ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

H. BRÉZIS
L. NIRENBERG

Characterizations of the ranges of some nonlinear operators and applications to boundary value problems

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 5, nº 2 (1978), p. 225-326
<http://www.numdam.org/item?id=ASNSP_1978_4_5_2_225_0>

Characterizations of the Ranges of Some Nonlinear Operators and Applications to Boundary Value Problems.

H. BRÉZIS (*) - L. NIRENBERG (**) (***)

dedicated to Jean Leray

Introduction.

This paper is concerned with techniques for attacking nonlinear partial differential equations, in particular, boundary value problems for semilinear equations.

One of the basic tools for treating such problems is the Leray-Schauder degree theory, in particular the Schauder fixed point theorem. This theory works within the category of compact operators. When compactness is not available, monotone operators have proved to be useful in treating certain classes of equations. By now there is a rich literature on this subject and its applications (see [Bré-2], [Bro-1] which also contain many further references). Depending on the applications in mind various attempts have been made to combine monotone operator theory with topological methods in case there is also some compactness (see Browder [Bro-1], Leray, Lions [Le-Li]). This paper may be regarded as a contribution in that direction though the only topological tool we use is the Schauder fixed point theorem.

Our paper is in some sense the outgrowth of two others: [La-La] by Landesman, Lazer and [Br-Ha] by Brézis, Haraux. [La-La] is concerned

(*) Université P. et M. Curie, Paris 5^e.

(**) Courant Institute of Mathematical Sciences, New York University.

(***) The second author was partially supported by National Science Foundation Grants No. NSF-MPS 73-08487 and NSF MCS 76-07039, and by the John Simon Guggenheim Memorial Foundation. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

Pervenuto alla Redazione il 2 Marzo 1977 ed in forma definitiva il 6 Maggio 1977.

with the Dirichlet problem for a function $u(x)$ in a bounded domain Ω in R^n with smooth boundary $\partial\Omega$:

$$Lu + g(u) = f(x) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where L is a second order linear elliptic operator. Assuming g to be bounded [La-La] presents sufficient conditions on g and f which are almost necessary for the existence of a solution. The results have been extended in various directions by quite a number of authors (see Williams [W], Hess [He-1-2], Fučik, Kučera, Nečas [F-K-N], Nirenberg [N-1-2], De Figueiredo [DeF-1-2], Kazdan, Warner [K-W], Dancer [Da-1-2]) where further references may be found. They seek sufficient conditions for solvability of nonlinear equations like (1) which are also close to being necessary.

In this paper we develop some general methods for attacking equations in a Hilbert space (with scalar product (,) and norm $|\ |$) of the form

$$Au + Bu = f \in H \tag{2}$$

where A is usually a linear operator: $D(A) \subset H \to H$, and B is a nonlinear map of H into H. These are then applied to semilinear boundary value problems. The papers cited above also treat equations of the form (2) (some in more general frameworks). We have tried to present results of sufficient generality so as to include many of the cited extensions of [La-La]. Some of the known results are however not contained here. In particular there are stronger results in case (2) arises from a variational problem, see for instance Ahmad, Lazer, Paul [A-L-P], Rabinowitz [Ra-3].

If A represents an elliptic partial differential operator under suitable boundary conditions then A^{-1}, if it exists, is compact, and one may rewrite (2) in the form $u + A^{-1}Bu = A^{-1}f$. The more interesting case however is that in which $N = N(A) = \ker A \neq 0$. For A elliptic, N is finite dimensional. In case A is self adjoint one also has

$$R(A) = N(A)^{\perp}. \tag{3}$$

We will seldom require A to be self adjoint but we will always suppose that (3) holds (see however Remark I.2); then we have the orthogonal decomposition:

$$H = H_1 \oplus H_2 = P_1 H \oplus P_2 H, \quad H_1 = R(A), \quad H_2 = N(A),$$

and $A^{-1}: H_1 \to H_1$ is then compact.

In characterizing the range $R(A + B)$ of $A + B$ most of the abstract results take the form

$$R(A+B) \simeq R(A) + R(B) \tag{4}$$

or more generally,

$$R(A+B) \simeq R(A) + \operatorname{conv} R(B) \tag{4'}$$

where « conv » denotes the convex hull, and $S \simeq T$ means the sets S and T have the same interiors and the same closures. In this respect our results are natural extensions of those of Brézis, Haraux [Br-Ha] which are concerned with relation (4) for A, B monotone. (It is not true in general that (4) holds for all monotone operators even in finite dimensions. For instance, in the plane, if $A = -B =$ the operation of clockwise rotation by $\pi/2$ then $R(A+B) = 0$ so that (4) does not hold.) In [Br-Ha] are presented various sufficient conditions for (4), together with applications to boundary value problems. Here we do not assume that A is monotone but we require, for instance as in Theorem I.1, that $A^{-1}\colon R(A) \to R(A)$ is compact. In fact (and this is important in the applications to hyperbolic problems) we usually do not assume that $N(A) = H_2$ is finite dimensional but we still require compactness of $A^{-1}\colon R(A) \to R(A)$.

Our method of proof of solvability of (2) for $f \in \operatorname{Int} R(A) + \operatorname{conv} R(B)$ begins in a rather customary manner. With the aid of results for maximal monotone operators, and the Schauder fixed point theorem, we solve $\forall \varepsilon > 0$

$$\varepsilon P_2 u_\varepsilon + A u_\varepsilon + B u_\varepsilon = f\,,$$

where P_2 is the orthogonal projection onto H_2. Then, with the aid of (essentially) energy estimates we obtain bounds for $|u_\varepsilon|$ independent of ε—the most difficult part being the estimate of $|P_2 u_\varepsilon|$. In obtaining the bounds for all solutions u_ε we adapt an argument of [Br-Ha] which makes use of the principle of uniform boundedness. The corresponding bounds for $|u_\varepsilon|$ are therefore not obtained constructively; we have no knowledge of their size, just of their existence. Our view is that the techniques and tricks in the proofs are perhaps of more interest than any of the particular results—which have been devised for certain applications and admit many variations.

The abstract results are presented in Chapters I-III, the applications in IV, V; we now give a brief description of the results. In Chapter I we treat nonlinear terms B which are monotone and, in our main result of the chapter, Theorem I.10, we permit A to have a nonlinear, monotone component A_2. A is not required to be monotone; its degree of non-monotonicity is measured in some sense by a positive number $\alpha \leqslant \infty$ which, in case A is linear, is defined as the largest positive constant α such that

$$(Au, u) \geqslant -\frac{1}{\alpha}|Au|^2 \qquad \forall u \in D(A)\,; \tag{5}$$

monotonicity corresponds to $\alpha = \infty$. The nonlinear terms we allow are not only to be monotone but are required to satisfy restrictive growth conditions as $|u| \to \infty$ (depending on α). For example, a simple case of the basic condition (1.14) is: for some positive $\gamma < \alpha$,

$$(Bu - Bw, u) \geqslant \frac{1}{\gamma}|Bu|^2 - C(w)\,, \qquad \forall u, w \in H \tag{6}$$

where $C(w)$ is independent of u. This is automatically satisfied if $R(B)$ is bounded; it implies in turn that $|Bu| = O(|u|)$ as $|u| \to \infty$. We say that a nonlinear operator B is bounded if it is bounded on bounded sets.

In order to give the reader some initial idea of the main result (which is somewhat technical) we begin, in § I.1, with a special case, Theorem I.1, which, though simple, still has interesting applications. In I.2 we present one—for the nonlinear wave equation

$$u_{tt} - u_{xx} + g(x, t, u) = 0$$

for which we seek solutions periodic in time. The main result is presented in I.3 together with a first variant Theorem I.14 and another in I.5. In I.4 we drop the compactness assumption on A^{-1}, and replace it by a kind of Lipschitz condition (1.28).

Chapter II introduces a device which is useful in determining whether a given $f \in H$ belongs to the interior or closure of $R(A) + \operatorname{conv} R(B)$—in particular when $\dim H_2 < \infty$. This is the recession function of B:

$$J_B(v) = \liminf_{\substack{t \to +\infty \\ u \to v}} (B(tu), u)\,, \tag{7}$$

which is defined for any B, not merely monotone. Various properties of J_B are described, in particular (Proposition II.3): if $B = \partial\psi$ is the gradient of a convex function ψ then

$$\begin{aligned} J_B(u) &= \lim_{t \to +\infty} (B(tu), u) \\ &= I^*_{\overline{R(B)}}(u) \underset{\text{defn}}{\equiv} \sup_{\varphi \in R(B)} (\varphi, u) \\ &= \lim_{t \to +\infty} \frac{\psi(tu)}{t}\,. \end{aligned} \tag{8}$$

Then, in II.2 the recession function is used to investigate $R(A) + \operatorname{conv} R(B)$. For example, (Cor. II.7) if A is linear with $N(A) = H_2$ finite dimensional, and B is monotone and satisfies (6), even in a weaker form, then, for given $f \in H$,

$$f \in \overline{R(A + B)} \Leftrightarrow J_B(v) \geqslant (f, v) \qquad \forall v \in N(A)\,, \tag{9}$$

$$f \in \operatorname{Int} R(A + B) \Leftrightarrow J_B(v) > (f, v) \quad \forall v \in N(A),\ v \neq 0\,. \tag{10}$$

Our methods for treating (2) apply also to non-monotone B, as we show in Chapter III assuming usually that A is linear and $\dim N(A) < \infty$—provided we require a condition like (6). We establish the arrows $\Leftarrow$ in (9) and (10) in a number of cases, Theorems III.1-2 and their corollaries. The results in Chapter III do not rely on those of Chapter 1.

In Section III.3 we consider a particular class of nonlinear operators B of the form

$$Bu = g(x, u(x))\,.$$

Here $H = L^2(\Omega)$ where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$; set

$$g_+(x) = \liminf_{u \to +\infty} g(x, u)\,, \qquad g_-(x) = \limsup_{u \to -\infty} g(x, u)\,.$$

If g has small linear growth as $|u| \to \infty$ we present sufficient conditions on f to be in the closure or interior of $R(A + B)$. These are like the right-hand sides of (9) and (10), except that $J_B(v)$ is replaced by

$$\int_{[v>0]} g_+ v + \int_{[v<0]} g_- v\,.$$

(For convenience we usually omit the element of volume dx in the integrals.) In case A is monotone, i.e. $\alpha = +\infty$, we also permit g to have arbitrary growth in u, in one direction, i.e. we require

$$ug(x, u) \geqslant -c(x)|u| - d \qquad \text{with } c(x) \in L^\infty,\ d(x) \in L^1\,,$$

$c, d \geqslant 0$. Under some additional conditions we obtain (Th'm. III.6) a solution $u \in L^1(\Omega)$ of

$$\bar{A}u + g(x, u) = f(x)$$

where $\bar{A}$ is the closure of A in $L^1 \times L^1$.

In Appendix A we collect a number of useful facts about monotone operators and gradients of convex functions. In particular, Proposition A.4

relates condition (6) with the growth of B as $|u| \to \infty$, while Proposition A.5 deals with a modified kind of Lipschitz condition as used in section I.4. In Appendix B we describe without application, or proof, since it is somewhat tedious, a still more general form of Theorem I.10 which includes both monotone and non-monotone nonlinearities.

Turning to the applications, Chapter IV presents several applications to semilinear elliptic boundary value problems

$$Lu + g(x, u) = f(x) \quad \text{in } \Omega$$

in which $u(x)$ may represent a vector $(u^1, \ldots, u^N)(x)$. L is a linear elliptic system. For simplicity we have confined ourselves only to the Dirichlet problem, supposing u has zero Dirichlet data on $\partial\Omega$. It will be clear that the results may be extended in various directions. To describe one result (see Theorem IV.8), consider the system

$$\begin{aligned} \Delta v + \lambda v + \varphi_v &= \xi(x) \in C^\infty(\bar{\Omega}) \\ -\Delta w - \lambda_1 w + \varphi_w &= \eta(x) \in C^\infty(\bar{\Omega}) \end{aligned} \qquad v = w = 0 \text{ on } \partial\Omega$$

where φ is the convex function

$$\varphi(v, w) = [1 + a^2v^2 + b^2v^4 + c^2w^2 + d^2w^4]^{\frac{1}{2}}$$

and a, b, c, d are constants, $d > 0$. Here λ_1 is the first eigenvalue of $-\Delta$ and λ is some other eigenvalue; let $\underline{\lambda}$ be the eigenvalue of $-\Delta$ just preceding λ. Then there is a solution $\begin{pmatrix} v \\ w \end{pmatrix} \in C^\infty(\bar{\Omega})$ of the system provided

(a) $0 < b < \frac{1}{4}(\lambda - \underline{\lambda})$,

or

(b) $b = 0$ and $a\int|v| > \int \xi v \;\; \forall v \in N(\Delta + \lambda),\; v \neq 0$.

In Chapter V we treat parabolic and hyperbolic equations confining ourselves to simple model problems. For parabolic equations of the form

$$u_t - \Delta_x u + g(x, t, u) = 0 \quad \text{in } \Omega,\; u = 0 \text{ on } \partial\Omega$$

we treat the initial boundary value problems as well as others. Section V.2 takes up a semilinear hyperbolic equation (with dissipation) in n-dimensions for which we find solutions which are periodic in time, with prescribed period.

Further bibliographical remarks, in addition to those in the text, are made after Appendix B.

CHAPTER I

MONOTONE NONLINEARITIES

In I.1 we describe—without proof—simple versions of our main result and we derive some corollaries. Their use is illustrated in I.2 where we solve nonlinear wave equations with periodic boundary conditions. Most of our applications are presented in Chapters IV and V. In I.4 we consider a variant of the main result, in which the compactness assumption is replaced by a Lipschitz condition. In contrast with most other proofs in the paper, the proof does not rely on the Schauder fixed point theorem. Another variant is given in I.5.

I.1. Simple versions of the main result and some corollaries.

Throughout the paper the following class of linear operators will play an important role. Let H be a real Hilbert space.

PROPERTY I. Let $D(A) \subset H \to H$ be a closed linear operator with dense domain and closed range. Assume that

$$N(A) = N(A^*) \qquad \text{(or equivalently } R(A) = N(A)^{\perp}) \, .$$

A is therefore a one-one map of $D(A) \cap R(A)$ onto $R(A)$. Assume furthermore that the inverse

$$A^{-1}: R(A) \to R(A) \quad \text{is compact}.$$

Operators A satisfying all these conditions will be said to have *Property I.*

H has an orthogonal decomposition $H = R(A) \oplus N(A)$. For $u \in H$ we set $u = u_1 + u_2 = P_1 u + P_2 u$ or sometimes $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with $u_1 \in R(A)$, $u_2 \in N(A)$. Since there is a positive constant α_0 such that $\alpha_0 |u_1| \leqslant |Au|$, $u \in D(A)$, we have

$$(Au, u) \geqslant - |Au||u_1| \geqslant - \frac{1}{\alpha_0} |Au|^2 \qquad u \in D(A).$$

Throughout the paper we denote by α the largest positive constant such that

$$(Au, u) \geqslant - \frac{1}{\alpha} |Au|^2 \qquad \forall u \in D(A).$$

(We have $\alpha = +\infty$ iff $(Au, u) \geqslant 0$ $\forall u \in D(A)$.)

In case $A = A^*$ then α is the first positive eigenvalue of $-A$.

Assume $B: H \to H$ is a (nonlinear) operator satisfying

$$\text{(1.1)} \qquad \begin{cases} \text{For some positive constant } \gamma < \alpha, \\ (Bu - Bw, u) \geqslant \dfrac{1}{\gamma} |Bu|^2 - C(w), \quad \forall u,\ w \in H \end{cases}$$

where $C(w)$ depends only on w.

This artificial looking hypothesis should be viewed as an assumption about the behavior of B at infinity and not as a coerciveness assumption. It implies

$$\limsup_{|u| \to \infty} \frac{|Bu|}{|u|} \leqslant \gamma$$

and, conversely, it can often be derived from the behavior of B at infinity. This is true in particular for gradients of convex functions (see Appendix A, Propositions A.1, A.4, A.5, A.6). Note also that any monotone operator with bounded range satisfies (1.1) since $(Bu - Bw, u) \geqslant (Bu - Bw, w)$.

THEOREM I.1. *Suppose A has Property I. Let B be a monotone demicontinuous* (i.e. B is continuous from strong H into weak H) *operator satisfying* (1.1).

Then

$$R(A + B) \simeq R(A) + \operatorname{conv} R(B)$$

(conv denotes convex hull).

THEOREM I.1′. *Suppose A has* Property I *and for some $r > 0$, $A + rI$ is invertible. Let B be a monotone demicontinuous operator such that*

$$\lim_{|u|\to\infty} \frac{|Bu - ru|}{|u|} = 0 \,. \tag{1.1′}$$

Then $A + B$ is onto.

We omit the proofs—Theorem I.1 and I.1′ are special cases of Theorem I.10—and proceed with some consequences. [To derive Theorem I.1′ from Theorem I.10 take $S = rI$ on H_1 and note that (1.14) holds since

$$\begin{aligned}(Bu - Bw, u - v) &= (Bu - ru + ru - Bw, u - v)\\ &\geqslant r|u|^2 - C(v, w)|u| - |Bu - ru||u - v|\\ &\geqslant \frac{r}{2}|u|^2 - C(v, w) \geqslant \frac{1}{\gamma}|Bu - ru|^2 - C(v, w, \gamma)\end{aligned}$$

for any $\gamma > 0$. Finally (1.1′) implies that B is onto, since B^{-1} maps bounded sets into bounded sets.] We study in I.5 the case where $N(A + rI) \neq \{0\}$.

COROLLARY I.2. *Under the assumptions of* Theorem I.1, *if $H_2 \subset R(B)$* (which is the case when $|Bu| \to \infty$ as $|u| \to \infty$), *then $A + B$ is onto.*

COROLLARY I.3. *Assume A has* Property I. *Suppose B is demicontinuous and $B = \partial\psi$ is the Gateaux derivative of a convex continuous function ψ, that is*

$$(Bu, v) = \lim_{t\to 0} \frac{\psi(u + tv) - \psi(u)}{t}\,, \quad u, v \in H \,.$$

Assume

$$\limsup_{|v|\to\infty} \frac{|Bv|}{|v|} < \frac{\alpha}{2}\,.$$

Then

$$R(A + B) \simeq R(A) + \operatorname{conv} R(B)\,.$$

Corollary I.3 follows from Theorem I.1 and Proposition A.4.

COROLLARY I.4. *Assume A has* Property I. *Suppose B is demicontinuous and $B = \partial\psi$, ψ convex. Assume*

$$\limsup_{|v|\to\infty} \frac{\psi(v)}{|v|^2} < \frac{\alpha}{4}. \tag{1.2}$$

Assume furthermore that for some $R > 0$ and $\delta > 0$

$$\psi(v) - \psi(0) \geqslant \delta , \qquad \forall v \in N(A),\ |v| = R . \tag{1.3}$$

Then there is a solution $u \in H$ of

$$Au + Bu = 0 .$$

In particular if $\lim\limits_{\substack{|v|\to\infty\\ v\in N(A)}} \psi(v)/|v| = +\infty$, then $A + B$ is onto.

PROOF. Assumption (1.2) implies (1.1) (see Appendix A, Proposition A.1). Since ψ is convex we see from (1.3) that

$$\psi(v) - \psi(0) \geqslant \frac{\delta|v|}{R} \qquad \text{for } v \in N(A) \text{ with } |v| \geqslant R .$$

Hence

$$\psi(v) \geqslant \frac{\delta|v|}{R} - C \qquad \text{for all } v \in N(A) .$$

Thus for any $f \in H$ with $|f| < \delta/R$, the convex function $\psi(v) - (f, v)$ has a minimum on $N(A)$ which is achieved at a point v_0.

Consequently $Bv_0 - f \in N(A)^\perp = R(A)$ which means $f \in R(A) + R(B)$. Thus $0 \in \operatorname{Int}[R(A) + R(B)] = \operatorname{Int}[R(A + B)]$. q.e.d.

A more general form of this is given in Corollary I.15.

COROLLARY I.5. *Assume A has* Property I. *Suppose B is monotone demicontinuous, $B = \partial\psi$, B is onto and*

$$\limsup_{|v|\to\infty} \frac{|Bv|}{|v|} < \infty .$$

Then $\exists \varepsilon_0 > 0$ such that for $0 < |\varepsilon| < \varepsilon_0$

$$A + \varepsilon B \quad \textit{is onto} .$$

Corollary I.5 follows from Corollary I.3.

REMARK I.1. In solving equations of the form

$$Au_\varepsilon + \varepsilon Bu_\varepsilon = 0 \tag{1.4}$$

one usually performs a bifurcation analysis about a solution $u_0 \in N(A)$ satisfying $Bu_0 \in N(A)^\perp$. Under our conditions, all solutions of (1.4) satisfy $|u_\varepsilon| \leqslant$ constant for $|\varepsilon|$ small. Through a suitable sequence $\varepsilon_n \to 0$, u_{ε_n} therefore converges weakly to some $u_0 \in N(A)$ satisfying $Bu_0 \in N(A)^\perp$. This will be clear from the estimates occurring in the proof of Theorem I.10.

If we strengthen condition (1.1) we also obtain a uniqueness result.

COROLLARY I.6. *Assume A has* Property I. *Suppose B is onto, and satisfies*

$$(Bu - Bw, u - w) \geqslant \frac{1}{\gamma} |Bu - Bw|^2, \qquad \forall u, w \in H \tag{1.5}$$

with $\gamma < \alpha$.

Then $\forall f \in H$ there exists a solution of

$$Au + Bu = f, \tag{1.6}$$

and the solution is unique mod $N(A)$. *If furthermore B is one-one the solution is unique.*

PROOF. Note first that (1.5) implies (1.1) with any $\gamma' > \gamma$. So existence of a solution u follows from Corollary I.2. If w is another solution of (1.6), then by (1.5) we have

$$\frac{1}{\gamma} |Bu - Bw|^2 \leqslant \frac{1}{\alpha} |Au - Aw|^2 \leqslant \frac{1}{\alpha} |Bu - Bw|^2 .$$

Since $\gamma < \alpha$, the desired result follows.

REMARK I.2. The condition $R(A) = N(A)^\perp$ can often be achieved by a change of scalar product: Let $A: D(A) \subset H \to H$ be a closed linear operator with dense domain and closed range. Assume that $H = R(A) \oplus N(A)$—a direct sum not necessarily orthogonal. Any $u \in H$ can be uniquely decomposed as $u = u_1 + u_2$ with $u_1 \in R(A)$ and $u_2 \in N(A)$. If we define on H the new scalar product

$$\langle u, v\rangle = (u_1, v_1) + (u_2, v_2)$$

then $R(A)$ and $N(A)$ become orthogonal. The conclusion of Theorem I.1 holds provided B is a monotone operator relative to $\langle\ ,\ \rangle$ and satisfies (1.1) with respect to $\langle\ ,\ \rangle$. In general the conclusion of Theorem I.1 is false if we assume only that $H = R(A) \oplus N(A)$ (in place of $R(A) = N(A)^{\perp}$) and that B satisfies the monotonicity assumption as well as (1.1) with respect to the original scalar product $(\ ,\ )$. Indeed in $H = R^2$ set

$$A\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x+y\\ 0\end{pmatrix}, \quad B\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}g(x+y)\\ g(x+y)\end{pmatrix}$$

where g is a continuous nondecreasing function on R such that $|g| \leqslant M$. B is monotone and in fact $B = \partial\psi$ (with $\psi(x, y) = G(x+y)$, $G' = g$). Here $R(A) + R(B) = \left\{\begin{pmatrix}x\\ y\end{pmatrix};\ y \in R(g)\right\}$ while $R(A+B)$ is the graph of the function $g((I+g)^{-1})$; thus $R(A+B)$ is «much smaller» than $R(A) + R(B)$.

REMARK I.3. Suppose A satisfies Property I. Let ψ be a *nonconvex* C^1 function on H such that $B = \partial\psi$ has bounded range and $\psi(v) \to +\infty$ as $|v| \to \infty$, $v \in N(A)$. It is natural to raise the question whether $0 \in R(A+B)$. The answer is positive when $A^* = A$ and (for simplicity) $\dim H < \infty$ (see [A-L-P]). The answer is negative in general. Here is an example. In $H \simeq R^3$ set

$$A\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}y\\ -x\\ 0\end{pmatrix}, \qquad \psi(x, y, z) = h(x)\cos z + h(y)\sin z + f(z)$$

where:

$f(t)$ is a smooth function satisfying

$|\dot f(t)| \leqslant \frac{1}{5}$, and $f(t) \to +\infty$ as $|t| \to \infty$,

$h(t)$ is a smooth odd function satisfying $0 \leqslant \dot h(t) \leqslant \frac{1}{2}$, $h(t) = t/2$ for $0 \leqslant t \leqslant 1$ and $h(t) =$ constant for $t \geqslant 2$. Clearly B is bounded on R^3 and $\psi(0, 0, z) \to +\infty$ as $|z| \to \infty$.

CLAIM. The equation $Au + Bu = 0$ has no solution. Suppose $u = \begin{pmatrix}x\\ y\\ z\end{pmatrix}$ is a solution. The first two equations state

$$y + \dot h(x)\cos z = 0$$
$$-x + \dot h(y)\sin z = 0\,.$$

These imply $x^2 + y^2 \leqslant \frac{1}{4}$ and hence $\dot{h}(x) = \dot{h}(y) = \frac{1}{2}$, $h(x) = x/2$, $h(y) = y/2$. Thus

$$x = \tfrac{1}{2}\sin z\,, \qquad y = -\tfrac{1}{2}\cos z\,.$$

Inserting these values in the third equation we find

$$-\tfrac{1}{4}\sin^2 z - \tfrac{1}{4}\cos^2 z + \dot{f}(z) = 0$$

contradicting the fact that $|\dot{f}| \leqslant \frac{1}{5}$.

To conclude this section we describe a result (without proof) in which A^{-1} is not compact. It is a special case of Theorem I.16 of Section I.4.

THEOREM I.7. *Assume A satisfies all the conditions of Property I except the condition that A^{-1} is compact. Assume $B\colon H \to H$ is demicontinuous, $B = \partial\psi$, ψ convex and satisfies*

$$(Bu - Bw, u - w) \leqslant \gamma |u - w|^2\,, \qquad \forall u, w \in H \ \textit{with} \ \gamma < \alpha\,.$$

Then

$$R(A + B) \simeq R(A) + \operatorname{conv} R(B)\,.$$

REMARK I.4. In Theorem I.7, that $B = \partial\psi$ cannot be replaced by the condition that B is merely monotone, even if we assume

$$|Bu - Bw| \leqslant \gamma |u - w|\,, \qquad \forall u,\ w \in H \ \text{with} \ \gamma < \alpha\,.$$

Indeed in $H = R^2$ consider

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & \varepsilon^2 \end{pmatrix}, \quad \varepsilon \neq 0\,.$$

Here B is monotone, one-one and onto. But $A + B$ is singular for every ε while $|Bu| \leqslant \gamma |u|$ for some $\gamma < \alpha = 1$ if $|\varepsilon|$ is small.

I.2. Applications to nonlinear wave equations.

To illustrate the results of I.1 we present a simple application to a nonlinear wave equation in one space variable. We seek solutions which are periodic in time.

Consider the equation

$$u_{tt} - u_{xx} + g(x, t, u) = 0 \tag{1.7}$$

in Ω: $0 \leqslant x \leqslant \pi$, $0 \leqslant t \leqslant 2\pi$ with the boundary conditions

$$u(0, t) = u(\pi, t) = 0 \,, \tag{1.8}$$

for which we wish to find solutions periodic in t of period 2π; g is assumed periodic of period 2π in t. (We may replace (1.8) by periodicity conditions in x, period π, and obtain similar results.)

We assume g is measurable in (x, t), continuous in u, furthermore either g or $-g$ is nondecreasing as a function of u and satisfies in our first application a.e. (x, t), $\forall u$

$$\eta|u| - h_1(x, t) \leqslant |g(x, t, u)| \leqslant \gamma|u| + h_2(x, t) \tag{1.9}$$

where $\eta > 0$, h_1, $h_2 \in L^2$. Assume $\gamma < 3$ or $\gamma < 1$ according as g or $-g$ is nondecreasing.

THEOREM I.8. *Under these conditions the problem possesses at least one solution in L^2. Furthermore if $g \in C^\infty$ and $\pm g_u \geqslant \varepsilon > 0$, then there is a C^∞ solution.*

In general if g is smooth, solutions need not be smooth, nor unique. For instance any function of the form

$$u = p(t + x) - p(t - x)$$

with $p \in L^\infty$, $\sup |p| \leqslant k/2$ is a solution in case $g \equiv 0$ for $|u| \leqslant k$. Under additional hypotheses on g we can get a uniqueness result:

THEOREM I.9. *In Theorem* I.8 *if we add the condition*

$$|g(x, t, u) - g(x, t, v)| \leqslant \gamma|u - v| \qquad \text{a.e. } (x, t),\ \forall u, v \tag{1.10}$$

with $\gamma < 3$ or $\gamma < 1$ respectively.

Then the solution of (1.7) *is unique* mod $N(A)$. *If, furthermore, g is strictly monotone in u for every (x, t), then the solution is unique.*

This contains Theorem II in DeSimon-Torelli [DeS-T] (see also Mawhin [M-2]) as a very special case. Note that in Theorem I.8 we obtain *existence* without assuming any Lipschitz condition on g.

Theorems I.8 and I.9 hold in fact for any period T which is a rational multiple of π (then the range of $\partial^2/\partial t^2 - \partial^2/\partial x^2$ is closed). This fact as well as other existence results for equations of this form will be presented in another paper devoted to periodic solutions for hyperbolic equations under various growth conditions on the nonlinearities [Br-N].

We give a brief description of the proof without carrying out all details.

PROOF OF THEOREMS I.8 AND I.9. Let $H = L^2(\Omega)$, $A = \pm(\partial^2/\partial t^2 - \partial^2/\partial x^2)$ with the boundary and periodicity conditions, where we choose the $+$ (resp. $-$) sign if g is nondecreasing (resp. nonincreasing) in u. Using Fourier series, in fact a sine series (in x) expansion for $u = \sum_{j<0} a_{jk} \sin jxe^{ikt}$, $a_{j,k} = \overline{a_{j,-k}}$, so $Au = \pm \sum_{\substack{j<0\\k}} (j^2 - k^2) a_{jk} \sin jxe^{ikt}$, as in [Ra-1], one sees easily that $H_2 = N(A)$ is spanned by functions of the form $\sin jx \cos jt$, $\sin jx \sin jt$, $j > 0$, that $H_1 = H_2^\perp$, and that A satisfies condition I with $\alpha = 3$ or $\alpha = 1$ respectively.

We apply Corollary I.2. Using the left-hand inequality of (1.9) one sees that $|Bu| \to \infty$ as $|u| \to \infty$. Assumption (1.1) follows from the right-hand inequality in (1.9) and Proposition A.6. Therefore $A + B$ is onto. Theorem I.9 follows from Corollary I.6, since (1.10) implies (1.5).

To see that $u \in C^\infty$ in Theorem I.8 under the additional conditions on g we rely on known regularity results (see for example [Ra-1] § 3): namely if $u = u_1 + u_2$, $u_1 \in R(A)$, $u_2 \in N(A)$ then

(i) If $u_1 \in H^k$ (i.e. has square integrable derivatives up to order k), then $u_2 \in H^{k+1}$.

(ii) If $u_2 \in H^k$ then $u_1 \in H^k$, $k = 0, 1, 2, \ldots$.

Repeated application of this yields the regularity result. Q.e.d.

REMARK I.5. If (1.9) (or (1.10)) holds with $\gamma = 3$ or 1 respectively, there need not be a solution of (1.7); for example the equations

$$u_{tt} - u_{xx} + 3u = \sin x \sin 2t \qquad \text{and} \qquad u_{tt} - u_{xx} - u = \sin x$$

have no solutions satisfying the boundary and periodicity conditions.

We have used Theorem I.1 in proving Theorems I.8 and I.9 and because of that we had to restrict γ to be small in (1.9). Let us suppose however that g satisfies (1.9) with $\eta > 0$ and some γ. For certain functions g (for convenience, assume g is nondecreasing in u) we may still solve (1.7)—with the aid of Theorem I.1′.

Consider g of the form

$$g(x, t, u) = ru + \hat{g}(x, t, u)$$

with some positive constant r about which we suppose $r \neq k^2 - j^2$ for all integers $j > 0$ and k.

Assume $\forall \delta > 0\ \exists h_\delta(x, t) \in L^2$ such that

$$|\hat{g}(x, t, u)| \leqslant \delta|u| + h_\delta(x, t) . \tag{1.9′}$$

THEOREM I.8′. *Under these conditions on g,* (1.7) *possesses at least one solution in L^2. Furthermore if $g \in C^\infty$ and $g_u \geqslant \varepsilon > 0$ then there is a C^∞ solution.*

PROOF. As before we take $A = (\partial^2/\partial t^2 - \partial^2/\partial x^2)$, $B = g$ and we apply now Theorem I.1′. This proves the existence. That u is in C^∞ under the additional conditions follows as in the proof of Theorem I.8.

Before leaving equation (1.7) we take up one more case. Consider again g (nondecreasing in u) of the form $g = ru + \hat{g}(x, t, u)$ with $r > 0$ but suppose now that $r = k^2 - j^2$ for some integers $j > 0$ and k. In this case r is an integer, and the set Σ of pairs of integers $j > 0$, k, for which $r = k^2 - j^2$, is finite. With $A = \partial^2/\partial t^2 - \partial^2/\partial x^2$ as before, let $H_2 = N(A)$, and let H_3 be the space spanned by the functions $\sin jx \cos kt$, $\sin jx \sin kt$ for j, k belonging to the set Σ; H_3 is finite dimensional.

Finally let H_1 be the orthogonal complement in $H = L_2$ of $H_2 \oplus H_3$. Denote by A_j the restrictions of A to the respective invariant subspaces H_j, $j = 1, 2, 3$. Clearly $A_2 = 0$, and $A_3 = -rI$.

Concerning $\hat{g}$ we now suppose $\forall \delta > 0\ \exists h_\delta(x, t) \in L^2$ such that

$$|\hat{g}(x, t, u)| \leqslant \delta|u| + h_\delta(x, t) \tag{1.9″}$$

and

$$u\hat{g}(x, t, u) \geqslant -c(x, t)|u| - d(x, t) \qquad \text{for } c \in L^2,\ d \in L^1 . \tag{1.9‴}$$

Set

$$\hat{g}_+(x, t) = \liminf_{u \to +\infty} \hat{g}(x, t, u) , \qquad \hat{g}_-(x, t) = \limsup_{u \to -\infty} \hat{g}(x, t, u) .$$

THEOREM I.8″. *Let g satisfy the preceding conditions and suppose*

$$\int_{\substack{0\\[v>0]}}^{2\pi}\int_{0}^{\pi}\hat{g}_{+}v\,dx\,dt+\int_{\substack{0\\[v<0]}}^{2\pi}\int_{0}^{\pi}\hat{g}_{-}v\,dx\,dt>0\,,\qquad \forall v\in H_3,\ |v|=1\,,$$

then (1.7) *has a solution in* L^2. *Furthermore if* $g\in C^\infty$ *and* $g_u\geqslant\varepsilon>0$ *then it is in* C^∞.

PROOF. The proof makes use of a variant of Theorem I.1, Theorem I.20 of section I.5. It also uses Prop. II.4 of Chapter II. Setting $Bu=g(x,t,u)$ we see that Theorem I.20 with $S=rP_1$, gives the desired result provided (1.33), (1.34) and (1.36) hold. Clearly (1.33) holds in view of (1.9″). In our case the operator N of that theorem is simply

$$Nu=\hat{g}(x,t,u)+rP_2u\,.$$

Thus (1.9″) implies (1.34). Finally for $\hat{B}=\hat{g}$ we see that $J_N(v)\geqslant J_{\hat{B}}(v)$ and consequently (1.36) holds in virtue of Prop. II.4 (which uses (1.9‴)). The regularity of the solution under the additional hypotheses is proved as before. Q.e.d.

Theorem I.8″ is related to Theorem III.4 in [Ra-3].

I.3. The main result.

In this section we prove Theorem I.1 in a much more general form; in particular A may be nonlinear. H is a Hilbert space with an orthogonal decomposition $H=H_1\oplus H_2$. For an element $u\in H$, we denote its decomposition by $u=u_1+u_2=P_1u+P_2u$ or sometimes write $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}$.

CONDITIONS:

(1.11) $S\colon H_1\to H_1$ is a demicontinuous operator with $|Su|\leqslant r|u|+C$, and A_1 is an operator: $D(A_1)\subset H_1\to H_1$ satisfying: $\tilde{A}_1=A_1+S$ is one-to-one, onto; $\tilde{A}_1^{-1}$ is assumed to be continuous from weak H_1 to strong H_1 and

$$(\tilde{A}_1u,u)\geqslant-\frac{1}{\alpha}|\tilde{A}_1u|^2-C\qquad\forall u\in D(A_1)\,,$$

$$\alpha_0|u|\leqslant|\tilde{A}_1u|+C\qquad\forall u\in D(A_1)$$

for some constants α, $\alpha_0>0$ and C.

(1.12) A_2 is a maximal monotone operator:

$$D(A_2) \subset H_2 \to H_2 \qquad \text{with } 0 \in D(A_2),\ A_2 0 = 0\,.$$

Set

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad D(A) = \begin{pmatrix} D(A_1) \\ D(A_2) \end{pmatrix}.$$

(1.13) $B\colon H \to H$ is monotone demicontinuous and $B(0) = 0$.
We write

$$Bu = \begin{pmatrix} B_1(u) \\ B_2(u) \end{pmatrix}.$$

(1.14) For some positive $\gamma < \alpha$, and some $\tau < \alpha_0^2(\gamma^{-1} - \alpha^{-1}) - r$, for every $v, w \in H$ and every $\delta > 0$, there exist $k(\delta)$ and $C(v, w)$ such that $\forall u \in H$

$$(Bu - Bw, u - v) \geqslant \frac{1}{\gamma}\, |B_1 u - S u_1|^2 - \tau |u_1|^2 - C(v, w)\big(\delta |u_2| + k(\delta)\big)\,.$$

THEOREM I.10. *Under the conditions* (1.11)-(1.14),

$$R(A + B) \simeq R(A) + \operatorname{conv} R(B)\,.$$

REMARK I.6. In Appendix A we present some conditions under which (1.14) holds with $S = 0$, $\tau = \delta = 0$. Since $(Bu - Bw, u - v) \geqslant (Bu - Bw, w - v)$ we observe then (with the help of Proposition A.2 in Appendix A) that (1.14) holds for any S and any τ, for any $\gamma > 0$, in case

$$\lim_{|u| \to \infty} \left| \frac{|B_1 u|^2}{|u_1|^2 + |u_2|} + \frac{|B_2 u|}{|u_1|^p + |u_2|} \right| = 0$$

for some $p < 2$.

REMARK I.7. In the situation of Theorem I.1 we take $H_1 = R(A)$,

$$H_2 = N(A),\ A_1 = A|_{D(A) \cap R(A)},\ A_2 = 0.$$

The conclusion of Theorem I.10 tells when the equation $Au + Bu = f$ is solvable or almost solvable. Our method of proof consists in treating an approximate equation: for $\varepsilon > 0$

$$\varepsilon u_{2\varepsilon} + Au_\varepsilon + Bu_\varepsilon = f\,. \tag{1.15}$$

With the aid of the Schauder fixed point theorem we first show that (1.15) has a solution for any $f \in H$. Next, for certain f, we establish bounds for $|u_\varepsilon|$, $|A_1 u_\varepsilon|$ etc. independent of ε. Finally we carry out a limit process as $\varepsilon \to 0$ using a variant of Minty's trick.

LEMMA I.11. *For every $f \in H$ and $\varepsilon > 0$ there exists a solution u_ε of* (1.15).

PROOF. There are several steps.

Step 1. For *fixed* $u_1 \in H_1$, there exists a unique solution $u_2 = \varphi(u_1)$ of

$$\varepsilon u_2 + A_2 u_2 + B_2(u_1 + u_2) = f_2 . \tag{1.16}$$

A_2 is maximal monotone and $B_2(u_1 + u_2)$ is monotone demicontinuous in u_2. Their sum is therefore still maximal monotone (see for example [Bré-2] Corollaire 2.7). Hence (1.16) has a unique solution for each fixed u_1 (see [Bré-2] Proposition 2.2).

Step 2. We claim that $\forall u_1 \in H$,

$$(B_1 u - S u_1, u_1) \geqslant \frac{1}{\gamma} |B_1 u - S u_1|^2 - (\tau + r)|u_1|^2 - C \tag{1.17}$$

where $u = u_1 + \varphi(u_1)$, and hence

$$|B_1 u| \leqslant C(|u_1| + 1) \tag{1.18}$$

where C is independent of u_1 but may depend on ε.

This is based on (1.14) with $v = w = 0$ and $\delta = 1$: with $u_2 = \varphi(u_1)$,

$$\begin{aligned}\frac{1}{\gamma} |B_1 u - S u_1|^2 &\leqslant (Bu, u) + \tau |u_1|^2 + |u_2| + C \\ &\leqslant (B_1 u, u_1) + (f_2 - \varepsilon u_2 - A_2 u_2, u_2) + \tau |u_1|^2 + |u_2| + C \\ &\leqslant (B_1 u, u_1) + \tau |u_1|^2 - \varepsilon |u_2|^2 + C|u_2| + C\end{aligned}$$

since A_2 is monotone. This yields (1.17), from which (1.18) follows easily.

Step 3. φ is continuous from H_1 into H_2 in the strong topologies. Indeed let $u_{1n} \to u_1$; we have

$$\begin{aligned}\varepsilon u_{2n} + A_2 u_{2n} + B_2(u_{1n} + u_{2n}) &= f_2 \\ \varepsilon u_2 + A_2 u_2 + B_2(u_1 + u_2) &= f_2 .\end{aligned}$$

Subtracting and taking the scalar product with $u_{2n} - u_2$ we find using the monotonicity of A_2

$$\varepsilon|u_{2n} - u_2|^2 + \left(B_2(u_{1n} + u_{2n}) - B_2(u_1 + u_2), \quad u_{2n} - u_2\right) \leqslant 0$$

or, adding an obvious term to both sides,

$$\varepsilon|u_{2n} - u_2|^2 + (Bu_n - Bu, u_n - u) \leqslant (B_1 u_n - B_1 u, u_{1n} - u_1) .$$

Since B is monotone we find

$$\varepsilon|u_{2n} - u_2|^2 \leqslant |B_1 u_n - B_1 u||u_{1n} - u_1| .$$

The desired result follows with the aid of (1.18).

Step 4. Conclusion of the proof of the lemma. To solve (1.15) we have to find a solution u_1 of

$$A_1 u_1 + B_1(u_1 + \varphi(u_1)) = f_1$$

or

$$\tilde{A}_1 u_1 + B_1(u_1 + \varphi) = f_1 + S u_1 ,$$

i.e.

$$u_1 = \tilde{A}_1^{-1}[f_1 + S u_1 - B_1(u_1 + \varphi(u_1))] \equiv T u_1 .$$

We shall prove that T has a fixed point.

We show first that T is continuous in the strong topologies: If $u_{1n} \to u_1$ then $u_{1n} + \varphi(u_{1n}) \to u_1 + \varphi(u_1)$ by Step 3. Because B is demicontinuous $B_1(u_{1n} + \varphi(u_{1n}))$ converges weakly to $B_1(u_1 + \varphi(u_1))$. The conclusion follows from the assumption on $\tilde{A}_1^{-1}$.

We note further from our assumptions that $\tilde{A}_1^{-1}$ maps bounded sets into precompact sets and hence T is a compact operator (here we use (1.18)).

Finally we shall verify that

$$Tu_1 \neq \lambda u_1 , \quad u_1 \in H_1, \ |u_1| = R, \ \forall \lambda \geqslant 1 \tag{1.19}$$

provided R is large enough.

From (1.19) it follows by the Schauder fixed point theorem (applied to $P_R T$ where P_R is the projection on the ball of radius R) that T has a fixed point with norm $< R$.

Suppose $Tu_1 = \lambda u_1$ with $\lambda \geqslant 1$, i.e.

$$\tilde{A}_1(\lambda u_1) + B_1(u_1 + \varphi(u_1)) = f_1 + S u_1 .$$

Taking the scalar product with λu_1 we find using (1.11) and (1.17)

$$\frac{\lambda}{\gamma}|B_1 u - Su_1|^2 \leqslant \lambda|f_1||u_1| + \frac{1}{\alpha}|\tilde{A}_1(\lambda u_1)|^2 + C + (\tau + r)\lambda|u_1|^2 + \lambda C$$

where $u = u_1 + \varphi(u_1)$.

Therefore for any $\tau' > \tau + r$,

$$\begin{aligned}\frac{1}{\gamma}|B_1 u - Su_1|^2 &\leqslant \frac{1}{\alpha}|\tilde{A}_1(\lambda u_1)|^2 + \tau'\lambda^2|u_1|^2 + C(\tau')\\ &\leqslant \frac{1}{\alpha}|\tilde{A}_1(\lambda u_1)|^2 + \frac{\tau'}{\alpha_0^2}(|\tilde{A}_1(\lambda u_1| + C)^2 + C(\tau')\\ &\leqslant \sigma|\tilde{A}_1(\lambda u_1)|^2 + C(\sigma)\end{aligned}$$

for any $\sigma > 1/\alpha + (\tau + r)\alpha_0^{-2}$ (choosing $\tau' - (\tau + r)$ small). Finally

$$\frac{1}{\gamma}|B_1 u - Su_1|^2 \leqslant \sigma|B_1 u - Su_1 - f_1|^2 + C$$

and since $\alpha^{-1} + (\tau + r)\alpha_0^{-2} < \gamma^{-1}$ this yields a bound for $|B_1 u - Su_1|$, $|\tilde{A}_1(\lambda u_1)|$ and then for $|u_1|$. The lemma is proved.

LEMMA I.12. *Under the hypothesis of* Theorem I.10, *for*

$$f \in R(A) + \operatorname{conv} R(B)$$

we have

$$\varepsilon u_{2\varepsilon} \to 0 \qquad \text{as } \varepsilon \to 0 .$$

PROOF. Write

$$f = Av + \sum_1^N t_i B w_i , \qquad t_i \geqslant 0,\ \sum t_i = 1 .$$

By (1.14)

$$\Big(Bu_\varepsilon - \sum t_i B w_i,\ u_\varepsilon - v\Big) \geqslant \frac{1}{\gamma}|B_1 u_\varepsilon - S\, u_{1\varepsilon}|^2 - \tau|u_{1\varepsilon}|^2 - \delta|u_{2\varepsilon}| - C$$

where C is independent of ε, but may depend on v, $w_1, \ldots, w_N, \gamma, \delta$.

Replacing Bu_ε by its expression determined from (1.15), and $f - \sum t_i B w_i$ by Av, we fiind

$$(-Au_\varepsilon + Av - \varepsilon u_{2\varepsilon},\ u_\varepsilon - v) + \tau|u_{1\varepsilon}|^2 + \delta|u_{2\varepsilon}| \geqslant \frac{1}{\gamma}|B_1 u_\varepsilon - Su_{1\varepsilon}|^2 - C .$$

Hence by monotonicity of A_2 and by (1.11)

$$\frac{1}{\gamma}|B_1u_\varepsilon - Su_{1\varepsilon}|^2 + \frac{\varepsilon}{2}|u_{2\varepsilon}|^2 \leqslant \frac{1}{\alpha}|\tilde{A}_1u_{1\varepsilon}|^2 + C|\tilde{A}_1u_{1\varepsilon}| + \\ + \tau|u_{1\varepsilon}|^2 + \delta|u_{2\varepsilon}| + C + (u_{1\varepsilon} - v_1, Su_{1\varepsilon} + A_1v)\,. \tag{1.20}$$

Looking at the H_1 component of (1.15) we have

$$\tilde{A}_1u_{1\varepsilon} + B_1u_\varepsilon - Su_{1\varepsilon} = f_1$$

and so

$$|\tilde{A}_1u_{1\varepsilon}| \leqslant |B_1u_\varepsilon - Su_{1\varepsilon}| + C\,,$$

$$\alpha_0|u_{1\varepsilon}| \leqslant |B_1u_\varepsilon - Su_{1\varepsilon}| + C\,.$$

Inserting these inequalities in (1.20) and recalling that $\alpha^{-1} + (\tau + r)\alpha_0^{-2} < \gamma^{-1}$ we find $\forall\delta > 0$

$$\frac{\varepsilon}{2}|u_{1\varepsilon}|^2 \leqslant \delta|u_{1\varepsilon}| + C(\delta)\,.$$

This implies that $\varepsilon|u_{2\varepsilon}| \to 0$ as $\varepsilon \to 0$, for otherwise, for some sequence $\varepsilon_n \to 0$

$$\tfrac{1}{2}\varepsilon_n|u_{2\varepsilon_n}| \geqslant c > 0$$

and so $|u_{2\varepsilon_n}| \to \infty$. But then if we take $\delta < c$ we find $c \leqslant \delta + C(\delta)/|u_{2\varepsilon_n}| \to \delta$, contradiction.

LEMMA I.13. *Under the assumptions of* Theorem I.10, *if*

$$f \in \operatorname{Int}[R(A) + \operatorname{conv} R(B)]$$

then $|u_\varepsilon|$, $|B_1u_\varepsilon|$, $|A_1u_{1\varepsilon}|$ *are bounded independent of* ε.

PROOF. As in [Br-Ha] we use the principle of uniform boundedness but the argument is a bit trickier. For any $h \in H$ with $|h| <$ some small number we may write

$$f + h = Av + \sum_1^N t_iBw_i\,, \qquad t_i \geqslant 0,\ \sum t_i = 1\,,$$

v, w_i, t_i and N depend on h. $C(h)$ will be used to denote various constants depending on h but independent of ε.

We have

$$\varepsilon u_{2\varepsilon} + Au_\varepsilon + Bu_\varepsilon - \left(Av + \sum t_i Bw_i - h\right) = 0 .$$

Taking scalar product with $u_\varepsilon - v$ we find, since A_2 is monotone,

$$(1.21)\qquad (h, u_\varepsilon) + \left(Bu_\varepsilon - \sum t_i Bw_i, u_\varepsilon - v\right) \leqslant -(A_1 u_{1\varepsilon} - A_1 v, u_{1\varepsilon} - v_1) + C(h)$$
$$\leqslant \frac{1}{\alpha}|\tilde{A}_1 u_{1\varepsilon}|^2 + C(h)|\tilde{A}_1 u_{1\varepsilon}| + C(h) + (u_{1\varepsilon} - v_1, Su_{1\varepsilon} + A_1 v)$$
$$\leqslant \frac{1}{\alpha'}|\tilde{A}_1 u_{1\varepsilon}|^2 + C(h, \alpha', r') + r'|u_{1\varepsilon}|^2$$

for any $\alpha' < \alpha$, $r' > r$.

From (1.14) we see that $\forall \delta > 0$

$$(1.21')\qquad (h, u_\varepsilon) + \frac{1}{\gamma}|B_1 u_\varepsilon - Su_{1\varepsilon}|^2 \leqslant \frac{1}{\alpha'}|\tilde{A}_1 u_\varepsilon|^2 + (\tau + r')|u_{1\varepsilon}|^2$$
$$+ C(h)\left(\delta|u_{2\varepsilon}| + k(\delta)\right) + C(h, \alpha', r') .$$

Using (1.11) and (1.15) we see that for any fixed $\sigma > 1/\alpha + (\tau + r)/\alpha_0^2$, by choosing α' close to α and r' close to r,

$$(1.22)\qquad (h, u_\varepsilon) + \left(\frac{1}{\gamma} - \sigma\right)|B_1 u_\varepsilon - Su_{1\varepsilon}|^2 \leqslant \left(\delta|u_{2\varepsilon}| + k(\delta)\right) C(h) .$$

We claim that

$$(1.23)\qquad |u_\varepsilon| \leqslant C \quad \text{independent of } \varepsilon .$$

Suppose not; then for a sequence $\varepsilon_n \to 0$,

$$|u_{\varepsilon_n}| \to \infty .$$

By (1.22) $\forall \delta > 0$ $\exists k(\delta)$ such that

$$(1.24)\qquad (h, u_{\varepsilon_n}) \leqslant \left(\delta|u_{\varepsilon_n}| + k(\delta)\right) C(h) .$$

For $r > 0$ set $\omega(r) = \operatorname*{Inf}_{\delta<0}\{\delta r + k(\delta)\}$ so that $\lim_{r\to+\infty}\omega(r)/r = 0$. Since $(h, u_{\varepsilon_n}) \leqslant \omega(|u_{\varepsilon_n}|)\, C(h)$ we see by the principle of uniform boundedness that $|u_{\varepsilon_n}|/\omega(|u_{\varepsilon_n}|) \leqslant C$, a contradiction.

Thus (1.23) is proved.

Choosing $h = 0$ in (1.22) we find

$$|B_1 u_\varepsilon| \leqslant C \quad \text{independent of } \varepsilon .$$

For (1.15) we obtain the desired estimate for $|A_1 u_{1\varepsilon}|$. Lemma I.13 is proved. Note that the proof gives no information at all about the size of the bound for $|u_\varepsilon|$, just its existence.

We are now in a position to complete the

PROOF OF THEOREM I.10.

(i) To prove $\overline{R(A+B)} = \overline{R(A) + \operatorname{conv} R(B)}$ we need only show that if $f \in R(A) + \operatorname{conv} R(B)$ then $f \in \overline{R(A+B)}$. This follows immediately from Lemma I.12 for our solution u_ε of (1.15).

(ii) To prove that $R(A+B)$ and $R(A) + \operatorname{conv} R(B)$ have the same interior, it suffices to show that any $f \in \operatorname{Int}\,(R(A) + \operatorname{conv} R(B))$ belongs to $R(A+B)$. Consider our solution u_ε of (1.15). We will study the passage to the limit in (1.15) as $\varepsilon \to 0$. For a suitable sequence $\varepsilon_n \to 0$ we may assert in view of the bounds of Lemma I.13 and the fact that $\tilde{A}_1^{-1}$ is continuous from weak H_1 to strong H_1 that

$$u_{2\varepsilon_n} \rightharpoonup u_2\,, \qquad u_{1\varepsilon_n} \to u_1\,, \qquad A_1 u_{1\varepsilon_n} \rightharpoonup A_1 u_1\,.$$

Set $u = u_1 + u_2$.

For any $\xi = \xi_1 + \xi_2$ with $\xi_1 \in H_1$, $\xi_2 \in D(A_2)$ we have, by monotonicity,

$$(Bu_{\varepsilon_n} - B\xi,\ u_{\varepsilon_n} - \xi) \geqslant 0$$

i.e.

$$(f - \varepsilon_n u_{2\varepsilon_n} - Au_{\varepsilon_n} - B\xi,\ u_{\varepsilon_n} - \xi) \geqslant 0\,.$$

Using the monotonicity of A_2 we find

$$(f - B\xi,\ u_{\varepsilon_n} - \xi) \geqslant \varepsilon_n(u_{2\varepsilon_n},\ u_{2\varepsilon_n} - \xi_2) + (A_1 u_{1\varepsilon_n},\ u_{1\varepsilon_n} - \xi) + (A_2\xi_2,\ u_{2\varepsilon_n} - \xi_2)$$

and going to the limit as $\varepsilon_n \to 0$,

$$(f - B\xi,\ u - \xi) \geqslant (A_1 u_1,\ u_1 - \xi_1) + (A_2\xi_2,\ u_2 - \xi_2)\,. \tag{1.25}$$

(a) Set $\xi_1 = u_1$; then we find

$$(f_2 - B_2(u_1 + \xi_2) - A_2\xi_2,\ u_2 - \xi_2) \geqslant 0\,.$$

Since the map $\xi_2 \to A_2\xi_2 + B_2(u_1 + \xi_2)$ is maximal monotone, it follows that

$$f_2 = A_2 u_2 + B_2 u\,. \tag{1.26}$$

(*b*) If we now set $\xi_2 = u_2$ in (1.25) we obtain

$$(f_1 - B_1(\xi_1 + u_2), u_1 - \xi_1) \geqslant (A_1 u_1, u_1 - \xi_1)\,.$$

Now use the Minty trick: for $w_1 \in H_1$ and t positive, set $\xi_1 = u_1 - tw_1$. After dividing by t we find

$$(f_1 - B_1(u - tw_1) - A_1 u_1, w_1) \geqslant 0\,.$$

Letting $t \to 0$ and using the fact that w_1 is arbirary in H_1 we may conclude that

$$f_1 - B_1 u - A_1 u = 0\,.$$

This together with (1.26) shows that

$$Au + Bu = f\,.$$

Theorem I.10 is proved.

One can present many variants of Theorem I.10 by slight modifications of the hypotheses. For example here is a result in which we weaken the conditions on B but strengthen those on A.

THEOREM I.14. *Assume* (1.11)-(1.13). *Assume furthermore that* A_2 *satisfies*

$$(1.27) \qquad (A_2 u - A_2 v, u - w) \geqslant - C(v, w)\,, \qquad u, v, w \in D(A_2)\,.$$

In place of (1.14) *assume for some positive* $\gamma < \alpha$, $\tau < \alpha_0^2(\gamma^{-1} - \alpha^{-1}) - r$ *and* $\forall \delta > 0$

$$(Bu - Bw, u) \geqslant \frac{1}{\gamma}\, |B_1 u - Su_1|^2 - \tau |u_1|^2 - C(w)\big(\delta |u_2| + k(\delta)\big)\,, \qquad \forall u, w$$

where $C(w)$ *depends only on* w, $k(\delta)$ *only on* δ. *Then*

$$R(A + B) \simeq R(A) + \operatorname{conv} R(B)\,.$$

PROOF. We indicate the modifications needed in the proof of Theorem I.10. These occur only in the proofs of Lemmas I.12 and I.13.

In the proof of Lemma I.12 we have now

$$\Big(Bu_\varepsilon - \sum t_i B w_i, u_\varepsilon\Big) \geqslant \frac{1}{\gamma}\, |B_1 u_\varepsilon - S u_{1\varepsilon}|^2 - \tau |u_{1\varepsilon}|^2 - \delta |u_{2\varepsilon}| - C$$

where C is independent of ε but may depend on $w_1, \ldots, w_N, \gamma, \delta$. Then we find

$$(Av - \varepsilon u_{2\varepsilon} - Au_\varepsilon, u_\varepsilon) \geqslant \frac{1}{\gamma} |B_1 u_\varepsilon - Su_{1\varepsilon}|^2 - \tau |u_{1\varepsilon}|^2 - \delta |u_{2\varepsilon}| - C .$$

Hence we obtain

$$\frac{1}{\gamma} |B_1 u - Su_{1\varepsilon}|^2 + \varepsilon |u_{2\varepsilon}|^2 \leqslant \frac{1}{\alpha} |\tilde{A}_1 u_{1\varepsilon}|^2 + C|u_{1\varepsilon}| + C + (r+\tau)|u_{1\varepsilon}|^2 + \delta |u_{2\varepsilon}|$$

since $(A_2 u_{2\varepsilon} - A_2 v, u_{2\varepsilon}) \geqslant C$ by (1.27). This is similar to (1.20) and we proceed as before.

Turning to Lemma I.13 we have, to begin with,

$$\left(\varepsilon u_{2\varepsilon} + Au_\varepsilon + Bu_\varepsilon - Av - \sum t_i Bw_i + h, u_\varepsilon\right) = 0 .$$

Hence

$$(h, u_\varepsilon) + (A_2 u_{2\varepsilon} - A_2 v, u_{2\varepsilon}) \leqslant \frac{1}{\alpha} |\tilde{A}_1 u_{1\varepsilon}|^2 + C(h)|u_1| + C + r|u_{1\varepsilon}|^2 - \left(Bu_\varepsilon - \sum t_i Bw_i, u_\varepsilon\right).$$

By (1.27) and (1.11) we find

$$(h, u_\varepsilon) \leqslant \frac{1}{\alpha} |\tilde{A}_1 u_{1\varepsilon}|^2 + C(h)|\tilde{A}_1 u_{1\varepsilon}| + C(h) + r|u_{1\varepsilon}|^2 - \left(Bu_\varepsilon - \sum t_i Bw_i, u_\varepsilon\right).$$

Hence, as before for any $\alpha' < \alpha$, $\tau' > \tau$

$$(h, u_\varepsilon) + \frac{1}{\gamma} |B_1 u_\varepsilon - Su_\varepsilon|^2 \leqslant \frac{1}{\alpha'} |\tilde{A}_1 u_{1\varepsilon}|^2 + \tau' |u_{1\varepsilon}|^2 + C(h)(\delta |u_{2\varepsilon}| + k(\delta)) + C(h, \alpha', \tau') .$$

This is like (1.21′) and the proof of Lemma I.13 proceeds as before. We consider Theorem I.14 proved.

Using Theorem I.10 we may prove a more general form of Cor. I.4:

COROLLARY I.15. *Let A satisfy conditions* (1.11), (1.12) *with $S = 0$, and let B be demicontinuous, $B = \partial\psi$, ψ convex. Assume that for some positive constant $\sigma < \alpha/4$,*

$$\psi(u_1 + u_2) \leqslant \sigma |u_1|^2 + C(u_2) , \quad \forall u_1 \in H_1,\ u_2 \in H_2 .$$

Assume furthermore that for some $R > 0$, $\delta > 0$

$$\psi(u_2) - \psi(0) \geqslant \delta \qquad \text{for } u_2 \in H_2,\ |u_2| = R\,.$$

Then there is a solution $u \in H$ *of*

$$Au + Bu = 0\,.$$

In particular if

$$\lim_{\substack{u_2 \in H_2\\ |u_2| \to \infty}} \frac{\psi(u_2)}{|u_2|} = +\infty$$

then $A + B$ *is onto.*

Proof. Proposition A.1 implies (1.14) with $S = 0$, $\delta = 0$ for some $\gamma < \alpha$. Then just follow the proof of Corollary I.4.

I.4. A « noncompact » variant of the main result.

We consider A_1 as in Theorem I.10, with slight modifications. As before $H = H_1 \oplus H_2$ is an orthogonal decomposition.

Conditions

(1.28) $A_1 \colon D(A_1) \subset H_1 \to H_1$ is one-one and onto, A_1^{-1} is demicontinuous and $\forall u, u' \in D(A_1)$ and some $\alpha > 0$:

$$(A_1 u - A_1 u', u - u') \geqslant -\frac{1}{\alpha}\,|A_1 u - A_1 u'|^2\,.$$

Theorem I.16. *Assume* A_1 *satisfies* (1.28), A_2 *satisfies* (1.12) *and set* $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$. *Assume* $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \colon H \to H$ *is demicontinuous,* $B = \partial\psi$, ψ *convex continuous, and*

$$(B_1 u - B_1 w, u - w) \leqslant \gamma |u - w|^2\,, \qquad \gamma < \alpha,\ \forall u, w \in H\,. \tag{1.29}$$

Then

$$R(A + B) \simeq R(A) + \operatorname{conv} R(B)\,.$$

Furthermore if $Au + Bu = Au' + Bu'$ *then* $u_1 = u_1'$ *and* $Bu = Bu'$.

REMARK I.8. Assumption (1.29) holds if

$$|B_1 u - B_1 w| \leqslant \gamma |u - w| \qquad \forall u, w \in H\,,$$

or if

$$(Bu - Bw, u - w) \leqslant \gamma |u - w|^2 \qquad \forall u, w \in H$$

(which is equivalent to $|Bu - Bw| \leqslant \gamma |u - w|$ $\forall u, w$ by Prop. A.5 and the remark following it).

COROLLARY I.17. *Under the assumptions of* Theorem I.16, $A + B$ *is onto if* B *is onto,* $A + B$ *is one-one if* B *is one-one.*

PROOF. If B is onto then $\mathrm{Int}\,[R(A + B)] = \mathrm{Int}\,[R(A) + \mathrm{conv}\ R(B)] = H$. Suppose u and u' are two solutions of

$$Au + Bu = f\,.$$

We deduce from (1.29) (see Prop. A.5) that

$$(Bu - Bu', u - u') \geqslant \frac{1}{\gamma}\, |B_1 u - B_1 u'|^2$$

and therefore, by (1.28),

$$\frac{1}{\gamma}\, |B_1 u - B_1 u'|^2 \leqslant \frac{1}{\alpha}\, |A_1 u_1 - A_1 u_1'|^2\,.$$

Thus

$$\frac{1}{\gamma} |A_1 u_1 - A_1 u_1'|^2 \leqslant \frac{1}{\alpha}\, |(A_1 u_1 - A_1 u_1')|^2$$

and so $A_1 u_1 = A_1 u_1'$, $B_1 u = B_1 u'$. Since A_1 is one-one, $u_1 = u_1'$. We also have $(Bu - Bu', u - u') = 0$, which yields $Bu = Bu'$ (see Proposition 2 in [Bré-B]). In case B is one-one it follows that $u = u'$.

COROLLARY I.18. *Let* A *be as in* Theorem I.16. *Assume* $B\colon H \to H$ *is demicontinuous,* $B = \partial\psi$, ψ *convex and*

$$\eta |u - w|^2 \leqslant (Bu - Bw, u - w) \leqslant \gamma |u - w|^2 \qquad \forall u, w \in H$$

with $\eta > 0$ *and* $\gamma < \alpha$. *Then* $A + B$ *is one-one and onto.*

REMARK I.9. If A is linear and satisfies all the conditions of property I in Section I.1 except that A^{-1} is compact, then it satisfies the condition of Theorem I.16 if we set $A_1 = A|_{D(A)\cap R(A)}$, $A_2 = 0$; we obtain Theorem I.7. In case A is also self adjoint and B is in C^1 then Cor. I.18 coincides with a result of Mawhin [M-4].

In the proof of Theorem I.16 we shall use

LEMMA I.19. *Under the assumptions of* Theorem I.16, *for every* $\varepsilon > 0$ *and* $f \in H$, *the equation*

$$\varepsilon u_2 + Au + Bu = f \tag{1.30}$$

has a solution.

PROOF. Set $A_\varepsilon = \varepsilon P_2 + A_1$ so that A_ε is one-one onto and $\forall v, v' \in H$

$$(A_\varepsilon^{-1}v - A_\varepsilon^{-1}v', v - v') \geqslant -\frac{1}{\alpha}|v_1 - v_1'|^2 + \frac{1}{\varepsilon}|v_2 - v_2'|^2 .$$

In addition A_ε^{-1} is demicontinuous. Equation (1.30) can be written as

$$u = A_\varepsilon^{-1}(f - \tilde{B}u) \qquad \text{where } \tilde{B}u = A_2 u_2 + Bu$$

or

$$Mv + A_\varepsilon^{-1}v \ni 0 .$$

with

$$v = f - \tilde{B}u , \qquad Mv = -\tilde{B}^{-1}(f - v) .$$

M is a maximal monotone operator (since $\tilde{B}$ is) and by Proposition A.5, (1.29) implies

$$(Mv - Mv', v - v') \geqslant \frac{1}{\gamma}|v_1 - v_1'|^2 \qquad \forall v, v' \in D(M)$$

(more precisely we have

$$(h - h', v - v') \geqslant \frac{1}{\gamma}|v_1 - v_1'|^2 \qquad \forall h \in Mv,\ \forall h' \in Mv' \Big).$$

Set $N = M + A_\varepsilon^{-1}$; N satisfies

$$(Nv - Nv', v - v') \geqslant \left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)|v_1 - v_1'|^2 + \frac{1}{\varepsilon}|v_2 - v_2'|^2 \geqslant \delta|v - v'|^2 \qquad (\delta > 0) .$$

Therefore N is monotone; in fact N is maximal monotone since $N + \lambda I = M + (\lambda I + A_\varepsilon^{-1})$ is onto for $\lambda > 1/\alpha$ (M is maximal monotone and $(1/\alpha)I + A_\varepsilon^{-1}$ is monotone demicontinuous). Consequently N is one-one onto.

PROOF OF THEOREM I.16. Since $B = \partial\psi$ and satisfies (1.29) it follows from Prop. A.5 that

$$(1.31) \qquad (Bu - Bw, u - v) \geqslant \frac{1}{\gamma'} |B_1 u|^2 - C(v, w) \qquad \forall u, v\, w \in H, \text{ some } \gamma' < \alpha .$$

Suppose first $f \in R(A) + \operatorname{conv} R(B)$ and write

$$f = Av + \sum t_i B w_i, \qquad t_i \geqslant 0, \ \sum t_i = 1 .$$

We deduce from (1.31) that

$$(1.32) \qquad (Bu - f + Av, u - v) \geqslant \frac{1}{\gamma'} |B_1 u|^2 - C \qquad \forall u \in H .$$

Let u_ε be the solution of

$$(1.30') \qquad \varepsilon u_{2\varepsilon} + Au_\varepsilon + Bu_\varepsilon = f .$$

Setting $u = u_\varepsilon$ in (1.32), using the monotonicity of A_2, and (1.28), we find

$$\frac{\varepsilon}{2} |u_{2\varepsilon}|^2 + \frac{1}{\gamma'} |B_1 u_\varepsilon|^2 \leqslant \frac{1}{\alpha} |A_1 u_{1\varepsilon} - A_1 v_1|^2 + C$$

or

$$\frac{\varepsilon}{2} |u_{2\varepsilon}|^2 + \frac{1}{\gamma'} |A_1 u_{1\varepsilon} - f_1|^2 \leqslant \frac{1}{\alpha} |A_1 u_{1\varepsilon} - A_1 v_1|^2 + C .$$

Since $\gamma' < \alpha$ we see that

$$\varepsilon |u_{2\varepsilon}|^2 \leqslant C , \qquad |A_1 u_{1\varepsilon}| \leqslant C .$$

It follows that $\varepsilon u_{2\varepsilon} \to 0$, and $f \in \overline{R(A+B)}$, proving that

$$R(A) + \operatorname{conv} R(B) \subset \overline{R(A+B)} .$$

Suppose now that some ball of radius $r > 0$ about f lies in $R(A) + \operatorname{conv} R(B)$. For any $h \in H$, $|h| < r$, we split

$$f + h = Av(h) + \sum t_i(h) \cdot Bw_i(h) , \qquad t_i(h) \geqslant 0, \ \sum t_i(h) = 1 .$$

We deduce from (1.32) that for our solution u_ε of (1.30')

$$(Bu_\varepsilon - f - h + Av(h), u_\varepsilon - v(h)) \geqslant \frac{1}{\gamma'} |B_1 u_\varepsilon|^2 - C(h)$$

or, as before,

$$\frac{1}{\gamma'}|B_1u_\varepsilon|^2 + (h, u_\varepsilon) \leqslant \frac{1}{\alpha}|A_1u_{1\varepsilon} - A_1v_1(h)|^2 + C(h)$$

i.e.

$$\frac{1}{\gamma'}|A_1u_{1\varepsilon} - f_1|^2 + (h, u_\varepsilon) \leqslant \frac{1}{\alpha}|A_1u_{1\varepsilon} - A_1v_1(h)|^2 + C(h)\,.$$

Hence

$$(h, u_\varepsilon) \leqslant C(h)\,, \qquad \forall h \in H \text{ with } |h| < r\,.$$

and by the principle of uniform boundedness

$$|u_\varepsilon| \leqslant C\,.$$

Finally, we have for ε, $\varepsilon' > 0$, by Prop. A.5,

$$\begin{aligned}\frac{1}{\gamma}|B_1u_\varepsilon - B_1u_{\varepsilon'}|^2 &\leqslant (Bu_\varepsilon - Bu_{\varepsilon'}, u_\varepsilon - u_\varepsilon) \\ &= -(Au_\varepsilon - Au_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}) - (\varepsilon u_\varepsilon - \varepsilon' u_{\varepsilon'}, u_\varepsilon - u_{\varepsilon'}) \\ &\leqslant \frac{1}{\alpha}|A_1u_{1\varepsilon} - A_1u_{1\varepsilon'}|^2 + C(\varepsilon + \varepsilon')\,.\end{aligned}$$

Therefore

$$\frac{1}{\gamma}|A_1u_{1\varepsilon} - A_1u_{1\varepsilon'}|^2 \leqslant \frac{1}{\alpha}|A_1u_{1\varepsilon'} - A_1u_{1\varepsilon'}|^2 + C(\varepsilon + \varepsilon')$$

which implies that $A_1u_{1\varepsilon}$ converges strongly as $\varepsilon \to 0$. Consequently $u_{1\varepsilon} \rightharpoonup u_1$. Suppose $u_{2_{\varepsilon_n}} \rightharpoonup u_2$ and set $u = u_1 + u_2$. We have $A_1u_{1\varepsilon} \to A_1u_1$ and we may now follow the argument for passing to the limit as $\varepsilon_n \to 0$ in the proof of Theorem I.10 and conclude that $Au + Bu = f$. We have proved that $\operatorname{Int}[R(A) + \operatorname{conv} R(B)] \subset R(A + B)$. In the proof of Cor. I.17 we already established the last assertion of Theorem I.16; the theorem is therefore proved.

I.5. Another variant.

In condition (1.11) of Theorem I.10 we required $A_1 + S$ to be invertible. With the aid of the recession function borrowed from the next chapter we will treat a case in which this condition is modified. For simplicity we replace condition (1.14) by a much more restrictive hypothesis.

Consider a Hilbert space H with an orthogonal decomposition

$$H = H_1 \oplus H_2 \oplus H_3 = P_1 H \oplus P_2 H \oplus P_3 H$$

where H_3 is finite dimensional. Let $S\colon H_1 \to H_1$ and $A_1\colon D(A_1) \subset H_1 \to H_1$ be operators satisfying conditions (1.11), and let $A_2\colon D(A_2) \subset H_2 \to H_2$ be as in (1.12). Let A_3 be a continuous mapping of H_3 into H_3. Set

$$A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}.$$

$B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}\colon H \to H$ is monotone demicontinuous, $B(0) = 0$, satisfying: for some constants $\varrho > 0$, $\forall \delta > 0$, $\exists C_\delta$ such that

$$(B_2 u, u_2) \geq \varrho |u_2|^2 - \delta |u|^2 - C \tag{1.33}$$

Furthermore we suppose the operator

$$Nu = Bu - Su_1 + A_3 u$$

satisfies

$$\frac{|Nu - B_2 u|}{|u|} \to 0 \quad \text{as } |u| \to \infty . \tag{1.34}$$

We wish to solve

$$Au + Bu = f \tag{1.35}$$

and we shall make use of the recession function (defined in (7) of the introduction) of the operator N.

THEOREM I.20. *Let A and B satisfy the preceding conditions. If*

$$J_N(v) > (f, v) \qquad \forall v \in H_3, \ v \neq 0 \tag{1.36}$$

then (1.35) *has a solution.*

The proof is similar to that of Theorem I.10 but much simpler in detail because of the strong hypotheses (1.33), (1.34). First we prove the analogue of Lemma I.11.

LEMMA I.21. *For every* $f \in H$ *and* $\varepsilon > 0$ *there exists a solution* $u = u_\varepsilon$ *of*

$$\varepsilon(u_2 + u_3) + Au + Bu = f . \tag{1.37}$$

PROOF. As in Steps 1-3 of the proof of Lemma I.11 (it is even simpler here) one sees that for fixed $u_1 \in H_1$, $u_3 \in H_3$ there exists a unique solution $u_2 = \varphi(u_1 + u_3)$ of

$$\varepsilon u_2 + A_2 u_2 + B_2(u_1 + u_2 + u_3) = f_2 .$$

Taking scalar product with u_2 we find, since A_2 is monotone,

$$\varepsilon |u_2|^2 + (B_2, u_2) = (f_2, u_2) .$$

and it follows from (1.33) that $\forall \delta > 0$, $\exists k_\delta$ independent of ε such that

$$|u_2| \leqslant \delta |u| + k_\delta . \tag{1.38}$$

To solve (1.37) we wish to solve

$$A_1 u_1 + B_1\big(u_1 + u_3 + \varphi(u_1 + u_3)\big) = f_1$$

$$\varepsilon u_3 + A_3 u_3 + B_3\big(u_1 + u_3 + \varphi(u_1 + u_3)\big) = f_3$$

or

$$u_1 = \tilde{A}_1^{-1}[f_1 - B_1(u_1 + u_3 + \varphi) + Su_1] \equiv T_1(u_1 + u_3)$$

$$u_3 = \varepsilon^{-1}[f_3 - A_3 u_3 - B_3(u_1 + u_3 + \varphi)] \equiv T_3(u_1 + u_3) .$$

The map $T = T_1 + T_3 \colon H_1 \oplus H_3 \to H_1 \oplus H_3$ is continuous in the strong topologies and compact (here we use the fact that H_3 is finite dimensional). By (1.34) and (1.38) we see that $|T(u_1 + u_3)|/|u_1 + u_3| \to 0$ as $|u_1 + u_3| \to \infty$. Using the Schauder fixed point theorem we infer that T has a fixed point. The lemma is proved.

Next we obtain estimates for the solution u_ε as in Lemma I.13.

LEMMA I.22. *If* (1.36) *holds, then*

$$|u_\varepsilon| , \qquad |B_1 u_\varepsilon| , \qquad |A_1 u_{1\varepsilon}|$$

are bounded independent of ε.

PROOF. Equation (1.37) takes the form

$$(1.37)_1 \qquad \tilde{A}_1 u_{1\varepsilon} + B_1 u_\varepsilon - S u_{1\varepsilon} = f_1$$

$$(1.37)_2 \qquad \varepsilon u_{2\varepsilon} + A_2 u_{2\varepsilon} + B_2 u_\varepsilon = f_2$$

$$(1.37)_3 \qquad \varepsilon u_{3\varepsilon} + A_3 u_{3\varepsilon} + B_3 u_\varepsilon = f_3 \,.$$

Using (1.34) and $(1.37)_1$ we see that $|\tilde{A}_1 u_{1\varepsilon}| \leqslant C$ and hence $|u_{1\varepsilon}| \leqslant C$ for some constants C independent of ε. Here we have also used the conditions (1.11). It follows again from $(1.37)_1$ that $|B_1 u_\varepsilon| \leqslant C$.

Now suppose $|u_\varepsilon|$ is not bounded. Then for some sequence of ε's we would have $t_\varepsilon = |u_\varepsilon| \to \infty$ and so from (1.38),

$$v_\varepsilon = \frac{u_\varepsilon}{t_\varepsilon} \to v \in H_3 \,, \qquad |v| = 1 \,.$$

If we now take the scalar product of (1.37) with v_ε we find, using the monotonicity of A_2 and (1.33)

$$\frac{\varepsilon\big(|u_{2\varepsilon}|^2 + |u_{3\varepsilon}|^2\big)}{t_\varepsilon} + \frac{(\tilde{A}_1 u_{1\varepsilon}, u_{1\varepsilon})}{t_\varepsilon} + \big(N(t_\varepsilon v_\varepsilon), v_\varepsilon\big) \leqslant (f_3, v_\varepsilon) \,.$$

Letting $\varepsilon \to 0$ through the sequence, we find, in view of the bounds for $|u_{1\varepsilon}|$, $|\tilde{A}_1 u_{1\varepsilon}|$, and the definition (7) of J_N,

$$J_N(v) \leqslant (f, v)$$

which contradicts (1.36). Therefore we must have $|u_\varepsilon| \leqslant C$ and the lemma is proved.

PROOF OF THEOREM I.20. We have only to carry out a limit process in (1.37) as $\varepsilon \to 0$. This proceeds exactly as in part (ii) of the proof of Theorem I.10. For a suitable sequence $\varepsilon_n \to 0$ we find $u_{2\varepsilon_n} \rightharpoonup u_2$, $u_{1\varepsilon_n} \to u_1$, $A_1 u_{1\varepsilon_n} \rightharpoonup A_1 u_1$, $u_{3\varepsilon_n} \to u_3$, and we obtain the analogue of (1.25): for $\xi = \xi_1 + \xi_2 + \xi_3$, $\xi_1 \in H_1$, $\xi_2 \in D(A_2)$ and $u = u_1 + u_2 + u_3$,

$$(f - B\xi, u - \xi) \geqslant (A_1 u_1, u_1 - \xi_1) + (A_2 \xi_2, u_2 - \xi_2) + (A_3 u_3, u_3 - \xi_3) \,.$$

We then just follow the remainder of the proof of Theorem I.10.

CHAPTER II

THE RECESSION FUNCTION

In Chapter I we have studied *geometrical* properties of $R(A+B)$. It is not always easy in a specific problem to decide whether a given f belongs to $R(A)+R(B)$ (or to $R(A)+\operatorname{conv} R(B)$). We introduce in Chapter II a new tool: the recession function of a nonlinear operator. It provides a convenient way of checking whether $f \in \overline{R(A)+\operatorname{conv} R(B)}$ or $f \in \operatorname{Int}[R(A)+ + \operatorname{conv} R(B)]$. Hence it leads to a simple *analytic* description of $R(A+B)$ which is especially useful when dealing with the solvability of nonlinear partial differential equations. The recession function also plays an important role when B is not monotone; in that case we no longer have $R(A+B) \simeq \simeq R(A)+R(B)$; nevertheless we can give *sufficient* conditions for f to lie in $\overline{R(A+B)}$ (or $\operatorname{Int} R(A+B)$), expressed in terms of the recession function of B.

II.1. Definition, examples, elementary properties.

Let H be a Hilbert space and let B be a nonlinear map from H into H. For $u \in H$ we define the *recession function*

$$J_B(u) = \liminf_{\substack{t\to+\infty \\ v\to u}} (B(tv), v) .$$

The term «recession function» has been used previously in convex analysis (see [Ro]) to denote $\lim_{t\to+\infty} \psi(tu)/t$, where ψ is a convex function. As we shall see, (Proposition II.3), the two quantities coincide when B is the gradient of ψ.

PROPOSITION II.1. *J_B is a lower semicontinuous function from H into $[-\infty, +\infty]$ and $J_B(\lambda u) = \lambda J_B(u)$ $\forall \lambda > 0$ and $u \in H$.*

PROOF. It is clear that J_B is positively homogeneous of order 1. We show now that for each $k \in [-\infty, +\infty)$ the set

$$E = \{u \in H | J_B(u) \leqslant k\}$$

is closed. Let $u_n \in E$ be a sequence such that $u_n \to u$. Since $J_B(u_n) \leqslant k$, for every n there exists $t_n > n$ and u_n' such that

$$|u_n' - u_n| < 1/n \qquad \text{and} \qquad (B(t_n u_n'), u_n') \leqslant k + 1/n$$

$\big($resp. $(B(t_n u_n'), u_n') \leqslant -n$ in case $k = -\infty\big)$. Therefore $t_n \to +\infty$, $u_n' \to u$ and thus $J_B(u) \leqslant k$.

REMARK II.1. Clearly $J_B(u) \leqslant \liminf_{t\to+\infty} (B(tu), u)$. In particular $J_B(0)$ is either 0 or $-\infty$, since J_B is positively homogeneous.

REMARK II.2. When B is monotone $\lim_{t\to+\infty} (B(tu), u)$ exists (possibly $+\infty$). Indeed if $t > s$, we have

$$(t-s)(B(tu) - B(su), u) \geqslant 0$$

and thus the function $t \mapsto (B(tu), u)$ is nondecreasing.

PROPOSITION II.2. *Assume B is monotone and sublinear i.e.* $\lim_{|v|\to\infty} |Bv|/|v| = 0$. *Then for each $u \in H$*

$$J_B(u) = \lim_{t\to+\infty} (B(tu), u) = I^*_{R(B)}(u)$$

where

$$I^*_{R(B)}(u) = \sup_{\varphi\in R(B)} (\varphi, u) = \sup_{\varphi\in\overline{\operatorname{conv} R(B)}} (\varphi, u)$$

denotes the support function of $\overline{\operatorname{conv} R(B)}$.

In particular, J_B is convex; in addition, if $R(B)$ is bounded, then J_B is continuous.

PROOF. From the monotonicity of B we have

$$\left(B(tv) - Bx, v - \frac{x}{t}\right) \geqslant 0 \qquad \forall x \in H,\ \forall v \in H,\ t > 0\,. \tag{2.1}$$

Since B is sublinear, for each $\delta > 0$ there is a C_δ such that $|Bw| \leqslant \delta|w| + C_\delta$ for all $w \in H$ (we use here the fact that B is bounded on bounded sets, see Appendix A, Proposition A.2). It follows that $\lim_{t\to+\infty} |B(tv)|/t = 0$ uniformly as v remains bounded. Passing to the lim inf in (2.1) as $v \to u$ and

$t \to +\infty$ we find

$$J_B(u) \geqslant (Bx, u) \qquad \forall x \in H, \ \forall u \in H .$$

Therefore

$$J_B \geqslant I^*_{R(B)} .$$

On the other hand we *always* have

$$\big(B(tu), u\big) \leqslant I^*_{R(B)}(u)$$

and therefore

$$J_B(u) \leqslant \lim_{t\to+\infty} \big(B(tu), u\big) \leqslant I^*_{R(B)}(u) . \tag{2.2}$$

PROPOSITION II.3. *Assume B satisfies*

$$(Bu - Bv, u) \geqslant C(v) \qquad \forall u \in H, \ \forall v \in H \tag{2.3}$$

where $C(v)$ is independent of u.

(Assumption (2.3) holds for example when $B = \partial\psi$ is a gradient of a convex function with $C(v) = (BO - Bv, v)\big)$. *Then, for each $u \in H$*

$$J_B(u) = \lim_{t\to+\infty} \big(B(tu), u\big) = I^*_{R(B)}(u) .$$

In addition if $B \subset \partial\psi$ with ψ convex continuous on H, then for each $u \in H$

$$J_B(u) = \lim_{t\to+\infty} \frac{\psi(tu)}{t} .$$

PROOF. From (2.3) we have

$$\big(B(tv) - Bx, v\big) \geqslant \frac{C(x)}{t} \qquad \forall x \in H, \ \forall v \in H, \ \forall t > 0$$

and consequently

$$J_B(u) \geqslant (Bx, u) \qquad \forall x \in H, \ \forall u \in H .$$

Hence

$$J_B(u) \geqslant I^*_{R(B)}(u) .$$

On the other hand we always have

$$J_B(u) \leqslant \liminf_{t\to+\infty} \big(B(tu), u\big) \leqslant \limsup_{t\to+\infty} \big(B(tu), u\big) \leqslant I^*_{R(B)}(u) .$$

Assume now that $B \subset \partial\psi$ (the subdifferential $\partial\psi$ of ψ is a multivalued operator and we assume that B is a section of $\partial\psi$).

We have

$$\psi(tu) - \psi(x) \geqslant (Bx, tu - x) \qquad \forall x \in H,\ \forall u \in H,\ \forall t > 0 .$$

Thus

$$\lim_{t\to+\infty} \frac{\psi(tu)}{t} \geqslant (Bx, u) \qquad \forall x \in H,\ \forall u \in H$$

(recall that the function $t \to (\psi(tu) - \psi(0))/t$ is nondecreasing and so $\lim\limits_{t\to+\infty} \psi(tu)/t$ exists). Consequently

$$\lim_{t\to+\infty} \frac{\psi(tu)}{t} \geqslant I^*_{R(B)}(u) .$$

On the other hand

$$\psi(0) - \psi(tu) \geqslant -\, (B(tu), tu) \qquad \forall u\ \delta\ H,\ \forall t > 0$$

and therefore

$$\lim_{t\to+\infty} \frac{\psi(tu)}{t} \leqslant \lim_{t\to+\infty} (B(tu), u) . \qquad \text{Q.E.D.}$$

REMARK II.3. In general if B is monotone, but not sublinear or a gradient, it may happen that $\lim\limits_{t\to+\infty} (B(tu), u) < I^*_{R(B)}(u)$. Consider for example in $H = R^2$, B a rotation by $+\pi/2$. Then $(B(tu), u) \equiv 0$ while

$$I^*_{R(B)}(u) = \begin{cases} 0 & \text{if } u = 0 \\ +\infty & \text{if } u \neq 0 \end{cases} .$$

An example.

Let $H = L^2(\Omega)$, Ω a measure space. Let $g(x, u)\colon \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ be measurable in x and continuous in u. Assume for a.e. $x \in \Omega$ and all $u \in R$:

$$(2.4) \qquad |g(x, u)| \leqslant a|u| + b(x) \qquad \text{with } a \in \boldsymbol{R},\ b \in L^2 ,$$

$$(2.5) \qquad u \cdot g(x, u) \geqslant -\, c(x)|u| - d(x) \qquad \text{with } c \in L^2,\ d \in L^1 .$$

Set

$$g_+(x) = \liminf_{u\to+\infty} g(x, u) , \qquad g_-(x) = \limsup_{u\to-\infty} g(x, u)$$

(so that $-\, c(x) \leqslant g_+(x) \leqslant +\infty$, $-\infty \leqslant g_-(x) \leqslant c(x)$).

PROPOSITION II.4. *Set* $(Bu)(x) = g(x, u(x))$ *for* $u \in H$. *Then for each* $u \in H$

$$J_B(u) \geqslant \int_{[u>0]} g_+(x)u(x)\,dx + \int_{[u<0]} g_-(x)u(x)\,dx\,. \tag{2.6}$$

REMARK II.4. The right-hand side in (2.6) can be written as (dropping the volume element dx, as we often do)

$$\int_{[u>0]} (g_+ + c)u + \int_{[u<0]} (g_- - c)u - \int c|u|$$

and thus belongs to $(-\infty, +\infty]$.

PROOF. Let $v_n \in L^2$, $t_n \to +\infty$ be such that $v_n \to u$ in L^2, $\int g(x, t_n v_n) v_n \to \to J_B(u)$. Extracting a subsequence we can always assume that $v_n \to u$ a.e. and that $|v_n| \leqslant h$, $\forall n$, for some fixed function $h \in L^2$. We have

$$t_n v_n g(x, t_n v_n) \geqslant -c(x) t_n |v_n| - d(x)$$

and so

$$v_n g(x, t_n v_n) \geqslant -c(x)|v_n| - \frac{d(x)}{t_n} \geqslant -k$$

where k is a fixed integrable function. We write

$$\int v_n g(x, t_n v_n) = \int_{[u>0]} v_n g(x, t_n v_n) + \int_{[u<0]} v_n g(x, t_n v_n) + \int_{[u=0]} v_n g(x, t_n v_n)$$

and note that

$$\text{on } [u>0]\colon\ \liminf_{n\to+\infty} v_n g(x, t_n v_n) \geqslant u g_+(x)\,,$$

$$\text{on } [u<0]\colon\ \liminf_{n\to+\infty} v_n g(x, t_n v_n) \geqslant u g_-(x)\,,$$

and

$$\int_{[u=0]} v_n g(x, t_n v_n) \geqslant \int_{[u=0]} -c(x)|v_n| - \frac{d(x)}{t_n}\,.$$

We conclude by Fatou's lemma that

$$J_B(u) \geqslant \int_{[u>0]} g_+ u + \int_{[u<0]} g_- u\,.$$

II.2. How to determine $R(A+B)$ using the recession function.

PROPOSITION II.5. *Let N be a closed subspace of H and let B be a nonlinear map in H. The following conditions are equivalent*

$$I^*_{R(B)}(v) \geqslant (f, v) \qquad \forall v \in N\,, \tag{2.7}$$

$$f \in \overline{N^\perp + \operatorname{conv} R(B)}\,. \tag{2.8}$$

REMARK II.5. Using (2.2) we see that

$$J_B(v) \geqslant (f, v) \qquad \forall v \in N,\ v \neq 0 \tag{2.9}$$

implies (2.8). It is equivalent to (2.8) in case B is monotone sublinear (Proposition II.2) or when $B = \partial\psi$ (Proposition II.3).

REMARK II.6. When B is monotone demicontinuous then

$$\overline{N^\perp + \operatorname{conv} R(B)} = \overline{N^\perp + R(B)} = \overline{R(P_{N^\perp} + B)}$$

(where $P_{N^\perp}$ denotes the orthogonal projection on $N^\perp$).

This follows from Theorem 4 in [Br-Ha] since $A = P_{N^\perp}$ is a gradient of a convex function.

PROOF. (2.7) $\Rightarrow$ (2.8).

Suppose by contradiction, that $f \notin \overline{N^\perp + \operatorname{conv} R(B)}$. By the Hahn-Banach theorem there exist $\xi \in H$ and $\alpha \in \boldsymbol{R}$ such that

$$(\xi, f) > \alpha \geqslant (\xi, g + h) \qquad \forall g \in N^\perp,\ \forall h \in R(B)\,.$$

It follows, since $N^\perp$ is a linear space that

$$(\xi, g) = 0 \qquad \forall g \in N^\perp$$

i.e. $\xi \in N$. In addition

$$I^*_{R(B)}(\xi) = \sup_{h \in R(B)} (\xi, h) \leqslant \alpha < (\xi, f)$$

—a contradiction.

(2.8) $\Rightarrow$ (2.7).

It suffices to show that (2.7) holds for $f \in N^\perp + \operatorname{conv} R(B)$. But, if

$v \in N$ we find

$$I^*_{R(B)}(v) = \sup_{h \in \operatorname{conv} R(B)} (h, v) \geq (f, v) .$$

PROPOSITION II.6. *Let N be a finite dimensional subspace of H and let B be a nonlinear map in H. The following are equivalent*

$$I^*_{R(B)}(v) > (f, v) \qquad \forall v \in N,\ v \neq 0 , \tag{2.10}$$

$$f \in \operatorname{Int}\left(N^\perp + \operatorname{conv} R(B)\right) . \tag{2.11}$$

PROOF. (2.10) $\Rightarrow$ (2.11).

Since $I^*_{R(B)}(v) - (f, v)$ is lower semicontinuous, it is bounded below by a positive c_0 on $\{v \in N, |v| = 1\}$. Thus

$$I^*_{R(B)}(v) \geq (f, v) + c_0|v| \qquad \forall v \in N .$$

Therefore it suffices to prove that $f \in N^\perp + \operatorname{conv} R(B)$, or, equivalently, that $P_N f \in P_N \operatorname{conv} R(B)$ (P_N denotes the orthogonal projection on N). Suppose by contradiction that $P_N f \notin P_N \operatorname{conv} R(B)$. Applying the Hahn-Banach theorem in N, we find $\exists \xi \in N$, $\xi \neq 0$ such that

$$(\xi, f) \geq (\xi, h) \qquad \forall h \in R(B)$$

(we are using here the assumption $\dim N < \infty$). Thus

$$(\xi, f) \geq I^*_{R(B)}(\xi)$$

—a contradiction.

(2.11) $\Rightarrow$ (2.10).

Let $f_0 \in \operatorname{Int}[N^\perp + \operatorname{conv} R(B)]$; since a ball $B(f_0, r)$ is contained in $N^\perp + \operatorname{conv} R(B)$, we have as in the proof of Proposition II.5

$$I^*_{R(B)}(v) \geq (f_0 + rx, v) \qquad \forall v \in N,\ \forall x \in H,\ |x| < 1 .$$

Hence

$$I^*_{R(B)}(v) \geq (f_0, v) + r|v| \qquad \forall v \in N .$$

REMARK II.7. When $\dim N = \infty$, the assumption

$$I^*_{R(B)}(v) \geq (f, v) + r|v| \qquad \forall v \in N,\ (r > 0) ,$$

does *not* imply in general that

$$f \in N^{\perp} + \operatorname{conv} R(B) .$$

Indeed, the conclusion would lead to

$$f \in \operatorname{Int} (N^{\perp} + \operatorname{conv} R(B)) .$$

But it may well happen that $N^{\perp} + \operatorname{conv} R(B)$ has an empty interior. For example we may have $N = H$, $\operatorname{conv} R(B)$ is dense but $\operatorname{conv} R(B)$ has an empty interior; in this case

$$I^*_{R(B)}(v) = \begin{cases} 0 & \text{if } v = 0 \\ +\infty & \text{if } v \neq 0 . \end{cases}$$

Combining Propositions II.5 and II.6 with the results of Chapter I leads to some interesting applications. We mention only one simple example:

COROLLARY II.7. *Let $A: D(A) \subset H \to H$ be a linear operator satisfying* Property I of § I.1. *Let $B: H \to H$ be a monotone demicontinuous operator satisfying*

$$(2.12) \qquad (Bu - Bw, u) \geqslant \frac{1}{\gamma} |Bu|^2 - C(w) ; \qquad \forall u \in H, \ \forall w \in H$$

with $\gamma < \alpha$. Given $f \in H$, the following conditions are equivalent

$$(2.13) \qquad J_B(v) \geqslant (f, v) \qquad \forall v \in N(A) ,$$

$$(2.14) \qquad f \in \overline{R(A + B)} .$$

In addition if $\dim N(A) < \infty$, *the following are equivalent*

$$(2.13') \qquad J_B(v) > (f, v) \qquad \forall v \in N(A), \ v \neq 0 ,$$

$$(2.14') \qquad f \in \operatorname{Int} [R(A + B)] .$$

PROOF. (2.13) $\Rightarrow$ (2.14).
It follows from Proposition II.3 that

$$I^*_{R(B)}(v) \geqslant (f, v) \qquad \forall v \in N(A) .$$

Applying Proposition II.5 with $N = N(A)$ we see that

$$f \in \overline{N(A)^{\perp} + \operatorname{conv} R(B)} = \overline{R(A) + \operatorname{conv} R(B)}\,.$$

By Theorem I.1 we know that $\overline{R(A) + \operatorname{conv} R(B)} = \overline{R(A + B)}$.

(2.14) $\Rightarrow$ (2.13).

This follows again from Propositions II.3 and II.5.

To prove that (2.13′) $\Leftrightarrow$ (2.14′) we use Proposition II.6 instead of II.5.

Example II.1. Let $H = L^2(\Omega)$ and let $A\colon D(A) \subset H \to H$ be a linear operator satisfying Property I. Let $g(x, u)\colon \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ be measurable in x, and continuous nondecreasing in u.

Set $Bu = g(x, u)$. Assume for a.e. $x \in \Omega$ and all $u \in \boldsymbol{R}$

$$|g(x, u)| \leqslant \gamma |u| + b(x) \qquad \text{with } \gamma < \alpha, \text{ and } b \in L^2\,. \tag{2.15}$$

Set $g_{\pm}(x) = \lim\limits_{u \to \pm\infty} g(x, u)$ (possibly $\pm\infty$).

Corollary II.8. *Given $f \in L^2$, the following conditions are equivalent*

$$\int\limits_{[v>0]} g_+(x)\, v(x)\, dx + \int\limits_{[v<0]} g_-(x)\, v(x)\, dx \geqslant \int\limits_{\Omega} f(x)\, v(x)\, dx \qquad \forall v \in N(A)\,, \tag{2.16}$$

$$f \in \overline{R(A + B)}\,. \tag{2.17}$$

In addition if $\dim N(A) < \infty$, *the following are equivalent*

$$\int\limits_{[v>0]} g_+(x) v(x) dx + \int\limits_{[v<0]} g_-(x) v(x) dx > \int\limits_{\Omega} f(x) v(x) dx \qquad \forall v \in N(A),\ v \neq 0\,, \tag{2.16′}$$

$$f \in \operatorname{Int} R(A + B)\,. \tag{2.17′}$$

Proof. We apply Corollary II.7. We know (see Proposition A.6) that (2.15) implies that (2.12) holds provided γ is replaced by any $\gamma' > \gamma$. It follows from Proposition II.3 that $J_B(v) = \lim\limits_{t \to +\infty} (B(tv), v)$ and the monotone convergence theorem implies that for every $v \in H$

$$J_B(v) = \int\limits_{[v>0]} g_+(x)\, v(x)\, dx + \int\limits_{[v<0]} g_-(x)\, v(x)\, dx\,;$$

the integral makes sense in $(-\infty, +\infty]$ since

$$g_+(x) \geqslant g(x, 0) \in L^2 \qquad \text{and} \qquad g_-(x) \leqslant g(x, 0) \in L^2\,.$$

CHAPTER III

NONMONOTONE OPERATORS B

In Section II.2 we have seen that for some classes of monotone operators B the condition $J_B(v) \geqslant (f, v)$ $\forall v \in N(A)$, $v \neq 0$ (resp. $J_B(v) > (f, v)$ $\forall v \in N(A)$, $v \neq 0$) is *equivalent* to $f \in \overline{R(A + B)}$ (resp. $f \in \operatorname{Int}[R(A + B)]$). In case B is not monotone, it is no longer true in general that $R(A + B) \simeq \simeq R(A) + R(B)$. However we shall prove for a large class of nonmonotone operators B that the condition $J_B(v) \geqslant (f, v)$ $\forall v \in N(A)$, $v \neq 0$ (resp. $J_B(v) > > (f, v)$ $\forall v \in N(A)$, $v \neq 0$) is still a *sufficient* condition for f to lie in $\overline{R(A + B)}$ (resp. $\operatorname{Int}[R(A + B)]$). In Section III.2-4 this is applied to $H = L^2$ of a measure space and B of the form $Bu = g(x, u)$ for $u(x) \in H$. In this chapter we do not use the results of Chapter I.

III.1. Abstract theory for nonmonotone B.

In a Hilbert space H let $A: D(A) \subset H \to H$ be a linear operator satisfying Property I with, in addition, $\dim N(A) < \infty$. In other words A is a linear densely defined closed operator such that $N(A) = N(A^*)$ (*) and

$$\{u \in D(A);\ |u| \leqslant 1 \text{ and } |Au| \leqslant 1\} \quad \text{is compact}. \tag{3.1}$$

[Clearly (3.1) implies that $\dim N(A) < \infty$. It also implies that $R(A)$ is closed and $A^{-1}: R(A) \to N(A)^{\perp}$ is compact. Indeed if $f_n \in R(A)$ is such that $f_n \to f$, let $u_n \in N(A)^{\perp}$ with $Au_n = f_n$. Then u_n is bounded; otherwise $v_n = u_n/|u_n|$ would satisfy $v_{n_k} \to v$ with $v \in N(A) \cap N(A)^{\perp}$ and $|v| = 1$, a contradiction. Hence $u_{n_k} \to u$ and $Au = f$, i.e., $f \in R(A)$.]

(*) Throughout this chapter one should always keep Remark I.2. in mind—in case H has a direct sum decomposition $H = R(A) \oplus N(A)$ which is not orthogonal. See the example added in proofs.

THEOREM III.1. *Assume B is demicontinuous and satisfies*

$$(3.2)\qquad (Bv, v) \geq \frac{1}{\gamma}|Bv|^2 - M|v| - C_\gamma \qquad \forall \gamma > 0,\ \forall v \in H$$

where C_γ depends only on γ, and M is independent of γ and v.
Let $f \in H$ be such that

$$(3.3)\qquad J_B(v) \geq (f, v) \qquad \forall v \in N(A),\ v \neq 0\,,$$

then $f \in \overline{R(A+B)}$.
In addition if

$$(3.4)\qquad J_B(v) > (f, v) \qquad \forall v \in N(A),\ v \neq 0$$

then $f \in \operatorname{Int}[R(A+B)]$.

REMARK III.1. One might ask whether in Theorem III.1 the condition (3.4) may be replaced by the weaker one:

$$(3.4')\qquad f \in \operatorname{Int}[R(A) + R(B)]\,.$$

This is not the case, as the following example with $H = R^2$ shows:

$$u_1 = 1 \qquad \beta(u_1) + \tfrac{1}{2}\beta(u_2) = \tfrac{1}{4}\,.$$

Here β is a C^∞ nondecreasing function: $\boldsymbol{R} \to \boldsymbol{R}$ with $\beta(r) \equiv 0$ for $r < 0$, $\beta(r) \equiv 2$ for $r > 1$. We have

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \beta(u_1) + \frac{1}{2}\beta(u_2) \end{pmatrix}, \quad f = \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix},$$

and $N(A)$ is spanned by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This example satisfies all the conditions of Theorem III.1—with (3.4′) in place of (3.4)—but it has no solution as one easily sees.

Theorem III.1 is an immediate consequence of our next result in which H has an orthogonal decomposition $H = H_1 \oplus H_2$ with $H_1 = R(A)$, $H_2 = N(A)$. For an element $u \in H$ we write $u = u_1 + u_2 = P_1 u + P_2 u$ or sometimes $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

THEOREM III.2. *Assume A has* Property I *with* $\dim N(A) < \infty$. *Let* $B: H \to H$ *be demicontinuous and bounded* (i.e. B maps bounded sets into bounded sets). *Assume for a given* $f \in H$, $\forall u \in H$

$$(Bu - f, u) \geqslant \frac{1}{\gamma}|Bu|^2 - \mu|u| - C(|u_1| + 1) \tag{3.5}$$

for some positive constants $\gamma < \alpha$, μ *and* C. *Suppose*

$$J_B(v) \geqslant (f, v) \qquad \forall v \in N(A),\ v \neq 0 \tag{3.6}$$

then $\operatorname{dist}(f, \overline{R(A + B)}) \leqslant \mu\gamma/\alpha$. *Suppose*

$$J_B(v) \geqslant (f, v) + c_0|v| \qquad \forall v \in N(A),\ v \neq 0 \tag{3.7}$$

with $c_0 > 0$ *and* $\gamma < \alpha(1 + \mu/c_0)^{-1}$, *then* $f \in \operatorname{Int}[R(A + B)]$.

PROOF. Given $\varepsilon > 0$ we shall prove that for every $f \in H$ the equation

$$\varepsilon u_2 + Au + Bu = f \tag{3.8}$$

has a solution and then let $\varepsilon \to 0$.

Step 1. (3.8) has a solution.

Set $A_\varepsilon = \varepsilon P_2 + A$, A_ε is one-one onto H and A_ε^{-1} is continuous from weak H into strong H. Equation (3.8) is equivalent to

$$u = A_\varepsilon^{-1}(f - Bu) \equiv Tu\,.$$

We shall verify that

$$Tu \neq \lambda u \qquad \forall u \in H,\ |u| = R,\ \forall \lambda \geqslant 1 \tag{3.9}$$

provided R is large enough. The Schauder fixed point theorem will then imply that T has a fixed point with norm $< R$ as in Step 4 in the proof of Lemma I.11. Suppose $Tu = \lambda u$ with $\lambda \geqslant 1$, i.e.

$$\varepsilon\lambda u_2 + \lambda Au + Bu = f\,.$$

Taking the scalar product of this with u and using (3.5) we find

$$\varepsilon|u_2|^2 + \frac{1}{\gamma}|Bu|^2 \leqslant \frac{\lambda^2}{\alpha}|Au|^2 + \mu|u_2| + C(|u_1| + 1)\,.$$

Finally we use the fact that

$$|u_1| \leqslant \frac{1}{\alpha_0} |Au| \tag{3.10}$$

and $\lambda Au = f - B_1 u$, to conclude that $|Bu| \leqslant C$, $|A(\lambda u)| \leqslant C$, $|u_1| \leqslant C$, $|u_2| \leqslant C$ (where C may depend on ε). Choosing R large enough we see that (3.9) is proved.

Step 2. For u_ε a solution of (3.8) we study now the behavior of u_ε as $\varepsilon \to 0$. Taking the scalar product of (3.8) with u_ε we find using (3.5)

$$\begin{aligned}\varepsilon |u_{2\varepsilon}|^2 + \frac{1}{\gamma} |Bu_\varepsilon|^2 &\leqslant \frac{1}{\alpha} |Au_\varepsilon|^2 + \mu |u_{2\varepsilon}| + C\big(|u_{1\varepsilon}| + 1\big) \\ &\leqslant \frac{1}{\alpha} |Au_\varepsilon|^2 + \mu |u_{2\varepsilon}| + C\big(|Au_\varepsilon| + 1\big) \leqslant \frac{1}{\alpha'} |Bu_\varepsilon|^2 + \mu |u_{2\varepsilon}| + C\end{aligned}$$

where we fix α' in $\gamma < \alpha' < \alpha$; hence

$$\varepsilon |u_{2\varepsilon}|^2 + \left(\frac{1}{\gamma} - \frac{1}{\alpha'}\right) |Bu_\varepsilon|^2 \leqslant \mu |u_{2\varepsilon}| + C\,. \tag{3.11}$$

Using (3.8) again we have

$$\varepsilon |u_{2\varepsilon}|^2 + (Bu_\varepsilon, u_\varepsilon) - (f, u_\varepsilon) \leqslant \frac{1}{\alpha} |Au_\varepsilon|^2 \leqslant \frac{1}{\alpha'} |Bu_\varepsilon|^2 + C\,. \tag{3.12}$$

Combining (3.11) and (3.12) we obtain

$$\frac{\varepsilon}{\gamma} |u_{2\varepsilon}|^2 + \left(\frac{1}{\gamma} - \frac{1}{\alpha'}\right) [(Bu_\varepsilon, u_\varepsilon) - (f, u_\varepsilon)] \leqslant \frac{\mu}{\alpha'} |u_{2\varepsilon}| + C\,. \tag{3.13}$$

If for a sequence $\varepsilon_n \to 0$, $|u_{2\varepsilon_n}|$ remains bounded, it follows from (3.11) that $|Bu_{\varepsilon_n}|$ remains bounded as well as $|Au_{\varepsilon_n}|$, $|u_{1\varepsilon_n}|$. We conclude (extracting another subsequence) that $u_{\varepsilon_n} \to u$ and that u satisfies

$$Au + Bu = f\,.$$

In what follows we may therefore assume that $|u_{2\varepsilon}| \to \infty$ as $\varepsilon \to 0$. From (3.11) and the inequalities $|u_{1\varepsilon}| \leqslant C|Au_\varepsilon| \leqslant C\big(|Bu_\varepsilon| + 1\big)$ we deduce that $\lim_{\varepsilon \to 0} |u_{1\varepsilon}|/|u_{2\varepsilon}| = 0$. We may extract a sequence $\varepsilon_n \to 0$ such that

$$\frac{u_{\varepsilon_n}}{|u_{2\varepsilon_n}|} \to v$$

with $v \in N(A)$ and $|v| = 1$.

Dividing (3.13) by $|u_{2\varepsilon}|$ and passing to the limit we obtain

$$\frac{1}{\gamma}\limsup \varepsilon_n|u_{2\varepsilon_n}| + \left(\frac{1}{\gamma}-\frac{1}{\alpha'}\right)[J_B(v)-(f,v)]\leqslant\frac{\mu}{\alpha'}\,.$$

If (3.6) holds we conclude that

$$\limsup \varepsilon_n|u_{2\varepsilon_n}|\leqslant\frac{\mu\gamma}{\alpha'}\qquad\text{for every } \alpha' \text{ with } \gamma<\alpha'<\alpha$$

and so $\operatorname{dist}(f,\overline{R(A+B)})\leqslant\mu\gamma/\alpha$.
If (3.7) holds we have

$$\left(\frac{1}{\gamma}-\frac{1}{\alpha'}\right)c_0\leqslant\frac{\mu}{\alpha'}\qquad\text{for every } \alpha' \text{ with } \gamma<\alpha'<\alpha$$

i.e. $\gamma\geqslant\alpha(1+\mu/c_0)^{-1}$, —a contradiction.

This means that when (3.7) holds there must be some sequence $\varepsilon_n\to 0$ such that $|u_{2\varepsilon_n}|\leqslant C$ and consequently $f\in R(A+B)$. Clearly (3.5) and (3.7) are stable under small perturbations of f so that in fact $f\in\operatorname{Int}[R(A+B)]$.

Condition (1.14) in our main result Theorem I.10 of Chapter I is in some sense weaker than (3.5) in that we permitted an additional term $-\tau|u_1|^2$ on the right-hand side. In fact we may include such a term here; it is easy to see that the proof of Theorem III.2 yields the following sharper form:

THEOREM III.2′. *Let A, B be as in* Theorem III.2 *except that in place of* (3.5) *we assume*

$$(3.5')\qquad (Bu-f,u)\geqslant\frac{1}{\gamma}|Bu|^2-\mu|u|-C-\tau|u_1|^2\qquad \textit{for all } u\in H\,,$$

for some positive constants $\gamma<\alpha$, μ and C, and some $\tau<\alpha_0^2(\gamma^{-1}-\alpha^{-1})$. Then (3.6) *implies*

$$\operatorname{dist}(f,\overline{R(A+B)})\leqslant\frac{\mu\gamma}{\alpha}\left(1-\frac{\gamma\tau}{\alpha_0^2}\right)^{-1},$$

while (3.7) *implies $f\in\operatorname{Int}[R(A+B)]$ provided*

$$\gamma<\alpha\left(1+\frac{\mu}{c_0}+\frac{\tau\alpha}{\alpha_0^2}\right)^{-1}.$$

Next we describe a corollary which is useful when dealing with an equation of the form

$$Au+Bu+Qu=f$$

where B is, for example, a gradient of a convex function, and Q is a perturbation « going to zero at infinity. » It bears some relation to Theorem I.1 (here B need not be monotone, but $\dim N(A) < \infty$).

COROLLARY III.3. *Assume A has* Property I *with* $\dim N(A) < \infty$. *Suppose B, $Q: H \to H$ are demicontinuous, B is bounded and $R(Q)$ is bounded. Assume further* $\forall u,\ w \in H$, $\forall \delta > 0$

$$(3.14) \qquad (Bu - Bw, u) \geqslant \frac{1}{\gamma}|Bu|^2 - \delta|u| - C(\delta, w)(|u_1| + 1) \qquad \textit{for some } \gamma < \alpha\,.$$

Finally assume that $\forall u \in H$, $\lim\limits_{\substack{t \to +\infty \\ z \to u}} (Q(tz), z) = 0$. *Then*

$$\overline{R(A) + \operatorname{conv} R(B)} \subset \overline{R(A + B + Q)}$$

and

$$\operatorname{Int}[R(A) + \operatorname{conv} R(B)] \subset \operatorname{Int}[R(A + B + Q)]\,.$$

In particular,

$$J_B(v) \geqslant (f, v) \qquad \forall v \in N(A),\ v \neq 0$$

implies $f \in \overline{R(A + B + Q)}$, *and*

$$J_B(v) > (f, v) \qquad \forall v \in N(A),\ v \neq 0$$

implies $f \in \operatorname{Int}[R(A + B + Q)]$.

PROOF. Let $f \in R(A) + \operatorname{conv} R(B)$, so that f can be written $f = Av + \sum t_i Bw_i$, $t_i \geqslant 0$, $\sum t_i = 1$. It follows from (3.14) that $\forall u \in H$, $\forall \delta > 0$

$$(3.15) \qquad (Bu - f, u) \geqslant \frac{1}{\gamma}|Bu|^2 - \delta|u| - C(\delta)(|u_1| + 1)\,.$$

On the other hand $\forall u \in H$, $\forall \delta > 0$ we have

$$(3.16) \qquad (Qu, u) \geqslant -\delta|u| - C(\delta)(|u_1| + 1)\,.$$

Indeed, suppose the contrary, then there exists $\delta_0 > 0$ and a sequence u_n such that

$$(Qu_n, u_n) < -\delta_0|u_n| - n(|u_{1n}| + 1)\,.$$

In particular

$$|u_{1n}| + 1 \leqslant \frac{C}{n}\,|u_n|$$

and thus $\lim\limits_{n\to\infty} |u_n| = \infty$, $\lim |u_{1n}|/|u_n| = 0$.

Extracting a subsequence we may assume that $u_n/|u_n| \to v \in N(A)$. Therefore $\delta_0 \leqslant |(Qu_n, u_n/|u_n|)| \to 0$, a contradiction.

Combining (3.15) and (3.16) we find $\forall u \in H$, $\forall \delta > 0$

$$(3.17)\qquad (Bu + Qu - f, u) \geqslant \frac{1}{\gamma'}\,|Bu + Qu|^2 - 2\delta|u| - C(\delta)(|u_1| + 1)$$

with $\gamma < \gamma' < \alpha$.

Clearly (3.17) implies $\forall \delta > 0$

$$J_{B+Q}(v) \geqslant (f, v) - 2\delta|v| \qquad \forall v \in N(A)$$

i.e.

$$J_{B+Q}(v) \geqslant (f, v) \qquad\qquad \forall v \in N(A)\,.$$

Therefore Theorem III.2 applies and we may conclude that $f \in \overline{R(A + B + Q)}$.

In case $f \in \operatorname{Int}[R(A) + \operatorname{conv} R(B)]$, we find

$$J_{B+Q}(v) \geqslant (f, v) + c_0|v| \qquad \forall v \in N(A),\ c_0 > 0\,.$$

Theorem III.2 yields $f \in \operatorname{Int}[R(A + B + Q)]$.

We conclude with a simple consequence of Theorem III.2.

COROLLARY III.4. *Assume A has* Property I *with* $\dim N(A) < \infty$. *Let $B\colon H \to H$ be demicontinuous and bounded. Assume $\forall u \in H$*

$$(Bu, u) \geqslant \frac{1}{\gamma}\,|Bu|^2 - \mu(|u| + 1)$$

for some positive constants γ and μ. Suppose

$$J_B(v) > 0 \qquad \forall v \in N(A),\ v \neq 0\,.$$

Then there exists $\lambda_0 > 0$ such that for every λ with $0 < |\lambda| < \lambda_0$ the equation $Au + \lambda Bu = 0$ has a solution.

PROOF. Clearly λB satisfies (3.5) with $f = 0$ and $\lambda\gamma$ in place of γ, $\lambda\mu$ in place of μ. Since $J_{\lambda B} = \lambda J_B$ we see that (3.7) holds with λc_0 in place of c_0.

Finally, we have $\lambda\gamma < \alpha(1 + \mu/c_0)^{-1}$ provided $\lambda > 0$ is small enough. Replacing A by $-A$ we may handle the case $\lambda < 0$.

III.2. $Au + g(x, u) = f(x)$ in Ω.

We now restrict ourselves to a special $H = L^2(\Omega)$, Ω a σ-finite measure space, and B of special form. Let $A: D(A) \subset H \to H$ be a linear operator satisfying Property I with $\dim N(A) < \infty$. Let $g(x, u): \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ be measurable in x, and continuous in u. Set

$$(Bu)(x) = g(x, u(x)) ,$$

$$g_+(x) = \liminf_{u \to +\infty} g(x, u) , \qquad g_-(x) = \limsup_{u \to -\infty} g(x, u) .$$

Assume

$$u \cdot g(x, u) \geqslant -c(x)|u| - d(x) \qquad \forall u \in \boldsymbol{R}, \text{ a.e. } x \in \Omega,\ c \in L^2,\ d \in L^1,\ c \geqslant 0,\ d \geqslant 0 \tag{3.18}$$

and

$$|g(x, u)| \leqslant \gamma |u| + b(x) \qquad \forall u \in \boldsymbol{R}, \text{ a.e. } x \in \Omega,\ b \in L^2 \text{ with } \gamma < \alpha . \tag{3.19}$$

We shall sometimes assume

$$g_-(x) \leqslant g_+(x) \qquad \text{a.e. } x \in \Omega \tag{3.20}$$

or

$$|g(x, u)| \leqslant \gamma |u| + b_\gamma(x) \qquad \forall \gamma > 0,\ \forall u \in \boldsymbol{R}, \text{ a.e. } x \in \Omega,\ b_\gamma \in L^2 . \tag{3.21}$$

Corollary III.5. *Assume* (3.18), (3.19) *and that one of the assumptions* (3.20) *or* (3.21) *holds. Let* $f \in L^2$. *If*

$$\int_{[v>0]} g_+ v + \int_{[v<0]} g_- v \geqslant \int f v \qquad \forall v \in N(A),\ v \neq 0 \tag{3.22}$$

then $f \in \overline{R(A + B)}$.

If

$$\int_{[v>0]} g_+ v + \int_{[v<0]} g_- v > \int f v \qquad \forall v \in N(A),\ v \neq 0 \tag{3.23}$$

then $f \in \operatorname{Int}[R(A + B)]$.

If g is independent of x, (3.22) clearly implies $g_- \leqslant g_+$.

REMARK III.2. Since g_+ (respectively g_-) can take the value $+\infty$ (respect. $-\infty$), assumptions (3.22) and (3.23) have to be verified only for the functions $v \in N(A)$, $v \neq 0$, such that $v \leqslant 0$ a.e. on $[g_+ = +\infty]$ and $v \geqslant 0$ a.e. on $[g_- = -\infty]$. In case, say, $g_+ \equiv +\infty$, assumption (3.22) on f should read

$$\int g_- v \geqslant \int fv \qquad \forall v \in N(A)\,, \qquad v \neq 0,\ v \leqslant 0 \text{ a.e.}$$

(respectively for (3.23), $\int g_- v > \int fv$ $\forall v \in N(A)$, $v \neq 0$, $v \leqslant 0$ a.e.)

In many instances the set $\{v \in N(A),\ v \neq 0,\ v \leqslant 0 \text{ a.e.}\}$ is empty; we can conclude then that $R(A + B) = H$ (since any $f \in L^2$ satisfies (3.23)). This is true for example when $Au = -\Delta u - \lambda u$, $D(A) = H^2 \cap H_0^1$ and $\lambda \neq \lambda_1$ (λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet data).

Without carrying out the proof, we remark, in addition, that if (3.18), (3.19) hold, and a stronger form of (3.23):

$$\text{(3.23')} \qquad \int_{[v>0]} g_+ v + \int_{[v<0]} g_- v > \int fv + c_0 \qquad \forall v \in N(A),\ |v| = 1\,,$$

for some $c_0 > (|f|_{L^2} + |c|_{L^2})\ (\alpha/\gamma - 1)^{-1}$, then $f \in \operatorname{Int}[R(A + B)]$.

PROOF OF COROLLARY III.5. It follows from (3.18) and (3.19) that

$$\text{(3.24)} \qquad u \cdot g(x, u) \geqslant \frac{1}{\gamma'} |g(x, u)|^2 - c(x)|u| - d(x)$$

$\forall u \in \boldsymbol{R}$, a.e. $x \in \Omega$, for some $\gamma' < \alpha$, $c \in L^2$, $d \in L^1$.
Indeed we have

$$\begin{aligned} u \cdot g(x, u) &= ug(x, u) + c|u| + d - c|u| - d \\ &= |ug(x, u) + c|u| + d| - c|u| - d \\ &\geqslant |u||g(x, u)| - 2c|u| - 2d \\ &\geqslant \frac{1}{\gamma} (|g(x, u)| - b)\,|g(x, u)| - 2c|u| - 2d \end{aligned}$$

which yields (3.24).

When we assume (3.21) the conclusion follows directly from Theorem III.1.

Turning now to the case where (3.20) holds, we shall prove that (3.5) is satisfied for some $\gamma < \alpha$ and all $\mu > 0$, so that we can apply Theorem III.2 to prove Cor. III.5.

We fix a function $\theta \in L^2$ such that $\|\theta\|_{L^2} = 1$, $\theta > 0$ a.e. on Ω and, more precisely, $\theta(x) \geqslant \theta_n > 0$ on Ω_n where $\Omega = \bigcup_n \Omega_n$ and meas $\Omega_n < \infty$.

Given $\mu > 0$, any f satisfying (3.22) can be split as

$$f = h_\mu + j_\mu$$

with $h_\mu \in L^2$, $j_\mu \in R(A)$ and

$$g_- - \mu\theta \leqslant h_\mu \leqslant g_+ + \mu\theta \qquad \text{a.e. on } \Omega .$$

Indeed set

$$K = \{h \in L^2;\ g_- - \mu\theta \leqslant h \leqslant g_+ + \mu\theta \text{ a.e. on } \Omega\} .$$

Note that $K \neq \emptyset$ since $g_+ \geqslant -c$ and $g_- \leqslant c$ and thus $\max\{g, -c\} \in K$. We wish to show that $P_2 f \in P_2 K$.

Suppose the contrary that $P_2 f \notin P_2 K$. Applying Hahn-Banach in $N(A)$ we find $\xi \in N(A)$, $\xi \neq 0$, such that

$$(f, \xi) \geqslant (h, \xi) \qquad \forall h \in K .$$

In particular

$$(f, \xi) \geqslant \int_{[\xi > 0]} (g_+ + \mu\theta)\xi + \int_{[\xi < 0]} (g_- - \mu\theta)\xi \geqslant \int f\xi + \mu \int \theta|\xi|$$

—a contradiction.

We have

$$(3.25) \qquad (Bu - f, u) = (Bu - h_\mu, u) - (j_\mu, u) \geqslant (Bu - h_\mu, u) - C_\mu |u_1| .$$

We now split

$$(Bu - h_\mu, u) = \int_{\Omega \setminus \Omega_n} (Bu - h_\mu) u + \int_{\Omega_n} (Bu - h_\mu) u .$$

On $\Omega \setminus \Omega_n$ we use (3.24) to find

$$\int_{\Omega \setminus \Omega_n} (Bu - h_\mu) u \geqslant \frac{1}{\gamma'} \int_{\Omega \setminus \Omega_n} |Bu|^2 - \int_{\Omega \setminus \Omega_n} (|c| + |h_\mu|)|u| - C .$$

Fixing n large enough (depending on μ) so that

$$\int_{\Omega \setminus \Omega_n} (|c| + |h_\mu|)^2 < \mu^2 ,$$

yields

$$\int_{\Omega\setminus\Omega_n}(Bu-h_\mu)u \geqslant \frac{1}{\gamma'}\int_{\Omega\setminus\Omega_n}|Bu|^2-\mu|u|-C\,. \tag{3.26}$$

We now consider $\int_{\Omega_n}(Bu-h_\mu)u$. Since $h_\mu-2\mu\theta\leqslant g_+-\mu\theta\leqslant g_+-\mu\theta_n$ on Ω_n, we find by applying Egorov's theorem to the sequence $g_k(x) = \operatorname*{Inf}_{u\geqslant k} g(x,u)$, that $\forall\delta>0$, $\exists E\subset\Omega_n$ with meas $E<\delta$, and there exists R such that

$$g(x,u)\geqslant h_\mu-2\mu\theta \qquad \text{a.e. } x\in\Omega_n\setminus E \text{ for } u\geqslant R\,.$$

Similarly we may assume

$$g(x,u)\leqslant h_\mu+2\mu\theta \qquad \text{a.e. } x\in\Omega_n\setminus E \text{ for } u\leqslant -R\,.$$

Hence

$$(g(x,u)-h_\mu)u\geqslant -2\mu\theta|u| \qquad \text{a.e. } x\in\Omega_n\setminus E, \text{ for } |u|\geqslant R\,.$$

Consequently a.e. $x\in\Omega_n\setminus E$, for $|u|\geqslant R$,

$$\begin{aligned}(g(x,u)-h_\mu)u &\geqslant |g(x,u)-h_\mu||u|-4\mu\theta|u|\\ &\geqslant\frac{1}{\gamma}|g(x,u)-h_\mu|(|g(x,u)|-b)-4\mu\theta|u|\\ &\geqslant\frac{1}{\gamma'}|g(x,u)|^2-4\mu\theta|u|-d'\end{aligned}$$

with $\gamma<\gamma'<\alpha$ and $d'\in L^1$.

By (3.19) a similar inequality holds for a.e. $x\in\Omega_n\setminus E$ and $|u|\leqslant R$, so that in fact we find for a.e. $x\in\Omega_n\setminus E$, $\forall u\in\mathbf{R}$,

$$(g(x,u)-h_\mu)\,u\geqslant\frac{1}{\gamma'}|g(x,u)|^2-4\mu\theta|u|-d'\,. \tag{3.27}$$

Finally we write

$$\int_{\Omega_n}(Bu-h_\mu)u=\int_{\Omega_n\setminus E}(Bu-h_\mu)u+\int_E(Bu-h_\mu)u\,.$$

We estimate the first integral using (3.27) and the second integral using (3.24) so that

$$\int\limits_{\Omega_n \setminus E} (Bu - h_\mu) u \geq \frac{1}{\gamma'} \int\limits_{\Omega_n \setminus E} |Bu|^2 - 4\mu |u| - C(\delta)$$

$$\int\limits_E (Bu - h_\mu) u \geq \frac{1}{\gamma'} \int\limits_E |Bu|^2 - \int\limits_E (|c| + |h_\mu|)|u| - C(\delta)\,.$$

Choosing $\delta > 0$ so small that meas $E < \delta$ implies $\int\limits_E (|c| + |h_\mu|)^2 < \mu^2$ we find

$$\int\limits_E (Bu - h_\mu) u \geq \frac{1}{\gamma'} \int\limits_E |Bu|^2 - \mu |u| - C\,.$$

Thus we obtain

$$\int\limits_{\Omega_n} (Bu - h_\mu) u \geq \frac{1}{\gamma'} \int\limits_{\Omega_n} |Bu|^2 - 5\mu |u| - C\,. \tag{3.28}$$

The desired result follows by combining (3.25), (3.26) and (3.28).

III.3. $Au + g(x, u) = f$, A monotone, g strongly nonlinear.

Let $H = L^2(\Omega)$ and suppose $A: D(A) \subset H \to H$ has Property I with $\dim N(A) < \infty$ and $\alpha = +\infty$, i.e. $(Au, u) \geq 0$ $\forall u \in D(A)$. If $g(x, u)$ satisfies (3.18) and

$$|g(x, u)| \leq \gamma |u| + b(x) \qquad \text{a.e. } x \in \Omega,\ \forall u \in \mathbf{R}$$

for *some* $\gamma < \infty$, then, as we know by Theorem III.2,

$$(3.22) \Rightarrow f \in \overline{R(A + B)}$$

$$(3.23) \Rightarrow f \in \operatorname{Int}[R(A + B)]\,.$$

When $\alpha = +\infty$, we may use a different technique to handle « strongly » nonlinear equations—that is $g(x, u)$ growing faster than linearly in u.

Assume now that meas. $\Omega < \infty$ and

(3.29) $A: D(A) \subset H \to H$ is a linear maximal monotone operator. Thus H has an orthogonal decomposition $H = \overline{R(A)} \oplus N(A)$; we write $u = u_1 + u_2$, $u_1 \in \overline{R(A)}$, $u_2 \in N(A)$.

(3.30) The set $\{u \in D(A);\ \|u\|_{L^1} \leqslant 1,\ \|Au\|_{L^1} \leqslant 1,\ (Au, u) \leqslant 1\}$ is relatively compact in L^1 (and therefore $\dim N(A) < \infty$).

(3.31) There is a constant C such that $\|u_1\|_{L^1} \leqslant C(\|Au\|_{L^1} + (Au, u)^{\frac{1}{2}})$ $\forall u \in D(A)$.

(3.32) $g(x, u)\colon \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is measurable in x, continuous in u and $\forall R > 0$

$$h_R(x) = \sup_{|u| \leqslant R} |g(x, u)| \in L^1(\Omega),$$

(3.33) $u \cdot g(x, u) \geqslant -c|u| - d$ a.e. $x \in \Omega$, $\forall u \in \boldsymbol{R}$ with $c \geqslant 0$, $d \geqslant 0$, $c \in L^\infty$, $d \in L^1$.

THEOREM III.6. *Assume* (3.29)-(3.33) *hold. Let* $f \in L^\infty$ *be such that* (3.23) *holds. Then there exists* $u \in L^1$ *with* $g(x, u) \in L^1$, $u \cdot g(x, u) \in L^1$, *which is a solution of*

$$\bar{A}u + g(x, u) = f(x)$$

where $\bar{A}$ *denotes the closure of* A *in* $L^1 \times L^1$.

REMARK III.3. If we strengthen (3.31) to

$$\|u_1\|_{L^2} \leqslant C(\|Au\|_{L^1} + (Au, u)^{\frac{1}{2}}) \qquad \forall u \in D(A), \tag{3.31'}$$

we may take $f \in L^2$ instead of L^∞, and assume (3.33) with $c \in L^2$ (instead of $c \in L^\infty$). In addition the solution u lies in L^2 and satisfies $\tilde{A}u + g(x, u) = f$ where $\tilde{A}$ denotes the closure of A in $L^2 \times L^1$ (the proof is essentially the same as the proof of Theorem III.6). (3.31') holds if $R(A)$ is closed in L^2 and $(Au, u - v) \geqslant -C(v)$ $\forall u, v \in D(A)$, by Proposition A.7 (for example, $A = A^*$ or A trimonotone).

PROOF OF THEOREM III.6. Set $g_n(x, u) = T_n g(x, u)$ where

$$T_n z = \begin{cases} n & z > n \\ z & \text{if } |z| \leqslant n \\ -n & \text{if } z < -n \end{cases} \qquad n > 0 \text{ integer}$$

Clearly (3.33) implies

$$u \cdot g_n(x, u) \geqslant -c|u| - d \quad \text{a.e. } x \in \Omega,\ \forall u \in \boldsymbol{R}.$$

There exists $u_n \in D(A)$ solution of

$$\frac{1}{n} u_n + A u_n + g_n(x, u_n) = f(x) \tag{3.35}$$

$\Bigg($the existence follows from the Schauder fixed point theorem applied in L^1 to the equation

$$u = \left(\frac{1}{n} I + A\right)^{-1} (f - g_n(x, u))\Bigg).$$

Multiplying (3.35) by u_n yields

$$\frac{1}{n}\int u_n^2 + (A u_n, u_n) + (u_n, g_n(x, u_n)) \leqslant \|f\|_{L^\infty} \|u_n\|_{L^1}. \tag{3.36}$$

On the other hand, by (3.33) we have

$$u \cdot g_n(x, u) \geqslant |u||g_n(x, u)| - 2c|u| - 2d \qquad \text{a.e. } x \in \Omega, \ \forall u \in \mathbf{R}.$$

Therefore

$$\frac{1}{n}\int u_n^2 + \int |u_n||g_n(x, u_n)| \leqslant C(\|u_n\|_{L^1} + 1).$$

Splitting $\int |u_n||g_n(x, u_n)|$ into $\int\limits_{|u_n| \leqslant R} + \int\limits_{|u_n| > R}$ we see using (3.32) that $\forall R > 0$

$$\frac{1}{n}\int u_n^2 + R\int |g_n(x, u_n)| \leqslant C\|u_n\|_{L^1} + C(R).$$

Thus

$$\frac{1}{n}\int u_n^2 + R\int |A u_n| \leqslant C\|u_n\|_{L^1} + C(R) + \frac{R}{n}\|u_n\|_{L^1}$$

and since $\|u_n\|_{L^1} \leqslant C\|u_n\|_{L^2}$ we obtain $\forall R > 0$

$$R\|A u_n\|_{L^1} \leqslant C\|u_n\|_{L^1} + C(R)$$

that is

$$\|A u_n\|_{L^1} \leqslant \varepsilon\|u_n\|_{L^1} + C(\varepsilon) \qquad \forall \varepsilon > 0. \tag{3.37}$$

Going back to (3.36) we find

$$(A u_n, u_n) \leqslant C\|u_n\|_{L^1} + C. \tag{3.38}$$

Combining (3.37) and (3.38) and using (3.31) leads to

$$\|u_{1n}\|_{L^1} \leqslant \varepsilon \|u_n\|_{L^1} + C(\varepsilon) \qquad \forall \varepsilon > 0 . \tag{3.39}$$

CLAIM. $\|u_{2n}\|_{L^1}$ remains bounded as $n \to \infty$. If not let $\|u_{2n_k}\|_{L^1} \to \infty$ and set $v_n = u_n / \|u_{2n}\|_{L^1}$. It follows from (3.39) that $v_{1n_k} \to 0$ in L^1; next $v_{2n_k} \to v$ in $N(A)$ with $\|v\|_{L^1} = 1$. Finally setting $t_n = \|u_{2n}\|_{L^1}$ we find

$$\int v_n g_n(x, t_n v_n) \leqslant \int f v_n .$$

Passing to the limit as in the proof of Proposition II.4 yields

$$\int\limits_{[v>0]} g_+ v + \int\limits_{[v<0]} g_- v \leqslant \int f v$$

—a contradiction.

Therefore $\|u_{2n}\|_{L^1}$ remains bounded, as well as $\|u_{1n}\|_{L^1}$, $\|Au_n\|_{L^1}$, (Au_n, u_n), $\|g_n(x, u_n)\|_{L^1}$ and $\|u_n g_n(x, u_n)\|_{L^1}$. It follows from (3.30) that $u_{n_k} \to u$ in L^1 and thus $g_{n_k}(x, u_{n_k}) \to g(x, u)$ a.e.

We shall deduce from Vitali's convergence theorem that $g_{n_k}(x, u_{n_k}) \to$ $\to g(x, u)$ in L^1. We have only to verify that the integrals $\int g_n(x, u_n)$ are uniformly absolutely continuous. By (3.32)

$$|g_n(x, u)| \leqslant h_R(x) + \frac{|u|}{R} |g_n(x, u)| \qquad \forall R > 0, \text{ a.e. } x \in \Omega, \ \forall u \in \mathbf{R}$$

and so

$$\int\limits_E |g_n(x, u_n)| \leqslant \int\limits_E h_R(x) + \frac{1}{R} \int |u_n||g_n(x, u_n)| \leqslant \int\limits_E h_R(x) + \frac{C}{R} < \varepsilon$$

provided meas. $E < \delta$ (first fix R large enough so that $C/R < \varepsilon/2$ and then δ so small that $\int\limits_E h_R(x) < \varepsilon/2$ for meas. $E < \delta$).

The conclusion follows directly.

Thus we find $\bar{A}u + g(x, u) = f$. Q.E.D.

III.4. $Au + g(x, u) = f$ for systems.

Corollary III.5 admits a partial extension to systems. Set $H = (L^2)^N$, $u(x) = (u^1(x), u^2(x), \ldots, u^N(x))$ and let $A: D(A) \subset (L^2)^N \to (L^2)^N$ be a linear operator having Property I with $\dim N(A) < \infty$. Let $g(x, u): \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ be measurable in x and continuous in u. Set $(Bu)(x) = g(x, u(x))$. In place of $g_{\pm}$ we consider now the recession function J_g of g, that is $J_g\colon \Omega\times\mathbf{R}^N\to\mathbf{R}^N$ is defined by

$$J_g(x,\xi)=\liminf_{\substack{t\to+\infty\\ \eta\to\xi}} g(x,t\eta)\cdot\eta\,.$$

Assume for some fixed $c\in L^2$ and for every $\gamma>0$

$$u\cdot g(x,u)\geqslant\frac{1}{\gamma}\,|g(x,u)|^2-c(x)|u|-d_\gamma(x)\qquad\text{a.e. } x\in\Omega,\ \forall u\in\mathbf{R}^N,\ d_\gamma\in L^1\,. \tag{3.40}$$

If $f\in(L^2)^N$ is such that

$$\int\limits_{[v\neq0]}J_g(x,v(x))\,dx\geqslant\int fv\qquad\forall v\in N(A),\ v\neq0 \tag{3.41}$$

then

$$f\in\overline{R(A+B)}\,;$$

if in addition

$$\int\limits_{[v\neq0]}J_g(x,v(x))\,dx>\int fv\qquad\forall v\in N(A),\ v\neq0 \tag{3.42}$$

then

$$f\in\operatorname{Int}[R(A+B)]\,.$$

PROOF. We apply Theorem III.1. Note that as in the proof of Proposition II.4, (3.40) implies

$$J_B(v)\geqslant\int\limits_{[v\neq0]}J_g(x,v(x))\,dx\qquad\forall v\in H$$

$\Big($the integral $\int_{[v\neq0]}J_g(x,v(x))\,dx$ makes sense in $(-\infty,+\infty]$ since $J_g(x,v(x))\geqslant$ $\geqslant-c(x)|v(x)|$ a.e. $x\in\Omega\Big)$.

Example III.1. Suppose $G(x,u)\colon\Omega\times\mathbf{R}^N\to\mathbf{R}$ is measurable in x, convex C^1 in u. Set $g(x,u)=D_uG(x,u)\colon\Omega\times\mathbf{R}^N\to\mathbf{R}^N$. Assume $\forall\theta>0$

$$|g(x,u)|\leqslant\theta|u|+h_\theta(x)\qquad\text{a.e. } x\in\Omega,\ \forall u\in\mathbf{R}^N \text{ with } h_\theta\in L^2 \tag{3.43}$$

then (3.40) holds (see e.g. the proof of Proposition A.4 with $w=0$). In this case it is in fact sufficient to assume that (3.43) holds for *some* $\theta<\alpha/2$ (apply Corollary II.7).

In case $\alpha=+\infty$ we may also handle «strongly nonlinear» systems. For example Theorem III.6 can be generalized as follows (here we assume meas. $\Omega<\infty$).

THEOREM III. 6′. *Assume* $A: D(A) \subset (L^2)^N \to (L^2)^N$ *is a linear maximal monotone operator satisfying* (3.29)-(3.31). *Assume* $g(x, u): \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ *is measurable in* x, *continuous in* u *and satisfies*

(3.44) *For some fixed* $c \in L^\infty$ *and for every* $R > 0$

$$u \cdot g(x, u) \geqslant R|g(x, u)| - c|u| - d_R(x) \qquad \text{a.e. } x \in \Omega,\ \forall u \in \mathbf{R}^N,\ d_R \in L^1 .$$

Let $f \in (L^\infty)^N$ *be such that* (3.42) *holds, then there exists* $u \in (L^1)^N$ *such that* $|g(x, u)| \in L^1$, $u \cdot g(x, u) \in L^1$, *solution of the system*

$$\bar{A}u + g(x, u) = f(x)$$

where $\bar{A}$ *denotes the closure of* A *in* $L^1 \times L^1$.

We omit the proof. It is quite similar to the proof of Theorem III.6.

REMARK III.4. Suppose $G(x, u): \Omega \times \mathbf{R}^N \to \mathbf{R}$ is measurable in x, convex C^1 in u. Set $g(x, u) = D_u G(x, u)$. Assume $g(x, 0) \in (L^\infty)^N$ and $\forall R > 0$

$$h_R(x) = \sup_{|u| \leqslant R} |g(x, u)| \in L^1$$

then it is easy to check that (3.44) holds (proceed as in the proof of Proposition A.4).

REMARK III.5. If we strengthen (3.31) as in Remark III.3, then we may take $f \in L^2$ and $c \in L^2$ in Theorem III.6′.

CHAPTER IV

ELLIPTIC EQUATIONS

In this chapter we shall apply our abstract results (except in Theorem IV.4) to semilinear elliptic equations of the form

$$Lu + g(x, u) = f .$$

In IV.1-3 we treat several scalar equations; it is clear that many variants and refinements of these models can be given. In the last section we treat elliptic systems.

In the scalar case we take L to be a strongly elliptic operator of order $2m$ with coefficients in $C^\infty(\bar{\Omega})$ though it is clear from the proofs that little regularity is required when dealing with generalized solutions. For simplicity we confine ourselves to zero Dirichlet data, but many other boundary conditions could be imposed. As usual we denote by H_0^m the H^m closure of $C_0^\infty(\Omega)$. We may consider L as an unbounded linear operator $L\colon D(L) \subset H \to H$ where $D(L) = H^{2m} \cap H_0^m$. Clearly L is closed, $R(L)$ is closed, $N(L)$ is finite dimensional.

In Section IV.1 we assume that L satisfies

$$(Lu, u) \geqslant 0 \qquad \forall u \in D(L) . \tag{4.1}$$

Therefore L is maximal monotone (since $\lambda I + L$ is onto for large positive λ, by Gårding's inequality). Also we have $R(L) = N(L)^\perp$ and thus L has property I of I.1 with $\alpha = +\infty$. Zero is therefore the first eigenvalue of L when $N(L) \neq \{0\}$.

We treat nonlinearities of the form $Bu = g(x, u(x))$ where $g\colon \Omega \times \mathbf{R} \to \mathbf{R}$ is measurable in x, continuous in u; set

$$g_+(x) = \liminf_{u \to +\infty} g(x, u) , \qquad g_-(x) = \limsup_{u \to -\infty} g(x, u) .$$

In IV.1 we consider two cases, g has at most linear growth in u, or g has unlimited but one sided growth (for example $g(x, u) \geqslant 0$ for $u \geqslant 0$, $g \leqslant 0$ for $u \leqslant 0$, but no other growth condition is imposed on g). In the second case we obtain only weak solutions and the results are related to those of McKenna, Rauch [McK-R] and De Figueiredo, Gossez [DeF-G].

In IV.2 a different method based on sub and supersolutions yields the existence of smooth solutions for second order equations. In IV.3 we treat non-monotone self adjoint L, for example $L = -\Delta u - \lambda_k u$, with λ_k an eigenvalue of $-\Delta$ other than the first one.

Finally in IV.4 we treat some special elliptic systems for pairs of scalars $u = \begin{pmatrix} v \\ w \end{pmatrix}$; for example with linear part of the form

$$\begin{pmatrix} \pm \Delta + \lambda I & 0 \\ 0 & \pm \Delta + \mu I \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -\Delta - \lambda I \\ \Delta + \lambda I & 0 \end{pmatrix} .$$

For the nonlinear term $B = g(x, u)$ we take a gradient operator or a monotone operator which is $o(|u|^{\frac{1}{2}})$. Four different types of examples are presented.

IV.1. L monotone; resonance at the first eigenvalue.

Assume

$$|g(x, u)| \leqslant \theta|u| + b(x) \qquad \text{a.e. } x \in \Omega, \ \forall u \in \mathbf{R} \tag{4.2}$$

for some $\theta > 0$ and $b \in L^2$,

$$u \cdot g(x, u) \geqslant -c(x)|u| - d(x) \qquad \text{a.e. } x \in \Omega, \ \forall u \in \mathbf{R} \tag{4.3}$$

with $c \in L^2$, $d \in L^1$, $c \geqslant 0$, $d \geqslant 0$.

PROPOSITION IV.1. *Assume* (4.1)-(4.3). *Let* $f \in L^2$.
If

$$\int_{[v>0]} g_+ v + \int_{[v<0]} g_- v \geqslant \int f v \qquad \forall v \in N(L), \ v \neq 0 \tag{4.4}$$

then $f \in \overline{R(L + B)}$.
If

$$\int_{[v>0]} g_+ v + \int_{[v<0]} g_- v > \int f v \qquad \forall v \in N(L), \ v \neq 0 \tag{4.5}$$

then $f \in \operatorname{Int}[R(L + B)]$.
In addition if $b \in L^\infty$, $f \in C^\infty$, $g \in C^\infty$ *then the solution* u *of* $Lu + Bu = f$ *is* C^∞.

PROOF. In the proof of Cor. III.5 we have seen that (3.18), (3.19), i.e. (4.3), (4.2), imply (3.24); existence therefore follows from Theorem III.2. The proof of the regularity of the solution is a standard bootstrap argument. Let $u \in H^{2m} \cap H_0^m$ be a solution of $Lu + Bu = f$. If $m > n/4$ it follows that u is Hölder continuous. Then all (x, u) derivatives of g are bounded. Continuing in this way we find easily with the aid of the Schauder estimates that $u \in C^\infty$. If $m < n/4$, it follows from the Sobolev inequalities that $u \in L^p$ for $1/p = \frac{1}{2} - 2m/n$ (if $m = n/4$, $u \in L^p$, $\forall p < \infty$). From (4.2) we deduce that $g(x, u) \in L^p$. By the elliptic regularity theory we infer that $u \in H^{2m,p}$. Applying Sobolev once more we find that either u is Hölder continuous or belongs to L^q with $1/q = 1/p - 2m/n = \frac{1}{2} - 4m/n$. Continuing this way we see that u is continuous and then as before u is necessarily in C^∞.

We now drop the growth restriction (4.2) and prove the existence of generalized solutions.

Assume

$$\forall R > 0 , \quad \sup_{|u| \leqslant R} |g(x, u)| \in L^1 . \tag{4.6}$$

PROPOSITION IV.2. *Assume* (4.1), (4.3), (4.6). *Let* $f \in L^2$. *If* (4.5) *holds, then there exists a weak solution* $u \in H_0^m$, of $Lu + Bu = f$ *with* $u \cdot g(x, u) \in L^1$ (so that $g(x, u) \in L^1$).

PROOF. It relies on Theorem III.6 (or more precisely Remark III.3). Here we have in fact

$$\|u\|_{H^m}^2 \leqslant C[(Lu, u) + \|Lu\|_{L^1}^2] \qquad \forall u \in D(L) \cap R(L) . \tag{4.7}$$

Indeed by Gårding's inequality

$$\|u\|_{H^m} \leqslant C[(Lu, u)^{\frac{1}{2}} + \|u\|_{L^2}] \qquad \forall u \in D(L) ,$$

and (4.7) then follows provided we can show that

$$\|u\|_{L^2}^2 \leqslant C((Lu, u) + \|Lu\|_{L^1}^2) \qquad \forall u \in D(L) \cap R(L) \tag{4.8}$$

for some constant C. Assume (4.8) does not hold. Then there is a sequence $u_n \in D(L) \cap R(L)$ with

$$\|u_n\|_{L^2} = 1 \quad \text{and} \quad (Lu_n, u_n) + \|Lu_n\|_{L^1}^2 \to 0 .$$

In virtue of Gårding's inequality above we see that

$$\|u_n\|_{H^m} \leqslant C \quad \text{independent of } n .$$

Consequently a subsequence, still denoted by u_n, converges strongly in L^2 to some function $u \in H_0^m$ and since $\|Lu_n\|_{L^1} \to 0$ we see that $u \in N(L)$, i.e. $u \perp R(L)$. But $u \in R(L)$ and $\|u\|_{L^2} = 1$-contradiction; (4.8) is proved and so is (4.7).

REMARK IV.1. Using (4.7) it is not difficult (via the same proof as in Theorem III.6) to verify that we need only assume $f \in L^p$ and $c \in L^p$ (c occurs in (4.3)) where p is such that $H_0^m \subset L^{p'}$ (i.e. $p = 1$ for $m > n/2$, $p > 1$ arbitrary for $m = n/2$, and $p = 2n/(n + 2m)$ for $m < n/2$).

COROLLARY IV.3. *Assume now that $g(x,u)$ is non decreasing in u, $g(x,0)=0$, $g(\cdot,u)\in L^1$, $\forall u\in\boldsymbol{R}$ and for some constant C, $|g(x,-u)|\leqslant \leqslant C|g(x,u)|$* a.e. $x\in\Omega$, $\forall u\in\boldsymbol{R}$ (for example g may be odd in u). *Then the operator $Au=Lu+Bu$ with*

$$D(A)=\{u\in H_0^m;\, ug(x,u)\in L^1, Lu+g(x,u)\in L^2\}$$

is maximal monotone in H. Here Lu is meant in the distribution sense; for $u\in D(A)$ we see that $g(x,u)\in L^1$ and hence $Lu\in L^1$.

PROOF. A is monotone, i.e. $\forall u,\ v\in D(A)$

$$(Lu-Lv,u-v)+\int_\Omega[g(x,u)-g(x,v)](u-v)\geqslant 0\,.$$

By hypothesis, the first term is non-negative and we have only to verify that the integral is well defined—which is easy to do using the conditions on g since $|g(x,u)\cdot v|\leqslant\max\big(g(x,u)\cdot u,\ g(x,v)\cdot v,\ |g(x,-v)v|\big)$. To prove that A is maximal, it suffices to show that given any $f\in L^2$ we can solve

$$u+Lu+g(x,u)=f\,.$$

We are reduced to Proposition IV.2 with $I+L$ in place of L. Since $N(I+L)=\{0\}$, (4.5) holds for any $f\in L^2$ Q.e.d.

REMARK IV.2. Assume g is C^∞. If g has a « mild » polynomial growth, it is a straightforward bootstrap argument (as in the proof of Proposition IV.1) to show that $f\in C^\infty$ implies $u\in C^\infty$. In the general case we don't know any regularity result, and in fact we believe that the solution need not be smooth except when L is of second order—see the next section.

IV.2. Resonance at the first eigenvalue for second order equations.

In this section we consider L of second order and permit g to have unlimited but one sided growth in u. Here we make use of sub and supersolutions of the equation and consequently we can treat the most general elliptic second order operator

$$Mu=-\sum a^{ij}u_{x_ix_j}+\sum a^iu_{x_i}+au$$

with coefficients in $C^\infty(\bar{\Omega})$; a^{ij} is positive definite. We consider Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. The first eigenvalue λ_1 of M (i.e. with smallest real part) is real and, as is well known, the eigenspace is spanned by a function $v_1 > 0$ in Ω. Let $w_1 > 0$ be the corresponding eigenfunction for the adjoint operator $M^* - \lambda_1$. Writing $M - \lambda_1 = L$, we wish to solve

$$Lu + g(x, u) = f(x)\,, \quad u = 0 \text{ on } \partial\Omega\,. \tag{4.9}$$

Since we may add a constant to λ_1 we may assume $a(x) \geqslant 0$ in Ω; then $\lambda_1 > 0$.

Assume

$$u \cdot g(x, u) \geqslant -c(x)|u| \quad \text{a.e. } x \in \Omega,\ \forall u \in \mathbf{R} \tag{4.10}$$
$$\text{with } c \in L^p,\ p > \nu,\ c \geqslant 0,\ \nu = \max(1, n/2)\,.$$

$$\forall R > 0\,, \quad \sup_{|u| \leqslant R} |g(x, u)| \in L^p \quad p > \nu\,. \tag{4.11}$$

THEOREM IV.4. *Assume* (4.10), (4.11). *Let* $f \in L^p$ *be such that*

$$\int g_+ w_1 > \int f w_1 > \int g_- w_1\,, \tag{4.12}$$

then there exists a solution $u \in C \cap H_0^1$ *of* (4.9). *Furthermore if* $g \in C^\infty$ *and* $f \in C^\infty$, *there is a solution* $u \in C^\infty$.

Condition (4.12) is very close to conditions $\beta_\pm$ in Kazdan, Warner [K-W]; consequently the theorem is almost contained in their Theorem 2.5.

PROOF. By considering $g(x, u) - f(x)$ we may always assume that $f = 0$. We will construct a supersolution $\bar{u} \geqslant 0$, i.e. $\bar{u}$ will satisfy

$$L\bar{u} + g(x, \bar{u}) \geqslant 0 \quad \text{in } \Omega,\ \bar{u} \geqslant 0 \text{ on } \partial\Omega\,.$$

Let

$$h_j(x) = \operatorname*{Inf}_{\substack{u \in C(\bar{\Omega}) \\ u \geqslant j v_1}} g(x, u) \quad j = 1, 2, \ldots\,.$$

Clearly h_j is an increasing sequence and as $j \to \infty$, $h_j(x) \to g_+(x)$ a.e. in Ω so that $\int h_j w_1 \to \int g_+ w_1$. Fix k so large that $\int h_k w_1 > 0$. Since $-c \leqslant h_k \leqslant g(x, kv_1)$ we have $h_k \in L^p$.

Setting

$$\bar{h}_k = \frac{\int h_k w_1}{\int w_1}$$

we find

i) $\bar{h}_k > 0$,

ii) there exists a solution $u_0 \in H_0^1 \cap C$ (since $p > \nu$) of

$$Lu_0 = -h_k + \bar{h}_k .$$

For $\bar{u} = u_0 + \varepsilon + \mu v_1$, with $\varepsilon > 0$, $\mu > 0$ to be chosen, we have

$$L\bar{u} = -h_k + \bar{h}_k + (a - \lambda_1)\varepsilon .$$

Now choose $\varepsilon = \bar{h}_k/\lambda_1$ and then μ so large that $\bar{u} \geqslant kv_1$ (this is always possible since $u_0 + \varepsilon > 0$ on a neighborhood of $\partial\Omega$). Thus $L\bar{u} + g(x, \bar{u}) \geqslant -h_k + a\varepsilon + h_k \geqslant 0$. Similarly we construct a subsolution $\underline{u} \leqslant 0$.

The existence of a solution u of (4.9) with $\underline{u} \leqslant u \leqslant \bar{u}$ is quite standard. We sketch a proof: Set

$$T(x, r) = \begin{cases} \bar{u}(x) & \text{if } \bar{u}(x) \leqslant r \\ r & \text{if } \underline{u}(x) \leqslant r \leqslant \bar{u}(x) \\ \underline{u}(x) & \text{if } r \leqslant \underline{u}(x) . \end{cases}$$

Using the Schauder fixed point theorem we see that there exists a solution $u \in H_0^1 \cap H^{2,p}$ (so u is continuous) of

$$Mu + f(x, T(x, u(x))) = 0 \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega , \tag{4.13}$$

where

$$f(x, r) = -\lambda_1 r + g(x, r) .$$

We claim that $u(x) \leqslant \bar{u}(x)$. Suppose $u > \bar{u}$ somewhere. In the set $\tilde{\Omega}$ where the function $w = u - \bar{u}$ is positive, w is a generalized solution of

$$-\sum a^{ij} w_{x_i x_j} + \sum a^j w_{x_i} = -\left(L\bar{u} + g(x, \bar{u})\right) - aw \leqslant 0 \text{ in } \tilde{\Omega},\ w = 0 \text{ on } \partial\tilde{\Omega} .$$

By the generalized form of the maximum principle, see for instance Theorem 2 in Littman [Li], it follows that $w \leqslant 0$ in $\tilde{\Omega}$, i.e. $u \leqslant \bar{u}$ in $\tilde{\Omega}$. Similarly $u \geqslant \underline{u}$ and therefore (4.13) leads to (4.9). In case g and f are in C^∞ one shows in the usual way that $u \in C^\infty$.

REMARK IV.3. The proof of Theorem IV.4 does not rely on our abstract results. If we restrict ourselves to a self adjoint case, say to $M = -\Delta$, we can give an alternative proof of the theorem. First we apply Proposition IV.2 to obtain the existence of a generalized solution $u \in H_0^1$ of (4.9) such that $ug(x, u) \in L^1$. Next, a bootstrap argument yields $u \in C$. Indeed, multiplying (4.9) by $|u|^{q-2}u$ we obtain

$$(q-1)\int |u|^{q-2}\left(\frac{\partial u}{\partial x}\right)^2 \leqslant \int (\lambda_1|u| + |c|)|u|^{q-1}.$$

(the integration is justified by a truncation of the function $|u|^{q-2}u$). Noting that

$$|u|^{q-2}\left(\frac{\partial u}{\partial x_i}\right)^2 = \left[\frac{2}{q}\frac{\partial}{\partial x_i}|u|^{q/2}\right]^2$$

and using Sobolev's inequality we find

$$\| |u|^{q/2}\|_{L^{2^*}}^2 \leqslant C\int (\lambda_1|u| + |c|)|u|^{q-1}$$

where

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}.$$

Thus for $r = q2^*/2$,

$$\|u\|_{L^r}^q \leqslant C(\lambda_1\|u\|_{L^q}^q + \|c\|_{L^q}\|u\|_{L^q}^{q-1})$$

(provided $q \leqslant p$). Starting with $q = 2$ we end up with $u \in L^p$. Finally we multiply (4.9) by $(u-k)^+$, $k \geqslant 0$ and obtain

$$\int |\mathrm{grad}\,(u-k)^+|^2 \leqslant \int (\lambda_1|u| + |c|)(u-k)^+.$$

Since $\lambda_1|u| + |c| \in L^p$ and $p > \nu$ we may proceed as in Stampacchia [Sta], Chap. 4, to conclude that $u \in L^\infty$. The second method is more involved than the first one, but it shows that *every* generalized solution is smooth provided all the data are smooth.

REMARK IV.4. As a direct consequence of Theorem IV.4 we obtain the following. Assume (4.10) (4.11) and that $f \in L^p$, $p > \nu$. Let $\mu(x) \in C(\overline{\Omega})$ with $\mu \leqslant \lambda_1$ in Ω, $\mu \not\equiv \lambda_1$. Then there exists a solution $u \in H_0^1 \cap H^{2,p}$ of

$$Mu - \mu u + g(x, u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Indeed we write the equation as

$$Mu - \lambda_1 u + \tilde{g}(x, u) = f$$

where $\tilde{g} = g + (\lambda_1 - \mu)u$, so that now $\int \tilde{g}_\pm w_1 = \pm\infty$ and (4.12) therefore holds for $\tilde{g}$.

IV.3. Resonance at any eigenvalue for self adjoint L.

We assume now that $+L$ *or* $-L$ is a (scalar) *self adjoint* elliptic operator of order $2m$ with zero Dirichlet boundary condition. Thus L can be viewed as a self adjoint (unbounded) operator $L: D(L) \subset H \to H$ where $H = L^2(\Omega)$ and $D(L) = H^{2m}(\Omega) \cap H_0^m(\Omega)$. Clearly L is closed $R(L) = N(L)^\perp$, $N(L)$ is finite dimensional and $L^{-1}: R(L) \to R(L)$ is compact (in other words L has Property I).

We denote by α the largest positive constant such that $(Lu, u) \geqslant -(1/\alpha)|Lu|^2$, $\forall u \in D(L)$ and we assume that $\alpha < +\infty$ (the case $\alpha = +\infty$, i.e. $(Lu, u) \geqslant 0$, $\forall u \in D(L)$, has been considered in previous sections).

Example (*a*). $Lu = -\Delta u - \lambda u$, $D(L) = H^2 \cap H_0^1$ for some $\lambda > \lambda_1$ (the first eigenvalue of $-\Delta$), then $\alpha = \lambda - \underline{\lambda}$ where $\underline{\lambda}$ is the nearest eigenvalue of $-\Delta$ strictly less than λ.

Example (*b*). $Lu = \Delta u + \lambda u$, $D(L) = H^2 \cap H_0^1$, λ real, then $\alpha = \bar{\lambda} - \lambda$ where $\bar{\lambda}$ is the nearest eigenvalue of $-\Delta$ strictly greater than λ.

Let $g(x, u): \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ be measurable in x and continuous in u. Set

$$(Bu)(x) = g(x, u(x)),\ g_+(x) = \liminf_{u \to +\infty} g(x, u),\ g_-(x) = \limsup_{u \to -\infty} g(x, u)\,.$$

Assume

$$|g(x, u)| \leqslant \gamma|u| + b(x) \quad \text{a.e. } x \in \Omega,\ \forall u \in \boldsymbol{R},\ \gamma < \alpha,\ b \in L^2, \tag{4.14}$$

$$u \cdot g(x, u) \geqslant -c(x)|u| - d(x) \quad \text{a.e. } x \in \Omega,\ \forall u \in \boldsymbol{R},\ c \in L^2,\ d \in L^1\,. \tag{4.15}$$

Given $f \in L^2$ our purpose is to solve $Lu + Bu = f$. We prove «roughly» that

$$\int_{[v>0]} g_+ v + \int_{[v<0]} g_- v \geqslant \int f v \qquad \forall v \in N(L),\ v \neq 0 \tag{4.16}$$

implies $f \in \overline{R(L+B)}$ and

$$\int_{[v>0]} g_+ v + \int_{[v<0]} g_- v > \int f v \qquad \forall v \in N(L),\ v \neq 0 \tag{4.17}$$

implies $f \in \operatorname{Int}[R(L+B)]$.

Unfortunately we have not been able to establish a result of this generality. What we prove is that the conclusions hold if we add *one* of the following conditions:

$$|g(x,u)| \leqslant \gamma |u| + b_\gamma(x) \qquad \text{a.e. } x \in \Omega,\ \forall u \in \mathbf{R},\ \forall \gamma > 0,\ b_\gamma \in L^2 . \tag{4.18}$$

$$g_-(x) \leqslant g_+(x) \qquad \text{a.e. in } \Omega . \tag{4.19}$$

THEOREM IV.5. *Assume* (4.14) *and* (4.15) *and one of the conditions* (4.18) *or* (4.19). *Then* (4.16) *implies* $f \in \overline{R(L+B)}$, *and* (4.17) *implies* $f \in \operatorname{Int}[R(L+B)]$. *In addition when* $f \in C^\infty$ *and* $g \in C^\infty$, *then* $u \in C^\infty$.

PROOF. The existence part is a straightforward consequence of Corollary III.5. Regularity is proved as in Proposition IV.1.

COROLLARY IV.6. *Assume* (4.14) *and* (4.15) *as well as one of the conditions* (4.18) *or* (4.19).

Assume in addition that there exists $u_0 \in L^2$ *such that* $g(x, u_0(x)) \in N(L)^\perp$ *and*

$$g_+(x) > g(x, u_0(x)),\qquad g_-(x) < g(x, u_0(x)) \qquad \text{a.e. } on\ \Omega . \tag{4.20}$$

Then there exists $u \in D(L)$ *solution of* $Lu + Bu = 0$; *furthermore if* $g \in C^\infty$, *then* $u \in C^\infty$.

PROOF. We apply Theorem IV.5. We have $\forall v \in N(L),\ v \neq 0$

$$\int_{[v>0]} g_+ v + \int_{[v<0]} g_- v = \int_{[v>0]} (g_+ - g(x,u_0))\, v + \int_{[v<0]} (g_- - g(x,u_0))\, v > 0 .$$

REMARK IV.5. In case L satisfies a unique continuation property, meaning that $v \equiv 0$ is the only $v \in N(L)$ which vanishes on a set of positive measure, then the result of Corollary IV.6 holds with (4.20) replaced by

$$g_+(x) > g(x, u_0(x)),\qquad g_-(x) < g(x, u_0(x)) \tag{4.20$'$}$$

a.e. on some set $E \subset \Omega$ of positive measure.

Example: $Bu = g(x, u) = a(x)g_1(u)$ with $a \geq 0$, $a \in L^\infty$, $a \not\equiv 0$, g_1 continuous nondecreasing in u, $g_1(\pm\infty) = \pm\infty$, $|g(x, u)| \leq \gamma|u| + C$, $\forall u \in \boldsymbol{R}$, $\gamma < \alpha$. Then $L + B$ is onto L^2. Cor. IV.6 contains (essentially) Theorem 2 of Berger, Schechter [B-S].

IV.4. Elliptic systems.

For elliptic systems of the form

$$Au + g(x, u) = f(x) \quad \text{in } \Omega$$

under suitable boundary conditions, with A an elliptic operator acting on $u = (u^1, \ldots, u^N)$, it is clear that we may find suitable extensions of the results of Sections IV.1 and IV.3 for functions g satisfying (3.40), (3.41) and (3.42). We may simply apply Cor. III.5 as adapted for systems at the end of § III.2, as well as Theorem III.6′ and the related remarks.

Rather than restate these results for systems we shall take up some others—confining ourselves to rather simple systems for a pair of scalar functions $u = \begin{pmatrix} v \\ w \end{pmatrix}$. It will be clear that these, in turn, permit a great variety of extensions and modifications.

Example IV.1. Assume $+L$ or $-L$, $+M$ or $-M$ are scalar, strongly elliptic, operators having Property I in $L^2(\Omega)$ with $D(L) = D(M) = H^{2m} \cap H_0^m$. (They might even have different orders.)

Set $A\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} Lv \\ Mw \end{pmatrix}$ so that A has Property I in $H = (L^2)^2$ with $D(A) = (H^{2m} \cap H_0^m)^2$ and $\alpha = \min(\alpha_L, \alpha_M)$ (where $(Lv, v) \geq -(1/\alpha_L)|Lv|^2$, $\forall v \in D(L)$, $(Mw, w) \geq -(1/\alpha_M)|Mw|^2$ $\forall w \in D(M)$).

Set $B\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \varphi_v \\ \varphi_w \end{pmatrix}$ where

$$\varphi(v, w) = \int_\Omega (1 + a^2v^2 + b^2v^4 + c^2w^2 + d^2w^4)^{\frac{1}{2}}\,dx$$

and a, b, c, d are nonnegative constants, i.e.

$$\varphi_v = \frac{a^2v + 2b^2v^3}{[\;]^{\frac{1}{2}}}, \quad \varphi_w = \frac{c^2w + 2d^2w^3}{[\;]^{\frac{1}{2}}}.$$

Assume

$$\max(b, d) < \tfrac{1}{4}\alpha\,. \tag{4.21}$$

THEOREM IV.7. *There exists a solution* $u = \begin{pmatrix} v \\ w \end{pmatrix} \in D(A)$ *of*

$$Au + Bu = f = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

in each of the following cases:

$$(4.22) \qquad b > 0 \quad \text{and} \quad d > 0\,,$$

$$(4.23) \qquad b > 0\,, \quad d = 0 \quad \text{and} \quad c\int |w| > \int \eta w \qquad \forall w \in N(M),\ w \neq 0\,,$$

$$(4.24) \qquad b = 0\,, \quad d > 0 \quad \text{and} \quad a\int |v| > \int \xi v \qquad \forall v \in N(L),\ v \neq 0\,,$$

$$(4.25) \qquad b = 0\,, \quad d = 0$$

and

$$\int \sqrt{a^2 v^2 + c^2 w^2} > \int \xi v + \eta w \qquad \forall v \in N(L),\ \forall w \in N(M),\ \begin{pmatrix} v \\ w \end{pmatrix} \neq 0\,.$$

Furthermore $v, w \in C^\infty$ if $\xi, \eta \in C^\infty$.

Example. $Lv = \Delta v + \lambda v$, $Mw = -\Delta w - \mu w$ so that $\alpha = \mathrm{Min}(\bar{\lambda} - \lambda, \mu - \underline{\mu})$; see examples (*a*) and (*b*) in § IV.3 (we set $\underline{\mu} = -\infty$ in case $\mu \leqslant \lambda_1$, the first eigenvalue of $-\Delta$).

PROOF. We have $\varphi(u) \leqslant \varrho |u|^2 + C$ for any $\varrho > \max(b, d)$. By Proposition A.1 we see that $B = \partial\varphi$ satisfies (2.12) with $\gamma < \alpha$. We may therefore apply Corollary II.7. Note that $J_B(u) = \lim_{t \to +\infty} \varphi(tu)/t$ (see Proposition II.3) and thus for $u = \begin{pmatrix} v \\ w \end{pmatrix} \neq 0$ we find

$$J_B(u) = \begin{cases} +\infty & \begin{array}{|l} \text{if } b > 0,\ d > 0 \\ \text{or } b = 0,\ d > 0,\ w \neq 0 \\ \text{or } b > 0,\ d = 0,\ v \neq 0\,, \end{array} \\ a\int |v| & \text{if } b = 0,\ d > 0,\ w = 0\,, \\ c\int |w| & \text{if } b > 0,\ d = 0,\ v = 0\,, \\ \int \sqrt{a^2 |v|^2 + c^2 |w|^2} & \text{if } b = 0,\ d = 0\,. \end{cases}$$

The smoothness of u is proved as in Proposition IV.1.

REMARK IV.6. Let

$$Q(x, v, w) = \begin{pmatrix} q(x, v, w) \\ r(x, v, w) \end{pmatrix} : \Omega \times \boldsymbol{R}^2 \to \boldsymbol{R}^2$$

be measurable in x, continuous in v, w and such that

$$\operatorname*{Sup}_{v,w\in\boldsymbol{R}^2} \big(|q(x, v, w)| + |r(x, v, w)|\big) \in L^2 .$$

Assume

$$\lim_{t\to+\infty} q(x, tv, tw) = 0 \qquad \text{a.e. } x \in \Omega,\ \forall v \in \boldsymbol{R},\ v \neq 0,\ \forall w \in \boldsymbol{R}$$

$$\lim_{t\to+\infty} r(x, tv, tw) = 0 \qquad \text{a.e. } x \in \Omega,\ \forall v \in \boldsymbol{R},\ \forall w \in \boldsymbol{R},\ w \neq 0 .$$

Then the equation $Au + Bu + Qu = f$ has a solution provided one of the conditions (4.22)-(4.25) holds. It suffices to apply Corollary III.3.

Example IV.2.

We discuss now an example where B still has linear growth—but not small with respect to one of the unknowns.

L and M are the same as in Example IV.1 and we assume in addition $(Mw, w) \geqslant 0$ $\forall w \in D(M)$ so that $\alpha = \alpha_L$ (since $\alpha_M = +\infty$).

Assume B is the same as in Example IV.1, but in place of (4.21) we assume only

$$b < \tfrac{1}{4}\alpha \tag{4.26}$$

(and *no assumption about* d).

THEOREM IV.8. *There exists a solution* $u = \begin{pmatrix} v \\ w \end{pmatrix} \in D(A)$ *of*

$$Au + Bu = f = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

provided one of the conditions (4.22)-(4.25) *holds.*

PROOF. Set

$$H_1 = \left\{ \begin{pmatrix} v \\ 0 \end{pmatrix}; \quad v \in R(L) \right\}$$

$$H_2 = \left\{ \begin{pmatrix} v \\ w \end{pmatrix}; \quad v \in N(L) \text{ and } w \in L^2 \right\}$$

so that $H = (L^2)^2$ has an orthogonal decomposition $H = H_1 \oplus H_2$.

Clearly $A_1 = L_{|D(L)\cap R(L)}$ is one-one onto with compact inverse, while $A_2\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ Mw \end{pmatrix}$ is maximal monotone.

On the other hand we have $\forall \varepsilon > 0$

$$\varphi(u) \leqslant (b+\varepsilon)|v|^2 + C_\varepsilon(w) \qquad \forall u \in H$$

and in particular

$$\varphi(u) \leqslant (b+\varepsilon)|u_1|^2 + C'_\varepsilon(u_2) \qquad \forall u \in H$$

which implies (see Proposition A.1) that

$$(Bu - Bw,\ u - v) \geqslant \frac{1}{\gamma}\,|B_1 u|^2 - C(v, w) \qquad \forall u, v, w \in H$$

for some $\gamma < \alpha$.

Thus Theorem I.10 applies and we find

$$R(A+B) \simeq R(A) + \operatorname{conv} R(B)\,.$$

Finally we conclude with the help of Propositions II.5 and II.6 that

$$J_B(u) \geqslant (f, u) \qquad \forall u \in N(A), \qquad u \neq 0 \Rightarrow f \in \overline{R(A+B)}$$

$$J_B(u) > (f, u) \qquad \forall u \in N(A)\,, \qquad u \neq 0 \Rightarrow f \in \operatorname{Int}[R(A+B)]\,.$$

REMARK IV.7. In Theorem IV.8 M need not be linear. For example suppose M is a (nonlinear) maximal monotone operator in L^2 with $M0 = 0$. Assume $d > 0$ and $0 \leqslant b < \frac{1}{4}\alpha$ then the equation $Au + Bu = f$ is solvable provided (4.22) or (4.24) holds.

PROOF. Theorem I.10 still applies and we find

$$R(A+B) \simeq R(A) + \operatorname{conv} R(B)\,.$$

If $b > 0$ we have $\varphi(u) \geqslant \delta|u|^2$ $\forall u \in H$ and some $\delta > 0$. Thus B is onto and so $A + B$ is onto.

We consider now the case $b = 0$. By assumption

$$a\int |v| > \int \xi v \qquad \forall v \in N(L),\ v \neq 0$$

and so

$$a\int|v| - \int\xi v \geqslant c_0\int|v| \qquad \forall v\in N(L), \text{ some } c_0>0\,.$$

Let $a - c_0 < a' < a$ so that

$$a'\int|v| - \int\xi v \geqslant c_0'\int|v| \qquad \forall v\in N(L)$$

with $c_0' = a' - a + c_0 > 0$.

On the other hand we have $\forall v\in L^2$, $\forall w\in L^2$

$$\varphi(v,w)\geqslant a'\int|v| + \delta\int|w|^2 \quad \text{for some } \delta>0\,.$$

Thus for any $\eta\in L^2$

$$\varphi(v,w) - \int\xi v - \int\eta w \geqslant c_0'\int|v| + \frac{\delta}{2}\int w^2 - C \qquad \forall v\in N(L),\ \forall w\in L^2\,.$$

Minimizing $\varphi(v,w) - \int\xi v - \int\eta w$ over $N(L)\times L^2$ we see that

$$f = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \in \begin{pmatrix}N(L)^\perp\\ 0\end{pmatrix} + R(B)\subset R(A) + R(B)$$

and in fact $f\in \operatorname{Int}[R(A)+R(B)]$ q.e.d.

Example IV.3.

We discuss now an elliptic system of the form $Au + Bu = f$ where B is still monotone—but is not a gradient operator.

Let $\varphi(v,w)$ be a C^2 function satisfying $\varphi_{vv}\geqslant 0$, $\varphi_{ww}\leqslant 0$ and

$$\lim_{|v|+|w|\to\infty}\frac{|\varphi_v|^2+|\varphi_w|}{|v|+|w|} = 0\,.$$

Set $B\begin{pmatrix}v\\ w\end{pmatrix} = \begin{pmatrix}\varphi_v\\ -\varphi_w\end{pmatrix}$.

Assume $A\begin{pmatrix}v\\ w\end{pmatrix} = \begin{pmatrix}Lv\\ Mw\end{pmatrix}$: $D(A)\subset H\to H$, $H=(L^2)^2$ where $L\colon D(L)\subset L^2\to L^2$ has Property I with $\dim N(L) < +\infty$, and $M\colon D(M)\subset L^2\to L^2$ is a linear maximal monotone operator with closed range, and $\dim N(M) < +\infty$.

THEOREM IV.9. *Let $f\in H$ be such that*

$$\lim_{t\to+\infty}(B(tu),u) > (f,u) \qquad \forall u\in N(A), \quad u\neq 0\,.$$

Then the equation $Au + Bu = f$ is solvable.

PROOF. We use the same orthogonal decomposition $H = H_1 \oplus H_2$ as in the proof of Theorem IV.8. B is monotone since the quadratic form associated with $\begin{pmatrix} \varphi_{vv} & \varphi_{vw} \\ -\varphi_{vw}, & -\varphi_{ww} \end{pmatrix}$ is positive semidefinite.

On the other hand $\forall \varepsilon > 0$ we have

$$|\varphi_v|^2 + |\varphi_w| \leqslant \varepsilon(|v| + |w|) + C_\varepsilon \qquad \forall v \in \boldsymbol{R},\ \forall w \in \boldsymbol{R}\,.$$

Thus in particular we find

$$\lim_{\substack{|u| \to \infty \\ u \in H}} \frac{|B_1 u|^2 + |B_2 u|}{|u|} = 0$$

and therefore Theorem I.10 yields $R(A + B) \simeq R(A) + \operatorname{conv} R(B)$ (see Remark I.6). Also $J_B(u) = I^*_{R(B)}(u) = \lim\limits_{t \to +\infty} (B(tu), u)$ by Proposition II.2.

From Proposition II.6 we conclude that

$$I^*_{R(B)}(u) > (f, u) \qquad \forall u \in N(A),\ u \neq 0$$

implies

$$f \in \operatorname{Int}[R(A) + \operatorname{conv} R(B)]\,.$$

Example IV.4.

Assume $+L$ or $-L$ is a strongly elliptic operator of order $2m$, $L\colon D(L) \subset L^2 \to L^2$ with $D(L) = H^{2m} \cap H^m_0$. Let $g, h\colon \boldsymbol{R} \to \boldsymbol{R}$ be continuous non-decreasing functions.

THEOREM IV.10. *Let* $f = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in (L^\infty)^2$ *be such that*

$$\left\{\begin{aligned} &\int\limits_{[v>0]} g_+ v + \int\limits_{[v<0]} g_- v > \int \xi v \qquad \forall v \in N(L^*),\ v \neq 0 \\ &\int\limits_{[w>0]} h_+ w + \int\limits_{[w<0]} h_- w > \int \eta w \qquad \forall w \in N(L),\ w \neq 0\,. \end{aligned}\right. \tag{4.27}$$

Then there exists a solution $u = \begin{pmatrix} v \\ w \end{pmatrix} \in (L^2)^2$ *of the system*

$$\left\{\begin{aligned} \bar{L} w + g(v) &= \xi \\ -\bar{L}^* v + h(w) &= \eta \end{aligned}\right.$$

with $vg(v) \in L^1$, $wh(w) \in L^1$, *where* $\bar{L}$ *and* $\bar{L}^*$ *denote the* $L^1 \times L^1$ *closures of* L *and* L^*.

PROOF. We apply Theorem III.6′ with

$$A\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} Lw \\ -L^* v \end{pmatrix} \qquad D(A) = (H^{2m} \cap H_0^m)^2 .$$

Clearly A is maximal monotone; indeed $A^* = -A$ so that $\langle Au, u\rangle = \langle A^* u, u\rangle = 0$. Hence A is maximal monotone since A is closed and A, A^* are monotone (see [Bré-1]).

We rely on the following known facts

a) The sets

$$\{v \in D(L^*);\ \|v\|_{L^1} \leqslant 1,\ \|L^* v\|_{L^1} \leqslant 1\}$$

and

$$\{w \in D(L);\ \|w\|_{L^1} \leqslant 1,\ \|Lw\|_{L^1} \leqslant 1\}$$

are relatively compact in L^1.

b) L^2 has orthogonal decompositions:

$$L^2 = R(L) \oplus N(L^*) \quad \text{and} \quad H = R(L^*) \oplus N(L) .$$

Set $v = v_{R(L)} + v_{N(L^*)}$, $w = w_{R(L^*)} + w_{N(L)}$.

Then we have

$$\|v_{R(L)}\|_{L^1} \leqslant C\|L^* v\|_{L^1} \qquad \forall v \in D(L^*)$$

$$\|w_{R(L^*)}\|_{L^1} \leqslant C\|Lw\|_{L^1} \qquad \forall w \in D(L) .$$

Properties (3.30)-(3.31) follow from *a*) and *b*).

Example. Given $\lambda \in \boldsymbol{R}$, $\xi, \eta \in L^\infty$, there exist $v \in L^6$ and $w \in L^4$ solutions of the system

$$\left\{\begin{aligned} &-\Delta w - \lambda w + v^5 = \xi && \text{on } \Omega \\ &\Delta v + \lambda v + w^3 = \eta && \text{on } \Omega \\ &v = w = 0 && \text{on } \partial\Omega . \end{aligned}\right. \tag{4.28}$$

System (4.28) can also be solved by a different method. Indeed let V and W be reflexive Banach spaces. Let $\Lambda: D(\Lambda) \subset W \to V'$ be a linear densely defined closed operator. Let $\Lambda^*: D(\Lambda^*) \subset V \to W'$ be its adjoint. Set

$$A\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \Lambda w \\ -\Lambda^* v \end{pmatrix} : D(A) \subset V \times W \to V' \times W'$$

where $D(A) = D(A^*) = D(\Lambda^*) \times D(\Lambda)$. Clearly $A^* = -A$ so that $\langle Au, u\rangle = \langle A^* u, u\rangle = 0$, $\forall u \in D(A)$. It follows (see [Bré-1]) that A is maximal monotone. Thus if B is any monotone demicontinuous coercive operator from $V \times W$ into $V' \times W'$, then $A + B$ is onto.

In particular if we choose $V = L^6$, $W = L^4$, $\Lambda w = -\Delta w - \lambda w$ with $D(\Lambda) = \{w \in L^4 \cap W^{2,6/5}$ and $v = 0$ on $\partial\Omega\}$ so that $v = -\Delta v - \lambda v$, with $D(\Lambda^*) = \{v \in L^6 \cap W^{2,4/3}$ and $v = 0$ on $\partial\Omega\}$, we see that (4.28) has a unique solution in $D(\Lambda^*) \times D(\Lambda)$ (uniqueness is obvious since B is strictly monotone).

Example IV.5.

Assume $+L$ or $-L$ is a strongly elliptic operator of order $2m$, $L\colon D(L) \subset L^2 \to L^2$ with $D(L) = H^{2m} \cap H^m_0$.

Let $g, h\colon \boldsymbol{R} \to \boldsymbol{R}$ be continuous nondecreasing functions such that

$$\lim_{|r|\to\infty} \frac{g(r)}{r} = \lim_{|r|\to\infty} \frac{h(r)}{r} = 0 .$$

THEOREM IV.11. *Let* $f = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \in (L^2)^2$ *be such that* (4.27) *holds. Then there exists* $u = \begin{pmatrix}v\\ w\end{pmatrix} \in D(L^*) \times D(L)$ *solution of the system*

$$\begin{cases} Lw + g(v) = \xi \\ L^* v + h(w) = \eta . \end{cases}$$

PROOF. Set $H = (L^2)^2$, $A\begin{pmatrix}v\\ w\end{pmatrix} = \begin{pmatrix}Lw\\ L^* v\end{pmatrix}$ with $D(A) = D(L^*) \times D(L)$ so that A has Property I with $\dim N(A) < \infty$.

We may apply Corollary II.7 (here (2.12) holds with any $\gamma > 0$). On the other hand

$$J_B\begin{pmatrix}v\\ w\end{pmatrix} = \int_{[v>0]} g_+ v + \int_{[v<0]} g_- v + \int_{[w>0]} h_+ w + \int_{[w<0]} h_- w .$$

CHAPTER V

PARABOLIC AND HYPERBOLIC APPLICATIONS

V.1. Parabolic equations.

V.2. Nonlinear telegraph equations.

In the preceding section we have always taken A to be an elliptic partial differential operator. However, our results apply equally well to parabolic equations and to hyperbolic time periodic equations with dissipation. A few simple illustrations are presented here.

If V.1 we consider

$$u_t - \Delta_x u - \lambda u + g(x, t, u) = 0$$

and treat first the initial boundary value problem. We then consider the problem of finding solutions periodic in time, assuming λ is an eigenvalue of $-\Delta$, the first or some other eigenvalue. Finally we treat a parabolic system with initial and terminal conditions.

V.2 is concerned with hyperbolic equations involving dissipation, such as the nonlinear telegraph equation

$$u_{tt} - (\Delta_x - \lambda)u + \sigma u_t + g(x, t, u) = 0\,, \quad \text{for } x \in \Omega;\ u = 0 \text{ on } \partial\Omega.$$

We seek solutions which are periodic in time.

V.1. Parabolic equations.

Example V.I. *Initial boundary value problem for a parabolic equation*

$$(5.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + g(x, t, u) = 0 & \text{in} \quad \Omega \times (0, T) = Q \\ u(x, t) = 0 & \text{on} \quad \partial\Omega \times (0, T) = \Sigma \\ u(x, 0) = 0 & \quad x \in \Omega\,. \end{cases}$$

Assume

$$(5.2)\qquad g(x, t, u)\colon \Omega \times (0, T) \times \mathbf{R} \to \mathbf{R} \text{ is measurable in } (x, t),$$

continuous in u and

$$(5.3)\qquad \begin{gathered} \forall R > 0 \qquad \operatorname*{Sup}_{|u|<R} |g(x, t, u)| \in L^1(Q) \\ u \cdot g(x, t, u) \geqslant -c(x, t)|u| - d(x, t) \\ \text{a.e. } (x, t),\ \forall u,\ c \in L^2(Q),\ d \in L^1(Q)\,. \end{gathered}$$

THEOREM V.1. *There exists* $u \in L^2(0, T; H^1) \cap L^\infty(0, T; L^2)$ *with* $ug(x, t, u) \in$ $\in L^1(Q)$ *which is a generalized solution of* (5.1).

PROOF. We apply Theorem III.6 (or rather Remark III.3) in $H = L^2(Q)$ with

$$Au = \frac{\partial u}{\partial t} - \Delta u\,,$$

$$D(A) = \left\{u \in L^2(0, T; H^2 \cap H_0^1), \qquad \frac{\partial u}{\partial t} \in L^2(Q), \qquad u(x, 0) = 0 \quad x \in \Omega\right\}.$$

Clearly A is maximal monotone, $N(A) = \{0\}$, and

$$(Au, u) \geqslant c_0 \|u\|^2_{L^2(0,T,H_0^1)} \qquad \forall u \in D(A)\,, \qquad \text{some } c_0 > 0.$$

On the other hand, the set

$$\{u \in D(A);\ \|u\|_{L^1(Q)} \leqslant 1,\ \|Au\|_{L^1(Q)} \leqslant 1,\ (Au, u) \leqslant 1\}$$

is relatively compact in $L^1(Q)$ (see [B-H-V]). Therefore (3.30) and (3.31) hold. Remark III.3 yields the existence of a generalized solution $u \in L^2(Q)$. The additional properties $u \in L^2(0, T; H_0^1)$ and $u \in L^\infty(0, T; L^2)$ are easily established by a direct argument.

REMARK V.1. Under slightly stronger assumptions we may construct bounded sub- and super-solutions; a bootstrap argument yields then $u \in C^\infty$ when all data are smooth. Assume

$$\text{(5.4)} \qquad \forall R > 0 \quad \operatorname*{Sup}_{|u| < R} |g(x, t, u)| \in L^p(Q) \qquad p > n + 1\,,$$

$$\text{(5.5)} \qquad u \cdot g(x, t, u) \geqslant -c(x, t)|u| \qquad \text{a.e. } (x, t),\ \forall u \in \mathbf{R},\ c \in L^p(Q),\ p > n + 1\,.$$

Then (5.1) has a solution $u \in L^\infty(Q)$. Indeed we construct a supersolution $\overline{u} \geqslant 0$. Let u_0 be the solution of

$$\begin{aligned} \frac{\partial u_0}{\partial t} - \Delta u_0 &= c(x, t) && \text{on } Q \\ u_0(x, t) &= 0 && \text{on } \Sigma \\ u_0(x, 0) &= 0 && \text{on } \Omega\,. \end{aligned}$$

Since $c \in L^p(Q)$, $p > n + 1$, $u_0 \in C(\overline{Q})$. Then $\overline{u} = u_0 + \mu$ (with μ chosen large enough so that $\overline{u} \geqslant 0$) satisfies

$$\frac{\partial \overline{u}}{\partial t} - \Delta \overline{u} + g(x, t, \overline{u}) = c(x, t) + g(x, t, \overline{u}) \geqslant 0$$

(by (5.5)).

Example V.2. *Periodic solutions in time; resonance at* λ_1.

Consider a simple model problem:

$$(5.6)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u - \lambda_1 u + g(x, t, u) = 0 & \text{in } Q \\ u(x, t) = 0 & \text{on } \Sigma \\ u(x, T) = u(x, 0) & \text{on } \Omega \end{cases}$$

where λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition.

THEOREM V.2. *Assume* (5.2), (5.3) *and*

$$(5.7)\qquad \int_Q g_+ v_1 > 0 > \int_Q g_- v_1 .$$

Then there exists a generalized solution

$$u \in L^2(0, T; H^1) \cap L^\infty(0, T; L^2) \text{ of (5.6) } \textit{with } ug(x, t, u) \in L^1(Q).$$

PROOF. We apply again Theorem III.6 (or rather Remark III.3) in $H = L^2(Q)$ with

$$Au = \frac{\partial u}{\partial t} - \Delta u - \lambda_1 u ,$$

$$D(A) = \left\{u \in L^2(0, T; H^2 \cap H_0^1) ,\quad \frac{\partial u}{\partial t} \in L^2(Q) ,\quad u(x, T) = u(x, 0),\ x \in \Omega\right\}$$

so that $N(A) = \{\lambda v_1;\ \lambda \in \mathbf{R}\}$.

REMARK V.2. Here again, as in the proof of Theorem IV.4 we may construct bounded sub- and super-solutions under the assumptions (5.4), (5.5) and (5.7), and thus obtain the existence of a bounded solution u for (5.6).

Example V.3. *Periodic solutions; resonance at* λ_k.

Let L be a self adjoint strongly elliptic operator of order $2m$, $L\colon D(L) \subset L^2 \to L^2$ with $D(L) = H^{2m} \cap H_0^m$, and let α be the largest constant such that $(Lu, u) \geqslant -(1/\alpha)|Lu|^2\ \forall u \in D(L)$.

Consider the problem

$$(5.8)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + Lu + g(x, t, u) = f(x, t) & \text{in } Q \\ u(\cdot, t) \in H_0^m & \forall t \in (0, T) \\ u(x, 0) = u(x, T) & x \in \Omega . \end{cases}$$

THEOREM V.3. *Assume* (4.14) *and* (4.15) *and one of the conditions* (4.18) *or* (4.19) (*where* x *is replaced by* (x, t)). *Assume*

$$\int_{[(x,t)\in Q;v>0]} g_+ v + \int_{[(x,t)\in Q;v<0]} g_- v > \int_Q f\cdot v \qquad \forall v \in N(L),\ v \neq 0\,.$$

Then (5.8) *has a solution. In addition if* $f \in C^\infty$ *and* $g \in C^\infty$, *then* $u \in C^\infty$.

PROOF. Set $H = L^2(Q)$, $Au = \partial u/\partial t + Lu$ with

$$D(A) = \left\{u \in L^2(0, T; H^{2m}, \cap H_0^m),\quad \frac{\partial u}{\partial t} \in L^2(\Omega),\quad u(x, T) = u(x, 0),\quad x \in \Omega\right\}$$

Clearly $N(A) = N(A^*) = N(L)$ (since L is self adjoint).

Also, the set $\{u \in H;\ |u|_H \leqslant 1,\ |Au|_H \leqslant 1\}$ is compact in H, so that A has Property I. In addition we have

$$(Au, u) \geqslant -\frac{1}{\alpha}|Au|_H^2 \qquad \forall u \in D(A)$$

(since $|Au|^2_{L^2(Q)} = |\partial u/\partial t|^2_{L^2(Q)} + |Lu|^2_{L^2(Q)}$) and in fact α cannot be increased since the inequality is true in particular for $u \in D(L)$ independent of t (and then it says $(Lu, u) \geqslant -(1/\alpha)|Lu|^2$). In other words we have proved that A and L have the « same α ». We may therefore proceed as in the proof of Theorem IV.5. The proof of the regularity of the solution is a standard bootstrap argument (as in the proof of Proposition IV.1 except that we rely now on the L^p estimates for parabolic equations (see [L-S-U]).

Example V.5. *A parabolic system.*

Consider the system

$$(5.9)\qquad \begin{cases} \dfrac{\partial w}{\partial t} - \Delta w + v^5 = \xi & \text{on } Q \\ \dfrac{\partial v}{\partial t} + \Delta v + w^3 = \eta & \text{on } Q \\ v(x, t) = w(x, t) = 0 & \text{on } \Sigma \\ v(x, T) = w(x, 0) = 0 & \text{on } \Omega\,. \end{cases}$$

Systems similar to this occur in control theory (J. L. Lions, personal communication).

THEOREM V.4. *Given* $\xi, \eta \in L^\infty(Q)$ *there exists* $v \in L^6(Q)$ *and* $w \in L^4(Q)$ *generalized solutions of the system* (5.9).

The proof is similar to the one in Example IV.4 and is left to the reader.

V.2. Nonlinear telegraph equations.

We now apply our results to a linear hyperbolic operator A with dissipation of the form

$$Au = u_{tt} + Lu + \sigma u_t$$

acting on time periodic (period 2π) functions $u(x, t)$. Here L is a linear self adjoint elliptic operator in space variables x with coefficient independent of t on a compact manifold Ω, or on a bounded domain $\Omega \subset R^n$ with suitable homogeneous boundary conditions on u (say Dirichlet boundary conditions); $\sigma \neq 0$ is a real constant. Furthermore $u(x, t)$ might be vector valued.

Let

$$\ldots \leqslant \lambda_{-1} < 0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_d < \lambda_{d+1} \leqslant \ldots$$

be the eigenvalues of L (zero may be an eigenvalue of multiplicity d) with corresponding orthonormal (in $L^2(\Omega)$) eigenfunctions $\ldots w_{-1}, w_1, w_2, \ldots$. Since we may expand 2π-time periodic functions on Ω in the form

$$u(x, t) = \sum a_{jk} w_j(x) \exp[ikt] \qquad a_{j,-k} = \overline{a_{j,k}} \tag{5.10}$$

the action of A on such functions is easily determined. Take $H = L^2(\Omega \times \times (0, 2\pi))$, 2π time periodic. Then

$$Au = \sum a_{jk}(\lambda_j - k^2 + i\sigma k) w_j(x) \exp[ikt]\,. \tag{5.11}$$

Clearly $N = N(A)$ is finite dimensional, spanned by $w_1(x), \ldots, w_d(x)$, and $R(A) = N(A)^\perp$. Furthermore for $f \perp N(A)$,

$$f = \sum c_{jk} w_j(x) \exp[ikt]\,, \tag{5.12}$$

we have

$$A^{-1} f = \sum \frac{c_{jk}}{\lambda_j - k^2 + i\sigma k} w_j(x) \exp[ikt]\,. \tag{5.13}$$

Since $|\lambda_j| \to \infty$ as $|j| \to \infty$ we see that A^{-1} is a compact operator; A satisfies all the conditions of Property I with $\alpha =$ the largest positive number such that

$$\alpha(\lambda_j - k^2) + (\lambda_j - k^2)^2 + \sigma^2 k^2 \geqslant 0 \qquad \forall j, k$$

i.e.

$$\alpha = \inf_{\substack{j,k \\ \lambda_j < k^2}} \left| k^2 - \lambda_j + \frac{\sigma^2 k^2}{k^2 - \lambda_j} \right| .$$

We may then apply our results to nonlinear equations of the form

$$Au + Bu \equiv u_{tt} + Lu + \sigma u_t + g(x, t, u) = f(x, t) \tag{5.14}$$

where g, f are periodic in t of period 2π. We assume $N \neq \{0\}$; the case $N = \{0\}$ was treated in a very extensive way by Prodi [P].

We describe some results for the scalar case; the extension of Cor. III.5 at the end of § III.2 may be applied to systems. In the following all conditions on integration refer to $\Omega \times (0, 2\pi)$. Suppose g measurable in x, continuous in u and

$$|g(x, t, u)| \leqslant \gamma |u| + b_\gamma(x, t), \qquad \text{some } \gamma > 0,\ b_\gamma \in L^2 . \tag{5.15}$$

$$u \cdot g(x, t, u) \geqslant - c(x, t)|u| - d(x, t), \qquad c \in L^2,\ d \in L^1 . \tag{5.16}$$

Assume also that for

$$g_+(x, t) = \liminf_{u \to +\infty} g(x, t, u), \qquad g_-(x, t) = \limsup_{u \to -\infty} g(x, t, u)$$

we have, for some $c_0 > 0$,

$$\iint_{[w>0]} g_+ w \, dx\, dt + \iint_{[w<0]} g_- w \, dx\, dt \geqslant \iint f w \, dx\, dt + c_0 \left[\iint |w|^2 dx\, dt \right]^{\frac{1}{2}} \qquad \forall w \in N . \tag{5.17}$$

THEOREM V.5. *In each of the following cases* (5.14) *has a generalized solution*:

(i) $g_- \leqslant g_+$ *and* (5.15) *holds with some* $\gamma < \alpha$.

(ii) (5.15) *holds for every* $\gamma > 0$.

PROOF. Apply Corollary III.5.

Example V.7. Consider $L = - \Delta - 5$ on a square Ω: $0 \leqslant x_j \leqslant \pi$, $j = 1, 2$ in the plane, acting on real functions $u(x, t)$ which vanish on $\partial\Omega$. The eigenvalues of L are all positive and of the form $r^2 + s^2 - 5$, $r, s \in Z_+$ (positive integers). Then N consists of functions of the form $a \sin 2x_1 \sin x_2 + b \sin x_1 \sin 2x_2$. If $\sigma = \pm 1$ it is not difficult to verify that $\alpha = 2$.

Consider the corresponding nonlinear telegraph equation on the square

$$(5.14') \qquad u_{tt} \pm u_t - \Delta_x u - 5u + g(x, t, u) = f(x, t) \qquad u = 0 \text{ on } \partial\Omega .$$

Then, *in each of the following examples, the equation* (5.14′) *has a solution*:

(i) $g = C \dfrac{u^3}{1 + u^2}$, $\quad 0 < C < 2$; f arbitrary in L^2,

(ii) $g = \tanh^{-1} u + \dfrac{u}{(1 + u^2)^{\frac{1}{4}}} (1 + \sin u)$; $\quad f \in N^\perp$.

(iii) $g = \dfrac{u^3}{1 + Cu^2} + \sin u^2$, $\quad C > \frac{1}{2}$, f arbitrary in L^2.

REGULARITY. It is natural to ask if the solutions in Theorem V.5 are smooth—in case g, f, the coefficients of L and $\partial\Omega$, are in C^∞. In low dimensions we can show under some mild additional assumptions that they are, and we sketch a proof—assuming zero Dirichlet boundary conditions on $\partial\Omega$. We shall suppose L of order two, but the discussion extends also to higher order operators. We shall make use of the fact that for functions of the form

$$u(x) = \sum u_j w_j(x)$$

the H^1 norm in Ω is equivalent to

$$\left[\sum |u_j|^2 (1 + |\lambda_j|)\right]^{\frac{1}{2}}.$$

First we investigate the regularizing properties of A^{-1} represented by (5.13). For $f = \sum c_{jk} w_j \exp[ikt] \in L^2$ we see easily from (5.13) that $u = A^{-1} f$ satisfies

$$(5.18) \qquad \int_0^{2\pi}\!\!\int_\Omega \left(|u_t|^2 + \sum |u_{x_j}|^2 + |u|^2\right) dx\,dt \leqslant c \|f\|^2_{L^2} .$$

But in fact we may assert that

$$(5.19) \qquad \int_\Omega \left(|u_t|^2 + \sum |u_{x_j}|^2 + |u|^2\right) dx \leqslant C \|f\|^2_{L^2} \qquad \forall t .$$

For the left hand side

$$\leqslant C \sum_j \sum_l |c_j|^2 \sum_{\substack{k=0 \\ |\lambda_j|+k\neq 0}}^{\infty} \frac{1+|\lambda_j|+k^2}{|\lambda_j - k^2 + i\sigma k|^2},$$

and the result then follows easily from the inequality

$$\sum_{k=1}^{\infty} \frac{1+|\lambda|+k^2}{(\lambda-k^2)^2+\sigma^2 k^2} \leqslant C(\sigma) = \text{constant independent of } \lambda \text{ for } \lambda \in \boldsymbol{R}, \tag{5.20}$$

whose proof we omit.

A similar argument yields the following: If $f \in H^m$, $m > 0$ an integer, then $u = A^{-1}f$ belongs to H^m and in fact

$$\int_{\Omega} \sum_{|\alpha| \leqslant m+1} |D^{\alpha}_{x,t} u|^2 dx \leqslant C \|f\|^2_{H^m} \qquad \forall t. \tag{5.9'}$$

In Theorem V.5 our solution $u \in L^2$, and hence $g(x, t, u) \in L^2$; u has the form $A^{-1}[f(x,t) - g(x,t,u)] (\bmod N)$. Since the functions in N are C^∞, it follows from the preceding that $u \in H^1(\Omega)$ for each t.

Claim. *For $n = 1$ our solution $u \in C^\infty(\bar{\Omega} \times [0, 2\pi])$.*

Proof. Since $u \in H^1(\Omega)$, $\forall t$, u is continuous in x for each t, and also bounded. It follows then that $g(x, t, u(x, t)) \in H^1$. Applying the preceding again we find $u \in H^2$ and so $u(x, t)$ is continuous.

We may differentiate the equation with respect to t and find that u_t satisfies

$$Au_t + g_t + g_u u_t = f_t \qquad u_t = 0 \text{ on } \partial\Omega. \tag{5.21}$$

Repeating the preceding argument we find u_t is continuous in x for each t, uniformly in t—in particular u_t is bounded. It then follows that $g_t + g_u u_t \in H^1$ and so $u_t \in H^2$. Differentiating the equation (5.14) with respect to x we may solve for u_{xxx} in terms of other third (and lower order) derivatives of u, and thus conclude that $u_x \in H^2$. Hence u_t, u_x are continuous. And so on—via repeated differentiations of the equation, first with respect to t, then with respect to x, and repeated application of this argument.

Claim. *Under the additional hypotheses:*

$$|g_t| \leqslant C(1 + |u|), \qquad |g_u| \leqslant C \tag{5.22}$$

our solution in Theorem V.5 *belongs to* C^∞ *in case* $n \leq 3$. *The same conclusion holds for* $n = 4$ *if in addition we require*

$$(5.23) \qquad |g_x|\,, \quad |g_{tt}|\,, \quad |g_{tu}| \leq C(1 + |u|)\,, \quad |g_{uu}| \leq C\,.$$

PROOF. We know that $u \in H^1$. Under the new conditions on g we conclude that $g(x, t, u(x, t)) \in H^1$ and hence $u \in H^2$. As before u_t satisfies (5.21) and hence by (5.19), u_{tt}, $u_{tx_j} \in L^2(\Omega)$ $\forall t$. Then, for each t,

$$\int_\Omega |Lu|^2\, dx \leq C \quad \text{independent of } t\,.$$

It follows from the standard estimates up to the boundary for elliptic operators, that

$$\int_\Omega |u_{x_i x_j}|^2 dx \leq C \qquad \forall t\,.$$

Thus

$$(5.24) \qquad \int |D^2 u|^2 dx \leq C \qquad \forall t\,.$$

For $n \leq 3$ we may infer that $u(x, t)$ is uniformly Hölder continuous in the x variables—in particular $|u|$ is bounded while for $n = 4$ we find that $u \in L^p(\Omega)$ $\forall p < \infty$, $\forall t$. Furthermore by the Sobolev inequalities we know that for $n \leq 4$ $u_t(x, t) \in L^4(\Omega)$ $\forall t$.

Differentiating the equation (5.14) once more we find u_{tt} satisfies

$$(5.25) \qquad Au_{tt} + g_{tt} + 2g_{tu} u_t + g_{uu} u_t^2 + g_u u_{tt} = f_{tt} \qquad u_{tt} = 0 \text{ on } \partial\Omega\,.$$

Since $u_t \in L^4$ it follows that

$$g_{tt} + 2g_{tu} u_t + g_{uu} u_t^2 + g_u u_{tt} \in L^2$$

and hence

$$(5.26) \qquad \int_\Omega \left(|u_{ttt}|^2 + \sum |u_{ttx_j}|^2\right) dx \leq C \qquad \forall t\,.$$

As before we find $\int_C |Lu_t|^2 dx \leq C$ $\forall t$ and so

$$(5.27) \qquad \int_\Omega |u_{tx_i x_j}|^2 dx \leq C \qquad \forall t\,.$$

Again from elliptic theory we infer from the inequality $\|Lu\|^2_{H^1(\Omega)} \leq C$, $\forall t$ (here we've used (5.26-27)) that

$$\int_\Omega |u_{x_i x_j x_k}|^2 dx \leq C \qquad \forall t\,.$$

For $n \leq 3$ we conclude that u_t and u_{x_i} are Hölder continuous in x, in particular they are bounded. And so on for $n \leq 3$.

For $n = 4$ we infer that u is bounded and uniformly Hölder continuous in x; also u_t, $u_{x_i} \in L^p(\Omega)$ $\forall p < \infty$, $\forall t$, and $u_{tt} \in L^4(\Omega)$, $\forall t$. Differentiating the equation again we find $u_{ttt} = 0$ on $\partial\Omega$ and

$$Au_{ttt} = f_{ttt} - [g_{ttt} + 3g_{ttu}u_t + 3g_{tuu}u_t^2 + g_{uuu}u_t^3 + 3g_{tu}u_{tt} + \\ + 3g_{uu}u_t u_{tt} + g_u u_{ttt}] \in L^2\,.$$

Hence

$$\int_\Omega (|u_{tttt}|^2 + |u_{tttx}|^2)\, dx \leq C \qquad \forall t\,.$$

From (5.25) it follows that

$$\int_\Omega |Lu_{tt}|^2 dx \leq C \qquad \forall t$$

and hence by elliptic theory

$$\int_\Omega |u_{ttx_i x_j}|^2 dx \leq C \qquad \forall t\,.$$

Differentiating (5.21) with respect to x we find

$$\|Lu_t\|^2_{H^1(\Omega)} \leq C \qquad \forall t$$

and again by elliptic theory

$$\int_\Omega |u_{tx_i x_j x_k}|^2 dx \leq C \qquad \forall t.$$

Differentiating (5.14) twice with respect to x variables we find

$$\|Lu\|_{H^2(\Omega)} \leq C \qquad \forall t$$

and hence

$$|u|_{H^4(\Omega)} \leq C \qquad \forall t\,.$$

It follows that u_t, u_{x_i} are continuous in x. And so on. We consider the claim to be proved.

For $n = 5$ we can prove that the solution is in C^∞ under a slight further strengthening of the conditions in the case $n = 4$, but the proof is more complicated. For $n \geqslant 6$ we do not know whether solutions are necessarily regular.

Appendix A.

Some properties of monotone operators and gradients of convex functions.

We have collected in this appendix some technical properties of monotone operators and gradients of convex functions.

Suppose H has an orthogonal decomposition $H = H_1 \oplus H_2$. We write $u = u_1 + u_2 = P_1 u + P_2 u$.

Proposition A.1. *Let $B \subset \partial\psi$, ψ convex continuous on H. Set $B_1 = P_1 B$. Assume for some $\theta < \alpha/4$*

$$\psi(u) \leqslant \theta|u_1|^2 + C(u_2) \qquad \forall u \in H$$

where $C(u_2)$ depends only on u_2. Then for some $\gamma < \alpha$ we have

$$\text{(A.1)} \qquad (Bu - Bw, u - v) \geqslant \frac{1}{\gamma}|B_1 u|^2 - C(v, w) \qquad \forall v, w, u \in H .$$

Proof. Supposing $B_1 u \neq 0$, set $\xi = B_1 u/|B_1 u|$. By the convexity of ψ, we have for $\lambda > 0$

$$\begin{aligned}
\psi(v + \lambda\xi) - \psi(u) &\geqslant (Bu, v - u) + \lambda|B_1 u|\\
\psi(u) \qquad - \psi(w) &\geqslant (Bw, u - w)\\
\psi(w) \qquad - \psi(v) &\geqslant (Bv, w - v) .
\end{aligned}$$

Adding

$$\psi(v + \lambda\xi) - \psi(v) \geqslant -(Bu - Bw, u - v) - C(v, w) + \lambda|B_1 u| .$$

Since

$$\psi(v + \lambda\xi) \leqslant \theta|v_1 + \lambda\xi|^2 + C(v_2)$$

we find

$$\lambda|B_1 u| \leqslant (Bu - Bw, u - v) + \theta\lambda^2 + 2\lambda\theta|v_1| + C(v, w)$$

that is

$$|B_1 u| - 2\theta|v_1| \leqslant \frac{(Bu - Bw, u - v) + C(v, w)}{\lambda} + \theta\lambda .$$

Minimizing the right-hand side with respect to $\lambda > 0$ yields

$$|B_1 u| - 2\theta|v_1| \leqslant 2\sqrt{\theta[(Bu - Bw, u - v) + C(v, w)]}$$

and (A.1) follows easily. Q.e.d.

Throughout this Appendix we denote by P an orthogonal projection operator in H.

PROPOSITION A.2. *Assume B is a monotone operator and set $B_1 = PB$. Then*

$$\sup_{|v| \leqslant R} |B_1 v| = \sup_{|v| = R} |B_1 v| \equiv \mu .$$

PROOF. For $v \in H$ with $|v| < R$ we wish to show that $|B_1 v| \leqslant \mu$. Assuming $B_1 v \neq 0$, set $\xi = B_1 v/|B_1 v|$. Then the function $r(\lambda) = \big(B(v + \lambda\xi), \xi\big)$ for $\lambda \in \boldsymbol{R}$ is nondecreasing. Let $0 < \lambda_1$ be such that $|v + \lambda_1 \xi| = R$.

We have $r(0) \leqslant r(\lambda_1)$ i.e. $|B_1 v| \leqslant |B(v + \lambda_1 \xi)| \leqslant \mu$.

PROPOSITION A.3. *Assume B is a map from H into itself and set $B_1 = PB$. Let $\alpha > 0$. The following are equivalent:*

$$\text{(A.2)} \qquad \begin{cases} \textit{for some } \gamma < \alpha \\ (Bu - Bw, u - v) \geqslant \dfrac{1}{\gamma} |B_1 u|^2 - C(v, w) \qquad \forall u, v, w \in H \end{cases}$$

$$\text{(A.3)} \qquad \begin{cases} (Bu - Bw, u - v) \geqslant - C(v, w) \qquad \forall u, v, w \in H \\ \textit{and for some } \gamma < \alpha \\ (Bu - Bw, u - w) \geqslant \dfrac{1}{\gamma} |B_1 u|^2 - C(w) \qquad \forall u, w \in H . \end{cases}$$

PROOF. Since (A.2) $\Rightarrow$ (A.3) trivially, we have only to show that (A.3) $\Rightarrow$ (A.2). For every $z \in H$,

$$\big(Bu - Bw, u - (w + z)\big) \geqslant - C(w, z)$$

i.e.

$$(Bu - Bw, z) \leqslant (Bu - Bw, u - w) + C(w, z) .$$

Choosing $z = (v - w)/\varepsilon$, with $0 < \varepsilon < 1$ we find

$$(Bu - Bw, v - w) \leqslant \varepsilon(Bu - Bw, u - w) + C(\varepsilon, v, w) . \tag{A.4}$$

On the other hand

$$\begin{aligned}(Bu - Bw, u - v) &= (Bu - Bw, u - w) + (Bu - Bw, w - v)\\ &= (1 - \varepsilon)(Bu - Bw, u - w) + \varepsilon(Bu - Bw, u - w) + (Bu - Bw, w - v)\\ &\geqslant \frac{1-\varepsilon}{\gamma} |B_1 u|^2 - (1 - \varepsilon) C(w) + (Bu - Bw, v - w) - C(\varepsilon, v, w)\\ &\qquad + (Bu - Bw, w - v)\end{aligned}$$

by (A.3) and (A.4),

$$= \frac{1-\varepsilon}{\gamma} |B_1 u|^2 - (1 - \varepsilon) C(w) - C(\varepsilon, v, w) .$$

This yields (A.2) for an appropriate choice of ε.

PROPOSITION A.4. *Assume $B \subset \partial\psi$ with ψ convex continuous and set $B_1 = PB$. Assume for some $\alpha > 0$*

$$\limsup_{|v|\to\infty} \frac{|B_1 v|}{|v|} < \frac{\alpha}{2} . \tag{A.5}$$

Then for some $\gamma < \alpha$ we have

$$(Bu - Bw, u - v) \geqslant \frac{1}{\gamma} |B_1 u|^2 - C(v, w) \qquad \forall u, v, w \in H .$$

PROOF. We have, by Proposition A.2,

$$|B_1 v| \leqslant \theta |v| + C \qquad \forall v \in H,\ \theta < \alpha/2 .$$

But for $u = u_1 + u_2$,

$$\begin{aligned}\psi(u) - \psi(u_2) &= \int_0^1 \frac{d}{dt} \psi(tu_1 + u_2)\, dt \leqslant \int_0^1 |B_1(tu_1 + u_2)||u_1| dt\\ &\leqslant |u_1| \int_0^1 [\theta t |u_1| + \theta |u_2| + C]\, dt = \frac{\theta}{2} |u_1|^2 + \theta |u_1||u_2| + C|u_1| .\end{aligned}$$

Therefore

$$\psi(u) \leqslant \theta'|u_1|^2 + C(u_2) \qquad \forall u,\ \theta' < \alpha/4\,, \tag{A.6}$$

and the result follows from Proposition A.1.

REMARK. We do not know whether the conclusion holds if (A.5) is replaced by $\limsup\limits_{|v|\to\infty} |B_1 v|/|v| < \alpha$.

Proposition A.4 and its proof are closely related to Proposition 4 in [B-B].

PROPOSITION A.5 (*). *Assume $B = \partial\psi$, ψ convex continuous B is demicontinuous, and set $B_1 = PB$. Assume for some $\gamma > 0$*

$$(B_1 u - B_1 w,\ u - w) \leqslant \gamma |u - w|^2 \qquad \forall u, w \in H\,. \tag{A.7}$$

Then

$$(Bu - Bw,\ u - w) \geqslant \frac{1}{\gamma}\, |B_1 u - B_1 w|^2 \qquad \forall u, w \in H \tag{A.8}$$

and also for any $\gamma' > \gamma$,

$$(Bu - Bw,\ u - v) \geqslant \frac{1}{\gamma'}\, |B_1 u|^2 - C_{\gamma'}(v, w) \qquad \forall u, v, w \in H\,. \tag{A.9}$$

REMARK. Proposition A.5 implies in particular that for $B = \partial\psi$, $\psi \in C^1$ convex, the following are equivalent

$$(Bu - Bw,\ u - w) \leqslant \gamma |u - w|^2 \qquad \forall u, w \in H$$

$$|Bu - Bw| \leqslant \gamma |u - w| \qquad \forall u, w \in H$$

$$(Bu - Bw,\ u - w) \geqslant \frac{1}{\gamma}\, |Bu - Bw|^2 \qquad \forall u, w \in H\,.$$

The equivalence of the last two conditions is due to Baillon, Haddad [Ba-Ha], see also Dunn [Du].

PROOF: Property (A.9) follows from (A.8) since B is trimonotone (see Proposition A.3). Thus we have only to establish (A.8). We decompose the proof in 4 steps.

(*) We thank J. B. BAILLON for some useful suggestions concerning this result.

Step 1. Suppose M is a bounded linear operator which is self-adjoint, positive semidefinite. Then

$$(PMu, u) \leqslant \gamma |u|^2 \qquad \forall u \in H$$

implies

$$(Mu, u) \geqslant \frac{1}{\gamma} |PMu|^2 \qquad \forall u \in H .$$

PROOF. By Cauchy-Schwarz we have

$$(Mu, Pv) \leqslant (Mu, u)^{\frac{1}{2}} (MPv, Pv)^{\frac{1}{2}} \qquad \forall u, v \in H .$$

Choosing $v = Mu$ yields

$$\begin{aligned} |PMu|^2 = (Mu, PMu) &\leqslant (Mu, u)^{\frac{1}{2}} (MPMu, PMu)^{\frac{1}{2}} \\ &= (Mu, u)^{\frac{1}{2}} (PM(PMu), PMu)^{\frac{1}{2}} \leqslant (Mu, u)^{\frac{1}{2}} \gamma^{\frac{1}{2}} |PMu| . \end{aligned}$$ Q.e.d.

Step 2. Assume (A.7) and furthermore $\dim H < \infty$, and ψ is C^2. Then (A.8) holds.

PROOF. Choosing $w = u - tv$ in (A.7), dividing by t^2 and passing to the limit as $t \to 0$ we find

$$(PMv, v) \leqslant \gamma |v|^2 \qquad \forall v \in H$$

where $M = B'(u)$ is positive semidefinite and self-adjoint.

Thus by Step 1,

$$(Mv, v) \geqslant \frac{1}{\gamma} |PMv|^2 \qquad \forall v \in H .$$

Finally

$$\begin{aligned} |B_1 u - B_1 w|^2 &= \left| \int_0^1 PB'\big(w + t(u - w)\big)(u - w)\, dt \right|^2 \\ &\leqslant \int_0^1 |PB'\big(w + t(u - w)\big)(u - w)|^2 dt \\ &\leqslant \gamma \int_0^1 \Big(B'\big(w + t(u - w)\big)(u - w), u - w\Big) dt \\ &= \gamma (Bu - Bw, u - w) . \end{aligned}$$

Step 3. Assume (A.7) and furthermore: $\dim H < \infty$ and ψ is C^1. Then (A.8) holds.

PROOF. Let ϱ_n be a sequence of mollifiers on H tending to the delta function. Set $\psi_n = \varrho_n * \psi$, $B_n = \partial\psi_n = \varrho_n * B$. Clearly

$$(PB_n u - PB_n w, u - w) \leqslant \gamma |u - w|^2 \qquad \forall u, w \in H$$

and thus Step 2 yields

$$(B_n u - B_n w, u - w) \geqslant \frac{1}{\gamma} |PB_n u - PB_n w|^2 \qquad \forall u, w \in H .$$

As $n \to \infty$ the desired result follows.

Step 4. The general case.

Let X denote the finite dimensional space spanned by u, w, Bu, Bw, Pu, Pw, PBu, PBw. Note that $P(X) \subset X$ and therefore $P(X^\perp) \subset X^\perp$. Set $P_X =$ orthogonal projection on X. Clearly P_X commutes with P. Set $\varphi = \psi|_X$ so that φ is convex and

$$\partial\varphi(x) = P_X Bx \qquad \text{for } x \in X .$$

For $x, y \in X$ we find since P and P_X commute

$$\begin{aligned}(P\,\partial\varphi(x) - P\,\partial\varphi(y), x - y) &= \\ &= (PP_X Bx - PP_X By, x - y) = (PBx - PBy, x - y) \leqslant \gamma |x - y|^2 .\end{aligned}$$

By Step 3 applied to $\partial\varphi$ and P in X we derive $\forall x, y \in X$

$$(\partial\varphi(x) - \partial\varphi(y),\ x - y) \geqslant \frac{1}{\gamma} |P\,\partial\varphi(x) - P\,\partial\varphi(y)|^2$$

i.e.

$$(Bx - By, x - y) \geqslant \frac{1}{\gamma} |PP_X Bx - PP_X By|^2 .$$

In particular

$$(Bu - Bw, u - w) \geqslant \frac{1}{\gamma} |PBu - PBw|^2 . \qquad \text{Q.e.d.}$$

PROPOSITION A.6. *Let Ω be a measure space. Let $\alpha > 0$. Assume $g(x, u)\colon \Omega \times \mathbf{R} \to \mathbf{R}$ is measurable in x and continuous nondecreasing in u.*

Suppose

$$|g(x, u)| \leqslant \theta |u| + h(x) \qquad \text{a.e. } x \in \Omega, \ \forall u \in \boldsymbol{R}$$

with $\theta < \alpha$ *and* $h \in L^2$.

Set $(Bu)(x) = g(x, u(x))$. *Then for some* $\gamma < \alpha$

$$(Bu - Bw, u - v) \geqslant \frac{1}{\gamma} |Bu|^2 - C(v, w) \qquad \forall u, v, w \in L^2 .$$

PROOF. Clearly it suffices to prove that for some $\gamma < \alpha$

$$(Bu - Bw, u - w) \geqslant \frac{1}{\gamma} |Bu|^2 - C(w) .$$

But

$$\begin{aligned}(Bu - Bw, u - w) &= \int_\Omega |g(x, u) - g(x, w)||u - w| dx \\ &\geqslant \int_\Omega |g(x, u) - g(x, w)| \left[\frac{|g(x, u)| - h(x)}{\theta} - |w|\right] dx \\ &\geqslant \frac{1}{\theta} |Bu|^2 - C(w)|Bu| - C(w) . \qquad \text{Q.e.d.}\end{aligned}$$

PROPOSITION A.7. *Assume* $A: D(A) \subset H \to H$ *is a linear maximal monotone operator with dense domain and closed range* $R(A)$.

Assume

$$(\text{A.10}) \qquad (Au, u - v) \geqslant C(v) \qquad \forall u, v \in D(A)$$

(this holds for example if A is trimonotone).

Then there exists $\beta > 0$ *such that*

$$(Au, u) \geqslant \beta |u_1|^2 \qquad \forall u \in D(A)$$

where u_1 *denotes the orthogonal projection of* u *on* $R(A)$.

PROOF. $A|_{D(A) \cap R(A)}$ is one-one and onto $R(A)$.

$$A^{-1}: R(A) \to R(A)$$

is a bounded operator and so there exists C such that

$$|v| \leqslant C|Av| \qquad \forall v \in D(A) \cap R(A) .$$

On the other hand (A.10) implies (see [Br-Ha] Proposition 1) that there exists C such that

$$|(Av, u)| \leqslant C(Au, u)^{\frac{1}{2}}[|v|^2 + |Av|^2]^{\frac{1}{2}} \qquad \forall u \in D(A),\ \forall v \in D(A)\,.$$

In particular for $v \in D(A) \cap R(A)$ we find

$$|(Av, u)| \leqslant C(Au, u)^{\frac{1}{2}}|Av|\,,$$

i.e.

$$|(w, u)| \leqslant C(Au, u)^{\frac{1}{2}}|w| \qquad \forall u \in D(A),\ \forall w \in R(A)\,.$$

This yields

$$|u_1| \leqslant C(Au, u)^{\frac{1}{2}}\,.$$

Appendix B.

More general form of the main result.

In Theorem I.10 the nonlinear terms A_2, B were required to be monotone. We now present a more general result in which we permit additional terms which need not be monotone but which are required to satisfy conditions somewhat like those in Chapter III. The results and proofs are then a mixture of those of Chapter III and of Theorem I.10. The conditions in the results are technical and rather complicated, and they are presented without applications, but with the thought that they may prove useful in later work.

The setup is the following: H is a real Hilbert space with a given orthogonal decomposition

$$H = H_1 \oplus H_2 = P_1 H \oplus P_2 H\,.$$

Conditions:

(i) $S\colon H_1 \to H_1$ is a demicontinuous operator with $|Su| \leqslant r|u| + C$ and A_1 is an operator: $D(A_1) \subset H_1 \to H_1$ satisfying $\tilde{A}_1 = A_1 + S$ is one-one and onto; $\tilde{A}_1^{-1}$ is continuous from weak to strong and $\forall u \in D(A_1)$:

$$(\tilde{A}_1 u, u) \geqslant -\frac{1}{\alpha}|\tilde{A}_1 u|^2 - C$$

$$\alpha_0|u| \leqslant |\tilde{A}_1 u| + C$$

for some constants α, $\alpha_0 > 0$, and C.

(ii) $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ is a monotone operator mapping $D(M) = H_1 \oplus S$ into H, $S \subset H_2$, $M(0) = 0$ satisfying: for each $u_1 \in H_1$ the map $u_2 \mapsto M_2(u_1 + u_2)$ is maximal monotone; for each $u_2 \in H_2$, M_1 is demicontinuous in u_1. Furthermore

$$\text{(B.1)} \quad \left\{ \begin{array}{l} \text{For some constant } \gamma_1 > 0,\ \tau_1 \geqslant 0,\ \forall \delta > 0,\ \forall v \in H,\ \exists k(\delta)\,, \\ C(v)\,, \quad \text{such that } \forall u \in H \\ \dfrac{1}{\gamma_1}\,|M_1 u|^2 \leqslant (Mu - Mv, u - v) + \tau_1 |u_1|^2 + C(v)\big(\delta|u| + k(\delta)\big) \end{array} \right.$$

(iii) $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}: H \to H$ is a monotone demicontinuous operator with $B(0) = 0$ satisfying

$$\text{(B.2)} \quad \left\{ \begin{array}{l} \text{For some } \gamma_2 > 0,\ \tau_2 \geqslant 0,\ \forall \delta > 0,\ \forall v, w \in H,\ \exists k(\delta),\ C(v, w) \\ \text{such that} \\ \dfrac{1}{\gamma_2}\,|B_1 u - S u_1|^2 \leqslant (Bu - Bw, u - v) + \tau_2 |u_1|^2 + C(v, w)\big(\delta|u| + k(\delta)\big) \\ \hfill \forall u \in H \end{array} \right.$$

(iv) $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}: H \to H$ with $G(0) = 0$ satisfies

$$\text{(B.3)} \qquad \forall \delta > 0,\ \exists C(\delta) \quad \text{such that } |Gu| \leqslant \delta |u| + C(\delta)\,.$$

G_1 is continuous from strong $H_1 \times$ weak $H_2 \to$ weak H_1;
G_2 is continuous from strong $H_1 \times$ weak $H_2 \to$ strong H_2.

Furthermore, G_2 is compact, while G_1 satisfies:
For some $\gamma_3 > 0$, $\tau_3 \geqslant 0$, $\forall \delta > 0$, $\forall z, v \in H$, $\exists k(\delta)$, $C(z, v)$ such that

$$\text{(B.4)} \qquad \frac{1}{\gamma_3}\,|G_1 u|^2 \leqslant (Gu - Gz, u - v) + \tau_3 |u_1|^2 + C(z, v)\big(\delta|u| + k(\delta)\big)\,, \quad \forall u\,.$$

We always assume that $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ and $\tau = r + \tau_1 + \tau_2 + \tau_3$ satisfy

$$\frac{1}{\alpha} + \frac{\tau}{\alpha_0^2} < \frac{1}{\gamma}\,.$$

(iv$'$) Same as (iv) except that (B.4) is assumed to hold only for $z = v = 0$.
With $N = M + G + B$, we are interested in solving

$$\text{(B.5)} \qquad A_1 u_1 + N u = f \in H\,.$$

THEOREM B.1. *Under hypotheses* (i)-(iv)

$$R(A_1 + M + G + B) \simeq R(A_1) + R(M) + R(G) + \operatorname{conv} R(B) .$$

THEOREM B.1′. *Under hypotheses* (i)-(iii) *and* (iv′),

$$R(A_1 + M + G + B) \simeq R(A_1) + R(M) + \operatorname{conv} R(B) .$$

Next we give some conditions on f so that (B.5) has a solution. Assume that we have an orthogonal decomposition of H_2:

$$H_2 = H_2' \oplus H_2'' \quad \text{with } \dim H_2'' < \infty .$$

Conditions on f:

(v) For every $h \in H_1 \oplus H_2'$ with $|h| < r$ for some $r > 0$,

$$f + h \in R(A_1) + R(M) + R(G) + \operatorname{conv} R(B) .$$

(v′) For every $h \in H_1 \oplus H_2'$ with $|h| < r$,

$$f + h \in R(A_1) + R(M) + \operatorname{conv} R(B) .$$

For $v \in H_2''$ recall the recession function

$$J_N(v) = \liminf_{\substack{u \to v \\ t \to +\infty}} (N(tu), u) .$$

(vi) For every $v \in H_2''$, $v \neq 0$,

$$J_N(v) > (f_2'', v) .$$

THEOREM B.2. *Under conditions* (i)-(vi),

$$f \in \operatorname{Int} R(A_1 + M + G + B) .$$

THEOREM B.2′. *Under conditions* (i)-(iii), (iv′), (v′) *and* (vi),

$$f \in \operatorname{Int} R(A_1 + M + G + B) .$$

We confine ourselves to a brief sketch of the proofs which are similar to that of Theorem I.10.

LEMMA B.3. *For any $\varepsilon > 0$ and any $f \in H$ there is a solution of*

$$\varepsilon u_{2\varepsilon} + A_1 u_\varepsilon + N u_\varepsilon = f . \tag{B.6}$$

The proof is a bit different from that of Lemma I.11; the solution u is obtained as a fixed point of a compact transformation T defined as follows: For $v = v_1 + v_2$ define $u_2 \in H_2$ as the solution of

$$\varepsilon u_2 + (M_2 + B_2)(v_1 + u_2) = f_2 - G_2 v .$$

Then set

$$u_1 = A_1^{-1}[f_1 - (M_1 + B_2)(v_1 + u_2) - G_1 v]$$

and finally define $w_2 \in H_2$ as the solution of

$$\varepsilon w_2 + (M_2 + B_2)(u_1 + w_2) = f_2 - G_2 v .$$

The mapping $v \to w = Tv$ is well defined and one easily sees that a fixed point of T is a solution of (B.6). Next one verifies that T is compact and continuous and finally that for R sufficiently large

$$Tv \neq \lambda v \qquad \forall v \in H , \quad |v| = R, \ \forall \lambda \geqslant 1 .$$

Then T has a fixed point in $|v| < R$. The proofs of these facts are similar but more involved than the corresponding proofs for Theorem I.10.

Next by an argument similar to the proof of Lemma I.12 one proves

LEMMA B.4. *Under the hypotheses* (i)-(iv), *if* $f \in R(A_1) + R(M) + R(G) + \operatorname{conv} R(B)$ *then for any solution* u_ε *of* (B.6), $\varepsilon u_{2\varepsilon} \to 0$ *as* $\varepsilon \to 0$. *If we assume* (i)-(iv′) *we obtain the same conclusion provided* $f \in R(A_1) + R(M) + \operatorname{conv} R(B)$.

Then one has the analogue of Lemma I.13 with similar proof:

LEMMA B.5. *Under conditions* (i)-(vi) (or conditions (i), (ii), (iii), (iv′), (v′) and (vi)) *any solution* u_ε *of* (B.6) *satisfies* $|u_\varepsilon|$, $|A_1 u_\varepsilon|$, $|M_1 u_\varepsilon|$, $|B_1 u_\varepsilon|$, $|G_1 u_\varepsilon| \leqslant C$ *independent of* ε.

Finally, to prove Theorems B.1 (and B.1′), the statement about the closures of the ranges follows immediately from Lemma B.4. The statement about the interiors is precisely the assertion of Theorem B.2 (and B.2′) in the case $H'' = 0$, in which case condition (vi) is empty.

By a passage to the limit argument as $\varepsilon \to 0$, similar to the proof of Theorem I.10, one proves Theorems B.2, B.2′.

BIBLIOGRAPHICAL REMARKS

Chapter I. Results concerning the almost equality $R(A+B) \simeq R(A) + R(B)$, for monotone operators A and B have been first proved in [Br-Ha]. Generalizations are given by BROWDER [Bro-2], GUPTA-HESS [G-H], CALVERT-GUPTA [C-G].

The decomposition device $H = H_1 \oplus H_2$ has been extensively used in bifurcation theory and nonlinear problems at resonance i.e. equations of the form $Au + Bu = f$ where A is a linear noninvertible operator; we refer to the expository papers and notes of CESARI [C] and MAWHIN [M-1], NIRENBERG [N-3] on the alternative methods, see also OSBORN-SATHER [O-S]. For other results concerning time periodic solutions of nonlinear wave equations we refer to RABINOWITZ [Ra 1-3], DE SIMON-TORELLI [De S-T], MAWHIN [M 2-4], VEJVODA [V], HALE [Ha], LOVICAROVA [Lo] and [Br-N] which contains an extensive bibliography.

Chapters II and III. Corollary II.7. and Theorem III.1. bear some similarity with results of FUČIK [F 1-2] and FUČIK-KUČERA-NEČAS [F-K-N]; their definition of the weak η-subasymptote for a nonlinear operator is related to the notion of recession function—however the exact relationship is not clear.

The measure theory argument we use in the proof of Theorem III.6. is originally due to STRAUSS [Str]; it has been widely applied.

Chapter IV. Nonlinear elliptic equations with resonance at the first eigenvalue have been considered by LIONS-STAMPACCHIA [L-S], SCHATZMAN [Sch], HESS [He-1] under the name of «semi-coercive» problems. Recent contributions include the works of DE FIGUEIREDO [DeF-1-2], DE FIGUEIREDO-GOSSEZ [DeF-G], MCKENNA-RAUCH [McK-R] (results related to Proposition IV.2.) and KAZDAN-WARNER [K-W] (for second order equations). Since A is monotone, the results of [Br-Ha] can also be applied. Nonlinear elliptic equations with resonance at any eigenvalue: the first result goes back to LANDESMAN-LAZER [La-La] (for a simple proof, see HESS [He-2]). Related results and generalizations have been given by many authors: L. NIRENBERG [N-1-2], DANCER [Da-1-2], BERGER-SCHECHTER [B-S], SCHECHTER [Sche], WILLIAMS [W]. FUČIK [F-2] (with an extensive bibliography), and AMBROSETTI, MANCINI [A-M].

Chapter V. Resonance problems for nonlinear telegraph and parabolic equations have also been studied by MAWHIN [M-2-4].

Added in proofs. – We describe a simple example showing how the results of Chap. III apply in case $H = R(A) \oplus N(A)$ is a direct sum which is not orthogonal.

Let $H = L^2(\Omega)$ and let $A\colon D(A) \subset H \to H$ be a densely defined closed operator with $R(A)$ closed, $\dim N(A) = 1$ (for simplicity), $\{u \in D(A),\ |u| \leqslant 1 \text{ and } |Au| \leqslant 1\}$ is compact and $H = R(A) \oplus N(A)$ (but not orthogonal). Assume $N(A)$ is spanned by v_0 with $\int v_0^2 = 1$ and $N(A^*)$ is spanned by w_0 with $\int v_0 w_0 = 1$. Assume $v_0 \neq 0$ a.e. in Ω.

Let $g(x, u)\colon \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ be measurable in x, continuous in u and satisfies:

$$|g(x, u)| \leqslant b(x) \text{ a.e. } x\,, \qquad \forall u \in \boldsymbol{R} \text{ with } b \in L^2\,,$$

$g_\pm(x) = \lim\limits_{u \to \pm\infty} g(x, u)$ exists for a.e. x.

If

$$\int_{[v_0>0]} g_- w_0 + \int_{[v_0<0]} g_+ w_0 < 0 < \int_{[v_0>0]} g_+ w_0 + \int_{[v_0<0]} g_- w_0$$

then the equation $Au + g(x, u) = 0$ has a solution. Indeed $R(A)$ and $N(A)$ become orthogonal for the new scalar product

$$\langle f, g \rangle = \int f_1 g_1 + \int f_2 g_2$$

where

$$f = f_1 + f_2, \qquad g = g_1 + f_2, \qquad f_1, g_1 \in R(A), \qquad f_2, g_2 \in N(A).$$

We may therefore apply Theorem III.1. Note that

$$J_B(v_0) = \liminf_{\substack{t\to+\infty \\ u\to v_0}} \langle g(x, tu), u \rangle = \liminf_{\substack{t\to+\infty \\ u\to v_0}} \langle g(x, tu), v_0 \rangle =$$
$$= \liminf_{\substack{t\to+\infty \\ u\to v_0}} \int g(x, tu)\, w_0 = \int_{[v_0>0]} g_+ w_0 + \int_{[v_0<0]} g_- w_0 .$$

BIBLIOGRAPHY

[A-L-P] S. Ahmad - A. C. Lazer - J. L. Paul, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J., **25** (1976), pp. 933-944.

[A-M] A. Ambrosetti - G. Mancini, *Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance. - I: The case of simple eigenvalue*, to appear.

[Ba-Ha] J. B. Baillon - G. Haddad, *Quelques propriétés des opérateurs anglebornés et n-cycliquement monotones*, Israel J. Math., **26** (1977), pp. 137-150.

[B-H-V] P. Baras - J. C. Hassan - L. Veron, *Compacité de l'opérateur définissant la solution d'une équation d'évolution non homogène*, C. R. Acad. Sc., **284** (1977), pp. 799-802.

[B-S] M. Berger - M. Schechter, *On the solvability of semilinear operator equations and elliptic boundary value problems*, Bull. A.M.S., **78** (1972), pp. 741-745.

[Bré-1] H. Brézis, *On some degenerate nonlinear parabolic equations, Nonlinear Functional Analysis*, Proc. Symp. Pure Math., Vol. **18** (part 1), F. Browder ed. AMS (1970), pp. 28-38.

[Bré-2] H. Brézis, *Opérateurs maximaux monotones*, Lecture Notes, Vol. 5, North-Holland, 1973.

[B-B] H. Brézis - F. Browder, *Nonlinear integral equations and systems of Hammerstein type*, Advances in Math., **18** (1975), pp. 115-147.

[Br-Ha] H. Brézis - A. Haraux, *Image d'une somme d'opérateurs monotones et applications*, Israel J. Math., **23** (1976), pp. 165-186.

[Br-N] H. BRÉZIS - L. NIRENBERG, *Forced vibrations for a nonlinear wave equation*, Comm. Pure Appl. Math., **31** (1978).

[Bro-1] F. BROWDER, *Nonlinear Operators in Banach Spaces*, Proc. Symp. Pure Math., Vol. **18** (part 2) AMS (1976).

[Bro-2] F. BROWDER, *On a principle of H. Brézis and its applications*, J. Funct. Anal., **25** (1977), pp. 356-365.

[C-G] B. CALVERT - C. GUPTA, *Nonlinear elliptic boundary value problems in L^p spaces and sums of ranges of accretive operators*, Nonlinear Analysis, TMA **2** (1978), pp. 1-26.

[C] L. CESARI, *Alternative methods in nonlinear analysis, International Conference on Differential Equations*, Antosiewicz ed. Acad. Press. (1975).

[Da-1] E. N. DANCER, *On a nonlinear elliptic boundary value problem*, Bull. Aust. Math. Soc., **12** (1975), pp. 399-405.

[Da-2] E. N. DANCER, *On the Dirichlet problem for weakly nonlinear elliptic partial differential equations*, to appear.

[DeF-1] D. G. DE FIGUEIREDO, *The Dirichlet problem for nonlinear elliptic equations: A Hilbert space approach, Partial differential equations and related topics*, Springer Verlag Lecture Notes 446 (1975), pp. 144-165.

[DeF-2] D. G. DE FIGUEIREDO, *Semilinear elliptic problems with nonlinearities near the first eigenvalue*, Proc. Symp. Dynamical Systems, Univ. of Florida (1976), to appear.

[DeF-G] D. G. DE FIGUEIREDO - J. P. GOSSEZ, *Nonlinear perturbations of a linear elliptic problem near its first eigenvalues*, to appear.

[DeS-T] L. DE SIMON - G. TORELLI, *Soluzione periodiche di equazioni a derivate parziali di tipo iperbolico nonlineari*, Rend. Sem. Math. Univ. Padova, **40** (1968), pp. 380-401.

[Du] J. C. DUNN, *Convexity, monotonicity and gradient processes in Hilbert space*, J. Math. Anal. Applic., **53** (1976), pp. 145-158.

[F-1] S. FUČIK, *Surjectivity of operators involving linear noninvertible part and nonlinear compact perturbation*, Funk. Ekv., **17** (1974), pp. 73-83.

[F-2] S. FUČIK, *Open problems in the solvability of nonlinear equations*, to appear.

[F-K-N] S. FUČIK - M. KUČERA - J. NEČAS, *Ranges of nonlinear asymptotically linear operators*, J. Diff. Eq., **17** (1975), pp. 375-394.

[G-H] C. GUPTA - P. HESS, *Existence theorems for nonlinear noncoercive operator equations and nonlinear elliptic boundary value problems*, J. Diff. Eq., **22** (1976), pp. 305-313.

[Ha] J. HALE, *Periodic solutions of a class of hyperbolic equations containing a small parameter*, Arch. Rat. Mech. Anal., **23** (1967), pp. 380-398.

[He-1] P. HESS, *On semicoercive nonlinear problems*, Indiana Univ. Math. J., **23** (1974), pp. 645-654.

[He-2] P. HESS, *On a theorem by Landesman and Lazer*, Indiana Univ. Math. J. **23** (1974), pp. 827-830.

[K-W] J. KAZDAN - F. WARNER, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math., **28** (1975), pp. 567-597.

[L-S-U] O. LADYZENSKAJA - V. SOLONNIKOV - N. URAL'CEVA, *Linear and quasilinear equations of parabolic type*, Translations of Math. Monographs, Vol. **23** AMS (1968).

[La-La] E. LANDESMAN - A. LAZER, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech., **19** (1970), pp. 609-623.

[Le-Li] J. LERAY - J. L. LIONS, *Quelques résultats de Visik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. Fr., **93** (1965), pp. 97-107.

[L-S] J. L. LIONS - G. STAMPACCHIA, *Variational inequalities*, Comm. Pure Appl. Math., **20** (1967), pp. 493-519.

[Li] W. LITTMAN, *A strong maximum principle for weakly L-subharmonic functions*, J. Math. Mech., **8** (1959), pp. 761-770.

[Lo] H. LOVICAROVA, *Periodic solutions of weakly nonlinear wave equation in one dimension*, Czech. Math. J., **19** (1969), pp. 324-342.

[McK-R] P. MCKENNA - J. RAUCH, *Strongly nonlinear perturbations of nonnegative boundary value problems with kernel*, Jour. Diff. Equa., to appear.

[M-1] J. MAWHIN, *Nonlinear perturbations of Fredholm mappings in normed spaces and applications to differential equations*, Univ. de Brasilia, Trabalho de Mat., No. 61 (1974).

[M-2] J. MAWHIN, *Periodic solutions of nonlinear telegraph equations*, Proc. Int. Symp. Dyn. Syst. Gainesville, Florida (1976).

[M-3] J. MAWHIN, *Solutions periodiques d'équations aux dérivées partielles hyperboliques nonlinéaires* (to appear in a volume dedicated to TH. VOGEL).

[M-4] J. MAWHIN, *Lecture at Oberwolfach*, January 1976.

[N-1] L. NIRENBERG, *An application of generalized degree to a class of nonlinear problems*, Coll. Analyse Fonctionnelle, Liège, 1970 (Math. Vander., pp. 57-74).

[N-2] L. NIRENBERG, *Generalized degree and nonlinear problems, Contribution to Nonlinear Functional Analysis*, Zarantonello ed. (1971), Acad. Press, pp. 1-9.

[N-3] L. NIRENBERG, *Topics in Nonlinear Functional Analysis*, New York Univ. Lecture Notes (1974).

[O-S] J. E. OSBORN - D. SATHER, *Alternative problems and monotonicity*, J. Diff. Eq'ns., **18** (1975), pp. 393-410.

[P] G. PRODI, *Soluzioni periodiche delle equazioni a derivate parziali di tipo iperbolico nonlineari*, Annali Mat. Pura Appl., **42** (1956), pp. 25-49.

[Ra-1] P. RABINOWITZ, *Periodic solutions of nonlinear hyperbolic partial differential equations*, Comm. Pure Appl. Math., **20** (1967), pp. 145-205.

[Ra-2] P. RABINOWITZ, *Time periodic solutions of nonlinear wave equations*, Manuscripta Math., **5** (1971), pp. 165-194.

[Ra-3] P. RABINOWITZ, *Some minimax theorems and applications to nonlinear partial differential equations*, to appear.

[Ro] R. T. ROCKAFELLAR, *Convex analysis*, Princeton Univ. Press, 1970.

[Scha] M. SCHATZMAN, *Problèmes aux limites, nonlinéaires, noncoercifs*, Ann. Scuola Norm. Sup. Pisa, **27** (1973), pp. 641-686.

[Sche] M. SCHECHTER, *A nonlinear elliptic boundary value problem*, Ann. Scuola Norm. Sup. Pisa, **27** (1973), pp. 707-716.

[Sta] G. STAMPACCHIA, *Equations elliptiques du second ordre à coefficients discontinus*, Presses Univ. Montreal, (1966).

[Str] W. STRAUSS, *On weak solutions of semilinear hyperbolic equations*, Anais Acad. Brasileira Ciencias, **42** (1970), pp. 645-651.

[V] O. VEJVODA, *Periodic solutions of nonlinear partial differential equations of evolution*, Proc. Symp. Diff. Eq. and Applications, Bratislava, (1969), pp. 293-300.

[W] S. WILLIAMS, *A sharp sufficient condition for solution of nonlinear elliptic boundary value problem*, J. Diff. Eq., **8** (1970), pp. 580-588.