence was just demonstrated, contains an element x that satisfies (9.1), (9.2) and that if $\mu_n(\alpha)$ is repeated k times, then the set of such elements in G_n has genus $\geqslant k$.

We define a set valued mapping $\sigma: H\setminus\{0\} \to H\setminus\{0\}$ by

$$\sigma(x) = \{y \in H: y = \alpha z/\|z\|_H, z \in \partial\psi(x)\} .$$

We shall show:

1) if $0 < \|x\|_H \leqslant \alpha$, then for all $y \in \sigma(x)$

$$\|y\|_H = \alpha \quad \text{and} \quad \psi(y) \geqslant \psi(x)$$

with $\psi(y) = \psi(x)$ only if $\|x\|_H = \alpha$ and $y = \lambda x$ for some real positive λ,

2) If $G \in \mathcal{G}_n(\alpha)$ then $\sigma(G) \in \mathcal{G}_n(\alpha)$.

To prove 1), let $x \neq 0$, $\|x\|_H < \alpha$, and let $y = \alpha z/\|z\|_H$ where $z \in \partial\psi(x)$, then

$$\begin{aligned}\psi(y) - \psi(x) &\geqslant \langle y - x, z\rangle \\ &\geqslant \langle \alpha z/\|z\|_H - x, z\rangle \\ &\geqslant \alpha\|z\|_H - \langle x, z\rangle \\ &\geqslant 0\,,\end{aligned}$$

and it is clear from the derivation that equality holds only under the condition stated.

To prove 2), let $\omega: H\setminus\{0\} \to H\setminus\{0\}$ be defined by

$$\omega(z) = \alpha z/\|z\|_H\,,$$

so that

$$\sigma = \omega\circ\partial\psi\,.$$

Let $G \in \mathcal{G}_n(\alpha)$, i.e. $G \in \mathcal{G}(\alpha)$ and

$$\gamma(iG) \geqslant n\,.$$

By Lemma 7.2

$$\partial\varphi = B\circ\partial\varphi\circ i\,.$$

By Lemmas 4.2, 6.1 and the fact that compactness of B is equivalent to the sequential continuity of B from (X', X) to H, we conclude that $B\partial\varphi$ is upper semi-continuous and therefore, by Lemma 8.1, the genus in H of $B\partial\psi(iG)$ is not less than that of iG in X. Since ω is continuous and odd we conclude that

$$\gamma\big(i\sigma(G)\big) \geqslant \gamma\big(i\omega(B\partial\psi(iG))\big)\,,$$

which was to be proved.

Having verified the properties 1) and 2) of the transformation σ, the proof can be completed by means of the standard arguments of the Lyusternik-Schnirelman theory, as for example in [6] or [7].

We shall not make a careful reinterpretation here of the results of Theorem 9.1 in terms of the original Hammerstein operator. Indeed in view of Lemma 7.2 and the special nature of the splitting

$$A = i\circ B$$

it is clear that, for example, an element x satisfying (9.2) is a function such that

$$x = \lambda\int_\Omega k(\cdot, s)\,h(s)\,d\mu$$

where λ is a real number and h satisfies (1.3), i.e. x is an eigenfunction of the operator (1.2).

We conclude with a theorem expressing the duality between the two variational problems considered above.

THEOREM 9.2. *Let* $\alpha, \beta > 0$ *then for* $n = 1, 2, \ldots$

$$\mu_n'(\mu_n(\alpha)) = \alpha$$
$$\mu_n(\mu_n'(\beta)) = \beta .$$

PROOF. We prove only the first formula, the proof of the second is similar. From the proof of Theorem 9.1 follows the existence, for a given n, of $G \in \mathcal{G}^n(\alpha)$ such that

$$\mu_n(\alpha) = \min\{\psi(x) : x \in G\} ,$$

we have, obviously, $G \in \mathcal{G}_n'(\mu_n(\alpha))$ and

$$\mu_n'(\mu_u(\alpha)) \leqslant \tfrac{1}{2} \dim G = \alpha .$$

If the equality does not hold then there is a $G' \in \mathcal{G}_n'(\mu_n(\alpha))$ with

$$\tfrac{1}{2} \operatorname{diam} G' < \alpha$$

and

$$\mu_n(\alpha) \leqslant \min\{\psi(x) : x \in G'\} .$$

If we take $G'' = \omega(G') = \{z : z = \alpha x / \|x\|_H,\ x \in G'\}$, then $\frac{1}{2} \operatorname{diam} G'' = \alpha$' $G'' \in \mathcal{G}_n(\alpha)$ and

$$\mu_n(\alpha) < \min\{\psi(x) : x \in G''\}$$

which is a contradiction.

REFERENCES

[1] H. AMANN, *Lyusternik-Schnirelman theory and non-linear eigenvalue problems*, Math. Ann., **199** (1972), pp. 55-72.

[2] H. AMANN, *Hammersteinsche Gleichungen mit kompakten Kernen*, Math. Ann., **186** (1970), pp. 334-340.

[3] C. BERGE, *Espaces topologiques, fonctions multivoques*, Dunod, Paris, 1959. (English translation: *Topological Spaces including a Treatment of Multi-valued Functions, Vector Spaces and Convexity*, Macmillan, New York, 1963).

[4] F. E. BROWDER - C. P. GUPTA, *Monotone operators and nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc., **75** (1969), pp. 1347-1353.

[5] C. V. COFFMAN, *A minimum-maximum principle for a class of non-linear integral equations*, J. Analyse Math., **22** (1969), pp. 391-418.

[6] C. V. COFFMAN, *Spectral theory of monotone Hammerstein operators*, Pacific J. Math., **36** (1971), pp. 303-322.

[7] C. V. COFFMAN, *Lyusternik-Schnirelman theory and eigenvalue problems for monotone potential operators*, J. Functional Analysis, **14** (1973), pp. 237-252.

[8] M. M. DAY, *On the basis problem in normed spaces*, Proc. Amer. Math. Soc., **13** (1962), pp. 655-662.

[9] J. P. DIAS, *Un théorème de Sturm-Liouville pour une classe d'opérateurs non linéaires maximaux monotones*, J. Math. Anal. Appl., **47** (1974), pp. 400-405.

[10] J. P. DIAS - J. HERNANDEZ, *A Sturm-Liouville for some odd multivalued maps*, Proc. Amer. Math. Soc., **53** (1975), pp. 72-74.

[11] I. EKELAND - R. TEMAM, *Analyse Convexe et Problèmes Variationnels*, Dunod, Paris, 1974.

[12] J. W. JAWOROWSKI, *Theorems on antipodes for multi-valued mappings and a fixed point theorem*, Bull. Acad. Polon. Sci., Cl. III, **4** (1956), pp. 187-192.

[13] W. A. J. LUXEMBURG, *Banach function spaces* (Thesis, Delft), Assen, The Netherlands.

[14] W. A. J. LUXEMBURG - A. C. ZAANEN, *Notes on Banach function spaces*, Note I, Proc. Acad. Sci. Amsterdam (Indag. Math.), A **66** (1963), pp. 135-147.

[15] W. A. J. LUXEMBURG - A. C. ZAANEN, *Compactness of integral operators in Banach function spaces*, Math. Ann., **149** (1963), pp. 150-180.

[16] M. A. KRASNOSEL'SKII, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan, New York, 1964.

[17] J. J. MOREAU, *Semi-continuité du sous-gradient d'une fonctionelle*, C. R. Acad. Sci. Paris Ser. A-B, **260** (1965), pp. 1067-1070.

[18] V. R. PORTNOV, *A contibution to the theory of Orlicz spaces generated by variable N-functions*, Dokl. Akad. Nauk SSSR, **175** (1967), pp. 296-299; Soviet Math. Dokl., **8** (1967), pp. 857-860.

[19] R. T. ROCKAFELLAR, *Integrals which are convex functionals II*, Pacific J. Math., **39** (1971), pp. 439-469.