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A Boundary Value Problem for Quasilinear Hyperbolic Systems in the Schauder Canonic Form (*).

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1. – Introduction.

In the present paper we take into consideration the following Schauder canonic form of quasilinear hyperbolic systems

$$(1.1)\qquad \sum_{j=1}^{m} A_{ij}(x, y, z)\Big[\partial z_j/\partial x + \sum_{k=1}^{r} \varrho_{ik}(x, y, z)\, \partial z_j/\partial y_k\Big] = f_i(x, y, z)\,,$$

$$i = 1, \ldots, m, \qquad \det(A_{ij}) \neq 0, \qquad z(x, y) = (z_1, \ldots, z_m), \qquad y = (y_1, \ldots, y_r),$$

in a slab $D_a = I_a \times E^r$, $I_a = [x|0 \leqslant x \leqslant a]$. Thus, whenever the $m \times m$ matrix $[A_{ij}]$ is the identity matrix, system (1.1) reduces to the Lax-Courant canonic form

$$(1.2)\qquad \partial z_i/\partial x + \sum_{k=1}^{r} \varrho_{ik}(x, y, z)(\partial z_i/\partial y_k) = f_i(x, y, z)\,,$$

$$i = 1, \ldots, m\,, \qquad z(x, y) = (z_1, \ldots, z_m)\,, \qquad y = (y_1, \ldots, y_r)\,.$$

Instead of usual Cauchy data at $x = 0$, we shall take into consideration here more general types of boundary data (I, II, III below).

I. For instance, we may assume that certain functions $\psi_i(y)$, $y \in E^r$, $i = 1, \ldots, m$, and an integer m', $0 \leqslant m' \leqslant m$, are assigned, and we may re-

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quest that

$$z_i(0, y) = \psi_i(y)\,, \qquad i = 1, \ldots, m',$$

$$z_i(a, y) = \psi_i(y)\,, \qquad i = m' + 1, \ldots, m,\ y \in E^r.$$

For $m' = m$ (as well as for $m' = 0$) we have the usual Cauchy problem.

II. More generally, we may assume that certain numbers a_i, $0 \leqslant a_i \leqslant a$, and functions $\psi_i(y)$, $y \in E^r$, $i = 1, \ldots, m$, are assigned, and we may request that

$$z_i(a_i, y) = \psi_i(y)\,, \qquad y \in E^r,\ i = 1, \ldots, m.$$

III. In a more general setting, we may assume that certain numbers a_i, $0 \leqslant a_i \leqslant a$, functions $\psi_i(y)$, $y \in E^r$, $i = 1, \ldots, m$, and an $m \times m$ matrix $[b_{ij}(y),$ $i, j = 1, \ldots, m]$, $y \in E^r$, are assigned, $\det(b_{ij}) \neq 0$, and we may request that

$$\sum_{j=1}^{m} b_{ij} z_j(a_i, y) = \psi_i(y)\,, \qquad y \in E^r,\ i = 1, \ldots, m\,.$$

If (b_{ij}) is the identity matrix, then this boundary condition III reduces to II. If furthermore, $a_i = 0$ for $i = 1, \ldots, m'$, $a_i = a$ for $i = m' + 1, \ldots, m$, $0 \leqslant m' \leqslant m$, then we have problem I.

In the present paper we prove a theorem of existence, uniqueness, and continuous dependence upon the data, for Schauder hyperbolic systems (1.1) with boundary conditions III when both the matrix $[A_{ij}]$ and the matrix $[b_{ij}]$ have « dominant » main diagonal. Thus, problems I and II ($[b_{ij}]$ the identity matrix) for system (1.2) ($[A_{ij}]$ the identity matrix) are always included.

In § 2 we give a new proof with needed estimates of the existence theorem for the Cauchy problem for Schauder's system (1.1), proof based on Banach's fixed point theorem (see [7, 8] for a previous proof). In § 3 we then prove the existence theorem for system (1.1) with boundary conditions III (thus, including boundary conditions I and II). The proof is also based on Banach's fixed point theorem, and on the precise estimates obtained in § 2.

We proved a slightly simpler theorem in [1, 3] for systems (1.2) with boundary conditions III (problems I and II being always included). When the « dominant main diagonal condition » is not satisfied, the conclusions of the same theorems may not hold, as simple counterexamples show [2].

Since we obtain the solution as the fixed point of transformations which are contractions in the uniform topology, the usual iterative method is uniformly convergent to the unique solution.

The boundary value problems under consideration, in the present generality, are new. However, problem I, for very particular systems, was considered by O. Niccoletti [11], and aspects of these problems were discussed anew later by different authors (see e.g. [12-21]).

Leaving aside Goursat problems and analogous ones, let us mention here that boundary value problems for linear symmetric systems have been studied by Friedrichs [9] and Sarason [13]. Finally, various periodicity requirements as boundary value problems for canonic forms of nonlinear hyperbolic systems in the plane, including the wave equation, have been studied by a number of authors, in particular by Cesari [5] and Hale [10] making use of alternative methods (see these two papers for further references).

2. – The main existence theorem for the Cauchy problem.

We consider here quasilinear hyperbolic systems of the Schauder canonic form. Thus, x is a scalar, $y=(y_1, \dots, y_r)$ is an r-vector, and $z(x,y) = (z_1, \dots, z_m)$ is the m-vector of unknown functions $z_i(x, y_1, \dots, y_r)$, $i=1, \dots, m$. We denote by $|y| = \mathrm{Max}_k |y_k|$ the norm of y in E^r and by $|z| = \max_i |z_i|$ the norm of z in E^m.

We consider first the Cauchy problem for the differential system

$$(2.1) \qquad \sum_{j=1}^{m} A_{ij}(x,y,z)\Big[\partial z_j/\partial x + \sum_{k=1}^{r} \varrho_{ik}(x,y,z)(\partial z_j/\partial y_k)\Big] = f_i(x,y,z)\,,$$

$$i=1,\dots,m\,, \qquad \det[A_{ij}] \neq 0\,, \qquad z(x,y)=(z_1,\dots,z_m)\,, \qquad y=(y_1,\dots,y_r)\,,$$

in an infinite strip $D_a = [(x,y)|0 \leqslant x \leqslant a,\ y \in E^r]$ with initial data

$$(2.2) \qquad z_i(0,y) = \varphi_i(y)\,, \qquad y \in E^r, \qquad i=1,\dots,m\,.$$

THEOREM I (an existence theorem for the Cauchy Problem (2.1), (2.2)). Let I_a denote an interval $I_a = [x|0 \leqslant x \leqslant a] \subset E^1$, and, if $\Omega > 0$ is a given constant, let Ω also denote the interval $[-\Omega, \Omega]^m \subset E^m$.

Let $A_{ij}(x,y,z)$, $i,j=1,\dots,m$, be continuous functions on $I_{a_0} \times E^r \times \Omega$, $a_0 > 0$, with $\det(A_{ij}) \geqslant \mu > 0$ in $I_{a_0} \times E^r \times \Omega$ for some constant μ, and let us assume that there are constants $H > 0$, $C \geqslant 0$ and a function $\mathring{m}(x) \geqslant 0$, $0 \leqslant x \leqslant a_0$, $\mathring{m} \in L_1[0,a_0]$, such that, for all (x,y,z), $(x,\bar{y},\bar{z})$, $(\bar{x},y,z) \in I_{a_0} \times E^r \times \Omega$, and all $i,j=1,\dots,m$, we have

$$(2.3) \qquad |A_{ij}(x,y,z)| \leqslant H\,,$$

$$(2.4) \qquad |A_{ij}(x,y,z) - A_{ij}(x,\bar{y},\bar{z})| \leqslant C[|y-\bar{y}| + |z-\bar{z}|]\,,$$

$$(2.5) \qquad |A_{ij}(x,y,z) - A_{ij}(\bar{x},y,z)| \leqslant \Big|\int_x^{\bar{x}} \mathring{m}(\alpha)\,d\alpha\Big|\,.$$

Let $\varrho_{ik}(x, y, z)$, $f_i(x, y, z)$, $i = 1, \dots, m$, $k = 1, \dots, r$, be functions defined in $I_{a_0} \times E^r \times \Omega$, measurable in x for every (y, z), continuous in (y, z) for every x, and let us assume that there are nonnegative functions $m(x)$, $l(x)$, $n(x)$, $l_1(x)$, $0 \leqslant x \leqslant a_0$, m, l, n, $l_1 \in L_1[0, a_0]$, such that, for all (x, y, z), $(x, \bar{y}, \bar{z}) \in I_{a_0} \times E^r \times \Omega$, $i = 1, \dots, m$, $k = 1, \dots, r$, we have

$$(2.6) \qquad |\varrho_{ik}(x, y, z)| \leqslant m(x), \qquad |f_i(x, y, z)| \leqslant n(x),$$

$$(2.7) \qquad |\varrho_{ik}(x, y, z) - \varrho_{ik}(x, \bar{y}, \bar{z})| \leqslant l(x)[|y - \bar{y}| + |z - \bar{z}|],$$

$$(2.8) \qquad |f_i(x, y, z) - f_i(x, \bar{y}, \bar{z})| \leqslant l_1(x)[|y - \bar{y}| + |z - \bar{z}|],$$

Let $\varphi_i(y)$, $y \in E^r$, $i = 1, \dots, m$, be given functions continuous in E^r, and let us assume that there are constants ω, Λ, $0 \leqslant \omega < \Omega$, $\Lambda \geqslant 0$, such that, for all $y, \bar{y} \in E^r$ and $i = 1, \dots, m$, we have

$$(2.9) \qquad |\varphi_i(y)| \leqslant \omega < \Omega, \qquad |\varphi_i(y) - \varphi_i(\bar{y})| \leqslant \Lambda |y - \bar{y}|.$$

Then, for a sufficiently small, $0 < a \leqslant a_0$, there are a constant $Q > 0$, a function $\chi(x) \geqslant 0$, $0 \leqslant x \leqslant a$, $\chi(x) \in L_1[0, a]$, and functions $z(x, y) = z(x, y_1, \dots, y_r) = (z_1, \dots z_m)$, continuous in $I_a \times E^r$, such that for all (x, y), $(x, \bar{y})$, $(\bar{x}, y) \in I_a \times E^r$, and $i = 1, \dots, m$, we have

$$(2.10) \qquad |z_i(x, y)| \leqslant \Omega, \qquad |z_i(x, y) - z_i(x, \bar{y})| \leqslant Q |y - \bar{y}|,$$

$$(2.11) \qquad |z_i(x, y) - z_i(\bar{x}, y)| \leqslant \left| \int_x^{\bar{x}} \chi(\alpha)\, d\alpha \right|,$$

satisfying (2.2) everywhere in E^r and satisfying (2.1) a.e. in D_a. Furthermore, $z(x, y)$ is unique and depends continuously on $\varphi = (\varphi_1, \dots, \varphi_m)$ in the classes which are described in the proof below.

PROOF. The proof is divided into parts $(a), \dots, (g)$.

(a) *Choice of constants p, Q, function χ, and estimates for a.* As usual we denote by α_{ij} the cofactor of A_{ij} in the $m \times m$ matrix (A_{ij}) divided by $\det[A_{ij}]$. Since $\det(A_{ij}) \geqslant \mu > 0$, relations (2.3-5) yield analogous relations for the elements α_{ij}. Thus, there are constants H', C', and a function $\mathring{m}'(x) \geqslant 0$, $0 \leqslant x \leqslant a_0$, $\mathring{m}' \in L_1[0, a_0]$, such that for all (x, y, z), $(x, \bar{y}, \bar{z})$, $(\bar{x}, y, z) \in D_{a_0} \times \Omega$ and $i, j = 1, \dots, m$, we have

$$(2.12) \qquad |\alpha_{ij}(x, y, z)| \leqslant H',$$

$$(2.13) \qquad |\alpha_{ij}(x, y, z) - \alpha_{ij}(x, \bar{y}, \bar{z})| \leqslant C'[|y - \bar{y}| + |z - \bar{z}|],$$

$$(2.14) \qquad |\alpha_{ij}(x, y, z) - \alpha_{ij}(\bar{x}, y, z)| \leqslant \left| \int_x^{\bar{x}} \mathring{m}'(\alpha)\, l\alpha \right| .$$

Note that the functions $A_{ij}(x, y, z)$, $\alpha_{ij}(x, y, z)$ are absolutely continuous in each single variable x, y_h, z_s with

$$|\partial A_{ij}/\partial x| \leqslant \mathring{m}(x)\,, \qquad |\partial A_{ij}/\partial y_h| \leqslant C\,, \qquad |\partial A_{ij}/\partial z_s| \leqslant C\,,$$
$$|\partial \alpha_{ij}/\partial x| \leqslant \mathring{m}'(x)\,, \qquad |\partial \alpha_{ij}/\partial y_h| \leqslant C', \qquad |\partial \alpha_{ij}/\partial z_s| \leqslant C',$$
$$\text{(a.e.)}\,, \quad i, j, s = 1, \ldots, m, \quad h = 1, \ldots, r\,.$$

Analogously, the functions $\varrho_{ik}(x, y, z)$, $f_i(x, y, z)$ are absolutely continuous in each y_h and in each z_s with

$$|\partial \varrho_{ik}/\partial y_h| \leqslant l(x)\,, \qquad |\partial \varrho_{ik}/\partial z_s| \leqslant l(x)\,,$$
$$|\partial f_i/\partial y_h| \leqslant l_1(x)\,, \qquad |\partial f_i/\partial z_s| \leqslant l_1(x)\,,$$
$$\text{(a.e.)}, \quad i, s = 1, \ldots, m, \quad h, k = 1, \ldots, r\,.$$

For every a, $0 < a \leqslant a_0$, we define the following constants:

$$M_a = \int_0^a m(x)\,dx\,, \qquad N_a = \int_0^a n(x)\,dx\,, \qquad \mathring{M}_a = \int_0^a \mathring{m}(x)\,dx\,,$$

$$L_a = \int_0^a l(x)\,dx\,, \qquad L_{1a} = \int_0^a l_1(x)\,dx\,.$$

Let us choose constants p, Q, k, R_0, R_1, R_2, R_3 with

$$0 < p < 1\,, \quad Q > \Lambda(1 + m^2 H' H(2 + p))\,, \quad 0 < k < 1\,, \tag{2.15}$$

$$R_0 > mH'\,, \quad R_1, R_2 > 0\,, \quad R_3 > m^2 H' H \Lambda (1 - k)^{-1}\,. \tag{2.16}$$

Let us take

$$\chi(x) = R_0 n(x) + R_1 \mathring{m}(x) + R_2 \mathring{m}'(x) + R_3 m(x)\,, \quad 0 \leqslant x \leqslant a_0\,, \tag{2.17}$$

and, for every a, $0 < a \leqslant a_0$, let us denote by Ξ_a the constant

$$\Xi_a = \int_0^a \chi(x)\,dx\,. \tag{2.18}$$

We first can choose a, $0 < a \leqslant a_0$, sufficiently small so that

$$L_a(1 + p)(1 + Q) \leqslant p\,, \quad L_a(1 + Q) \leqslant k < 1\,, \tag{2.19}$$

and we denote by λ the constant $\lambda = (1 - L_a(1+Q))^{-1}$, so that $1 < \lambda \leqslant \leqslant (1-k)^{-1}$ and certainly $R_3 > m^2 H' H \Lambda \lambda$. We shall have to impose on a further limitations from above. Though this could well be done at this stage, we prefer to mention the further restrictions on the size of a as need comes in the course of the argument.

(*b*) *The classes* $\mathcal{K}_0$ *and* $\mathcal{K}_1$. We denote by D_a and Δ_a the regions

$$D_a = I_a \times E^r = [(x, y) | 0 \leqslant x \leqslant a, \; -\infty < y_k < +\infty, \; k = 1, \ldots, r] \subset E^{r+1},$$

$$\Delta_a = I_a \times I_a \times E^r = [(\xi, x, y) | 0 \leqslant \xi, \; x \leqslant a, \; -\infty < y_k < +\infty, \; k = 1, \ldots, r] \subset E^{r+2}.$$

Let $\mathcal{K}_0$ be the set of all systems

$$g = [g_{ik}(\xi; x, y), \; i = 1, \ldots, m, \; k = 1, \ldots, r], \tag{2.20}$$

of continuous functions g_{ik} in Δ_a satisfying the following conditions

$$g_{ik}(x; x, y) = y_k \qquad \text{for all } (x, y) \in D_a, \tag{2.21}$$

$$|g_{ik}(\xi; x, y) - g_{ik}(\bar{\xi}; x, y)| \leqslant \left| \int_{\xi}^{\bar{\xi}} m(\alpha)\, d\alpha \right|, \tag{2.22}$$

$$|g_{ik}(\xi; x, y) - g_{ik}(\xi; x, \bar{y}) - y_k + \bar{y}_k| \leqslant p |y - \bar{y}| \tag{2.23}$$

$$\text{for all } (\xi; x, y), \; (\bar{\xi}; x, y), \; (\xi; x, \bar{y}) \in \Delta_a .$$

Thus, each function g_{ik} is absolutely continuous in ξ for every (x, y), and we have

$$|\partial g_{ik}(\xi; x, y) / \partial \xi| \leqslant m(\xi)$$

a.e. in Δ_a, $i = 1, \ldots, m$, $k = 1, \ldots, r$. For every $i = 1, \ldots, m$, we denote by $\breve{g}_i(\xi; x, y)$ the r-vector $\breve{g}_i(\xi; x, y) = (g_{ik}, k = 1, \ldots, r)$. We shall denote by $\widetilde{\mathcal{K}}_0$ the set of all systems

$$h = [h_{ik}(\xi; x, y), \; i = 1, \ldots, m, \; k = 1, \ldots, r],$$

with $h_{ik} = g_{ik}(\xi; x, y) - y_k$, $(\xi; x, y) \in \Delta_a$, where $g = [g_{ik}] \in \mathcal{K}_0$. Thus, if $\breve{h}_i = [h_{ik}, k = 1, \ldots, r]$, we have $\breve{h}_i = \breve{g}_i(\xi; x, y) - y$, $(\xi; x, y) \in \Delta_a$, $g = [g_{ik}] \in \mathcal{K}_0$. Then relations (2.21-23) become

$$h_{ik}(x; x, y) = 0 \qquad \text{for all } (x, y) \in D_a, \tag{2.24}$$

$$|h_{ik}(\xi; x, y) - h_{ik}(\bar{\xi}; x, y)| \leqslant \left| \int_{\xi}^{\bar{\xi}} m(\alpha)\, d\alpha \right|, \tag{2.25}$$

$$|h_{ik}(\xi; x, y) - h_{ik}(\xi; x, \bar{y})| \leqslant p |y - \bar{y}| \tag{2.26}$$

$$\text{for all } (\xi; x, y), \; (\bar{\xi}; x, y), \; (\xi; x, \bar{y}) \in \Delta_a.$$

Thus, for $(\xi; x, y) \in \Delta_a$ we have

$$|h_{ik}(\xi; x, y)| = |h_{ik}(x; x, y) + [h_{ik}(\xi; x, y) - h_{ik}(x; x, y)]| \leq M_a ,$$

that is, the functions h_{ik} are uniformly bounded in Δ_a. Also

$$|\breve{h}_i(\xi; x, y) - \breve{h}_i(\xi; x, \bar{y})| = \operatorname{Max}_k |h_{ik}(\xi; x, y) - h_{ik}(\xi; x, \bar{y})| \leq p|y - \bar{y}| .$$

Finally, for the r-vector functions $\breve{g}_i(\xi; x, y) = (g_{ik},\ k = 1, \dots, r)$, we also have

$$\begin{aligned} &\breve{g}_i(x; x, y) = y , \\ (2.27) \qquad &|g_{ik}(\xi; x, y) - g_{ik}(\xi; x, \bar{y})| \leq (1 + p)|y - \bar{y}| , \qquad k = 1, \dots, r , \\ &|\breve{g}_i(\xi; x, y) - \breve{g}_i(\xi; x, \bar{y})| \leq (1 + p)|y - \bar{y}| . \end{aligned}$$

Note that $\tilde{\mathcal{K}}_0$ is a subset of the Banach space $\big(C(\Delta_a) \cap L_\infty(\Delta_a)\big)^{mr}$ with norm

$$\|h\| = \max_i \|\breve{h}_i\| , \qquad \breve{h}_i = [h_{ik},\ k = 1, \dots, r] ,$$

$$\|\breve{h}_i\| = \max_k \|h_{ik}\| , \qquad \|h_{ik}\| = \operatorname{Sup}_{\Delta_a} |h_{ik}(\xi; x, y)| .$$

We also consider the set $\mathcal{K}_1$ of all systems

$$(2.28) \qquad z = [z_i(x, y),\ i = 1, \dots, m] ,$$

of continuous bounded functions z_i in D_a satisfying the following conditions

$$(2.29) \qquad -\Omega \leq z_i(x, y) \leq \Omega ,$$

$$(2.30) \qquad |z_i(x, y) - z_i(x, \bar{y})| \leq Q|y - \bar{y}| ,$$

$$(2.31) \qquad |z_i(x, y) - z_i(\bar{x}, y)| \leq \left| \int_x^{\bar{x}} \chi(\alpha)\, d\alpha \right| ,$$

for all (x, y), $(x, \bar{y})$, $(\bar{x}, y) \in D_a$, $i = 1, \dots, m$. Thus, each z_i is absolutely continuous in x for every y, Lipschitzian in y for every x, and we have

$$|\partial z_i(x, y)/\partial x| \leq \chi(x) , \qquad |\partial z_i(x, y)/\partial y_k \leq Q$$

a.e. in $D_a = I_a \times E^r$, $i = 1, \dots, m$, $k = 1, \dots, r$. Moreover, for $z(x, y) =$

$= (z_1, \dots, z_m)$, we also have

$$(2.32)\qquad |z(x, y)| \leqslant \Omega\,, \qquad |z(x, y) - z(x, \bar{y})| \leqslant Q|y - \bar{y}|\,,$$
$$|z(x, y) - z(\bar{x}, y)| \leqslant \left|\int_x^{\bar{x}} \chi(\alpha)\, d\alpha\right|,$$

for all (x, y), $(x, \bar{y})$, $(\bar{x}, y) \in D_a$. Here, $\mathcal{K}_1$ is a subset of the Banach space $\big(C(D_a) \cap L_\infty(D_a)\big)^m$ with norm

$$\|z\| = \max_i \|z_i\|\,, \qquad \|z_i\| = \sup_{D_a} |z_i(x, y)|\,.$$

(*c*) *The transformation* T_z. For every fixed $z \in \mathcal{K}_1$, let us consider the transformation T_z defined on $\mathcal{K}_0$, say $G = T_z g$, $g \in \mathcal{K}_0$, or $[g_{ik}] \to [G_{ik}]$, by taking

$$(2.33)\qquad G_{ik}(\xi; x, y) = y_k - \int_\xi^x \varrho_{ik}\big(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y))\big)\, d\alpha\,,$$
$$(\xi; x, y) \in \Delta_a = I_a \times I_a \times E^r, \qquad i = 1, \dots, m, \;\; k = 1, \dots, r\,.$$

Note that the functions G_{ik} are obviously continuous, and that

$$(2.34)\qquad G_{ik}(x; x, y) = y_k \qquad \text{for all } (x, y) \in I_a \times R;$$

$$(2.35)\qquad |G_{ik}(\xi; x, y) - G_{ik}(\bar{\xi}; x, y)| \leqslant \left|\int_\xi^{\bar{\xi}} m(\alpha)\, d\alpha\right|;$$

$$(2.36)\qquad |G_{ik}(\xi; x, y) - G_{ik}(\xi; x, \bar{y}) - y_k + \bar{y}_k| \leqslant$$
$$\leqslant \left|\int_\xi^x \Big|\varrho_{ik}\big(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y))\big) -\right.$$
$$\left. - \varrho_{ik}\big(\alpha, \breve{g}_i(\alpha; x, \bar{y}), z(\alpha, \breve{g}_i(\alpha; x, \bar{y}))\big)\Big| d\alpha\right| \leqslant$$
$$\leqslant \left|\int_\xi^x \big|l(\alpha)(1 + Q)|\breve{g}_i(\alpha; x, y) - \breve{g}_i(\alpha; x, \bar{y})| d\alpha\right| \leqslant$$
$$\leqslant L_a(1 + p)(1 + Q)|y - \bar{y}| \leqslant p|y - \bar{y}|$$

for all $(\xi; x, y)$, $(\xi; x, \bar{y})$, $(\bar{\xi}; x, y) \in \Delta_a$, $i = 1, \dots, m$, $k = 1, \dots, r$. We have used here inequalities (2.6), (2.7), (2.10), (2.19), (2.27), (2.32).

By comparison of (2.34-36) with (2.21-23) we conclude that $G = T_z g$ belongs to $\mathcal{K}_0$. In other words, for every $z \in \mathcal{K}_1$, the transformation T_z defined above is a map $T_z\colon \mathcal{K}_0 \to \mathcal{K}_0$. Considering the differences $h_{ik} = g_{ik} - y_k$, $H_{ik} = G_{ik} - y_k$, we may well think of T_z as a map $\tilde{T}_z\colon \tilde{\mathcal{K}}_0 \to \tilde{\mathcal{K}}_0$ with $\tilde{\mathcal{K}}_0$ a subset of a Banach space. Let us prove that $\tilde{T}_z\colon \tilde{\mathcal{K}}_0 \to \tilde{\mathcal{K}}_0$ is a contraction. Indeed, if $g, g' \in \mathcal{K}_0$, $G = T_z g$, $G' = T_z g'$, and h, h', H, H' are the corresponding elements in $\tilde{\mathcal{K}}_0$, then

$$\begin{aligned}
|H_{ik} - H'_{ik}| \leqslant \Big| \int_\xi^x \Big| & \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y))) - \\
& - \varrho_{ik}(\alpha, \breve{g}'_i(\alpha; x, y), z(\alpha, \breve{g}'_i(\alpha; x, y))) \,|d\alpha| \leqslant \\
& \leqslant \int_0^a l(\alpha)[|\breve{g}_i(\alpha; x, y) - \breve{g}'_i(\alpha; x, y)| + \\
& + |z(\alpha, \breve{g}_i(\alpha; x, y)) - z(\alpha, \breve{g}'_i(\alpha; x, y))|]\, d\alpha \leqslant \\
& \leqslant L_a(1 + Q) \sup_{\Delta_a} |\breve{g}_i(\alpha; x, y) - \breve{g}'_i(\alpha; x, y)| = \\
& = L_a(1 + Q) \sup_{\Delta_a} |\breve{h}_i(\alpha; x, y) - \breve{h}'_i(\alpha; x, y)| \leqslant L_a(1 + Q) \|\breve{h}_i - \breve{h}'_i\| .
\end{aligned}$$

By the definition of norm $\|h\|$ we obtain, by force of (2.19),

$$\|H - H'\| \leqslant L_a(1 + Q)\|h - h'\| \leqslant k\|h - h'\|,$$

where $k < 1$. Thus, for every $z \in \mathcal{K}_1$, the map $\tilde{T}_z\colon \tilde{\mathcal{K}}_0 \to \tilde{\mathcal{K}}_0$ is a contraction of constant $k < 1$.

We conclude that $\tilde{T}_z\colon \tilde{\mathcal{K}}_0 \to \tilde{\mathcal{K}}_0$ has a fixed point $h \in \tilde{\mathcal{K}}_0$, and the corresponding element $g \in \mathcal{K}_0$ is a fixed point of the transformation $T_z\colon \mathcal{K}_0 \to \mathcal{K}_0$. We shall denote this fixed element by $g = g[z] \in \mathcal{K}_0$, or $g(\xi; x, y) = [g_{ik}, i = 1, \dots, m,\ k = 1, \dots, r]$, and $g[z]$ satisfies the integral equations

$$\begin{aligned}
(2.37) \qquad g_{ik}(\xi; x, y) = y_k - \int_\xi^x \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y)))\, d\alpha\,, \\
k = 1, \dots, r, \quad i = 1, \dots, m, \quad (\xi; x, y) \in \Delta_a.
\end{aligned}$$

Note that each component $g_{ik}(\xi; x, y)$ of the fixed element $g = T_z g$, is certainly an absolute continuous function in ξ for every (x, y), is Lipschitzian in y of constant $1 + p$ for every (ξ, x), and satisfies

$$\begin{aligned}
& |\partial g_{ik}(\xi; x, y)/\partial\xi| \leqslant m(\xi)\,, \\
& (\xi; x, y) \in I_a \times I_a \times E^r, \qquad \text{(a.e.)}, \quad i = 1, \dots, m, \quad k = 1, \dots, r\,.
\end{aligned}$$

Moreover, for every $i = 1, \ldots, m$, the r-vector function $\breve{g}_i(\xi; x, y) = (g_{ik}, k = 1, \ldots, r)$, thought of as a function of ξ, is a Carathéodory solution of the system of ordinary differential equations

$$(2.38) \qquad dg_{ik}(\xi; x, y)/d\xi = \varrho_{ik}(\xi, \breve{g}_i(\xi; x, y), z(\xi, \breve{g}_i(\xi; x, y))) , \quad 0 \leqslant \xi \leqslant a, \ \text{(a.e.)} ,$$

$$(2.39) \qquad g_{ik}(x; x, y) = y_k , \qquad k = 1, \ldots, r .$$

Let us prove that each component $g_{ik}(\xi; x, y)$ of the fixed element $g[z]$ is absolutely continuous in x for every (ξ, y). Indeed, for any two $(\xi; x, y)$, $(\xi; \bar{x}, y) \in \Delta_a$, we have

$$(2.40) \qquad \begin{aligned} &|g_{ik}(\xi; x, y) - g_{ik}(\xi; \bar{x}, y)| = \\ &= \Big| \int_\xi^x \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y)))\, d\alpha - \\ &- \int_\xi^{\bar{x}} \varrho_{ik}(\alpha, \breve{g}_i(\alpha; \bar{x}, y), z(\alpha, \breve{g}_i(\alpha; \bar{x}, y)))\, d\alpha \Big| \leqslant \\ &\leqslant \Big| \int_x^{\bar{x}} m(\alpha)\, d\alpha \Big| + \Big| \int_\xi^x l(\alpha)(1 + Q) |\breve{g}_i(\alpha; x, y) - \breve{g}_i(\alpha; \bar{x}, y)| \Big|\, d\alpha . \end{aligned}$$

Since

$$\delta = \max_k \max \left[|g_{ik}(\xi; x, y) - g_{ik}(\xi; \bar{x}, y)|,\ 0 \leqslant \xi \leqslant a \right] ,$$

is certainly attained for some k and some ξ, (δ depends on $x, \bar{x}, y, i$), we derive from (2.40) that

$$\delta \leqslant \Big| \int_x^{\bar{x}} m(\alpha)\, d\alpha \Big| + L_a(1 + Q)\, \delta ,$$

or

$$(2.41) \qquad |g_{ik}(\xi; x, y) - g_{ik}(\xi; \bar{x}, y)| \leqslant (1 - L_a(1 + Q))^{-1} \Big| \int_x^{\bar{x}} m(\alpha)\, d\alpha \Big| ,$$

$$0 \leqslant \xi \leqslant a, \quad k = 1, \ldots, r, \quad i = 1, \ldots, m .$$

This proves that each $g_{ik}[z](\xi; x, y)$ is an absolutely continuous function of x for every (ξ, y) with

$$|\partial g_{ik}[z](\xi; x, y)/\partial x| \leqslant (1 - L_a(1 + Q))^{-1} m(x) = \lambda m(x) ,$$

$$\text{(a.e.)}, \quad i = 1, \ldots, m, \quad k = 1, \ldots, r .$$

Because of (2.6-7) and (2.29-30) we know that $\breve{g}_i(\xi; x, y)$ is the unique solution of problem (2.38-39). Thus, $\breve{g}_i$ satisfies the groupal property

$$\breve{g}_i(\xi'; \xi, \breve{g}_i(\xi; x, y)) = \breve{g}_i(\xi'; x, y) , \qquad 0 \leqslant \xi,\ \xi' \leqslant a . \tag{2.42}$$

For ξ', ξ, x, y replaced by ξ, x, 0, η, or in particular by 0, x, 0, η, we have

$$\breve{g}_i(\xi; x, \breve{g}_i(x; 0, \eta)) = \breve{g}_i(\xi; 0, \eta) ,$$

$$\breve{g}_i(0; x, \breve{g}_i(x; 0, \eta)) = \breve{g}_i(0; 0, \eta) = \eta .$$

Thus, for $y = \breve{g}_i(x; 0, \eta)$, the symmetric relations hold

$$y = \breve{g}_i(x; 0, \eta) , \qquad \eta = \breve{g}_i(0; x, y) . \tag{2.43}$$

For any fixed $z \in \mathcal{K}_1$ and $x \in I_a$, these relations represent a $1-1$ transformation of the y-space E^r into the η-space E^r. Indeed, if

$$y_1 = \breve{g}_i(x; 0, \eta_1) = \eta_1 + \breve{h}_i(x; 0, \eta_1) , \qquad y_2 = \breve{g}_i(x; 0, \eta_2) = \eta_2 + \breve{h}_i(x; 0, \eta_2) ,$$

then

$$|y_1 - y_2| = |\eta_1 - \eta_2 + \breve{h}_i(x; 0, \eta_1) - \breve{h}_i(x; 0, \eta_2)| ,$$

and hence

$$(1-p)|\eta_1 - \eta_2| \leqslant |y_1 - y_2| \leqslant (1+p)|\eta_1 - \eta_2| ,$$

where $0 < p < 1$. Analogously, we could prove that

$$(1-p)|y_1 - y_2| \leqslant |\eta_1 - \eta_2| \leqslant (1+p)|y_1 - y_2| .$$

By adding equation $x = x$ to relations (2.43), we obtain a 1-1 transformation of the slab $I_a \times E^r$ of the xy-space E^{r+1} onto the slab $I_a \times E^r$ of the $x\eta$-space E^{r+1}.

Finally, we consider the operation $z \to g[z]$, or $\mathcal{K}_1 \to \mathcal{K}_0$, mapping each element $z \in \mathcal{K}_1$ into the corresponding element $g = g[z] \in \mathcal{K}_0$. By taking as usual $\breve{g}_i = y + \breve{h}_i$, we have a transformation $z \to h[z]$, or $\mathcal{K}_1 \to \widetilde{\mathcal{K}}_0$, mapping each element $z \in \mathcal{K}_1$ into the fixed point $h = \tilde{T}_z h$, or $h[z]$, of the transformation $\tilde{T}_z$. Let us prove that $z \to h[z]$ is a continuous map.

To this effect, let $z, z' \in \mathcal{K}_1$ and let us denote by h, h' the corresponding elements in $\tilde{\mathcal{K}}_0$, or fixed points $h = \tilde{T}_z h$, $h' = \tilde{T}_{z'} h'$. From (2.37) we derive now

$$|h_{ik}(\xi; x, y) - h'_{ik}(\xi; x, y)| = \left| \int_\xi^x [\varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y))) - \right.$$

$$\left. - \varrho_{ik}(\alpha, \breve{g}'_i(\alpha; x, y), z'(\alpha, \breve{g}'_i(\alpha; x, y)))] \, d\alpha \right| \leq$$

$$\leq \left| \int_\xi^x l(\alpha)\big((1+Q)\|h - h'\| + \|z - z'\|\big) \, d\xi \right| .$$

Hence,

$$\|h - h'\| \leq L_a(1+Q)\|h - h'\| + L_a\|z - z'\| ,$$

where $L_a(1+Q) < 1$, and this yields

$$\|h - h'\| \leq \big(1 - L_a(1+Q)\big)^{-1} L_a \|z - z'\| = \lambda L_a \|z - z'\| .$$

It is correct to write this relation in the form

$$\|g - g'\| \leq \lambda L_a \|z - z'\| . \tag{2.44}$$

(*d*) *The transformation* $T^*_{z\varphi}$. Here z denotes any element of $\mathcal{K}_1$ and $g = g[z] \in \mathcal{K}_0$ the unique fixed element $g = T_z g \in \mathcal{K}_0$ of the transformation T_z.

Let $\mathfrak{J}$ denote the class of all functions $\varphi(y) = (\varphi_1, \ldots, \varphi_m)$, $y \in E^m$, such that, for all $y, \bar{y} \in E^r$ and $i = 1, \ldots, m$, we have

$$|\varphi_i(y)| \leq \omega , \qquad |\varphi_i(y) - \varphi_i(\bar{y})| \leq \Lambda |y - \bar{y}| . \tag{2.45}$$

For every $\varphi \in \mathfrak{J}$ let us consider the set $\mathcal{K}_{1\varphi}$ of all systems

$$z = [z_i(x, y), i = 1, \ldots, m] \in \mathcal{K}_1 , \tag{2.46}$$

of continuous bounded functions z_i in $D_a = I_a \times E^r$ satisfying the following conditions

$$z_i(0, y) = \varphi_i(y) , \qquad y \in E^r , \qquad i = 1, \ldots, m , \tag{2.47}$$

Thus, $\mathcal{K}_{1\varphi} \subset \mathcal{K}_1$, and $\mathcal{K}_{1\varphi}$ is, as $\mathcal{K}_1$, a subset of the Banach space $\big(C(D_a) \cap \cap L_\infty(D_a)\big)^m$, with the norm stated in part (*b*).

For every fixed $z \in \mathcal{K}_1$ and corresponding $g = g[z] \in \mathcal{K}_0$, we consider now the linear transformation $U = T^*_{z\varphi} u$, $u \in \mathcal{K}_{1\varphi}$, or $[u_i] \to [U_i]$, defined by

$$(2.48)\quad U_i(x, y) = \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y)) \cdot \Big\{ \sum_{h=1}^{m} A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y)))\, \varphi_h(\breve{g}_s(0; x, y)) + \int_0^x \Big[\sum_{h=1}^{m} \big(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\big) u_h(\xi, \breve{g}_s(\xi; x, y)) + f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))\Big] d\xi \Big\},$$
$$(x, y) \in I_a \times E^r, \qquad i = 1, \ldots, m .$$

Note that $A_{sh}(x, y, z)$ is absolutely continuous in x and Lipschitzian in y and z; $\breve{g}_i(\xi; x, y)$ is absolutely continuous in ξ; $z_h(x, y)$, $u_h(x, y)$ are absolutely continuous in x and Lipschitzian in y. Hence, the composite functions $A_{sh}\big(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))\big)$, $u_h(\xi, \breve{g}_s(\xi; x, y))$ are absolutely continuous in ξ.

First, note that

$$S_i = \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y)) \sum_{h=1}^{m} A_{sh}(x, \breve{g}_s(x; x, y), z(x, \breve{g}_s(x; x, y)))\, \varphi_h(\breve{g}_s(x; x, y))$$
$$= \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y)) \sum_{h=1}^{m} A_{sh}(x, y, z(x\ y))\, \varphi_h(y) = \varphi_i(y) ,$$

so that, by adding and subtracting S_i in the second member of (2.48), we have

$$(2.49)\quad U_i(x, y) = \varphi_i(y) + \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y)) \cdot \Big\{ \sum_{h=1}^{m} A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y)))\, \varphi_h(\breve{g}_s(0; x, y)) - - \sum_{h=1}^{m} A_{sh}(x, \breve{g}_s(x; x, y), z(x, \breve{g}_s(x; x, y)))\, \varphi_h(\breve{g}_s(x; x, y)) + + \int_0^x \Big[\sum_{h=1}^{m} \big(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\big) u_h(\xi, \breve{g}_s(\xi; x, y)) + f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))\Big] d\xi \Big\} .$$

From here it is apparent that

$$(2.50)\qquad U_i(0, y) = \varphi_i(y) \qquad \text{for all } y \in E^r,\ i = 1, \ldots, m .$$

Note that, because of the absolute continuity of the composite function $A_{sh}(\ldots)$ in (2.49), we have

$$(2.51)\quad \Delta = A_{sh}(x, \breve{g}_s(x; x, y), z(x, \breve{g}_s(x; x, y))) - A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y)))$$
$$= \int_0^x \left(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\right) d\xi \,.$$

In addition, the relation

$$dA_{sh}/d\xi = dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi$$
$$= \partial A_{sh}/\partial\xi + \sum_{k=1}^{r} (\partial A_{sh}/\partial y_k)(\partial g_{sk}/\partial\xi) +$$
$$+ \sum_{j=1}^{m} (\partial A_{sh}/\partial z_j)\Big(\partial z_j/\partial\xi + \sum_{k=1}^{r} (\partial z_j/\partial y_k)(\partial g_{sk}/\partial\xi)\Big)$$

holds a.e. by force of usual chain rule differentiation statements of real analysis. For instance, by applying the chain rule lemma of no. 4,(*b*) of [1], or analogous statement in [6], we can say that, for every fixed $x \in [0, a]$, the relation above holds for almost all (ξ, y). By force of (2.49), (2.51) and manipulations we have

$$U_i(x, y) = \varphi_i(y) + \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y))(\Delta_{s1} + \Delta_{s2} + \Delta_{s3}) \,,$$

$$(2.52)\quad \begin{cases} \Delta_{s1} = \Delta_{s1}(x, y) = \int_0^x f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))\, d\xi \,, \\ \Delta_{s2} = \Delta_{s2}(x, y) = \sum_{h=1}^{m} A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) \\ \qquad\qquad [\varphi_h(\breve{g}_s(0; x, y)) - \varphi_h(\breve{g}_s(x; x, y))] \,, \\ \Delta_{s3} = \Delta_{s3}(x, y) = \int_0^x \sum_{h=1}^{m} \left(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\right) \\ \qquad\qquad [u_h(\xi, \breve{g}_s(\xi; x, y)) - \varphi_h(\breve{g}_s(x; x, y))]\, d\xi \,. \end{cases}$$

By using the bounds for the derivatives we already have, we obtain

$$(2.53)\quad |dA_{sh}/d\xi| \leqslant \mathring{m}(\xi) + rC(1 + mQ)m(\xi) + mC\chi(\xi) \,,$$

and hence

$$(2.54)\quad \int_0^x |dA_{sh}/d\xi|\, d\xi < \mathring{M}_a + rC(1 + mQ)\, M_a + mC\Xi_a \,.$$

Analogously, we have,

$$du_h/d\xi = du_h(\xi, \breve{g}_s(\xi; x, y))/d\xi = \\ = \partial u_h/\partial\xi + \sum_{k=1}^{r} (\partial u_h/\partial y_k)(\partial g_{sk}/\partial\xi) . \tag{2.55}$$

Again, by using the bounds for the derivatives we have, we obtain

$$|du_h/d\xi| \leqslant \chi(\xi) + rQm(\xi) . \tag{2.56}$$

Note that, for every $(\bar{\xi}, x, y) \in I_a \times I_a \times E^r$, we have

$$\begin{aligned} \Delta' &= u_h(\bar{\xi}, \breve{g}_s(\bar{\xi}; x, y)) - \varphi_h(\breve{g}_s(x; x, y)) \\ &= [u_h(\bar{\xi}, \breve{g}_s(\bar{\xi}; x, y)) - u_h(0, \breve{g}_s(\bar{\xi}; x, y))] + \\ &+ [u_h(0, \breve{g}_s(\bar{\xi}; x, y)) - u_h(0, \breve{g}_s(x; x, y))] , \end{aligned}$$

and hence

$$|\Delta'| \leqslant \int_0^{\bar{\xi}} \chi(\xi)\, d\xi + Q\left|\int_x^{\bar{\xi}} m(\xi)\, d\xi\right| \leqslant \Xi_a + QM_a . \tag{2.57}$$

Finally, we have, by using (2.3), (2.6), (2.9), (2.22), (2.52), (2.53), (2.57),

$$\left\{\begin{aligned} &|\Delta_{s1}| \leqslant \int_0^x n(\xi)\, d\xi \leqslant N_a , \\ &|\Delta_{s2}| \leqslant mH\Lambda \int_0^x m(\xi)\, d\xi \leqslant mH\Lambda\, M_a , \\ &|\Delta_{s3}| \leqslant m\int_0^x (\mathring{m}(\xi) + rC(1 + mQ)m(\xi) + mC\chi(\xi))(\Xi_a + QM_a)\, d\xi \\ &\qquad \leqslant m(\mathring{M}_a + rC(1 + mQ)\,M_a + mC\Xi_a)(\Xi_a + QM_a)\, d\xi. \end{aligned}\right. \tag{2.58}$$

From relations (2.12), (2.52), (2.58), we have now

$$|U_i(x, y)| \leqslant |\varphi_i(y)| + \left|\sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y))(\Delta_{s1} + \Delta_{s2} + \Delta_{s3})\right| \leqslant \tag{2.59}$$

$$\leqslant \omega + mH'[(N_a + mH\Lambda\, M_a) + m(\mathring{M}_a + rC(1 + mQ)\,M_a + mC\Xi_a)(\Xi_a + QM_a)] \leqslant$$
$$\leqslant \omega + (\Omega - \omega) = \Omega ,$$

provided a is assumed sufficiently small in order that

$$(2.60)\quad mH'[N_a + mHAM_a + m(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a)(\Xi_a + QM_a)] \leqslant \leqslant \Omega - \omega .$$

For any two points (x, y), $(x, \bar{y}) \in I_a \times E^r$, and by using (2.52), we see that the difference $U_i(x, y) - U_i(x, \bar{y})$ can be written as the sum of terms δ_0, δ_0', δ_1, δ_2, δ_3, which we write and estimate below one by one:

$$|\delta_0| = |\varphi_i(y) - \varphi_i(\bar{y})| < A|y - \bar{y}|;$$

$$|\delta_0'| = \left|\sum_{s=1}^{m} [\alpha_{si}(x, y, z(x, y)) - \alpha_{si}(x, \bar{y}, z(x, \bar{y}))](A_{s1}(x, y) + A_{s2}(x, y) + A_{s3}(x, y))\right|$$

$$\leqslant mC'(1 + Q)[N_a + mHAM_a + m(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a)$$

$$(\Xi_a + QM_a)]|y - \bar{y}|;$$

$$|\delta_1| = \left|\sum_{s=1}^{m} \alpha_{si}(x, \bar{y}, z(x, \bar{y}))(A_{s1}(x, y) - A_{s1}(x, \bar{y}))\right| =$$

$$= \left|\sum_{s=1}^{m} \alpha_{si}(x, \bar{y}, z(x, \bar{y}))\int_0^x [f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))) -\right.$$

$$\left. - f_s(\xi, \breve{g}_s(\xi; x, \bar{y}), z(\xi, \breve{g}_s(\xi; x, \bar{y})))]\, d\xi\right| \leqslant$$

$$\leqslant mH'\int_0^x l_1(\xi)(1 + p)(1 + Q)|y - \bar{y}|\, d\xi \leqslant mH'(1 + p)(1 + Q)\, L_{1a}|y - \bar{y}|;$$

$$|\delta_2| = \left|\sum_{s=1}^{m} \alpha_{si}(x, \bar{y}, z(x, \bar{y}))(A_{s2}(x, y) - A_{s2}(x, \bar{y}))\right| =$$

$$= \left|\sum_{s=1}^{m} \bar{\alpha}_{si} \sum_{h=1}^{m} \{A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y)))\cdot\right.$$

$$\cdot[\varphi_h(\breve{g}_s(0; x, y) - \varphi_h(\breve{g}_s(0; x, \bar{y}) - \varphi_h(y) + \varphi_h(\bar{y}))] +$$

$$+ [A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) - A_{sh}(0, \breve{g}_s(0; x, \bar{y}), z(0, \breve{g}_s(0; x, \bar{y})))]\cdot$$

$$\left.\cdot[\varphi_h(\breve{g}_s(0; x, \bar{y}) - \varphi_h(\breve{g}_s(x; x, \bar{y})))]\}\right| \leqslant$$

$$\leqslant m^2H'[HA(2 + p) + CA(1 + Q)(1 + p)M_a]|y - \bar{y}|.$$

Finally, by manipulations and integration by parts, we write and estimate δ_3 as follows:

$$
\begin{aligned}
|\delta_3| &= \Big|\sum_s \alpha_{si}(x,\bar{y},z(x,\bar{y}))(\Delta_{s3}(x,y)-\Delta_{s3}(x,\bar{y}))\Big| = \\
&= \Big|\sum_s \bar{\alpha}_{si}\int_0^x \sum\nolimits_h [(dA_{sh}/d\xi)(u_h-\varphi_h)-(d\bar{A}_{sh}/d\xi)(\bar{u}_h-\bar{\varphi}_h)]\,d\xi\Big| = \\
&= \Big|\sum_s \bar{\alpha}_{si}\sum_h \Big[\int_0^x (dA_{sh}/d\xi - d\bar{A}_{sh}/d\xi)(u_h-\varphi_h)\,d\xi + \\
&\quad + \int_0^x (d\bar{A}_{sh}/d\xi)(u_h-\bar{u}_h-\varphi_h+\bar{\varphi}_h)\,d\xi\Big]\Big| = \\
&= \Big|\sum_s \bar{\alpha}_{si}\sum_h \{[A_{sh}(x,y,z(x,y)) - A_{sh}(x,\bar{y},z(x,\bar{y}))](u_h(x,y)-u_h(0,y)) - \\
&\quad - [A_{sh}(0,\breve{g}_s(0;x,y),z(0,\breve{g}_s(0;x,y))) - \\
&\quad - A_{sh}(0,\breve{g}_s(0;x,\bar{y}),z(0,\breve{g}_s(0;x,\bar{y})))](u_h(0,\breve{g}_s(0;x,y)) - u_h(0,\breve{g}_s(x;x,y))) - \\
&\quad - \int_0^x [A_{sh}(\xi,\breve{g}_s(\xi;x,y),z(\xi,\breve{g}_s(\xi;x,y))) - \\
&\quad - A_{sh}(\xi,\breve{g}_s(\xi;x,\bar{y}),z(\xi,\breve{g}_s(\xi;x,\bar{y})))](du_h(\xi,\breve{g}_s(\xi;x,y))/d\xi)\,d\xi + \\
&\quad + \int_0^x (d\bar{A}_{sh}/d\xi)[u_h(\xi,\breve{g}_s(\xi;x,y)) - u_h(\xi,\breve{g}_s(\xi;x,\bar{y}) - \varphi_h(y) + \varphi_h(\bar{y})]\,d\xi\}\Big| \leqslant \\
&\leqslant m^2H'\Big\{C(1+Q)\int_0^x \chi(\xi)\,d\xi + CQ(1+Q)(1+p)\int_0^x m(\xi)\,d\xi + \\
&\quad + \int_0^x C(1+Q)(1+p)(\chi(\xi)+rQm(\xi))\,d\xi + \\
&\quad + \int_0^x [\mathring{m}(\xi) + rC(1+mQ)m(\xi) + mC\chi(\xi)](Q(1+p)+\Lambda)\,d\xi\Big\}|y-\bar{y}| \leqslant \\
&\leqslant m^2H'\{C(1+Q)\Xi_a + CQ(1+Q)(1+p)M_a + \\
&\quad + C(1+Q)(1+p)(\Xi_a + rQM_a) + \\
&\quad + (\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a)(Q(1+p)+\Lambda)\}|y-\bar{y}|\,.
\end{aligned}
$$

Combining the previous estimates we have

$$
\begin{aligned}
|U_i(x,y) - U_i(x,\bar{y})| &\leqslant |\delta_0| + |\delta_0'| + |\delta_1| + |\delta_2| + |\delta_3| \leqslant \\
&\leqslant [\Lambda(1 + m^2H'H(2+p)) + \\
&\quad + \gamma_1 N_a + \gamma_2 \mathring{M}_a + \gamma_3 L_{1a} + \gamma_4 M_a + \gamma_5 \Xi_a]|y-\bar{y}|\,,
\end{aligned}
$$

where

$$\gamma_1 = mC'(1+Q)\,,$$
$$\gamma_2 = m^2[C'(1+Q)(\Xi_a + QM_a) + H'(Q(1+p)+\Lambda)]\,,$$
$$\gamma_3 = mH'(1+Q)(1+p)\,,$$
$$\gamma_4 = m^2[C'\Lambda H(1+Q) + rCC'(1+Q)(1+mQ)(\Xi_a+QM_a) + + H'C\Lambda(1+Q)(1+p) + (r+1)H'CQ(1+Q)(1+p) + + rH'C(1+mQ)(Q(1+p)+\Lambda)]\,,$$
$$\gamma_5 = m^2[mCC'(1+Q)(\Xi_a+QM_a) + H'C(1+Q)(2+p) + + mH'C(Q(1+p)+\Lambda)]\,.$$

If we assume a sufficiently small so that

$$\gamma_1 N_a + \gamma_2 \mathring{M}_a + \gamma_3 L_{1a} + \gamma_4 M_a + \gamma_5 \Xi_a \leqslant Q - \Lambda(1 + m^2 H' H(2+p))\,, \tag{2.61}$$

then we have, for all (x, y), $(x, \bar{y}) \in D_a$, and $i = 1, \ldots, m$,

$$|U_i(x, y) - U_i(x, \bar{y})| \leqslant Q|y - \bar{y}|\,. \tag{2.62}$$

For any two points (x, y), $(\bar{x}, y) \in I_a \times E^r$, and by using (2.52), we see that the difference $U_i(x, y) - U_i(\bar{x}, y)$ can be written as the sum of terms

$$U_i(x, y) - U_i(\bar{x}, y) = \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3\,,$$

which we write and estimate below one by one:

$$|\sigma_0| = \left|\sum_{s=1}^{m} [\alpha_{si}(x, y, z(x, y)) - \alpha_{si}(\bar{x}, y, z(\bar{x}, y))](\Delta_{s1} + \Delta_{s2} + \Delta_{s3})\right| \leqslant$$
$$\leqslant m[N_a + mH\Lambda M_a + m(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a)(\Xi_a + QM_a)] \cdot$$
$$\cdot \left(\left| \int_x^{\bar{x}} \mathring{m}'(\xi)\, d\xi \right| + C' \left| \int_x^{\bar{x}} \chi(\xi)\, d\xi \right| \right);$$

$$|\sigma_1| = \left| \sum_{s=1}^{m} \alpha_{si}(\bar{x}, y, z(\bar{x}, y)) (\Delta_{s1}(x, y) - \Delta_{s1}(\bar{x}, y)) \right| =$$
$$= \left| \sum_{s=1}^{m} \alpha_{si}(\bar{x}, y, z(\bar{x}, y)) \left[\int_0^x f_s(\xi, \check{g}_s(\xi; x, y), z(\xi, \check{g}_s(\xi; x, y)))\, d\xi - \right.\right.$$
$$\left.\left. - \int_0^{\bar{x}} f_s(\xi, \check{g}_s(\xi; \bar{x}, y), z(\xi, \check{g}_s(\xi; \bar{x}, y)))\, d\xi \right] \right| \leqslant$$
$$\leqslant mH' \left[(1+Q)\lambda L_{1a} \left| \int_x^{\bar{x}} m(\xi)\, d\xi \right| + \left| \int_x^{\bar{x}} n(\xi)\, d\xi \right| \right];$$

$$|\sigma_2| = \left|\sum_{s=1}^{m} \alpha_{si}(\bar{x}, y, z(\bar{x}, y))(\Delta_{s2}(x, y) - \Delta_{s2}(\bar{x}, y))\right| =$$

$$= \Big|\sum_{s=1}^{m} \bar{\alpha}_{si} \sum_{h=1}^{m} \{A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y)))\cdot$$

$$\cdot [\varphi_h(\breve{g}_s(0; x, y)) - \varphi_h(\breve{g}_s(0; \bar{x}, y))] +$$

$$+ [A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) -$$

$$- A_{sh}(0, \breve{g}_s(0; \bar{x}, y), z(0, \breve{g}_s(0; \bar{x}, y)))][\varphi_h(\breve{g}_s(0; \bar{x}, y)) - \varphi_h(\breve{g}_s(\bar{x}; \bar{x}, y))]\}\Big| \leqslant$$

$$\leqslant m^2 H' \left[H\Lambda\lambda\left|\int_x^{\bar{x}} m(\xi)\, d\xi\right| + C\Lambda\lambda(1 + Q) M_a \left|\int_x^{\bar{x}} m(\xi)\, d\xi\right|\right].$$

We have used here (2.6), (2.8), (2.13), (2.14), (2.30), (2.41). By manipulation and integration by parts, we write and estimate σ_3 as follows:

$$|\sigma_3| = \left|\sum_s \alpha_{si}(\bar{x}, y, z(\bar{x}, y))(\Delta_{s3}(x, y) - \Delta_{s3}(\bar{x}, y))\right| =$$

$$= \left|\sum_s \bar{\alpha}_{si} \sum_h \left[\int_0^x (dA_{sh}/d\xi)(u_h - \varphi_h)\, d\xi - \int_0^{\bar{x}} (d\bar{A}_{sh}/d\xi)(\bar{u}_h - \bar{\varphi}_h)\, d\xi\right]\right| =$$

$$= \Big|\sum_s \bar{\alpha}_{si} \sum_h \Big\{-\int_x^{\bar{x}} (d\bar{A}_{sh}/d\xi)(\bar{u}_h - \bar{\varphi}_h)\, d\xi + \int_0^x (dA_{sh}/d\xi - d\bar{A}_{sh}/d\xi)(u_h - \varphi_h)\, d\xi +$$

$$+ \int_0^x (d\bar{A}_{sh}/d\xi)(u_h - \bar{u}_h - \varphi_h + \bar{\varphi}_h)\, d\xi\Big\}\Big| =$$

$$= \Big|\sum_s \bar{\alpha}_{si} \sum_h \Big\{-\int_x^{\bar{x}} (d\bar{A}_{sh}/d\xi)[u_h(\xi, \breve{g}_s(\xi; \bar{x}, y)) - \varphi_h(\breve{g}_s(\bar{x}; \bar{x}, y))]\, d\xi +$$

$$+ [A_{sh}(x, \breve{g}_s(x; x, y), z(x, \breve{g}_s(x; x, y))) -$$

$$- A_{sh}(x, \breve{g}_s(x; \bar{x}, y), z(x, \breve{g}_s(x; \bar{x}, y)))](u_h(x, y) - u_h(0, y)) -$$

$$- [A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) -$$

$$- A_{sh}(0, \breve{g}_s(0; \bar{x}, y), z(0, \breve{g}_s(0; \bar{x}, y)))]\cdot(u_h(0, \breve{g}_s(0; x, y)) - u_h(0, \breve{g}_s(x; x, y))) -$$

$$- \int_0^x [A_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))) -$$

$$- A_{sh}(\xi, \breve{g}_s(\xi; \bar{x}, y), z(\xi, \breve{g}_s(\xi; \bar{x}, y)))](du_h(\xi, \breve{g}_s(\xi; x, y))/d\xi)\, d\xi +$$

$$+ \int_0^x (d\bar{A}_{sh}/d\xi)[u_h(\xi, \breve{g}_s(\xi; x, y)) - u_h(\xi, \breve{g}_s(\xi; \bar{x}, y))]\, d\xi\Big\}\Big| \leqslant$$

$$\leqslant m^2H'\Big\{\Big|\int_x^{\bar{x}}(\mathring{m}(\xi)+rC(1+mQ)m(\xi)+mC\chi(\xi))(\Xi_a+QM_a)\,d\xi\Big|+$$

$$+C(1+Q)\lambda\Big|\int_x^{\bar{x}}m(\xi)\,d\xi\Big|\Xi_a+C(1+Q)\lambda\Big|\int_x^{\bar{x}}m(\xi)\,d\xi\Big|QM_a+$$

$$+\int_0^x C(1+Q)\lambda\Big|\int_x^{\bar{x}}m(\beta)\,d\beta\Big|(\chi(\xi)+rQm(\xi))\,d\xi+$$

$$+\int_0^x(\mathring{m}(\xi)+rC(1+mQ)m(\xi)+mC\chi(\xi))\Big(Q\lambda\Big|\int_x^{\bar{x}}m(\beta)\,d\beta\Big|\Big)\,d\xi\Big\}\leqslant$$

$$\leqslant m^2H'\Big\{(\Xi_a+QM_a)\Big|\int_x^{\bar{x}}(\mathring{m}(\xi)+rC(1+mQ)m(\xi)+mC\chi(\xi))\,d\xi\Big|+$$

$$+2C(1+Q)\lambda(\Xi_a+rQM_a)\Big|\int_x^{\bar{x}}m(\xi)\,d\xi\Big|+$$

$$+Q\lambda(\mathring{M}_a+rC(1+mQ)M_a+mC\Xi_a)\Big|\int_x^{\bar{x}}m(\xi)\,d\xi\Big|\Big\}.$$

We have used here (2.12), (2.31), (2.41), (2.53), (2.56), (2.57).

Combining the previous estimates we have

$$|U_i(x,y)-U_i(\bar{x},y)|\leqslant|\sigma_0|+|\sigma_1|+|\sigma_2|+|\sigma_3|$$

$$\leqslant mH'\Big|\int_x^{\bar{x}}n(\xi)\,d\xi\Big|+m^2H'HA\lambda\Big|\int_x^{\bar{x}}m(\xi)\,d\xi\Big|+\gamma_1'\Big|\int_x^{\bar{x}}\mathring{m}(\xi)\,d\xi\Big|+$$

$$+\gamma_2'\Big|\int_x^{\bar{x}}\mathring{m}'(\xi)\,d\xi\;+\gamma_3'\Big|\int_x^{\bar{x}}m(\xi)\,d\xi\Big|+\gamma_0'\Big|\int_x^{\bar{x}}\chi(\xi)\,d\xi\Big|,$$

$$(2.63)\quad\begin{cases}\gamma_1'=m^2H'(\Xi_a+QM_a),\\ \gamma_2'=m[N_a+mHAM_a+m(\mathring{M}_a+rC(1+mQ)M_a+\\ \quad+mC\Xi_a)(\Xi_a+QM_a)],\\ \gamma_3'=mH'(1+Q)\lambda L_{1a}+m^2H'CA\lambda(1+Q)M_a+\\ \quad+rm^2H'C(\Xi_a+QM_a)(1+mQ)+2m^2H'C(1+Q)\lambda(\Xi_a+rQM_a)+\\ \quad+m^2H'Q\lambda(\mathring{M}_a+rC(1+mQ)M_a+mC\Xi_a),\\ \gamma_0'=mC'[N_a+mHAM_a+m(\mathring{M}_a+rC(1+mQ)M_a+mC\Xi_a)\cdot\\ \quad\cdot(\Xi_a+QM_a)]+m^3H'C(\Xi_a+QM_a).\end{cases}$$

From relations (2.16), or $R_0 > mH'$, $R_1 > 0$, $R_2 > 0$, $R_3 > m^2H'HA(1-k)^{-1}$ and consequent relation $R_3 > m^2H'HA\lambda$, we derive

$$1 - R_0^{-1}mH' > 0\,, \qquad 1 - R_3^{-1}m^2H'HA\lambda > 0\,,$$

We shall take a sufficiently small so that

$$\begin{aligned}(2.64)\quad & \gamma_0' < 1 - R_0^{-1}mH'\,, \qquad \gamma_0' < 1 - R_3^{-1}m^2H'HA\lambda\,, \qquad \gamma_1' \leqslant (1-\gamma_0')R_1\,,\\ & \gamma_2' \leqslant (1-\gamma_0')R_2\,, \qquad \gamma_3' \leqslant (1-\gamma_0')R_3 - m^2H'HA\lambda\,.\end{aligned}$$

Then $mH' + R_0\gamma_0' \leqslant R_0$, and using (2.17), (2.63), (2.64), we derive

$$\begin{aligned}(2.65)\qquad |U_i(x, y) - U_i(\bar{x}, y)| &\leqslant mH'\left|\int_x^{\bar{x}} n(\xi)\,d\xi\right| + m^2H'HA\lambda\left|\int_x^{\bar{x}} m(\xi)\,d\xi\right| +\\ &+ (1-\gamma_0')\left|\int_x^{\bar{x}}\big(R_1\mathring{m}(\xi) + R_2\mathring{m}'(\xi)\big)\,d\xi\right| +\\ &+ [(1-\gamma_0')R_3 - m^2H'HA\lambda]\left|\int_x^{\bar{x}} m(\xi)\,d\xi\right| +\\ &+ \gamma_0'\left|\int_x^{\bar{x}}\big(R_0 n(\xi) + R_1\mathring{m}(\xi) + R_2\mathring{m}'(\xi) + R_3 m(\xi)\big)\,d\xi\right| \leqslant\\ &\leqslant \left|\int_x^{\bar{x}}\big(R_0 n(\xi) + R_1\mathring{m}(\xi) + R_2\mathring{m}'(\xi) + R_3 m(\xi)\big)\,d\xi\right| = \left|\int_x^{\bar{x}}\chi(\xi)\,d\xi\right|.\end{aligned}$$

Comparing (2.50), (2.59), (2.62), (2.65), with (2.29), (2.30), (2.31), (2.47), we see that, for every fixed $z \in \mathcal{K}_1$, corresponding $g = g[z] \in \mathcal{K}_0$, and every fixed $\varphi \in \mathfrak{J}$, the transformation $T^*_{z\varphi}$, or $u \to U$, maps $\mathcal{K}_{1\varphi}$ into itself.

Let us prove that $T^*_{z\varphi}$ is a contraction. Indeed for any $z \in \mathcal{K}_1$, corresponding $g = g[z] \in \mathcal{K}_0$, and any two elements $u, u' \in \mathcal{K}_{1\varphi}$, we have from (2.48), for $U = T^*_{z\varphi}u$, $U' = T^*_{z\varphi}u'$,

$$\begin{aligned}|U_i(x, y) - U_i'(x, y)| &= \Big|\sum_{s=1}^{m}\alpha_{si}(x, y, z(x, y))\cdot\\ &\cdot\int_0^x\sum_{h=1}^{m}\big(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\big)\cdot\\ &\cdot[u_h(\xi, \breve{g}_s(\xi; x, y)) - u_h'(\xi, \breve{g}_s(\xi; x, y))]\,d\xi\Big| \leqslant\\ &\leqslant m^2H'\int_0^x\big(\mathring{m}(\xi) + rC(1 + mQ)\,m(\xi) + mC\chi(\xi)\big)\,\|u - u'\|\,d\xi \leqslant\\ &\leqslant m^2H'(\mathring{M}_a + rC(1 + mQ)\,M_a + mC\Xi_a)\,\|u - u'\|\,.\end{aligned}$$

We shall take a sufficiently small so that

$$\gamma = m^2 H'(\overset{\circ}{M}_a + rC(1+mQ)\,M_a + mC\Xi_a) \leqslant k < 1 , \tag{2.66}$$

and then the previous estimate yields

$$\| U - U' \| \leqslant k \| u - u' \| .$$

Thus $T^*_{z\varphi}\colon \mathcal{K}_{1\varphi} \to \mathcal{K}_{1\varphi}$ is a contraction. There exists, therefore, a unique fixed element $u = u[z, \varphi] \in \mathcal{K}_{1\varphi}$ with $u = T^*_{z\varphi} u$. For this fixed element, we derive from (2.48) the integral equations

$$\begin{aligned} u_i(x, y) = \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y)) \cdot &\Big\{ \sum_{h=1}^{m} A_{sh}\big(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))\big)\, \varphi_h\big(\breve{g}_s(0; x, y)\big) + \\ &+ \int_0^x \Big[\sum_{h=1}^{m} \Big(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi \Big) \cdot \\ &\cdot u_h(\xi, \breve{g}_s(\xi; x, y)) + f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))) \Big] d\xi \Big\} , \\ &(x, y) \in D_a = I_a \times E^r, \qquad i = 1, \ldots, m , \end{aligned} \tag{2.67}$$

and from (2.50) we have $u_i(0, y) = \varphi_i(y)$, $i = 1, \ldots, m$, $y \in E^r$.

From (2.52) we have for $u(x, y) = (u_1, \ldots, u_m)$ also the equivalent integral equations:

$$\left\{ \begin{aligned} u_i(x, y) &= \varphi_i(y) + \sum_{s=1}^{m} \alpha_{si}(x, y, z(x, y))(\Delta_{s1} + \Delta_{s2} + \Delta_{s3}) , \\ \Delta_{s1} = \Delta_{s1}(x, y) &= \int_0^x f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))\, d\xi , \\ \Delta_{s2} = \Delta_{s2}(x, y) &= \sum_{h=1}^{m} A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) \cdot \\ &\quad \cdot [\varphi_h(\breve{g}_s(0; x, y)) - \varphi_h(\breve{g}_s(x; x, y))] , \\ \Delta_{s3} = \Delta_{s3}(x, y) &= \int_0^x \sum_{h=1}^{m} \Big(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi \Big) \cdot \\ &\quad \cdot [u_h(\xi, \breve{g}_s(\xi; x, y)) - \varphi_h(\breve{g}_s(x; x, y))] d\xi , \end{aligned} \right. \tag{2.68}$$

where $z \in \mathcal{K}_1$, and $g = g[z] \in \mathcal{K}_0$ is the unique element in $\mathcal{K}_0$ with $g = T_z g$, $\varphi \in \mathfrak{J}$, and $u = u[z, \varphi] \in \mathcal{K}_{1\varphi} \subset \mathcal{K}_1$.

(*e*) *The element* $u[z, \varphi]$ *as the solution of the linear Cauchy problem.* Let us prove that this element $u = u[z, \varphi] \in \mathcal{K}_{1\varphi} \subset \mathcal{K}_1$ is the unique solution of the Cauchy problem for the linear system, in the unknowns $u_1, \ldots, u_m$,

$$(2.69) \qquad \sum_{j=1}^{m} A_{ij}(x, y, z(x, y))\Big[\partial u_j/\partial x + \sum_{k=1}^{r} \varrho_{ik}(x, y, z(x, y))(\partial u_j/\partial y_k)\Big] = \\ = f_i(x, y, z(x, y)), \qquad (x, y) \in D_a,$$

$$(2.70) \qquad u_i(0, y) = \varphi_i(y), \qquad y \in E^r, \qquad i = 1, \ldots, m.$$

(Cfr. [6] for this proof). If we write j instead of i in relations (2.67), then, by multiplication by $A_{ij}(x, y, z(x, y))$, summation with respect to j, and usual simplifications, we have

$$\sum_{j=1}^{m} A_{ij}(x, y, z(x, y))\, u_j(x, y) = \\ = \sum_{j=1}^{m} A_{ij}(x, y, z(x, y)) \sum_{s=1}^{m} \alpha_{sj}(x, y, z(x, y))\{\ldots\} = \\ = \sum_{h=1}^{m} A_{ih}(0, \breve{g}_i(0; x, y), z(0, \breve{g}_i(0; x, y)))\, \varphi_h(\breve{g}_i(0; x, y)) + \\ + \int_0^x \Big[\sum_{h=1}^{m} \big(dA_{ih}(\xi, \breve{g}_i(\xi; x, y), z((\xi, \breve{g}_i(\xi; x, y)))/d\xi\big)\, u_h(\xi, \breve{g}_i(\xi; x, y)) + \\ + f_i(\xi, \breve{g}_i(\xi; x, y), z(\xi, \breve{g}_i(\xi; x, y)))\Big]\, d\xi.$$

By integration by parts, and further simplifications, we obtain

$$(2.71) \qquad \int_0^x \Big[-\sum_{h=1}^{m} A_{ih}(\xi, \breve{g}_i(\xi; x, y), z(\xi, \breve{g}_i(\xi; x, y)))\big(du_h(\xi, \breve{g}_i(\xi; x, y))/d\xi\big) + \\ + f_i(\xi, \breve{g}_i(\xi; x, y), z(\xi, \breve{g}_i(\xi; x, y)))\Big]\, d\xi = 0,$$

and this relation holds for all $(x, y) \in I_a \times E^r$, $i = 1, \ldots, m$. By taking $y = \breve{g}_i(x; 0, \eta)$ and making use of (2.42), relation (2.71) is transformed into

$$(2.72) \qquad \int_0^x \Big[-\sum_{h=1}^{m} A_{ih}(\xi, \breve{g}_i(\xi; 0, \eta), z(\xi, \breve{g}_i(\xi; 0, \eta)))\big(du_h(\xi, \breve{g}_i(\xi; x, y))/d\xi\big)_{y=\breve{g}_i(x;0,\eta)} + \\ + f_i(\xi, \breve{g}_i(\xi; 0, \eta), z(\xi, \breve{g}_i(\xi; 0, \eta)))\Big]\, d\xi = 0,$$

and this relation holds for all (x, η) of the region $I_a \times E^r$ (in the $x\eta$-space). By force of (2.38) and (2.55), the derivative in (2.72) becomes

$$D_{hi}(\xi; 0, \eta) = \big(du_h(\xi, \breve{g}_i(\xi; x, y))/d\xi\big)_{y=\breve{g}_i(x;0,\eta)} = \\ = \Big[\partial u_h/\partial \xi + \sum_{l=1}^{r} \varrho_{il}(\xi, \breve{g}_i(\xi; x, y), z(\xi, \breve{g}_i(\xi; x, y)))(\partial u_h/\partial y_l)\Big]_{y=\breve{g}_i(x;0,\eta)} = \\ = \partial u_h/\partial \xi + \sum_{l=1}^{r} \varrho_{il}(\xi, \breve{g}_i(\xi; 0, \eta), z(\xi, \breve{g}_i(\xi; 0, \eta)))(\partial u_h/\partial y_l),$$

where the arguments of $\partial u_h/\partial y_l$ are $(\xi, \breve{g}_i(\xi; 0, \eta))$, and this relation holds a.e. in the region $I_a \times E^r$ of the $\xi\eta$-space. By differentiating (2.72) with respect to x we obtain the relation

$$\sum_{h=1}^{m} A_{ih}(x, \breve{g}_i(x; 0, \eta), z(x, \breve{g}_i(x; 0, \eta)))\, D_{hi}(x; 0, \eta) =$$
$$= f_i(x, \breve{g}_i(x; 0, \eta), z(x, \breve{g}_i(x; 0, \eta))) , \qquad (x, \eta) \in I_a \times E^r, \text{ (a.e.)}, \ i = 1, \dots, m,$$

and this relation holds a.e. in $I_a \times E^r$. Finally, by taking here $\eta = \breve{g}_i(0; x, y)$, that is, returning to the variables xy, we obtain

$$\text{(2.73)} \qquad \sum_{h=1}^{m} A_{ih}(x, y, z(x, y)) \Big[\partial u_h/\partial x + \sum_{l=1}^{r} \varrho_{il}(x, y, z(x, y)) (\partial u_h/\partial y_l) \Big] =$$
$$= f_i(x, y, z(x, y)) , \qquad (x, y) \in I_a \times E^r, \text{ (a.e.)}, \ i = 1, \dots, m .$$

Since the transformation $\eta = \breve{g}_i(0; x, y)$, or (2.43), preserves sets of measure zero, we conclude that (2.73) holds a.e. in $I_a \times E^r$ as stated.

We have proved that the element $u = u[z, \varphi]$ is a solution of the linear Cauchy problem (2.69), (2.70).

(*f*) *The element $u[z, \varphi]$ is a continuous function of z and φ.* We need to show that $u[z, \varphi]$ is a continuous function of z and φ. Let z, z' be any two elements of $\mathcal{K}_1$ and let $g = g[z]$, $g' = g[z']$ be the corresponding elements of $\mathcal{K}_0$, $g = T_z g$, $g' = T_{z'} g'$. Let φ, φ' be any two elements of $\mathcal{J}$, and let $u = u[z, \varphi]$, $u' = u[z', \varphi']$ be the corresponding elements $u = T^*_{z\varphi} u$, $u' = T^*_{z'\varphi'} u'$, $u \in \mathcal{K}_{1\varphi}$, $u' = \mathcal{K}_{1\varphi'}$. Then from (2.68) we derive

$$u_i(x, y) - u'_i(x, y) = (\varphi_i(y) - \varphi'_i(y)) + \varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 ,$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ have the expressions given below, and we shall estimate them one by one. First we have

$$\varepsilon_0 = \sum_{s=1}^{m} [\alpha_{si}(x, y, z(x, y)) - \alpha_{si}(x, y, z'(x, y))] \cdot [\Delta_{s1}(x, y) + \Delta_{s2}(x, y) + \Delta_{s3}(x, y)] ,$$

with $\Delta_{s1}, \Delta_{s2}, \Delta_{s3}$ given by (2.52) and for which we gave in (2.58) the estimate

$$|\Delta_{s1} + \Delta_{s2} + \Delta_{s3}| \leqslant N_a + mH\Lambda M_a + m(\mathring{M}_a + rC(1 + mQ) M_a +$$
$$+ mC\Xi_a)(\Xi_a + QM_a) .$$

By force of (2.13) we have now

$$|\varepsilon_0| \leqslant mC'[N_a + mH\Lambda M_a + m(\mathring{M}_a + rC(1 + mQ) M_a + mC\Xi_a)(\Xi_a + QM_a)] \|z - z'\| .$$

Then, we have with obvious notations

$$\varepsilon_1 = \sum_{s=1}^{m} \alpha_{si}(x, y, z'(x, y))(\Delta_{s1}(x, y) - \Delta'_{s1}(x, y)) ,$$

$$\Delta_{s1}(x, y) - \Delta'_{s1}(x, y) = \int_0^x [f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))) - \\ - f_s(\xi, \breve{g}'_s(\xi; x, y), z'(\xi, \breve{g}'_s(\xi; x, y)))] \, d\xi .$$

By force of (2.8), (2.12), (2.30) and (2.44) we have

$$|\Delta_{s1}(x, y) - \Delta'_{s1}(x, y)| \leqslant \int_0^x l_1(\xi)[(1 + Q)\|\breve{g}_s - \breve{g}'_s\| + \|z - z'\|] \, d\xi \leqslant \\ \leqslant L_{1a}(1 + (1 + Q)\lambda L_a)\|z - z'\| ,$$

$$|\varepsilon_1| \leqslant mH' L_{1a}(1 + (1 + Q)\lambda L_a)\|z - z'\| .$$

Analogously, we have

$$\varepsilon_2 = \sum_{s=1}^{m} \alpha_{si}(x, y, z'(x, y))(\Delta_{s2}(x, y) - \Delta'_{s2}(x, y)) ,$$

$$\Delta_{s2}(x, y) - \Delta'_{s2}(x, y) = \sum_{h=1}^{m} A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) \cdot \\ \cdot [\varphi_h(\breve{g}_s(0; x, y)) - \varphi_h(\breve{g}_s(x; x, y)) - \\ - \varphi'_h(\breve{g}'_s(0; x, y)) + \varphi'_h(\breve{g}'_s(x; x, y))] + \\ + \sum_{h=1}^{m} [A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) - \\ - A_{sh}(0, \breve{g}'_s(0; x, y), z'(0, \breve{g}'_s(0; x, y)))] \cdot \\ \cdot [\varphi'_h(\breve{g}'_s(0; x, y)) - \varphi'_h(\breve{g}'_s(x; x, y))] .$$

By force of (2.4), (2.12), (2.30), (2.44), (2.45) we have

$$|\Delta_{s2}(x, y) - \Delta'_{s2}(x, y)| \leqslant mH[2\|\varphi - \varphi'\| + 2\Lambda\|g - g'\|] + \\ + mC[(1 + Q)\|g - g'\| + \|z - z'\|]\Lambda M_a = \\ = 2mH\|\varphi - \varphi'\| + m\Lambda[2H\lambda L_a + C(1 + (1 + Q)\lambda L_a) M_a]\|z - z'\| ,$$

$$|\varepsilon_2| \leqslant 2m^2 H' H\|\varphi - \varphi'\| + m^2 H' \Lambda[2H\lambda L_a + C(1 + (1 + Q)\lambda L_a) M_a]\|z - z'\| .$$

Finally, we have, by manipulation and integration by parts,

$$\varepsilon_3 = \sum_{s=1}^{m} \alpha_{si}(x, y, z'(x, y))(\Delta_{s3}(x, y) - \Delta'_{s3}(x, y)) ,$$

$$\Delta_{s3}(x, y) - \Delta'_{s3}(x, y) = \sum_{h=1}^{m} \int_0^x [dA_{sh}/d\xi)(u_h - \varphi_h) - (dA'_{sh}/d\xi)(u'_h - \varphi'_h)]\, d\xi =$$

$$= \sum_{h=1}^{m} \int_0^x [dA_{sh}/d\xi - dA'_{sh}/d\xi)(u_h - \varphi_h) +$$

$$+ (dA'_{sh}/d\xi)(u_h - \varphi_h - u'_h + \varphi'_h)]\, d\xi =$$

$$= \sum_{h=1}^{m} \Big\{ \big[A_{sh}(x, y, z(x, y)) - A_{sh}(x, y, z'(x, y))\big] \cdot$$

$$\cdot (u_h(x, y) - u_h(0, y)) -$$

$$- \big[A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) -$$

$$- A_{sh}(0, \breve{g}'_s(0; x, y), z'(0, \breve{g}'_s(0; x, y)))\big] \cdot$$

$$\cdot (u_h(0, \breve{g}_s(0; x, y)) - u_h(0, \breve{g}_s(x; x, y))) -$$

$$- \int_0^x \big[A_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))) -$$

$$- A_{sh}(\xi, \breve{g}'_s(\xi; x, y), z'(\xi, \breve{g}'_s(\xi; x, y)))\big] \cdot$$

$$\cdot (du_h(\xi, \breve{g}_s(\xi; x, y))/d\xi)\, d\xi + \int_0^x (dA'_{sh}/d\xi) \cdot$$

$$\cdot \big[u_h(\xi, \breve{g}_s(\xi; x, y)) - u'_h(\xi, \breve{g}'_s(\xi; x, y)) - \varphi_h(y) + \varphi'_h(y)\big]\, d\xi \Big\}.$$

By force of (2.4), (2.12), (2.22), (2.30), (2.31), (2.44), (2.53), (2.57) we have

$$|\Delta_{s3}(x, y) - \Delta'_{s3}(x, y)| \leqslant$$

$$\leqslant m\Big[C\|z - z'\| \int_0^x \chi(\xi)\, d\xi + 2\big(C(1 + Q)\|g - g'\| + C\|z - z'\|\big) \cdot$$

$$\cdot \int_0^x (\chi(\xi) + rQm(\xi))\, d\xi + \int_0^x [\mathring{m}(\xi) + rC(1 + mQ)m(\xi) + mC\chi(\xi)] \cdot$$

$$\cdot (Q\|g - g'\| + \|u - u'\| + \|\varphi - \varphi'\|)\, d\xi\Big] \leqslant$$

$$\leqslant m[C\Xi_a + 2C(1 + (1 + Q)\lambda L_a)(\Xi_a + rQM_a) +$$

$$+ (\mathring{M}_a + rC(1 + mQ)\, M_a + mC\Xi_a) Q\lambda L_a] \|z - z'\| +$$

$$+ m(\mathring{M}_a + rC(1 + mQ)\, M_a + mC\Xi_a)(\|u - u'\| + \|\varphi - \varphi'\|)\,,$$

$$|\varepsilon_3| \leqslant m^2 H'[C\Xi_a + 2C(1 + (1 + Q)\lambda L_a)(\Xi_a + rQM_a) +$$

$$+ (\mathring{M}_a + rC(1 + mQ)\, M_a + mC\Xi_a) Q\lambda L_a] \|z - z'\| +$$

$$+ m^2 H'(\mathring{M}_a + rC(1 + mQ)\, M_a + mC\Xi_a)(\|u - u'\| + \|\varphi - \varphi'\|)\,.$$

Combining the previous estimates we have

$$
(2.74)\quad \begin{cases}
|u_i(x, y) - u_i'(x, y)| \leqslant |\varepsilon_0| + |\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| + \|\varphi - \varphi'\| \leqslant \\
\qquad \leqslant \gamma\|u - u'\| + (1 + \gamma + 2m^2 H' H)\|\varphi - \varphi'\| + \bar{\gamma}\|z - z'\|, \\
\gamma = m^2 H'(\mathring{M}_a + rC(1 + mQ)M_a + mC\Xi_a), \\
\bar{\gamma} = mC'[N_a + mH\Lambda M_a + m(\mathring{M}_a + rC(1 + mQ)M_a + mC\Xi_a)\cdot \\
\qquad \cdot(\Xi_a + QM_a)] + mH'L_{1a}(1 + (1 + Q)\lambda L_a) + \\
\qquad + m^2 H'\Lambda[2H\lambda L_a + C(1 + (1 + Q)\lambda L_a)M_a] + \\
\qquad + m^2 H'[C\Xi_a + 2C(1 + (1 + Q)\lambda L_a)(\Xi_a + rQM_a) + \\
\qquad + (\mathring{M}_a + rC(1 + mQ)M_a + mC\Xi_a)Q\lambda L_a],
\end{cases}
$$

and finally

$$
\|u - u'\| \leqslant \gamma\|u - u'\| + (1 + \gamma + 2m^2 H' H)\|\varphi - \varphi'\| + \bar{\gamma}\|z - z'\|,
$$

where γ is the same constant we have encountered in (2.66), $0 < \gamma \leqslant k < 1$. We shall assume $a > 0$ sufficiently small so that

$$
(2.75)\qquad \gamma < 1, \qquad (1 - \gamma)^{-1}\bar{\gamma} \leqslant k < 1 .
$$

Then, the estimates above yield

$$
(2.76)\qquad \|u[z, \varphi] - u[z', \varphi']\| \leqslant (1 - \gamma)^{-1}(1 + \gamma + 2m^2 H' H)\|\varphi - \varphi'\| + k\|z - z'\| .
$$

(*g*) *The transformation* $\mathcal{T}_\varphi$. For each $z \in \mathcal{K}_1$ we have first determined a unique element $g = g[z] \in \mathcal{K}_0$ with $g = T_z g$, and for each $\varphi \in \mathfrak{I}$ we have determined a unique element $u = u[z, \varphi] \in \mathcal{K}_{1\varphi}$, $u = T^*_{z\varphi} u$, satisfying (2.68). Since $\mathcal{K}_{1\varphi} \subset \mathcal{K}_1$, we may take $z \in \mathcal{K}_{1\varphi}$, and then we have actually defined a map $u = \mathcal{T}_\varphi z$, or $z \to u$, or $\mathcal{K}_{1\varphi} \to \mathcal{K}_{1\varphi}$. This transformation is a contraction. Indeed, for any two elements $z, z' \in \mathcal{K}_{1\varphi}$, $g = g[z]$, $g' = g[z']$, and $u = u[z, \varphi]$, $u' = u[z', \varphi]$, we have from (2.76)

$$
\|u - u'\| = \|u[z, \varphi] - u[z', \varphi]\| \leqslant k\|z - z'\| ,
$$

where $k < 1$. Thus, for every $\varphi \in \mathfrak{I}$, there is a fixed element $z = \mathcal{T}_\varphi z$ with

$z \in \mathcal{K}_{1\varphi}$ such that the following integral equations hold:

$$(2.77)\quad \begin{cases} g_{ik}(\xi; x, y) = y_k - \displaystyle\int_\xi^x \varrho_{ik}(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y)))\, d\alpha\,, \\ \qquad\qquad (\xi; x, y) \in I_a \times I_a \times E^r, \quad i = 1, \ldots, m,\ k = 1, \ldots, r\,, \\ z_i(x, y) = \varphi_i(y) + \displaystyle\sum_{s=1}^m \alpha_{si}(x, y, z(x, y))(\Delta_{s1} + \Delta_{s2} + \Delta_{s3})\,, \\ \Delta_{s1} = \Delta_{s1}(x, y) = \displaystyle\int_0^x f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))\, d\xi\,, \\ \Delta_{s2} = \Delta_{s2}(x, y) = \displaystyle\sum_{h=1}^m A_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y)))\cdot \\ \qquad\qquad \cdot[\varphi_h(\breve{g}_s(0; x, y)) - \varphi_h(\breve{g}_s(x; x, y))]\,, \\ \Delta_{s3} = \Delta_{s3}(x, y) = \displaystyle\int_0^x \sum_{h=1}^m \big(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\big)\cdot \\ \qquad\qquad \cdot[z_h(\xi, \breve{g}_s(\xi; x, y)) - \varphi_h(\breve{g}_s(x; x, y))]d\xi\,, \\ \qquad\qquad (x, y) \in I_a \times E^r, \quad z_i(0, y) = \varphi_i(y),\ y \in E^r, \quad i = 1, \ldots, m\,. \end{cases}$$

Here $z \in \mathcal{K}_{1\varphi}$ and $g = g[z] \in \mathcal{K}_0$. We shall denote this element z by $z[\varphi]$. From (2.67) we derive for z also the equivalent equations

$$(2.78)\quad \begin{aligned} z_i(x, y) = \sum_{s=1}^m \alpha_{si}(x, y, z(x, y)) \Big\{ &\sum_{h=1}^m A_{sh}\,(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y)))\cdot \\ &\cdot \varphi_h(\breve{g}_s(0; x, y)) + \int_0^x \Big[\sum_{h=1}^m \big(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\big)\cdot \\ &\cdot z_h(\xi, \breve{g}_s(\xi; x, y)) + f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))\Big] d\xi \Big\}, \\ &\qquad (x, y) \in I_a \times E^r, \quad i = 1, \ldots, m\,. \end{aligned}$$

From (e) we derive that $z[\varphi]$ is a solution of the Cauchy problem which we obtain from (2.69), (2.70) by taking $z = u$; that is, $z[\varphi]$ is a solution of the original Cauchy problem (2.1), (2.2). We have already seen that $z[\varphi]$ is the unique element in the class $\mathcal{K}_{1\varphi}$ having this property. Let us prove that $z[\varphi]$ depends continuously on φ.

Since $\mathcal{K}_{1\varphi} \subset \mathcal{K}_1$, the map $\varphi \to z$ is actually a map from $\mathfrak{J}$ into $\mathcal{K}_1$, and this map is continuous. Indeed, if $\varphi, \varphi' \in \mathfrak{J}$ and $z = z[\varphi]$, $z' = z[\varphi']$, then, from (2.76), we derive

$$\|z-z'\| = \|z[\varphi]-z[\varphi']\| = \|u[z,\varphi]-u[z',\varphi']\| \leqslant k\|z-z'\| + (1-\gamma)^{-1}(1+\gamma+2m^2H'H)\|\varphi-\varphi'\|$$

where $k<1$. Hence, we have

$$\|z[\varphi]-z[\varphi']\| \leqslant (1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)\|\varphi-\varphi'\| . \tag{2.79}$$

Theorem I is thereby proved.

Note that the only restrictions we had to impose on the size of a, $0<a\leqslant a_0$, are relations (2.19), (2.60), (2.61), (2.64), (2.66), (2.75). These are not meant to give, however, the best possible estimate for a. Improved estimates on the size of a will be discussed elsewhere.

3. – The existence theorem for the boundary value problem.

We consider here hyperbolic systems of the same Schauder canonic form (1.1), or

$$\sum_{j=1}^{m} A_{ij}(x,y,z)\left[\partial z_j/\partial x + \sum_{k=1}^{r} \varrho_{ik}(x,y,z)\,\partial z_j/\partial y_k\right] = f_i(x,y,z) , \tag{3.1}$$
$$(x,y)\in I_a\times E^r, \qquad i=1,\dots,m , \qquad \det[A_{ij}]\neq 0 ,$$

with boundary conditions III, that is,

$$\sum_{j=1}^{m} b_{ij}(y)\,z_j(a_i,y) = \psi_i(y) , \qquad y\in E^r, \qquad i=1,\dots,m , \tag{3.2}$$

where $\psi_i(y)$, $b_{ij}(y)$ are given functions of y in E^r with $\det[b_{ij}]\neq 0$, and where $0\leqslant a_1\leqslant\dots\leqslant a_m\leqslant a$ are given numbers (between 0 and a). As mentioned in the introduction, we assume here that both the $m\times m$ matrix $[b_{ij}]$ and the $m\times m$ matrix $[A_{ij}]$ have « dominant » diagonal terms. By possibly multiplying each equation (3.1) and (3.2) by suitable nonzero factors, we shall simply assume that

$$A_{ij}(x,y,z) = \delta_{ij} + \tilde{A}_{ij}(x,y,z) , \qquad (x,y,z)\in I_a\times E^r\times\Omega , \tag{3.3}$$
$$b_{ij}(y) = \delta_{ij} + \tilde{b}_{ij}(y) , \qquad y\in E^r, \qquad i,j=1,\dots,m ,$$

where $\delta_{ii}=1$, $\delta_{ij}=0$ for $i\neq j$, and $\tilde{A}_{ij}$, $\tilde{b}_{ij}$ are «small» in the sense we state below.

Let

$$\sigma_0 = \mathrm{Max}_i \,\mathrm{Sup} \sum_{h=1}^{m} |\tilde{b}_{ih}(y)| \,, \tag{3.4}$$

where Sup is taken for all $y \in E^r$. If $[b_{ij}]$ is the identity matrix, then $\sigma_0 = 0$. Thus, the smallness of σ_0 gives an indication of the closeness of $[b_{ij}]$ to the identity matrix.

We proceed in some way analogously with the $m\times m$ matrix $[A_{ij}]$. As in Section **2** we denote by α_{ij} the cofactor of A_{ij} divided by $\det[A_{ij}]$, and we take

$$\alpha_{ij}(x, y, z) = \delta_{ij} + \tilde{\alpha}_{ij}(x, y, z)\,, \qquad (x, y, z) \in I_{a_0}\times E^r \times \Omega, \qquad i, j = 1, \ldots, m\,.$$

Now let

$$\begin{aligned}
\sigma_1 &= \mathrm{Max}_i \,\mathrm{Sup} \sum_{h=1}^{m} |\tilde{A}_{ih}(x, y, z)| \,,\\
\sigma_2 &= \mathrm{Max}_i \,\mathrm{Sup} \sum_{h=1}^{m} |\tilde{\alpha}_{hi}(x, y, z)| \,,\\
\sigma_3 &= \mathrm{Max}_i \,\mathrm{Sup} \sum_{s=1}^{m} \sum_{h=1}^{m} |\tilde{\alpha}_{si}(x, y, z)|\,|\tilde{A}_{sh}(x, y, z)| \,,
\end{aligned} \tag{3.5}$$

where Sup is taken for all $(x, y, z) \in I_{a_0}\times E^r\times\Omega$, and let

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3\,.$$

Note that, for $[A_{ij}]$ the identity matrix, we have $\sigma = 0$. Thus, the smallness of σ gives an indication of the closeness of $[A_{ij}]$ to the identity matrix.

We shall assume below that

$$\sigma + \sigma_0 + \sigma\sigma_0 < 1\,.$$

Note that, if $[A_{ij}]$ is the identity matrix, $\sigma = 0$, and all we need is that $\sigma_0 < 1$. If $[b_{ij}]$ is the identity matrix, $\sigma_0 = 0$, and all we need is that $\sigma < 1$.

THEOREM II (an existence theorem for boundary value problem (3.2) and Schauder's canonic system (3.1)). Let Ω be a given positive number, and let Ω also denote the interval $[-\Omega, \Omega]^m$ in E^m. Let $A_{ij}(x, y, z)$, $i, j = 1, \ldots, m$, be continuous functions on $I_{a_0}\times E^r\times\Omega$, $a_0 > 0$, with $\det[A_{ij}(x, y, z)] \geqslant \mu > 0$ in $I_{a_0}\times E^r \times \Omega$ for some constant μ. Let us assume that there are constants

$H>0$, $C\geqslant 0$ and a function $\mathring{m}(x)\geqslant 0$, $0\leqslant x\leqslant a_0$, $\mathring{m}\in L_1[0, a_0]$, such that, for all (x, y, z), $(x, \bar{y}, \bar{z})$, $(\bar{x}, y, z)\in I_{a_0}\times E^r\times\Omega$, and all $i, j=1, \dots, m$, we have

$$
\begin{aligned}
&|A_{ij}(x, y, z)|\leqslant H\,,\\
(3.6)\qquad &|A_{ij}(x, y, z)-A_{ij}(x, \bar{y}, \bar{z})|\leqslant C[|y-\bar{y}|+|z-\bar{z}|]\,,\\
&|A_{ij}(x, y, z)-A_{ij}(\bar{x}, y, z)|\leqslant\left|\int_x^{\bar{x}}\mathring{m}(\alpha)(\alpha)\,d\alpha\right|.
\end{aligned}
$$

If α_{ij} denotes the cofactor of A_{ij} in the $m\times m$ matrix $[A_{ij}]$ divided by $\det[A_{ij}]$, then certainly there are constants $H'>0$, $C'\geqslant 0$ and a function $\mathring{m}'(x)\geqslant 0$, $0\leqslant x\leqslant a_0$, $\mathring{m}'\in L_1[0, a_0]$, such that, as above

$$
\begin{aligned}
&|\alpha_{ij}(x, y, z)|\leqslant H'\,,\\
(3.7)\qquad &|\alpha_{ij}(x, y, z)-\alpha_{ij}(x, \bar{y}, \bar{z})|\leqslant C'[|y-\bar{y}|+|z-\bar{z}|]\,,\\
&|\alpha_{ij}(x, y, z)-\alpha_{ij}(\bar{x}, y, z)|\leqslant\left|\int_x^{\bar{x}}\mathring{m}'(\alpha)\,d\alpha\right|.
\end{aligned}
$$

Let $\varrho_{ik}(x, y, z)$, $f_i(x, y, z)$, $i=1, \dots, m$, $k=1, \dots, r$, be functions defined in $I_{a_0}\times E^r\times\Omega$, measurable in x for every (y, z), continuous in (y, z) for every x, and let us assume that there are nonnegative functions $m(x)$, $l(x)$, $n(x)$, $l_1(x)$, $0\leqslant x\leqslant a_0$, $m, l, n, l_1\in L_1[0, a_0]$, such that, for all (x, y, z), $(x, \bar{y}, \bar{z})\in I_{a_0}\times E^r\times\Omega$, $i=1, \dots, m$, $k=1, \dots, r$, we have

$$
\begin{aligned}
&|\varrho_{ik}(x, y, z)|\leqslant m(x)\,,\qquad |f_i(x, y, z)|\leqslant n(x)\,,\\
(3.8)\qquad &|\varrho_{ik}(x, y, z)-\varrho_{ik}(x, \bar{y}, \bar{z})|\leqslant l(x)[|y-\bar{y}|+|z-\bar{z}|]\,,\\
&|f_i(x, y, z)-f_i(x, \bar{y}, \bar{z})|\leqslant l_1(x)[|y-\bar{y}|+|z-\bar{z}|]\,.
\end{aligned}
$$

Let $\psi_i(y)$, $b_{ij}(y)$, $y\in E^r$, $i, j=1, \dots, m$, be given continuous functions in E^r, and let us assume that there are constants ω_0, Λ_0, τ_0, $0<\omega_0<\Omega$, $\Lambda_0\geqslant 0$, $\tau_0\geqslant 0$, such that, for all $y, \bar{y}\in E^r$ and $i=1, \dots, m$, we have

$$
\begin{aligned}
(3.9)\qquad &|\psi_i(y)|\leqslant\omega_0\,,\qquad |\psi_i(y)-\psi_i(\bar{y})|\leqslant\Lambda_0|y-\bar{y}|\,,\\
&\sum_{j=1}^{m}|b_{ij}(y)-b_{ij}(\bar{y})|\leqslant\tau_0|y-\bar{y}|\,.
\end{aligned}
$$

With the notations (3.3), (3.4), (3.5) let us assume that $\sigma+\sigma_0+\sigma\sigma_0<1$. Then, for a, ω_0, τ_0, C, C' sufficiently small, $0<a\leqslant a_0$, ω_0, τ_0, C, $C'>0$, and for every system of numbers a_i, $0\leqslant a_i\leqslant a$, $i=1, \dots, m$, there are a constant $Q>0$, a function $\chi(x)\geqslant 0$, $0\leqslant x\leqslant a$, $\chi\in L_1[0,a]$, and a vector function $z(x,y)=(z_1,\dots,z_m)$, $(x,y)=(x,y_1,\dots,y_r)\in I_a\times E^r$, continuous in $I_a\times E^r$, satisfying (3.2) everywhere in E^r, satisfying (3.1) a.e. in $I_a\times E^r$, and such that for all (x,y), $(x,\bar y)$, $(\bar x,y)\in I_a\times E^r$ and $i=1,\dots,m$, we have

$$\text{(3.10)}\qquad |z_i(x,y)|\leqslant\Omega\,,\qquad |z_i(x,y)-z_i(x,\bar y)|\leqslant Q|y-\bar y|\,,$$
$$|z_i(x,y)-z_i(\bar x,y)|\leqslant\left|\int_x^{\bar x}\chi(\alpha)\,d\alpha\right|.$$

The function $z(x,y)=(z_1,\dots,z_m)$ above is unique and depends continuously on $\psi(y)=(\psi_1,\dots,\psi_m)$ for z and φ in classes which are described in the proof which follows.

Also, computable estimates of ω_0, C, C', τ_0, a are given which depend only on the constants Ω, H, H', Λ_0, σ, σ_0, on the constants $\tilde H$, $\tilde H'$ in (3.11) below, and on the functions $\mathring{m}$, $\mathring{m}'$ m, n, l, l_1, but not on the numbers a_i, $0\leqslant a_i\leqslant a$, $i=1,\dots,m$.

PROOF. The proof is divided into parts (*a*), (*b*), (*c*), (*d*).

(*a*) *Choice of constants.* First, let us denote by $\tilde H$, $\tilde H'$ constants such that

$$\text{(3.11)}\qquad |\tilde A_{ij}(x,y,z)|\leqslant\tilde H\,,\qquad |\tilde a_{ij}(x,y,z)|\leqslant\tilde H'$$

for all $(x,y,z)\in I_a\times E^r\times\Omega$ and $i,j=1,\dots,m$. Thus, we can take H, $H'\leqslant 1+\sigma$, $\tilde H$, $\tilde H'\leqslant\sigma$.

Let us choose any number ω, $0<\omega<\Omega$, as close to Ω as we want, and let us choose ω_0, $0<\omega_0<\omega<\Omega$, so small that $\omega_0<[1-(\sigma+\sigma_0+\sigma\sigma_0)]\omega$. Also, let us choose some number $\Lambda>\Lambda_0$ as large as we want, so as to satisfy $\Lambda_0<[1-(\sigma+\sigma_0+\sigma\sigma_0)]\Lambda$. We shall write these relations in the form

$$\text{(3.12)}\qquad (\sigma+\sigma_0+\sigma\sigma_0)\omega<\omega-\omega_0\,,\qquad \Lambda_0+(\sigma+\sigma_0+\sigma\sigma_0)\Lambda<\Lambda\,.$$

Let Q be any number

$$Q>\Lambda(1+3m^2HH')\,,$$

and let S_0, S_1, S_2, T_1, T_2 denote the numbers

$$S_0=1+\sigma\,,$$
$$S_1=m(1+\sigma_0)(1+Q)(1+m\tilde H'+2(1+\sigma)mH')\,,$$

$$S_2 = m(1+\sigma_0)(1+Q)(1+m\tilde{H}),$$

$$T_1 = (1+\sigma_0)[m(1+m\tilde{H}') + 2m^2H'(1+\sigma)](1+2m^2H'H),$$

$$T_2 = (1+\sigma_0)m(1+m\tilde{H})(1+2m^2H'H).$$

We shall assume that C, C', τ_0 are so small that

$$\Lambda_0 + (\sigma+\sigma_0+\sigma\sigma_0)\Lambda + S_0(\tau_0\,\omega) + S_1(C\omega) + S_2(C'\omega) < \Lambda,$$

$$(\sigma+\sigma_0+\sigma\sigma_0) + T_1(C\omega) + T_2(C'\omega) < 1.$$

Let k' denote any number such that

$$(\sigma+\sigma_0+\sigma\sigma_0) + T_1(C\omega) + T_2(C'\omega) < k' < 1.$$

It is possible to satisfy these relations because of $\sigma+\sigma_0+\sigma\sigma_0<1$ and of (3.12).

If $0<k$, p, $\gamma<1$ denote arbitrary numbers, let

$$S_0' = (1+\sigma)(1-\gamma)^{-1},$$

$$S_1' = m(1+\sigma_0)(1+Q)[(1+p)(1+m\tilde{H}') + mH'(1+\sigma)(1-\gamma)^{-1}(2+p)],$$

$$S_2' = S_2,$$

$$T_1' = (1+\sigma_0)[m(1+m\tilde{H}') + 2m^2H'(1-\gamma)^{-1}(1+\sigma)]\cdot$$
$$\cdot(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H),$$

$$T_2' = (1+\sigma_0)m(1+m\tilde{H})(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H).$$

These numbers approach $S_0, \dots, T_2$, respectively, as $k, p, \gamma \to 0^+$. Thus, we can determine numbers $0<\bar{k}, \bar{p}, \bar{\bar{\gamma}}<1$, such that we also have

$$\Lambda_0 + (1+p)(\sigma+\sigma_0+\sigma\sigma_0)\Lambda + S_0'(\tau_0\omega) + S_1'(C\omega) + S_2'(C'\omega) < (1+p)^{-1}\Lambda,$$

$$(\sigma+\sigma_0+\sigma\sigma_0) + T_1'(C\omega) + T_2'(C'\omega) < k' < 1$$

for all $0\leqslant k\leqslant\bar{k}<1$, $0\leqslant p\leqslant\bar{p}<1$, $0\leqslant\gamma\leqslant\bar{\bar{\gamma}}<1$. Note that for any such p we certainly have

$$Q > \Lambda(1+m^2H'H(2+p)).$$

We now take $k=\bar{k}$, $p=\bar{p}$, and we choose constants R_0, R_1, R_2, R_3, such that

$$R_0 > mH', \quad R_1 > 0, \quad R_2 > 0, \quad R_3 > m^2 H' H \Lambda (1-k)^{-1}.$$

Thus, relations (2.15) and (2.16) are all satisfied.

We are now in a position to define $\chi(x)$ as in (2.17), and to determine a preliminary value, say $\bar{a}$, for the constant a, $0<\bar{a}\leqslant a_0$, so as to satisfy relations (2.19), (2.60), (2.61), (2.64), (2.66), (2.75). Moreover, in determining $\bar{a}$, we shall require, furthermore, that the numbers γ, $\bar{\gamma}$ defined in (2.74) satisfy the following relations.

$$\gamma = m^2 H'(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a) \leqslant \bar{\bar{\gamma}} < 1, \tag{3.13}$$
$$(1-\gamma)^{-1}\bar{\gamma} \leqslant k < 1, \quad \gamma\omega < (1-\gamma)(\omega-\omega_0) - (\sigma+\sigma_0+\sigma\sigma_0)\omega,$$

and thus relation (2.75) is certainly satisfied. We now proceed to define the classes $\mathcal{K}_0$, $\mathcal{K}_1$, $\mathfrak{J}$ as in Section **2**, parts (*b*) and (*d*), in connection with the choice of the constants p, Q, k, ω, Λ, R_0, R_1, R_2, R_3 already made.

Now let

$$R_a = m[\sigma_2\omega\mathring{M}_a + \sigma_2(1+Q)(C\omega)M_a + H'N_a + \sigma_2\omega C\Xi_a],$$

$$\begin{aligned} S_a = {} & m\sigma_2\Lambda(1+p)\big(\mathring{M}_a + C(1+Q)M_a + C\Xi_a\big) + \\ & + m^2C'(1+Q)(1-\gamma)^{-1}(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a)\big((1+\sigma)\omega + R_a\big) + \\ & + mC'(1+Q)N_a + mH'(1+Q)(1+p)L_{1a} + \\ & + m^2H'C(1+Q)(1-\gamma)^{-1}(2+p)R_a + \\ & + m^2H'C(1+Q)(1+p)(\Xi_a + rQM_a) + \\ & + m^2H'Q(1+p)\big(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a\big), \end{aligned}$$

$$\begin{aligned} T_a = {} & m\sigma_2\big(\mathring{M}_a + C\Xi_a + C(1+Q)M_a\big) + (1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)\cdot \\ & \cdot\Big[\Big(\sigma + m\sigma_2\big(\mathring{M}_a + C\Xi_a + C(1+Q)M_a\big)\Big)\Lambda\lambda L_a + \\ & + m(1+Q)(1+m\tilde{H}')(C\omega)\lambda L_a + mC'N_a + \\ & + m^2C'(1-\gamma)^{-1}(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a)\big((1+\sigma)\omega + R_a\big) + \\ & + mH'\big(1+(1+Q)\lambda L_a\big)L_{1a} + m^2H'(1-\gamma)^{-1}CR_a + \\ & + m^2H'(1-\gamma)^{-1}(1+\sigma)(C\omega)(1+Q)\lambda L_a + m^2H'C(1-\gamma)^{-1}\cdot \\ & \cdot\big(1+(1+Q)\lambda L_a\big)R_a + m^2H'C\big(1+(1+Q)\lambda L_a\big)(\Xi_a + rQM_a) + \\ & + m^2H'\big(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a\big)(1+Q\lambda L_a)\Big]. \end{aligned}$$

We can finally determine a, $0 < a \leqslant \bar{a} \leqslant a_0$, sufficiently small so that

$$(3.14)\quad \gamma\omega + (1+\sigma_0)R_a \leqslant (1-\gamma)(\omega-\omega_0) - (\sigma+\sigma_0+\sigma\sigma_0)\omega\,,$$

$$\Lambda_0 + (1+p)(\sigma+\sigma_0+\sigma\sigma_0)\Lambda + S_0'(\tau_0\omega) + S_1'(C\omega) + S_2'(C'\omega) + \\ + (1+\sigma_0)S_a + \tau_0(1-\gamma)^{-1}R_a \leqslant (1+p)^{-1}\Lambda\,,$$

$$(3.15)\quad (\sigma+\sigma_0+\sigma\sigma_0) + T_1'(C\omega) + T_2'(C'\omega) + \\ + (1+\sigma_0)(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)\Lambda\lambda L_a + (1+\sigma_0)T_a \leqslant k'\,.$$

We shall write the first of these relations in the form

$$(3.16)\quad (1-\gamma)^{-1}\big((1+\sigma_0)R_a + \gamma\omega\big) \leqslant \omega - \omega_0 - (1-\gamma)^{-1}(\sigma+\sigma_0+\sigma\sigma_0)\omega\,.$$

(*b*) *The transformation* T^{**}. In Section **2**, for every $\varphi \in \mathfrak{J}$, we have determined a unique element $z = z[\varphi]$, and corresponding element $g = g[z]$, $z \in \mathcal{K}_{1\varphi} \subset \mathcal{K}_1$, $g \in \mathcal{K}_0$, satisfying (2.77), (2.78), or

$$g_{ik}(\xi; x, y) = y_k - \int_\xi^x \varrho_{ik}\big(\alpha, \breve{g}_i(\alpha; x, y), z(\alpha, \breve{g}_i(\alpha; x, y))\big)\, d\alpha\,,$$

$$(\xi; x, y) \in I_a \times I_a \times E^r, \qquad k = 1, \ldots, r\,, \quad i = 1, \ldots, m\,,$$

$$(3.17)\quad z_i(x, y) = \sum_{s=1}^m \alpha_{si}\big(x, y, z(x, y)\big)\Big\{\sum_{h=1}^m A_{sh}\big(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))\big)\cdot \\ \cdot\varphi_h\big(\breve{g}_s(0; x, y)\big) + \int_0^x \Big[\sum_{h=1}^m \Big(dA_{sh}\big(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))\big)/d\xi\Big)\cdot \\ \cdot z_h\big(\xi, \breve{g}_s(\xi; x, y)\big) + f_s\big(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))\big)\Big]\, d\xi\Big\}\,,$$

$$(x, y) \in I_a \times E^r, \qquad i = 1, \ldots, m\,,$$

$$z_i(0, y) = \varphi_i(y)\,, \qquad y \in E^r, \qquad i = 1, \ldots, m\,,$$

where we have written z in the form (2.78).

Because of $\alpha_{ij} = \delta_{ij} + \tilde{\alpha}_{ij}$, $A_{ij} = \delta_{ij} + \tilde{A}_{ij}$, $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$, $i, j = 1, \ldots, m$, with obvious simplifications we derive from (3.17) that

$$(3.18)\qquad z_i(x, y) = \varphi_i\big(\breve{g}_i(0; x, y)\big) + \\ + \sum_{h=1}^m \tilde{A}_{ih}\big(0, \breve{g}_i(0; x, y), z(0, \breve{g}_i(0; x, y))\big)\varphi_h\big(\breve{g}_i(0; x, y)\big) +$$

$$+\sum_{h=1}^{m}\tilde{a}_{hi}(x,y,z(x,y))\,\varphi_h(\breve{g}_h(0;x,y))+$$
$$+\sum_{s=1}^{m}\sum_{h=1}^{m}\tilde{a}_{si}(x,y,z(x,y))\,\tilde{A}_{sh}(0,\breve{g}_s(0;x,y),z(0,\breve{g}_s(0;x,y)))\cdot$$
$$\cdot\varphi_h(\breve{g}_s(0;x,y))+\sum_{s=1}^{m}\alpha_{si}(x,y,z(x,y))\cdot$$
$$\cdot\int_0^x\Big[\sum_{h=1}^{m}\big(dA_{sh}(\xi,\breve{g}_s(\xi;x,y),z(\xi,\breve{g}_s(\xi;x,y)))/d\xi\big)\cdot$$
$$\cdot z_h(\xi,\breve{g}_s(\xi;x,y))+f_s(\xi,\breve{g}_s(\xi;x,y),z(\xi,\breve{g}_s(\xi;x,y)))\Big]\,d\xi\,,$$
$$(x,y)\in I_a\times E^r,\qquad i=1,\dots,m\,.$$

We shall write (3.18) in the form

$$z_i(x,y)=\varphi_i(\breve{g}_i(0;x,y))+\mathfrak{z}_i(x,y))\,, \tag{3.19}$$

and we have for $\mathfrak{z}_i(x,y)$ the equivalent expression

$$\mathfrak{z}_i(x,y)=\sum_{h=1}^{m}\tilde{A}_{ih}(0,\breve{g}_i(0;x,y),z(0,\breve{g}_i(0;x,y)))\varphi_h(\breve{g}_i(0;x,y))+ \tag{3.20}$$
$$+\sum_{h=1}^{m}\tilde{a}_{hi}(x,y,z(x,y))\varphi_h(\breve{g}_h(0;x,y))+$$
$$+\sum_{s=1}^{m}\sum_{h=1}^{m}\tilde{a}_{si}(x,y,z(x,y))\tilde{A}_{sh}(x,y,z(x,y))\,\varphi_h(\breve{g}_s(0;x,y))+$$
$$+\sum_{s=1}^{m}\sum_{h=1}^{m}\tilde{a}_{si}(x,y,z(x,y))[\tilde{A}_{sh}(0,\breve{g}_s(0;x,y),z(0,\breve{g}_s(0;x,y)))-$$
$$-\tilde{A}_{sh}(x,y,z(x,y))]\varphi_h(\breve{g}_s(0;x,y))+$$
$$+\sum_{s=1}^{m}\alpha_{si}(x,y,z(x,y))\int_0^x\Big[\sum_{h=1}^{m}\big(dA_{sh}(\xi,\breve{g}_s(\xi;x,y),z(\xi,\breve{g}_s(\xi;x,y)))/d\xi\big)\cdot$$
$$\cdot z_h(\xi,\breve{g}_s(\xi;x,y))+f_s(\xi,\breve{g}_s(\xi;x,y),z(\xi,\breve{g}_s(\xi;x,y)))\Big]\,d\xi\,,$$
$$(x,y)\in I_a\times E^r,\qquad i=1,\dots,m\,.$$

By force of (2.22), (2.45), (3.5), (3.6), (3.10) the fourth term in (3.20) is in absolute value not larger than

$$\sum_s\sum_h|\tilde{a}_{si}|[\mathring{M}_a+C\Xi_a+C(1+Q)|\breve{g}_s(0;x,y)-\breve{g}_s(x;x,y)|]\,|\varphi_h(\breve{g}_s(0;x,y)|)\leqslant$$
$$\leqslant\sum_s\sum_h|\tilde{a}_{si}|[\mathring{M}_a+C\Xi_a+C(1+Q)\,M_a]\omega\leqslant m\sigma_2\,\omega(\mathring{M}_a+C\Xi_a+C(1+Q)\,M_a)\,.$$

By force of (2.53), (3.7), (3.8) the fifth term in (3.20) is in absolute value not larger than

$$m^2 H'\big(\mathring{M}_a + rC(1+mQ)\,M_a + mC\Xi_a\big)\|z\| + mH'N_a\,.$$

By these partial estimates, and (2.45), (3.5), and (3.20), we derive now the following estimate for $\mathfrak{z}_i$:

$$|\mathfrak{z}_i(x, y)| \leqslant \sigma_1\omega + \sigma_2\omega + \sigma_3\omega + m\sigma_2\omega\big(\mathring{M}_a + C\Xi_a + C(1+Q)\,M_a\big) + \\ + m^2H'\big(\mathring{M}_a + rC(1+mQ)\,M_a + mC\Xi_a\big)\|z\| + mH'N_a\,.$$

By using the numbers $0<\gamma<1$ and $R_a>0$ mentioned in part (*a*), we also have

$$|\mathfrak{z}_i(x, y)| \leqslant \sigma\omega + \gamma\|z\| + R_a\,,$$

so that, by force of (2.45) and (3.19) we have

$$\|z\| \leqslant \omega + \sigma\omega + \gamma\|z\| + R_a\,, \tag{3.21}$$
$$\|z\| \leqslant (1-\gamma)^{-1}\big((1+\sigma)\omega + R_a\big)\,,$$

and finally

$$|\mathfrak{z}_i(x, y)| \leqslant \sigma\omega + \gamma(1-\gamma)^{-1}\big((1+\sigma)\omega + R_a\big) + R_a = \\ = (1-\gamma)^{-1}(\sigma\omega + \gamma\omega + R_a)\,. \tag{3.22}$$

We consider now the transformation T^{**}, or $\Phi = T^{**}\varphi$, $\varphi \in \mathfrak{J}$, or $\varphi \to \Phi$, $\varphi(y) = (\varphi_1, \dots, \varphi_m)$, $\Phi(y) = (\Phi_1, \dots, \Phi_m)$, defined by

$$\Phi_i(\eta) = \big[\Phi_i(\breve{g}_i(0; a_i, y))\big]_{y=\breve{g}_i(a_i;\,0,\,\eta)}\,, \qquad \eta \in E^r, \qquad i = 1, \dots, m\,,$$

$$\Phi_i(\breve{g}_i(0; a_i, y)) = \varphi_i(y) - \sum_{j=1}^{m} \tilde{b}_{ij}(y)\, z_j(a_i, y) - \mathfrak{z}_i(a_i, y)\,, \tag{3.23}$$
$$y \in E^r, \qquad i = 1, \dots, m\,.$$

By force of (3.4), (3.9), (3.16), (3.21), and (3.22) we have now

$$|\Phi_i(\breve{g}_i(0; a_i, y))| \leqslant |\varphi_i(y)| + \sum_{j=1}^{m} |\tilde{b}_{ij}(y)||z_j(a_i, y)| + |\mathfrak{z}_i(a_i, y)| \leqslant \\ \leqslant \omega_0 + \sigma_0(1-\gamma)^{-1}\big((1+\sigma)\omega + R_a\big) + \tag{3.24}$$

$$
\begin{aligned}
&+(1-\gamma)^{-1}(\sigma\omega+\gamma\omega+R_a)=\\
&=\omega_0+(1-\gamma)^{-1}(\sigma+\sigma_0+\sigma\sigma_0)\omega+\\
&+(1-\gamma)^{-1}\big((1+\sigma_0)R_a+\gamma\omega\big)\leqslant\\
&\leqslant\omega_0+(1-\gamma)^{-1}(\sigma+\sigma_0+\sigma\sigma_0)\omega+\\
&+(\omega-\omega_0)-(1-\gamma)^{-1}(\sigma+\sigma_0+\sigma\sigma_0)\omega=\\
&=\omega\,.
\end{aligned}
$$

(*c*) *Properties of the transformation* T^{**}. For any two $y, \bar{y} \in E^r$ and $i=1, \dots, m$, by force of (3.20) and manipulations we have

$$
\begin{aligned}
&\mathfrak{z}_i(x,y)-\mathfrak{z}_i(x,\bar{y})=\\
&=\Big\{\sum_{h=1}^{m}\tilde{A}_{ih}\big(0,\breve{g}_i(0;x,y),z(0,\breve{g}_i(0;x,y))\big)\,[\varphi_h(\breve{g}_i(0;x,y))-\varphi_h(\breve{g}_i(0;x,\bar{y}))]+\\
&\quad+\sum_{h=1}^{m}\tilde{a}_{hi}(x,y,z(x,y))\,[\varphi_h(\breve{g}_h(0;x,y))-\varphi_h(\breve{g}_h(0;x,\bar{y}))]+\\
&\quad+\sum_{s=1}^{m}\sum_{h=1}^{m}\tilde{a}_{si}(x,y,z(x,y))\,\tilde{A}_{sh}(x,y,z(x,y))\,[\varphi_h(\breve{g}_s(0;x,y))-\varphi_h(\breve{g}_s(0;x,\bar{y}))]+\\
&\quad+\sum_{s=1}^{m}\sum_{h=1}^{m}\tilde{a}_{si}(x,y,z(x,y))\,\big[\tilde{A}_{sh}\big(0,\breve{g}_s(0;x,y),z(0,\breve{g}_s(0;x,y))\big)-\\
&\qquad-\tilde{A}_{sh}(x,y,z(x,y))\big]\,[\varphi_h(\breve{g}_s(0;x,y))-\varphi_h(\breve{g}_s(0;x,\bar{y}))]\Big\}+\\
&\quad+\Big\{\sum_{h=1}^{m}\big[\tilde{A}_{ih}\big(0,\breve{g}_i(0;x,y),z(0,\breve{g}_i(0;x,y))\big)-\\
&\qquad-\tilde{A}_{ih}\big(0,\breve{g}_i(0;x,\bar{y}),z(0,\breve{g}_i(0;x,\bar{y}))\big)\big]\varphi_h(\breve{g}_i(0;x,\bar{y}))+\\
&\quad+\sum_{h=1}^{m}[\tilde{a}_{hi}(x,y,z(x,y))-\tilde{a}_{hi}(x,\bar{y},z(x,\bar{y}))]\varphi_h(\breve{g}_h(0;x,\bar{y}))+\\
&\quad+\sum_{s=1}^{m}\sum_{h=1}^{m}\big[\tilde{a}_{si}(x,y,z(x,y))\,\tilde{A}_{sh}\big(0,\breve{g}_s(0;x,y),z(0,\breve{g}_s(0;x,y))\big)-\\
&\qquad-\tilde{a}_{si}(x,\bar{y},z(x,\bar{y}))\,\tilde{A}_{sh}\big(0,\breve{g}_s(0;x,\bar{y}),z(0,\breve{g}_s(0;x,\bar{y}))\big)\big]\varphi_h(\breve{g}_s(0;x,\bar{y}))\Big\}+\\
&\quad+\sum_{s=1}^{m}[\alpha_{si}(x,y,z(x,y))-\alpha_{si}(x,\bar{y},z(x,\bar{y}))]\cdot\\
&\qquad\cdot\int_0^x\Big[\sum_{h=1}^{m}\big(dA_{sh}(\xi,\breve{g}_s(\xi;x,y),z(\xi,\breve{g}_s(\xi;x,y)))/d\xi\big)z_h(\xi,\breve{g}_s(\xi;x,y))+\\
&\quad+f_s\big(\xi,\breve{g}_s(\xi;x,y),z(\xi,\breve{g}_s(\xi;x,y))\big)\Big]\,d\xi+
\end{aligned}
$$

$$+\sum_{s=1}^{m}\alpha_{si}(x,\bar y,z(x,\bar y))\int_0^x[f_s(\xi,\breve g_s(\xi;x,y),z(\xi,\breve g_s(\xi;x,y)))-$$

$$-f_s(\xi,\breve g_s(\xi;x,\bar y),z(\xi,\breve g_s(\xi;x,\bar y)))]\,d\xi+$$

$$+\sum_{s=1}^{m}\alpha_{si}(x,\bar y,z(x,\bar y))\int_0^x\sum_{h=1}^{m}\Big[\Big(dA_{sh}(\xi,\breve g_s(\xi;x,y),z(\xi,\breve g_s(\xi;x,y)))/d\xi\Big)\cdot$$

$$\cdot z_h(\xi,\breve g_s(\xi;x,y))-$$

$$-\Big(dA_{sh}(\xi,\breve g_s(\xi;x,\bar y),z(\xi,\breve g_s(\xi;x,\bar y)))/d\xi\Big)z_h(\xi,\breve g_s(\xi;x,\bar y))\Big]\,d\xi=$$

$$=\delta_1+\delta_2+\delta_3+\delta_4+\delta_5\,.$$

By force of (2.27), (2.45), (2.53), (3.5), (3.6), (3.7), (3.8), and (3.21) we have now

$$|\delta_1|\leqslant[(\sigma_1+\sigma_2+\sigma_3)\Lambda(1+p)+$$

$$+\sum_s\sum_h|\bar\alpha_{si}|(\mathring{M}_a+C\Xi_a+C(1+Q)M_a)\Lambda(1+p)]|y-\bar y|\leqslant$$

$$\leqslant[\sigma+m\sigma_2(\mathring{M}_a+C\Xi_a+C(1+Q)M_a)]\Lambda(1+p)|y-\bar y|\,;$$

$$|\delta_2|\leqslant[mC(1+Q)(1+p)\omega+mC'(1+Q)\omega+m^2\tilde H'C(1+Q)(1+p)\omega+$$

$$+m^2\tilde H C'(1+Q)\omega]|y-\bar y|=$$

$$=[m(1+Q)(1+p)(1+m\tilde H')(C\omega)+m(1+Q)(1+m\tilde H)(C'\omega)]|y-\bar y|\,;$$

$$|\delta_3|\leqslant mC'(1+Q)[m(\mathring{M}_a+rC(1+mQ)M_a+mC\Xi_a)(1-\gamma)^{-1}\cdot$$

$$\cdot((1+\sigma)\omega+R_a)+N_a]|y-\bar y|\,;$$

$$|\delta_4|\leqslant mH'L_{1a}(1+Q)(1+p)|y-\bar y|\,.$$

By integration by parts and the use of (2.27), (2.53), (2.56), (3.6), (3.7), and (3.21) we also have

$$|\delta_5|=\Big|\sum_s\alpha_{si}(x,\bar y,z(x,\bar y))\int_0^x\sum_{h=1}^{m}[(dA_{sh}/d\xi)z_h-(d\bar A_{sh}/d\xi)\bar z_h]\,d\xi\Big|$$

$$=\Big|\sum_s\bar\alpha_{si}\sum_h\Big[\int_0^x(dA_{sh}/d\xi-d\bar A_{sh}/d\xi)z_h\,d\xi+\int_0^x(d\bar A_{sh}/d\xi)(z_h-\bar z_h)\Big]\,d\xi\Big|$$

$$=\Big|\sum_s\bar\alpha_{si}\sum_h\Big\{[A_{sh}(x,y,z(x,y))-A_{sh}(x,\bar y,z(x,\bar y))]z_h(x,y)-$$

$$
\begin{aligned}
&-\big[A_{sh}(0,\breve{g}_s(0;x,y),z(0,\breve{g}_s(0;x,y)))-\\
&\quad -A_{sh}(0,\breve{g}_s(0;x,\bar{y}),z(0,\breve{g}_s(0;x,\bar{y})))\big]z_h(0,\breve{g}_s(0;x,y))-\\
&-\int_0^x\big[A_{sh}(\xi,\breve{g}_s(\xi;x,y),z(\xi,\breve{g}_s(\xi;x,y)))-\\
&\quad -A_{sh}(\xi,\breve{g}_s(\xi;x,\bar{y}),z(\xi,\breve{g}_s(\xi;x,\bar{y})))\big]\big(dz_h(\xi,\breve{g}_s(\xi;x,y))/d\xi\big)\,d\xi+\\
&+\int_0^x\big(dA_{sh}(\xi,\breve{g}_s(\xi;x,\bar{y}),z(\xi,\breve{g}_s(\xi;x,\bar{y})))/d\xi\big)\cdot\\
&\quad\cdot\big[z_h(\xi,\breve{g}_s(\xi;x,y))-z_h(\xi,\breve{g}_s(\xi;x,\bar{y}))\big]d\xi\Big\}\Big|\\
\leqslant\, & m^2H'\big[C(1+Q)(1-\gamma)^{-1}((1+\sigma)\omega+R_a)+\\
&+C(1+Q)(1+p)(1-\gamma)^{-1}((1+\sigma)\omega+R_a)+\\
&+C(1+Q)(1+p)(\Xi_a+rQM_a)+\\
&+(\mathring{M}_a+rC(1+mQ)M_a+mC\Xi_a)Q(1+p)\big]|y-\bar{y}|\,.
\end{aligned}
$$

Thus, combining the estimates above, we have

$$
\begin{aligned}
(3.25)\qquad |\mathfrak{z}_i(x,y)-\mathfrak{z}_i(x,\bar{y})|&\leqslant|\delta_1|+|\delta_2|+|\delta_3|+|\delta_4|+|\delta_5|\leqslant\\
&\leqslant[\sigma\Lambda(1+p)+m(1+Q)(1+p)(1+m\tilde{H}')(C\omega)+\\
&\quad+m(1+Q)(1+m\tilde{H})(C'\omega)+\\
&\quad+m^2H'(1+Q)(1+\sigma)(1-\gamma)^{-1}(2+p)(C\omega)]|y-\bar{y}|+\\
&\quad+\big[m\sigma_2\Lambda(1+p)(\mathring{M}_a+C\Xi_a+C(1+Q)M_a)+\\
&\quad+m^2C'(1+Q)(1-\gamma)^{-1}(\mathring{M}_a+rC(1+mQ)M_a+mC\Xi_a)\cdot\\
&\qquad\cdot((1+\sigma)\omega+R_a)+mC'(1+Q)N_a+\\
&\quad+mH'(1+Q)(1+p)L_{1a}+\\
&\quad+m^2H'C(1+Q)(1-\gamma)^{-1}(2+p)R_a+\\
&\quad+m^2H'C(1+Q)(1+p)(\Xi_a+rQM_a)+\\
&\quad+m^2H'Q(1+p)(\mathring{M}_a+rC(1+mQ)M_a+mC\Xi_a)\big]|y-\bar{y}|=\\
&=(K'+S_a)|y-\bar{y}|\,,
\end{aligned}
$$

where K' is the expression in the first brackets above, and S_a, the expression in the second brackets, was introduced in part (a). By force of (2.27), (2.45), (3.19), and (3.25) we have also

$$(3.26)\quad |z_i(x,y)-z_i(x,\bar y)|\leqslant|\varphi_i(\breve g_i(0;x,y)-\varphi_i(\breve g_i(0;x,\bar y)|+|\mathfrak{z}_i(x,y)-\mathfrak{z}_i(x,\bar y)|\leqslant \\ \leqslant\big(\Lambda(1+p)+K'+S_a\big)|y-\bar y|\,.$$

We have now, by using (3.4), (3.9), (3.14), (3.21), (3.23), (3.25), (3.26),

$$(3.27)\quad |\Phi_i(\breve g_i(0;a_i,y))-\Phi_i(\breve g_i(0;a_i,\bar y))\leqslant|\psi_i(y)-\psi_i(\bar y)|+ \\ +\sum_j|\tilde b_{ij}(y)-\tilde b_{ij}(\bar y)||z_j(a_i,y)|+ \\ +\sum_j|\tilde b_{ij}(\bar y)||z_j(a_i,y)-z_j(a_i,\bar y)|+|\mathfrak{z}_i(a_i,y)-\mathfrak{z}_i(a_i,\bar y)|\leqslant \\ \leqslant[\Lambda_0+\tau_0(1-\gamma)^{-1}\big((1+\sigma)\omega+R_a\big)+ \\ +\sigma_0\big(\Lambda(1+p)+K'+S_a\big)+(K'+S_a)]|y-\bar y|= \\ =\{\Lambda_0+(1+p)(\sigma+\sigma_0+\sigma\sigma_0)\Lambda+ \\ +(1+\sigma_0)[m(1+Q)(1+p)(1+m\tilde H')+ \\ +m^2H'(1+Q)(1+\sigma)(1-\gamma)^{-1}(2+p)](C\omega)+ \\ +(1+\sigma_0)m(1+Q)(1+m\tilde H)(C'\omega)+ \\ +(1-\gamma)^{-1}(1+\sigma)(\tau_0\omega)+ \\ +(1+\sigma_0)S_a+\tau_0(1-\gamma)^{-1}R_a\}|y-\bar y|= \\ =\{\Lambda_0+(1+p)(\sigma+\sigma_0+\sigma\sigma_0)\Lambda+ \\ +S_0'(\tau_0\omega)+S_1'(C\omega)+S_2'(C'\omega)+(1+\sigma_0)S_a+ \\ +\tau_0(1-\gamma)^{-1}R_a\}|y-\bar y|\leqslant(1+p)^{-1}\Lambda|y-\bar y|\,.$$

We note here that, if a function $F(y)$, $y\in E^r$, satisfies $|F(y)-F(\bar y)|\leqslant$ $\leqslant K|y-\bar y|$, $y,\bar y\in E^r$, then, for all $\eta,\bar\eta\in E^r$,

$$|F(\breve g_i(a_i;0,\eta))-F(\breve g_i(a_i;0,\bar\eta))\leqslant K|\breve g_i(a_i;0,\eta)-\breve g_i(a_i;0,\bar\eta)|\leqslant \\ \leqslant K(1+p)|\eta-\bar\eta|\,.$$

By force of (3.23) and (3.27), we now have, for all $\eta,\bar\eta\in E^r$ and $i=1,\dots,m$,

$$(3.28)\qquad |\Phi_i(\eta)-\Phi_i(\bar\eta)|\leqslant(1+p)^{-1}\Lambda(1+p)|\eta-\bar\eta|=\Lambda|\eta-\bar\eta|\,.$$

From (3.24) and (3.28) we see that the transformation T^{**}, or $\varphi\to\Phi$, defined by (3.23), maps $\mathfrak{J}$ into $\mathfrak{J}$.

(*d*) *The transformation T^{**} is a contraction.* Let us prove that the transformation $T^{**}\colon \mathfrak{J} \to \mathfrak{J}$ is a contraction. To this effect, let φ, φ' be elements of $\mathfrak{J}$, and $z \in \mathcal{K}_{1\varphi}$, $z' \in \mathcal{K}_{1\varphi'}$, $g = g[z]$, $g' = g[z']$, $\Phi = T^{**}\varphi$, $\Phi' = T^{**}\varphi'$, $\mathfrak{z}_i(x, y)$, $\mathfrak{z}_i'(x, y)$ be the corresponding elements. Then, we have

$$
\begin{aligned}
&\mathfrak{z}_i(x, y) - \mathfrak{z}_i'(x, y) = \\
&= \Big\{ \sum_{h=1}^{m} \tilde{A}_{ih}(0, \breve{g}_i(0; x, y), z(0, \breve{g}_i(0; x, y))) [\varphi_h(\breve{g}_i(0; x, y)) - \varphi_h'(\breve{g}_i'(0; x, y))] + \\
&\quad + \sum_{h=1}^{m} \tilde{a}_{hi}(x, y, z(x, y)) [\varphi_h(\breve{g}_h(0; x, y)) - \varphi_h'(\breve{g}_h'(0; x, y))] + \\
&\quad + \sum_{s=1}^{m} \sum_{h=1}^{m} \tilde{a}_{si}(x, y, z(x, y)) \tilde{A}_{sh}(x, y, z(x, y)) [\varphi_h(\breve{g}_s(0; x, y)) - \varphi_h'(\breve{g}_s'(0; x, y))] + \\
&\quad + \sum_{s=1}^{m} \sum_{h=1}^{m} \tilde{a}_{si}(x, y, z(x, y)) [\tilde{A}_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) - \\
&\quad - \tilde{A}_{sh}(x, y, z(x, y))] [\varphi_h(\breve{g}_s(0; x, y) - \varphi_h'(\breve{g}_s'(0; x, y))] \Big\} + \\
&\quad + \Big\{ \sum_{h=1}^{m} [\tilde{A}_{ih}(0, \breve{g}_i(0; x, y), z(0, \breve{g}_i(0; x, y))) - \\
&\quad - \tilde{A}_{ih}(0, \breve{g}_i'(0; x, y), z'(0, \breve{g}_i'(0; x, y)))] \varphi_h'(\breve{g}_i'(0; x, y)) + \\
&\quad + \sum_{h=1}^{m} [\tilde{a}_{hi}(x, y, z(x, y)) - \tilde{a}_{hi}(x, y, z'(x, y))] \varphi_h'(\breve{g}_h'(0; x, y)) + \\
&\quad + \sum_{s=1}^{m} \sum_{h=1}^{m} [\tilde{a}_{si}(x, y, z(x, y)) \tilde{A}_{sh}(0, \breve{g}_s(0; x, y), z(0, \breve{g}_s(0; x, y))) - \\
&\quad - \tilde{a}_{si}(x, y, z'(x, y)) \tilde{A}_{sh}(0, \breve{g}_s'(0; x, y), z'(0, \breve{g}_s'(0; x, y)))] \varphi_h'(\breve{g}_s'(0; x, y)) \Big\} + \\
&\quad + \sum_{s=1}^{m} [\alpha_{si}(x, y, z(x, y)) - \alpha_{si}(x, y, z'(x, y))] \cdot \\
&\quad \cdot \int_0^x \Big[\sum_{h=1}^{m} \big(dA_{hs}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\big) z_h(\xi, \breve{g}_s(\xi; x, y)) + \\
&\quad + f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))) \Big] d\xi + \\
&\quad + \sum_{s=1}^{m} \alpha_{si}(x, y, z'(x, y)) \int_0^x [f_s(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y))) - \\
&\quad - f_s(\xi, \breve{g}_s'(\xi; x, y), z'(\xi, \breve{g}_s'(\xi; x, y)))] d\xi + \\
&\quad + \sum_{s=1}^{m} \alpha_{si}(x, y, z'(x, y)) \int_0^x \Big[\sum_{h=1}^{m} \big(dA_{sh}(\xi, \breve{g}_s(\xi; x, y), z(\xi, \breve{g}_s(\xi; x, y)))/d\xi\big) \cdot \\
&\quad \cdot z_h(\xi, \breve{g}_s(\xi; x, y)) - \\
&\quad - \big(dA_{sh}(\xi, \breve{g}_s'(\xi; x, y), z'(\xi, \breve{g}_s'(\xi; x, y)))/d\xi\big) z_h'(\xi, \breve{g}_s'(\xi; x, y)) \Big] d\xi = \\
&= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 \,.
\end{aligned}
$$

We now evaluate each of these expressions one by one:

$$|\varepsilon_1| \leqslant [(\sigma_1 + \sigma_2 + \sigma_3) + \sum_s \sum_h |\tilde{\alpha}_{si}|(\mathring{M}_a + C\Xi_a + C(1+Q)M_a)](\|\varphi - \varphi'\| + \Lambda\|g - g'\|) \leqslant$$

$$\leqslant [\sigma + m\sigma_2(\mathring{M}_a + C\Xi_a + C(1+Q)M_a)](\|\varphi - \varphi'\| + \Lambda\|g - g'\|);$$

$$|\varepsilon_2| \leqslant mC[(1+Q)\|g - g'\| + \|z - z'\|]\omega + mC'\|z - z'\|\omega +$$

$$+ m^2\tilde{H}'C[(1+Q)\|g - g'\| + \|z - z'\|]\omega + m^2\tilde{H}C'\|z - z'\|\omega =$$

$$= m(1+Q)(1 + m\tilde{H}')(C\omega)\|g - g'\| +$$

$$+ m[(1 + m\tilde{H}')(C\omega) + (1 + m\tilde{H})(C'\omega)]\|z - z'\|;$$

$$|\varepsilon_3| \leqslant mC'\|z - z'\|[m(\mathring{M}_a + rC(1 + mQ)M_a +$$

$$+ mC\Xi_a)(1-\gamma)^{-1}((1+\sigma)\omega + R_a) + N_a] =$$

$$= mC'[m(1-\gamma)^{-1}(\mathring{M}_a + rC(1 + mQ)M_a +$$

$$+ mC\Xi_a)((1+\sigma)\omega + R_a) + N_a]\|z - z'\|;$$

$$|\varepsilon_4| \leqslant mH'L_{1a}((1+Q)\|g - g'\| + \|z - z'\|).$$

By integration by parts and analogous evaluations we have also

$$|\varepsilon_5| = \Big|\sum_{s=1}^m \alpha_{si}(x, y, z'(x,y)) \int_0^x \sum_{h=1}^m [(dA_{sh}/d\xi)z_h - (dA'_{sh}/d\xi)z'_h]\, d\xi\Big|$$

$$= \Big|\sum_s \alpha'_{si} \sum_h \Big[\int_0^x (dA_{sh}/d\xi - dA'_{sh}/d\xi)z_h\, d\xi + \int_0^x (dA'_{sh}/d\xi)(z_h - z'_h)\Big]\, d\xi\Big|$$

$$= \Big|\sum_s \alpha'_{si} \sum_h \Big\{[A_{sh}(x, y, z(x,y)) - A_{sh}(x, y, z'(x,y))]\, z_h(x,y) -$$

$$-[A_{sh}(0, \breve{g}_s(0; x,y), z(0, \breve{g}_s(0; x,y))) - A_{sh}(0, \breve{g}'_s(0; x,y), z'(0, \breve{g}'_s(0; x,y)))]\cdot$$

$$\cdot z_h(0, \breve{g}_s(0; x,y)) -$$

$$-\int_0^x [A_{sh}(\xi, \breve{g}_s(\xi; x,y), z(\xi, \breve{g}_s(\xi; x,y))) -$$

$$-A_{sh}(\xi, \breve{g}'_s(\xi; x,y), z'(\xi, \breve{g}'_s(\xi; x,y)))](dz_h(\xi, \breve{g}_s(\xi; x,y))/d\xi)\, d\xi +$$

$$+\int_0^x \big(dA_{sh}(\xi, \breve{g}'_s(\xi; x,y), z'(\xi, \breve{g}'_s(\xi; x,y)))/d\xi\big)\cdot$$

$$\cdot\left[z_h(\xi,\breve{g}_s(\xi;x,y)) - z'_h(\xi,\breve{g}'_s(\xi;x,y))\right]d\xi\Big\}\Big|$$

$$\leqslant m^2H'C\|z-z'\|(1-\gamma)^{-1}\big((1+\sigma)\omega+R_a\big)+$$
$$+m^2H'C[(1+Q)\|g-g'\|+\|z-z'\|](1-\gamma)^{-1}\big((1+\sigma)\omega+R_a\big)+$$
$$+m^2H'C[(1+Q)\|g-g'\|+\|z-z'\|](\Xi_a+rQM_a)+$$
$$+m^2H'\big(\mathring{M}_a+rC(1+mQ)M_a+mC\Xi_a\big)\big(Q\|g-g'\|+\|z-z'\|\big).$$

We have now, by using the estimates above and (2.44),

$$|\mathfrak{z}_i(x,y)-\mathfrak{z}'_i(x,y)|\leqslant|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|+|\varepsilon_4|+|\varepsilon_5|\leqslant$$
$$\leqslant[\sigma+m\sigma_2(\mathring{M}_a+C\Xi_a+C(1+Q)M_a)]\|\varphi-\varphi'\|+$$
$$+[\sigma+m\sigma_2(\mathring{M}_a+C\Xi_a+C(1+Q)M_a)]\Lambda\lambda L_a\|z-z'\|+$$
$$+\{m(1+Q)(1+m\tilde{H}')(C\omega)\lambda L_a+$$
$$+m[(1+m\tilde{H}')(C\omega)+(1+m\tilde{H})(C'\omega)]\}\|z-z'\|+$$
$$+mC'[m(1-\gamma)^{-1}(\mathring{M}_a+rC(1+mQ)M_a+$$
$$+mC\Xi_a)\big((1+\sigma)\omega+R_a\big)+N_a]\|z-z'\|+$$
$$+mH'L_{1a}\big((1+Q)\lambda L_a+1\big)\|z-z'\|+$$
$$+m^2H'C(1-\gamma)^{-1}\big((1+\sigma)\omega+R_a\big)\|z-z'\|+$$
$$+m^2H'C(1-\gamma)^{-1}\big((1+\sigma)\omega+R_a\big)\cdot$$
$$\cdot\big((1+Q)\lambda L_a+1\big)\|z-z'\|+$$
$$+m^2H'C(\Xi_a+rQM_a)\big((1+Q)\lambda L_a+1\big)\|z-z'\|+$$
$$+m^2H'\big(\mathring{M}_a+rC(1+mQ)M_a+$$
$$+mC\Xi_a)(Q\lambda L_a+1)\|z-z'\|\big).$$

By using (2.79) and manipulations, we obtain

$$(3.29)\qquad |\mathfrak{z}_i(x,y)-\mathfrak{z}'_i(x,y)|\leqslant$$
$$\leqslant\{\sigma+[m(1+m\tilde{H}')(C\omega)+m(1+m\tilde{H})(C'\omega)+$$
$$+2m^2H'(1-\gamma)^{-1}(1+\sigma)(C\omega)]\cdot$$
$$\cdot(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)\}\|\varphi-\varphi'\|+$$
$$+\Big\{m\sigma_2(\mathring{M}_a+C\Xi_a+C(1+Q)M_a)+\Big[\big(\sigma+m\sigma_2(\mathring{M}_a+C\Xi_a+$$
$$+C(1+Q)M_a)\big)\Lambda\lambda L_a+m(1+Q)(1+m\tilde{H}')(C\omega)\lambda L_a+mC'N_a+$$

$$+ m^2C'(1-\gamma)^{-1}\big(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a\big)\big((1+\sigma)\omega + R_a\big) +$$
$$+ mH'\big(1+(1+Q)\lambda L_a\big)L_{1a} + m^2H'(1-\gamma)^{-1}CR_a +$$
$$+ m^2H'(1-\gamma)^{-1}(1+\sigma)(C\omega)(1+Q)\lambda L_a +$$
$$+ m^2H'C(1-\gamma)^{-1}\big(1+(1+Q)\lambda L_a\big)R_a +$$
$$+ m^2H'C\big(1+(1+Q)\lambda L_a\big)(\Xi_a + rQM_a) +$$
$$+ m^2H'\big(\mathring{M}_a + rC(1+mQ)M_a + mC\Xi_a\big)(1+Q\lambda L_a)\Big]\cdot$$
$$\cdot(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)\Big\}\|\varphi-\varphi'\| =$$
$$= (L+T_a)\|\varphi-\varphi'\|,$$

where L is the expression in the first braces, and T_a, the expression in the second braces, was introduced in part (a).

Finally, by (2.44), (2.45), (2.79), (3.4), (3.19), (3.23), (3.29) we have

$$(3.30) \qquad |\Phi_i(\breve{g}_i(0;a_i,y)) - \Phi'_i(\breve{g}'_i(0;a_i,y))| \leqslant$$
$$\leqslant \sum_{j=1}^{m} |\tilde{b}_{ij}(y)|[|\varphi_j(\breve{g}_j(0;a_i,y)) - \varphi'_j(\breve{g}'_j(0;a_i,y))| + |\mathfrak{z}_j(a_i,y) - \mathfrak{z}'_j(a_i,y)|] +$$
$$+ |\mathfrak{z}_i(a_i,y) - \mathfrak{z}'_i(a_i,y)| \leqslant$$
$$\leqslant \sigma_0[\|\varphi-\varphi'\| + \Lambda\|g-g'\| + (L+T_a)\|\varphi-\varphi'\|] + (L+T_a)\|\varphi-\varphi'\| \leqslant$$
$$\leqslant \sigma_0[\|\varphi-\varphi'\| + \Lambda\lambda L_a\|z-z'\|] + (1+\sigma_0)(L+T_a)\|\varphi-\varphi'\| \leqslant$$
$$\leqslant [\sigma_0 + (1+\sigma_0)L + \sigma_0\Lambda\lambda L_a(1-k)^{-1}(1-\gamma)^{-1}\cdot$$
$$\cdot(1+\gamma+2m^2H'H) + (1+\sigma_0)T_a]\|\varphi-\varphi'\|.$$

Also,

$$|\Phi_i(\breve{g}_i(0;a_i,y)) - \Phi'_i(\breve{g}_i(0;a_i,y))| \leqslant$$
$$\leqslant |\Phi_i(\breve{g}_i(0;a_i,y)) - \Phi'_i(\breve{g}'_i(0;a_i,y))| + |\Phi'_i(\breve{g}'_i(0;a_i,y)) - \Phi'_i(\breve{g}_i(0;a_i,y))| \leqslant$$
$$\leqslant |\Phi_i(\breve{g}_i(0;a_i,y)) - \Phi'_i(\breve{g}'_i(0;a_i,y))| + \Lambda\lambda L_a\|z-z'\|,$$

and by using (3.30) we obtain

$$\|\Phi-\Phi'\| \leqslant [\sigma_0 + (1+\sigma_0)L + (1+\sigma_0)\Lambda\lambda L_a(1-k)^{-1}(1-\gamma)^{-1}\cdot$$
$$\cdot(1+\gamma+2m^2H'H) + (1+\sigma_0)T_a]\|\varphi-\varphi'\|.$$

By the expression of L above, by manipulations, and the use of (3.15) we have

$$\begin{aligned}\|\Phi-\Phi'\| \leqslant [(\sigma+\sigma_0+\sigma\sigma_0)+(1+\sigma_0)[m(1+m\tilde{H}')+\\ +2m^2H'(1-\gamma)^{-1}(1+\sigma)](C\omega)(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)+\\ +(1+\sigma_0)m(1+m\tilde{H})(C'\omega)(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)+\\ +(1+\sigma_0)(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)\Lambda\lambda L_a+\\ +(1+\sigma_0)T_a]\|\varphi-\varphi'\|=\\ =[(\sigma+\sigma_0+\sigma\sigma_0)+T_1'(C\omega)+T_2'(C'\omega)+\\ +(1+\sigma_0)(1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)\Lambda\lambda L_a+\\ +(1+\sigma_0)T_a]\|\varphi-\varphi'\|\leqslant\\ \leqslant k'\|\varphi-\varphi'\|\end{aligned}$$

where $k'<1$. Thus, $T^{**}\colon \mathfrak{J}\to\mathfrak{J}$ is a contraction. By Banach's fixed point theorem there is an element $\varphi\in\mathfrak{J}$ with $\varphi=T^{**}\varphi$, and this element φ is unique in the class $\mathfrak{J}$.

For this element $\varphi=T^{**}\varphi$ we derive from (3.19), (3.23)

$$\varphi_i(\breve{y}_i(0;a_i,y))=\psi_i(y)-\sum_{j=1}^m\tilde{b}_{ij}(y)z_j(a_i,y)-\mathfrak{z}_i(a_i,y)\,,$$

or

$$z_i(a_i,y)=\psi_i(y)-\sum_{j=1}^m\tilde{b}_{ij}(y)z_j(a_i,y)\,,$$

and because of (3.3) also

$$\sum_{j=1}^m b_{ij}(y)z_j(a_i,y)=\psi_i(y)\,,\qquad y\in E^r,\qquad i=1,\dots,m\,.$$

The element $\varphi=T^{**}\varphi$ is also a continuous function of ψ. Indeed, if ψ, ψ' are any two elements satisfying (3.1-2), and z, z' the corresponding elements, by repeating the argument above we find

$$\|\varphi-\varphi'\|\leqslant\|\psi-\psi'\|+k'\|\varphi-\varphi'\|,$$

and by using (2.79) also

$$\|z - z'\| \leqslant (1-k)^{-1}(1-\gamma)^{-1}(1+\gamma+2m^2H'H)(1-k')^{-1}\|\psi-\psi'\|.$$

Theorem II is thereby proved.

In particular situations, the restrictions imposed on the data in Theorem II can be reduced. These particularizations, together with improved estimates on the size of a, will be discussed in forthcoming papers. It will be also shown that the results of Niccoletti and others are particular cases of the present line of research.

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