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A simple proof of the mean fourth power estimate for $\zeta(\frac{1}{2}+it)$ and $L(\frac{1}{2}+it,\chi)$

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A Simple Proof of the Mean Fourth Power Estimate for $\zeta(\frac{1}{2}+it)$ and $L(\frac{1}{2}+it, \chi)$.

K. RAMACHANDRA (*)

To the memory of Professor ALBERT EDWARD INGHAM

1. – Introduction.

The main object of this paper is to prove the following four well-known theorems by a simple method.

THEOREM 1. *If* $3 \leqslant t_1 < t_2 < \ldots < t_R \leqslant T$, $(R \geqslant 2)$ *and* $t_{j+1} - t_j \geqslant 1$, *then*

$$\sum_{r=1}^{R} |\zeta(\tfrac{1}{2}+it_r)|^4 \ll T(\log T)^{50} .$$

THEOREM 2. *If* $T \geqslant 3$, $R \geqslant 2$ *and* $T \leqslant t_1 < t_2 < \ldots < t_R \leqslant T + T^{\frac{1}{2}}$ *and also* $t_{j+1} - t_j \geqslant 1$, *then*

$$\sum_{r=1}^{R} |\zeta(\tfrac{1}{2}+it_r)|^2 \ll T^{\frac{1}{2}}(\log T)^{50} .$$

THEOREM 3. *Let* χ *be a character* mod q *(*q *fixed),* $T \geqslant 3$, $-T \leqslant t_{\chi,1} < t_{\chi,2} \ldots < t_{\chi,R_\chi} \leqslant T$, $(R_\chi \geqslant 2)$, *and* $t_{\chi,j+1} - t_{\chi,j} \geqslant 1$. *If with each* χ *we associate such points* $t_{\chi,j}$ *then,*

$$\sum_{\chi \bmod q}{}^{*} \sum_{r=1}^{R_\chi} |L(\tfrac{1}{2}+it_{\chi,r}, \chi)|^4 \ll qT(\log(qT))^{50} ,$$

where $*$ *denotes the sum over primitive character* mod q.

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THEOREM 4. *Let a be a positive constant satisfying $0 < a < 4$. Then with the notation of theorem 3, we have*

$$\sum_{\chi \bmod q} \sum_{r=1}^{R_\chi} |L(\tfrac{1}{2} + it_{\chi,r}, \chi)|^a \ll qT(\log(qT))^A$$

where A is a constant depending only on a.

REMARK 1. The constants implied by $\ll$ in all these theorems are absolute except in Theorem 4 where it may depend on a.

REMARK 2. Our method allows us to work out the generalization, of Theorem 2 to L-series, corresponding to Theorems 3 and 4.

REMARK 3. In these Theorems we have not tried to get the best powers of $\log T$ or $\log(qT)$.

For other remarks and Theorems see § 4 and § 5, and also the appendix.

2. – Notation.

The letters a, A, B, k denote positive constants, δ and ε denote arbitrary but small positive constants. All the 0 and the $\ll$ constants and also $C_1, C_2, \ldots$ depend if at all only on these constants. Finally $d_k(n)$ is the coefficient of n^{-s} in $(\zeta(s))^k$, and $d(n) = d_2(n)$. The other symbols will be explained in the body of the paper.

3. – Proof of the Theorems.

I should begin by saying that an expert in the field who looks at the special case $q = 1$ of Lemma 3 (below) will certainly be able to imagine at once my proof of all these Theorems (1 to 4 above). However, we prove Theorem 3, which is a generalization of Theorem 1. The method of proof enables one to work out a proof of Theorem 2. From Theorem 3 we deduce Theorem 4.

The crucial lemma for the proof is the following simple

LEMMA 1. *Let $s = \sigma + it$ be a complex variable, $q \geqslant 1$ a natural number and a_j ($j = 1$ to N) be complex numbers. For each character χ $\bmod q$ put*

$f_\chi(s) = \sum_{n=1}^{N} \chi(n) a_n n^{-s}$. *Then*

$$\sum_\chi \int_{T_0}^{T_0+T} |f_\chi(s)|^2 dt \ll (qT + N\log N) \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma},$$

where T_0 *is any real number* $T \geqslant 3$ *and the constant implied by* $\ll$ *is absolute. We also need the sharper estimate*

$$\ll qT \sum_{n=1}^{N} |a_n|^2 n^{-2\sigma} + \Big(\sum_n |a_n|^2 n^{-2\sigma}\Big)^{\frac{1}{2}} \Big(\sum_n |a_n|^2 n^{2(1-\sigma)}\Big)^{\frac{1}{2}} \log N.$$

PROOF. Lemma is trivially true for $N = 1$. So we assume that $N \geqslant 2$. The sum in question is

$$\sum_\chi \sum_{m,n\,(mn,q)=1} a_m \bar{a}_n\, \chi(m)\overline{\chi(n)}(mn)^{-\sigma} \left(\frac{m}{n}\right)^{iT_0} \left(\left(\frac{m}{n}\right)^{iT} - 1\right)\left(i \log \frac{m}{n}\right)^{-1} = \sum\nolimits_1 + \sum\nolimits_2$$

where in $\sum_1$, $m = n$ and $\sum_2$ is the remaining sum. We have easily $|\sum_1| \ll qT \sum |a_n|^2 n^{-2\sigma}$. Using the identity $\sum_\chi \chi(m)\overline{\chi(n)} = \varphi(q)$ ir zero according as $m \equiv n \pmod q$ and $(m, q) = 1$ or not, and

$$\left|\log \frac{m}{n}\right|^{-1} \ll \left|\frac{m+n}{m-n}\right|$$

we have, by symmetry,

$$|\sum\nolimits_2| \ll \varphi(q) \sum_{\substack{m \equiv n \pmod q \\ m \neq n}} |a_m \bar{a}_n (mn)^{-\sigma} m(m-n)^{-1}|$$

$$\ll \varphi(q)\big(\sum |a_n|^2 n^{-2\sigma} |m-n|^{-1}\big)^{\frac{1}{2}} \big(\sum |a_m|^2 m^{2(1-\sigma)} |m-n|^{-1}\big)^{\frac{1}{2}}$$

and this leads to the lemma.

From Lemma 1 we deduce the following corollary.

LEMMA 2. *Let q be a natural number. With each character χ* mod q *suppose we associate $R_\chi (\geqslant 2)$ distinct complex numbers $s_{\chi,r} = \sigma_{\chi,r} + it_{\chi,r}$* $(r = 1, 2, \ldots, R_\chi)$. *Let* $\delta_1 = \min_\chi \min_{r \neq r'} |t_{\chi,r} - t_{\chi,r'}|$, *and* $\delta = \min(\delta_1, (2\log N^{-1}))$. *As before, let* $f_\chi(s) = \sum_{n=1}^{N} \chi(n) a_n n^{-s}$, $\sigma = \min_{\chi,r} \sigma_{\chi,r}$ *and* $T = \max t_{\chi,r} - \min t_{\chi,r} + 3$. *Then*

$$\sum_\chi \sum_r |f_\chi(s_{\chi,r})|^2 \ll \delta^{-2}(qT + N\log N) \sum |a_n|^2 n^{-2\sigma}.$$

We also need the sharper estimate

$$\ll \delta^{-2}(qT) \sum |a_n|^2 n^{-2\sigma} + \delta^{-2} \log N\left(\sum |a_n|^2 n^{-2\sigma}\right)^{\frac{1}{2}}\left(\sum |a_n|^2 n^{2(1-\sigma)}\right)^{\frac{1}{2}}.$$

PROOF. The lemma is trivial if $N = 1$ and for this reason we can assume $N \geqslant 2$ and $a_1 = 0$. Now

$$|f_\chi(s_{\chi,r})|^2 \ll \delta^{-2} \int |f_\chi(s)|^2 p\, dp\, d\theta$$

where the integral is over a disc of radius δ and with centre $s_{\chi,r}$. We can then sum up over r (observing that for fixed χ the respective discs are disjoint) and then over all characters χ mod q. The lemma now follows on applying Lemma 1.

LEMMA 3. *We have firstly*

$$\sum_{|t_r| \leqslant (\log(qT))^2} |\zeta(\tfrac{1}{2} + it_r)|^4 \ll (\log(qT))^6$$

and so we can suppose that $|t_r| \geqslant (\log(qT))^2$ *if* χ *is principal. Next define* $\psi(s, \chi)$ *by* $L(s, \chi) = \psi(s, \chi) L(1-s, \chi)$. *Let* $s = \frac{1}{2} + it$ *and* $|t| > (\log(qT))^2$ *if* χ *is principal. Also, let* $2 \leqslant X \leqslant (qT)^{100}$. *Then if* $w = u + iv$ *and* c *a constant satisfying* $0 < c < \frac{1}{2}$, *we have*

$$\begin{aligned}
(L(s, \chi))^2 = &\sum_{n=1}^{\infty} \chi(n) d(n) n^{-s} \exp(-n/X) - \\
&- \frac{1}{2\pi i} \int\limits_{u=-c-\frac{1}{2}} (\psi(s+w, \chi))^2 \Big(\sum_{n>X} \chi(n) d(n) n^{-1+s+w}\Big) \Gamma(w) X^w dw - \\
&- \frac{1}{2\pi i} \int\limits_{u=-(\log(qT))^{-1}} (\psi(s+w, \chi))^2 \Big(\sum_{n\leqslant X} \chi(n) d(n) n^{-1+s+w}\Big) \Gamma(w) X^w dw + O((qT)^{-2}). \\
&= S(s) - I_1(s) - I_2(s) + 0((qT)^{-2}) \text{ say}.
\end{aligned}$$

PROOF. We start with

$$\sum_{n=1}^{\infty} \chi(n) d(n) n^{-s} e^{-n/X} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (L(s+w, \chi))^2 \Gamma(w) X^w dw$$

and move the line of integration first to $u = -c - \frac{1}{2}$, use the functional equation for $L(s, \chi)$, retain the portion $n > X$ of the series for $(L(1-s-w, \chi))^2$

and move the line of integration in the integral containing the portion $n \leqslant X$ of the series to $u = -(\log(qT))^{-1}$. These two integrals are precisely $I_1(s)$ and $I_2(s)$. However, when we move the line of integration to $u = -c - \frac{1}{2}$ we encounter the pole at $w = 0$ of $\Gamma(w)$, where the contribution from the residue is $(L(s, \chi))^2$. We have also the possible pole at $w = 1 - s$ (in case χ is principal) in which case the contribution is

$$\frac{1}{2\pi i} \int_{|w-(1-s)| = (100 \log(qT))^{-1}} (L(s+w, \chi))^2 \Gamma(w) X^w \, dw$$

which is easily estimated to be $0((qT)^{-2})$.

REMARK. The method of proof also leads to an asymptotic formula for $\int_1^T |\zeta(\frac{1}{2} + it)|^4 \, dt$ and this will be published elsewhere.

LEMMA 4. *We have*

$$\sum_{\chi}^{*} \sum_{r} |S(\tfrac{1}{2} + it_{\chi,r})|^2 \ll (qT + X)(\log(qT))^{12}.$$

PROOF. Follows by an easy application of Lemma 2. (Note that we may break off the series for $S(s)$ at $n = [X(\log X)^2]$ with a small error).

We next state two easy lemmas and indicate their proof

LEMMA 5. *We have*

$$\sum_{\chi}^{*} \sum_{r} |I_1(\tfrac{1}{2} + it_{\chi,r})|^2 \ll (qT + X)(qT)^{2+4c} X^{-2-4c} (\log(qT))^{50}.$$

LEMMA 6. *We have*

$$\sum_{\chi}^{*} \sum_{r} |I_2(\tfrac{1}{2} + it_{\chi,r})|^2 \ll (qT + X)(\log(qT))^{12}.$$

PROOFS OF LEMMAS 5 AND 6. We may break off the portion $|v| \geqslant (\log(qT))^2$ of the integrals I_1 and I_2 with a small error. In series for $(L(1-s-w, \chi))^2$ in I_1 we may break off for instance, the portion $n > \operatorname{Exp}(\log(qT))^3$ with a small error. Observe that for primitive characters $\chi \bmod q$, $\psi(w, \chi) = O((q(|v|+2))^{\frac{1}{2}-u})$ uniformly for u in any fixed strip provided $|w - n| \geqslant \delta$ $(n = 1, 2, 3, \ldots)$. We next break up the portion $X \leqslant n \leqslant \operatorname{Exp}(\log(qT))^3$ into at most $O\big((\log(qT))^3\big)$ intervals of the form $(U, U+V)$ with $V \leqslant U$ and apply Hölders inequality and then Lemma 2. We have also to use $\sum_{n\leqslant X} n^{-1}\big(d(n)\big)^2 \ll (\log X)^4$. This proves Lemma 5 and the proof of Lemma 6 is similar.

Combining Lemmas 3 to 6 and putting $X = qT$ we see that Theorem 3 is proved. Theorem 1 is a special case of Theorem 3. The proof of Theorem 2 follows that of Theorem 3 with obvious changes. We now deduce Theorem 4 from Theorem 3. If χ^* denotes the primitive character which induces χ, then we see that

$$\sum_{\chi \bmod q} \sum_r |L(\tfrac{1}{2} + it_{\chi,r}, \chi^*)|^4 \ll T\big(\log(qT)\big)^{50} \sum_{d/q} d \ll qT\big(\log(qT)\big)^{51}.$$

However, $L(s, \chi) = L(s, \chi^*) \prod_{p/q} \big(1 - \chi^*(p)p^{-s}\big)$ and so

$$\sum_{\chi \bmod q} \sum_r |L(\tfrac{1}{2} + it_{\chi,r}, \chi)|^a \ll \Big(\sum_\chi \sum_r |L(\tfrac{1}{2} + it_{\chi,r}, \chi^*)|^4\Big)^{a/4} \Big(\sum_\chi \sum_r \Big|\prod_{p/q} \big(1 - \chi^*(p)p^{-\frac{1}{2}-it_{\chi,r}}\big)\Big|^{4a/(4-a)}\Big)^{1-a/4}.$$

Hence it suffices to prove that the quantity in the second bracket is $\ll qT\big(\log(qT)\big)^B$ where B depends only on a. Let $2m$ be the least even integer greater than $4a(4-a)^{-1}$. Let d be a divisor of q. We now estimate

$$\sum_{\chi \bmod d} \sum_r \Big|\prod_{p/q} \big(1 - \chi(p)p^{-\frac{1}{2}-it_{\chi,r}}\big)\Big|^{2m}$$

we have $\Big|\prod_{p/q} \big(1 - \chi(p)p^{-\frac{1}{2}-it_{\chi,r}}\big)^m\Big|$ (χ a character mod d)

$$= \Big|\prod_{p/q} \Big(1 - m\chi(p)p^{-\frac{1}{2}-it_{\chi,r}} + 0\Big(\frac{2^m}{p}\Big)\Big)\Big|$$

$$\ll \Big|\prod_{p/q\, p\leqslant 2^{m^2}} (\ldots) \prod_{p/q, p> 2^{m^2}} \Big\{\big(1 - m\chi(p)p^{-\frac{1}{2}-it_{\chi,r}}\big)\Big(1 + O\Big(\frac{2^m}{p}\Big)\Big)\Big\}\Big|$$

$$\ll (\log q) \Big|\prod_{p/q\, p>2^{m^2}} (1 - m\chi(p)p^{-\frac{1}{2}-it_{\chi,r}}\Big|$$

Let

$$\prod_{p/q, p>2^{m^2}} \big(1 - m\chi(p)p^{-s}\big) = \sum \mu(n) b_n n^{-s}$$

where obviously $|b_n| \leqslant \prod_{p/(n,q)} m$.

Hence the sum under question is by Lemma 2,

$$\ll (\log q)^2 \left(Td(\log (qT))^2 \sum \frac{|\mu(n) b_n^2|}{n} + (\log (qT))^3 \left(\sum \frac{|b_n^2 \mu(n)|}{n} \right)^{\frac{1}{2}} \left(\sum n |b_n^2 \mu(n)| \right)^{\frac{1}{2}} \right)$$

$$\ll (\log (qT))^6 dT \prod_{p/q} \left(1 + \frac{m^2}{p}\right) + (\log (qT))^5 \prod_{p/q} \left\{ \left(1 + \frac{m^2}{p}\right)(1 + m^2 p) \right\}^{\frac{1}{2}}$$

$$\ll (\log (qT))^8 (dT + q^{\frac{3}{4}}) .$$

Summing over all divisors d of q we are led to the estimate

$$\ll (\log (qT))^9 (qT + q) \ll qT (\log (qT))^9 .$$

This proves theorem 4 with $A = 56$.

4. – An asymptotic formula.

Just as we proved in [9] Theorems of the type

$$\sum_r \left| \frac{\zeta(\sigma_r + it_r)}{C(\log t_r)^4} \right|^{2/(1-\sigma_r)} < T \qquad (T \geqslant 3)$$

in the notation of [9], we can prove the following theorems as a corollary to Theorems 3 and 4.

THEOREM 5. *In the notation of Lemma 3 and further with* $\frac{1}{2} \leqslant \sigma_{\chi,r} < 1$ *and* $|t_{\chi,r}| \geqslant 1$ *if* χ *is principal the following results hold*

$$\sum_{\chi \bmod q} \sum_r \left| \frac{L(\sigma_{\chi,r} + it_{\chi,r}, \chi)}{C_1 (\log (qT))^{100}} \right|^{2/(1-\sigma_{\chi,r})} \ll qT \ (T \geqslant 3)$$

$$\sum_{\chi \bmod q} \sum_r \left| \frac{L(\sigma_{\chi,r} + it_{\chi,r}, \chi)}{C_2 (\log (qT))^{100}} \right|^{(2-\varepsilon)/(1-\sigma_{\chi,r})} \ll qT \ (T \geqslant 3, \ C_2 = C_2(\varepsilon)) .$$

It must be remarked that there is not much difference between Theorems 1 to 4 and the corresponding theorems where the sum over r is replaced by integrals. In fact one can pass from one to the other by a slight loss e.g. a power of $\log (qT)$. We next record here a Theorem which will be of use in the proof of Theorem 7.

THEOREM 6. *Let $k \geqslant 2$ be a natural number, q as before and $T \geqslant 3$. Then uniformly in σ,* $|\sigma - \frac{1}{2}| \leqslant (100 \log (qT))^{-1}$ we have

$$\sum_{\chi \bmod q} \int_{-T}^{T} |L(\sigma + it, \chi)|^{2k} dt \ll (qT)^{k/2} (\log (qT))^{100k^2} \operatorname{Exp}\left(\frac{k}{2} (\log q)^{\frac{1}{2}}\right).$$

PROOF. We have only to note the relation

$$\prod_{p/q} (1 + O(p^{-\frac{1}{2}})) = O\left(\operatorname{Exp}\left(\frac{C_3 (\log q)^{\frac{1}{2}}}{\log \log q}\right)\right)$$

and the result

$$\sum_{\chi \bmod d}^{*} \int_{-T}^{T} |L(\sigma + it, \chi)|^{2k} dt \ll (dT)^{k/2} (\log (dT))^{50k^2}$$

where d/q, (uniformly for σ as in Theorem 6) which can be proved in much the same way as Theorem 3. This completes the proof of Theorem 6.

We next state

LEMMA 7. *In the notation of Lemma* 1, *and with* $\frac{1}{2} \leqslant \sigma \leqslant 1$ *we have uniformly,*

$$\sum_{\chi} \int_{T_0}^{T_0 + 2T} |f_\chi(s)|^2 dt = 2T\varphi(q) \Big(\sum_{(n,q)=1,\, n \leqslant N} |a_n|^2 n^{-2\sigma} \Big) + O\left(\varphi(q) \Big(\sum_{n \leqslant q,\, (n,q)=1} |a_n|^2 n^{-2\sigma} \Big) + \frac{\varphi(q)}{q} (\log N)^2 N^{2(1-\sigma)} \Big(\sum_{n \leqslant N} |a_n|^2 n^{-1} \Big) \right).$$

PROOF. Left hand side is

$$\sum_{\chi} \sum_{m,n \leqslant N} \frac{a_m \bar{a}_n \chi(m) \overline{\chi(n)}}{i(mn)^\sigma \log m/n} \left(\frac{m}{n}\right)^{iT_0} \left(\left(\frac{m}{n}\right)^{2iT} - 1 \right)$$

Plainly we can impose the condition $(mn, q) = 1$ and we shall adopt this convention in the reminder of this proof. In this sum the terms with $m = n$ contribute the main term. The other terms are

$$O\left(\varphi(q) \sum_{\substack{(mn,q)=1, m \equiv n (\bmod q) \\ m,n \leqslant N}} |a_m a_n| (mn)^{-\sigma} \left| \log \frac{m}{n} \right|^{-1} \right).$$

Here we treat the terms with $|\log m/n| \geqslant \frac{1}{2}$ and the rest separately. Denoting then by Σ_1 and Σ_2 we have

$$\begin{aligned}\Sigma_1 &= O\left(\varphi(q)\sum_{r=1}^{q}\Bigg(\sum_{\substack{n\equiv r(\mathrm{mod}\, q)\\ n\leqslant N}}|a_n|n^{-\sigma}\Bigg)^2\right)\\
&= O\left(\varphi(q)\Bigg(\sum_{r=1}^{q}\Bigg(\sum_{\substack{n\equiv r(\mathrm{mod}\, q)\\ q<n\leqslant N}}|a_n|^2 n^{-1}\Bigg)\Bigg(\sum_{\substack{n\equiv r(\mathrm{mod}\, q)\\ q<n\leqslant N}} n^{1-2\sigma}\Bigg)+\sum_{n\leqslant q}|a_n|^2 n^{-2\sigma}\Bigg)\right)\\
&= O\left(\varphi(q)\Bigg(\sum_{n\leqslant q, (n,q)=1}|a_n|^2 n^{-2\sigma}\Bigg)+\varphi(q)\Bigg(\sum_{n\leqslant N}\frac{|a_n|^2}{n}\Bigg)\Bigg(\sum_{qj\leqslant N+q}(qj)^{-2\sigma+1}\Bigg)\right)\\
&= O\left(\varphi(q)\Bigg(\sum_{n\leqslant q, (n,q)=1}|a_n|^2 n^{-2\sigma}\Bigg)+\varphi(q)\Bigg(\sum_{n\leqslant N}\frac{|a_n|^2}{n}\Bigg)\frac{1}{q^{2\sigma-1}}\left(\frac{N+q}{q}\right)^{2(1-\sigma)}\right).\end{aligned}$$

(The second term is present only if $N > q$)

$$= O\left(\varphi(q)\Bigg(\sum_{n\leqslant q}|a_n|^2 n^{-2\sigma}\Bigg)+\frac{\varphi(q)}{q}N^{2(1-\sigma)}\Bigg(\sum_{n\leqslant N}\frac{|a_n|^2}{n}\Bigg)\right).$$

Next letting U run over non-negative integral powers of 2, we have (using $|\log m/n| \gg |(m+n)/(m-n)|^{-1}$)

$$\Sigma_2 = O\Bigg(\varphi(q)\sum_{U\leqslant N}\ \sum_{\substack{U<m\leqslant 2U\ n\neq m\\ \frac{1}{2}U\leqslant n\leqslant 4U\ n\equiv m(\mathrm{mod}\, q)}}|a_m\bar{a}_n|m^{1-\sigma}n^{-\sigma}|m-n|^{-1}\Bigg).$$

Using $|a_m\bar{a}_n| \leqslant |a_m|^2 + |a_n|^2$ we see that this is

$$O\left(\frac{\varphi(q)}{q}\sum_{U\leqslant N}\frac{\log U}{U^{2\sigma-1}}\sum_{\frac{1}{2}U\leqslant n\leqslant 4U}|a_n|^2\right) = O\left(\frac{\varphi(q)}{q}N^{2(1-\sigma)}(\log N)^2\sum_{n\leqslant N}\frac{|a_n|^2}{n}\right)$$

since we can always restrict to $n \leqslant N$ in the last sum. This proves Lemma 7.

We next confine to $\frac{1}{2}+\delta \leqslant \sigma \leqslant 1$ and fix a small constant δ (possibly depending on k and only on k) once for all. (of course we can get our final result in a slightly more general form valid uniformly for $\sigma \geqslant \frac{1}{2}+\delta$; but to avoid some complications of a trivial nature we have done this). We now modify Lemma 3 as follows.

LEMMA 8. *Let* $\frac{1}{2}+\delta \leqslant \sigma \leqslant 1$ *(δ is a small positive constant which may depend on k). We denote the principal character* mod q *by* χ_0. *We have firstly*

$$\int_{1\leqslant|t|\leqslant(\log(qT))^2}|L(s,\chi_0)|^{2k}\,dt = O\Big(\big(\mathrm{Exp}\,(2k(\log q))^{\frac{1}{2}}\big)(\log(qT))^{2k+2}\Big).$$

Let χ *be any character* $\bmod q$ *and* $|t| \geqslant (\log(qT))^2$ *if* $\chi = \chi_0$. *Then we have (this time we suppose* $2 \leqslant X \leqslant (qT)^{100k}$*)*

$$(L(s, \chi))^k = \sum_{n=1}^{\infty} \chi(n) d_k(n) n^{-s} e^{-\left(\frac{n}{X}\right)^2}$$

$$-\frac{1}{2\pi i} \int_{u=\frac{1}{2}-\sigma} (L(s+w, \chi))^k \Gamma\left(\frac{w}{2}+1\right) X^w \frac{dw}{w} +$$

$$+ O(\varepsilon(\chi)(qT)^{-2}) = S_\chi - I_\chi + E_\chi \quad \text{say}$$

where $\varepsilon(\chi)$ *and hence* E_χ *are zero if* $\chi \neq \chi_0$ *and* $\varepsilon(\chi_0) = 1$.

PROOF. We leave the proof of this as an exercise to the reader.

Next it is not hard to see that

$$|L(s, \chi)|^{2k} = |S_\chi|^2 + O(|S_\chi|(|I_\chi| + |E_\chi|) + |I_\chi|^2 + |E_\chi|^2)$$

and so if $\frac{1}{2} + \delta \leqslant \sigma \leqslant 1$

$$\int_{1 \leqslant |t| \leqslant T} |L(S, \chi_0)|^{2k} dt + \sum_{\chi \neq \chi_0} \int_{|t| \leqslant T} L(s, \chi)|^{2k} dt = \int_{(\log(qT))^2 \leqslant |t| \leqslant T} |\ldots|^{2k} dt +$$

$$+ \sum_{\chi \neq \chi_0} \int_{|t| \leqslant T} |\ldots|^{2k} dt + O\left(\left(\operatorname{Exp}\left(2k(\log q)\right)^{\frac{1}{2}}\right)(\log(qT))^{2k+2}\right) =$$

$$= \sum_{\chi} \int_{|t| \leqslant T} |S_\chi|^2 dt + O\left(\int_{|t| \leqslant (\log qT)^2} |S_{\chi_0}|^2 dt + \left(\operatorname{Exp}\left(2k(\log q)^{\frac{1}{2}}\right)\right)(\log(qT))^{2k+2}\right) +$$

$$+ O\left(\sum_{\chi} \int_{|t| \leqslant T} \left(|S_\chi|(|I_\chi| + |E_\chi|) + |I_\chi|^2 + |E_\chi|^2\right) dt\right).$$

LEMMA 9. *We have uniformly in* $\frac{1}{2} + \delta \leqslant \sigma \leqslant 1$,

$$|S_\chi| = O((X(\log X)^2)^{1-\sigma} (\log qT)^{k^2})$$

and so

$$\int_{|t| \leqslant (\log(qT))^2} |S_{\chi_0}|^2 dt = O\left(X^{2(1-\sigma)} (\log(qT))^{2k^2+6}\right).$$

Also

$$\int_{|t| \leqslant T} |E_\chi|^2 dt = O\left(\frac{1}{q^2 T}\right).$$

PROOF. Trivial.

LEMMA 10. *We have uniformly in* $\frac{1}{2}+\delta\leqslant\sigma\leqslant 1$,

$$\sum_{\chi}\int_{|t|\leqslant T}|S_{\chi}|^2\,dt = 2T\varphi(q)\sum_{(n,q)=1}\frac{(d_k(n))^2}{n^{2\sigma}} + O\Big(T\varphi(q)X^{1-2\sigma}(\log(qT))^{k^2+6}\Big) +$$

$$+ O\left(\varphi(q)+\frac{\varphi(q)}{q}X^{2(1-\sigma)}(\log(qT))^{k^2+6}\right) +$$

$$+ O\left(X^{-20}\Big(\varphi(q)T+\varphi(q)+\big(X^{2(1-\sigma)}+T\varphi(q)X^{1-2\sigma}\big)\log(qT)\big)^{k^2+4}\Big)\right).$$

PROOF. Let $S_{\chi}(X(\log X)^2)$ denote the sum $n\leqslant X(\log X)^2$ in the series for S_{χ}. Then it is easy to see that

$$|S_{\chi}|^2 = |S_{\chi}(X(\log X)^2)|^2 + O\Big(X^{-20}\big(|S_{\chi}(X(\log X)^2)|+1\big)\Big).$$

By Lemma 7, (with $T_0=-T$), we have

$$\sum_{\chi}\int_{|t|\leqslant T}|S_{\chi}(X(\log X)^2)|^2\,dt = 2T\varphi(q)\sum_{(n,q)=1,\ n\leqslant X(\log X)^2}(d_k(n))^2 n^{-2\sigma}e^{-2\left(\frac{n}{X}\right)^2}$$

$$+ O\left(\varphi(q)+\big(X(\log X)^2\big)^{2(1-\sigma)}(\log X)^2(\log(qT))^{k^2}\right).$$

The O-term is $O\big(\varphi(q)+X^{2(1-\sigma)}(\log(qT))^{k^2+4}\big)$. In the main term we can replace $\exp(-2(n/X)^2)$ by 1 with an error

$$O\left(T\varphi(q)\sum_{n\leqslant X(\log X)^2}(d_k(n))^2 n^{-2\sigma}\left(\frac{n}{X}\right)^2\right)$$

$$= O\Big(T\varphi(q)\big(X(\log X)^2\big)^{3-2\sigma}X^{-2}(\log(qT))^{k^2}\Big)$$

$$= O\Big(T\varphi(q)X^{1-2\sigma}(\log(qT))^{k^2+4}\Big)$$

in the main term. Next we can replace $n\leqslant X(\log X)^2$ by $n\geqslant 1$ with an error

$$O\Big(T\varphi(q)\sum_{n\geqslant X(\log X)^2}(d_k(n))^2 n^{-2\sigma}\Big)$$

$$= O\Big(T\varphi(q)\sum_{n\geqslant X(\log X)^2}(d_k(n))^2 n^{-1-C_3(\log X)^{-1}}n^{1-2\sigma+C_3(\log X)^{-1}}\Big)$$

(Here $C_3=C_3(\delta)$ is a small positive constant)

$$= O\Big(T\varphi(q)\big(X(\log X)^2\big)^{1-2\sigma+C_3(\log X)^{-1}}(\log(qT))^{k^2}\Big)$$

$$= O\Big(T\varphi(q)X^{1-2\sigma}(\log(qT))^{k^2}\Big).$$

Thus

$$\sum_{\substack{\chi \\ |t|\leqslant T}} \int |S_\chi(X(\log X)^2)|^2\, dt = 2T\varphi(q) \sum_{(n,q)=1} (d_k(n))^2 n^{-2\sigma} +$$

$$+ O\Big(\varphi(q) + X^{2(1-\sigma)}(\log(qT))^{k^2+4} + T\varphi(q) X^{1-2\sigma}(\log(qT))^{k^2+4}\Big).$$

This proves Lemma 10.

LEMMA 11. *We have uniformly in* $\frac{1}{2} + \delta \leqslant \sigma \leqslant 1$

$$\sum_{\substack{\chi \\ |t|\leqslant T}} \int |I_\chi|^2\, dt = O\left((qT)^{k/2}(\log(qT))^{100k^2}\left(\operatorname{Exp}\left(\frac{k}{2}(\log q)^{\frac{1}{2}}\right)\right) X^{-2\sigma+1}\right).$$

PROOF. In the integral for I_χ we can break off the portion $|\operatorname{Im} w| \geqslant \geqslant (\log(qT))^2$ with a small error. We then apply Hölders inequality and use Theorem 6. This completes the proof of Lemma 11.

Putting $X = (qT)^{k/2}$ and combining the result stated just before Lemma 9, Lemma 9, 10 and 11 we arrive at the following theorem

THEOREM 7. *We have uniformly in* $\frac{1}{2} + \delta \leqslant \sigma \leqslant 1$,

$$\int_{1\leqslant|t|\leqslant T} |L(s,\chi_0)|^{2k}\, dt + \sum_{\substack{\chi\neq\chi_0 \\ |t|\leqslant T}} \int |L(s,\chi)|^{2k}\, dt = 2T\varphi(q) \sum_{(n,q)=1} \frac{(d_k(n))^2}{n^{2\sigma}} +$$

$$+ O\Big(\varphi(q) + (qT)^{k/2(1-\sigma)}\big(\operatorname{Exp}(4k(\log q)^{\frac{1}{2}})\big)(\log(qT))^{150k^2}\big((qT)^{\frac{1}{2}} + (qT)^{k/2(1-\sigma)}\big)\Big).$$

REMARK. To prove Theorem 7 it is not essential to prove Theorem 6. Something like Theorem 5 which is a consequence of Theorem 3 is sufficient. It (Theorem 5) gives a mean $(2k)^{\text{th}}$ power on the line $\sigma = 1 - 1/k$. We can instead of Lemma 8 make a slight change and move the line of integration to $u = 1 - 1/k - \sigma$ to obtain an asymptotic formula for $(2k)^{\text{th}}$ powers of L series in $\sigma \geqslant 1 - 1/k + \delta$. So what is really important for the results of § 4 is a mean fourth power estimate on the critical line, which is a consequence of the functional equation. For some Dirichlet series we may not have a functional equation but still it may have for instance a mean square or a mean fourth power estimate.

For example we state the following Theorems which are not difficult to prove

THEOREM 8. *Let* $\{a_n\}$ *be a monotonic sequence of real numbers with* $a_n = O(n^\varepsilon)$ *for every* $\varepsilon > 0$. *Let* $d_0(n)$ *denote the coefficient of* n^{-s} *in*

$((1-2^{1-s})\zeta(s))^2$. *Then the function*

$$F(s)=\sum_{n=1}^{\infty}\frac{a_n d_0(n)}{n^s} \qquad (\sigma>1)$$

can be continued as an analytic function in $\sigma>\frac{1}{3}$ *and we have*

$$\int_{|t|\leqslant T}|F(\tfrac{1}{2}+it)|^2 dt \ll T^{1+\varepsilon}$$

THEOREM 9. *Let* $\{a_n\}$ *be a sequence of complex numbers with* $\sum_{n\leqslant x} a_n=O(x^\varepsilon)$. *Then the function* $F(s)=\sum_{n=1}^{\infty} a_n n^{-s}(\sigma>1)$ *can be continued as an analytic function in* $\sigma>0$ *and we have*

$$\int_{|t|\leqslant T}|F(\tfrac{1}{2}+it)|^2 dt \ll T^{1+\varepsilon}.$$

REMARK. We can get an asymptotic formula for $\int|F(\sigma+it)|^{2k}dt$ in $\sigma\geqslant 1-1/2k+\delta$ by the method of this Section. Here $F(s)$ can be either the function of Theorem 8 or 9 and $k\geqslant 1$.

THEOREM 10. *Let* $b(n)$ *be a periodic sequence of complex numbers with* $|\sum_{n\leqslant x} b_n|\ll 1$, a_n *a sequence of complex numbers with* $a_n=0(n^\varepsilon)$ *and* $\sum_{n\leqslant x}|a_n-a_{n-1}|\ll x^\varepsilon$ *then* $G(s)=\sum a_n b_n/n^s$ *is convergent in* $\sigma>0$ *and*

$$\int_0^T|G(\tfrac{1}{2}+it)|^4 dt \ll T^{1+\varepsilon}.$$

5. – Concluding remarks.

As is well known an important application of Theorems 1, 3 and 4 is to the study of the zeros of $\zeta(s)$ and $L(s,\chi)$. Ingham studied (see Theorem 9.19(B), page 203 of [10]) the zeros of $F_1(s)=\zeta(s)$ and Montgomery studied ([7], [8]) the zeros of $F_2(s)=\prod_{\chi \bmod q} L(s,\chi)$ and $F_3(s)=\prod_{q\leqslant Q}\ \prod^{*}_{\chi \bmod q} L(s,\chi)$ (* denotes the omission of improper characters). Let $\frac{1}{2}\leqslant\alpha\leqslant 1$. Denote by $R(\alpha,T)$ the rectangle $\alpha\leqslant\sigma\leqslant 1$ and $|t|\leqslant T(T\geqslant 3)$. Let $N_j(\alpha,T)$ $(j=1,2,3)$ denote the number of zeros of $F_j(s)$ in $R(\alpha,T)$. (In the results to be stated below the implied constants are independent of α, q, and T). Ingham proved that $N_1(\alpha,T)\ll T^{3(1-\alpha)/(2-\alpha)}(\log T)^{100}$ and Montgomery proved the theorem $N_2(\alpha,T)\ll(qT)^{3(1-\alpha)/(2-\alpha)}(\log(qT))^{100}$ and $N_3(\alpha,T)\ll(Q^2T)^{3(1-\alpha)/(2-\alpha)}(\log(QT))^{100}$ Montgomery also proved the theorems $N_2(\alpha,T)\ll(qT)^{2(1-\alpha)/\alpha}(\log(qT))^{100}$ and $N_3(\alpha,T)\ll(Q^2T)^{2(1-\alpha)/\alpha}(\log(QT))^{100}$. Huxley [3] proved that $N_1(\alpha,T)\ll$

$\ll T^{3(1-\alpha)/(3\alpha-1)}(\log T)^{100}$. All these results depended (besides other things) on Theorems 1, 3 and 4 of ours, which in turn depended on "the approximate functional equation for $(L(s, \chi))^2$".

Next Jutila proved [5] that $N_2(\alpha, T) \ll (qT)^{\lambda(1-\alpha)}(\log(qT))^{\mu}$ where $\lambda = \varepsilon + \max\big((6\alpha-3)/(6\alpha-4), 2\big)$ and $\mu = \mu(\varepsilon)$ provided $\alpha \geqslant \frac{2}{3} + \delta$. It is clear that Jutila's method gives $N_3(\alpha, T) \ll (Q^2T)^{\lambda(1-\alpha)}(\log(QT))^{\mu}$. Jutila's method also gives $N_2(\alpha, T) \ll (qT)^{(3+\varepsilon)(1-\alpha)/(2-\alpha)}(\log(qT))^{\mu}$ and $N_3(\alpha, T) \ll (Q^2T)^{((3+\varepsilon)(1-\alpha))/(2-\alpha)}(\log(QT))^{\mu}$ and also $N_1(\alpha, T) \ll T^{((3+\varepsilon)(1-\alpha))/(3\alpha-1)}(\log T)^{\mu}$ (the last result still depends on the method of [3]). One important feature of Jutila's method is a device which helps us to dispense with the use of Theorems like 1, 3 and 4 at the cost of obtaining a slightly weaker result than what would otherwise have been possible. While these results of Jutila are good enough for application to difference between consecutive primes and such other problems, it is desirable to have a more precise result in the neighbourhood of $\alpha = \frac{1}{2}$: These will be of some use in Bombieri (E)-Vinogradov (A.I) mean-value theorem (Theorem 15.2, page 136 of [8]) and Vinogradov (I.M)-Montgomery (H.L) theorem (Theorem 16.1, page 141 of [8]). We may note that Bombieri-Vinogradov mean value theorem has been proved in a simpler way by P.X. Gallagher [2]. Theorem 4 together with Lemma 2 could be used to prove, for instance, that $N_2(\alpha, T) \ll (qT)^{(1-\alpha)/(g-2h\alpha)}(\log(qT))^{\mu}$ where $g-h = \frac{1}{2}$ and $g-2h = (3+\varepsilon)^{-1}$. However, to prove something like $N_3(\alpha, T) \ll (Q^2T)^{3(1-\alpha)/(2-\alpha)}(\log(QT))^{100}$ we also require the Theorem 2 of [6]. But we have dispensed with the use of approximate functional equations in the proofs of Theorems 1, 2, 3 and 4 stated in § 1 and theorems of § 4 are easy consequences of these theorems.

It must be mentioned that I got the start for this paper by trying to simplify the approach given in chapters 19 to 22 of Huxley's book [4]. Huxley proves an approximate functional equation to prove something like Theorem 3 of ours.

Next a few words about the work of K. Chandrasekharan and R. Narasimhan [1] on approximate functional equations. Our method is certainly applicable to almost all the problems of mean values considered there. For instance consider an algebraic number field K of degree $n \geqslant 2$ and let $\zeta_K(s)$ denote the Dedekind zeta function of K. We can prove their result that if $1/n \leqslant \sigma \leqslant 1-1/n$, $\int_1^T |\zeta_K(\sigma+it)|^2 dt \ll T^{n(1-\sigma)}(\log T)^B$ (but B may be a little larger than their value). Also we can prove the asymptotic formula for the mean value if $\sigma > 1-1/n$, with an error of the type $O\big(T^{\frac{1}{2}(n(1-\sigma)+1)}(\log T)^B\big)$ where B may as before be larger than their value. (If $\sigma > 1$, we have to start with the integral $1/2\pi i \int_{2-i\infty}^{2+i\infty} \zeta_K(s+w)\Gamma(w/k_0+1)X^w(dw/w)$ where k_0 is some large positive constant). We can handle $\int_1^T |\zeta_K(\sigma+it)|^2 dt$ by dealing first with $\sigma \geqslant \frac{1}{2}$ and then applying functional equation to deal with $\sigma \geqslant \frac{1}{2}$.

In his letter dated 9.9.73, Professor P. X. Gallagher asked me whether it is possible to get by my method, Barban's 8-th power mean estimate

$$\sum_{q \leqslant Q} \sum_{\chi \bmod q} |L(\tfrac{1}{2}+it, \chi)|^8 \ll Q^2 (\log Q)^B \qquad (\ll \text{constant depending on } t).$$

It is not hard to see that my method gives a slightly stronger result

$$\sum_{q \leqslant Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{|t| \leqslant T} |L(\tfrac{1}{2}+it, \chi)|^8 dt \ll Q^2 T^2 (\log (QT))^B .$$

This gives (*) $N_3(\alpha, T) \ll (Q^5 T^3)^{2(1-\alpha)/(3-\alpha)} (\log (QT))^B$.

However, to prove these results one has also to use Theorem 2 of [6]. I take this opportunity to express my indebtedness to Professor P. X. Gallagher for his interest in my work and encouragement. My thanks are also due to Dr. M. K. V. Murthy for some help.

Appendix.

By slight variations in the method of the paper it is possible to prove in a simple way, the following theorem due to H. L. Montgomery (Theorem 10.1 of [8])

THEOREM. *We have uniformly in* $\sigma, q, T\big(q \geqslant 1, T \geqslant 30, |\sigma - \frac{1}{2}| \leqslant (\log (qT))^{-1}\big)$,

$$\sideset{}{^*}\sum_{\chi \bmod q} \int_{|t| \leqslant T} |L(\sigma+it, \chi)|^4 dt \ll \varphi(q)\, T (\log (qT))^4 .$$

In this appendix we indicate the alterations necessary. This can be done best by stating three lemmas (which can be used at appropriate places) and indicating their proofs.

Let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |a_n|^2 n^{-1-d}$ converges for $d > 0$ and is $\ll d^{-4}$ ad d tends to zero. Then we have the following three lemmas.

(*) We can also prove in $\alpha \geqslant \frac{2}{3} + \delta$, that $N_3(\alpha, T) \ll (Q^{2+\varepsilon})^{\lambda(1-\alpha)} (\log B)^B$ with $\lambda = \max(2, (10\alpha - 5)/(12\alpha - 8))$, if $T \ll Q^{\varepsilon}$, following the method of [5].

LEMMA 1. *Let* $s = \sigma + it$ $0 \leqslant \sigma_1 \leqslant \frac{1}{2}$, *and* $2 \leqslant X \leqslant (qT)^{100}$. *Then we have*

$$\int_{T_0+T/10}^{T_0+T} \sum_{\chi \bmod q} \left| \sum_{n \leqslant X} \frac{a_n \chi(n)}{n^s} \right|^2 dt \ll \varphi(q)\, T(\log (qT))^4 \left(1 + \left(\frac{X}{qT}\right)^2\right) X^{1-2\sigma_1}.$$

PROOF. Consider

$$\int_{T/100}^{T/10} du \int_{T_0+u}^{T_0+T+u} \sum_{\chi \bmod q} |\ldots|^2 \, dt.$$

We can estimate this by the usual method. Also this is

$$\gg T \int_{T_0+T/10}^{T_0+T} \sum_{\chi \bmod q} |\ldots|^2 \, dt.$$

LEMMA 2. *Let* $s = \sigma + it$, $k(\geqslant 2)$ *an integer and* $2 \leqslant X \leqslant (qT)^{100}$. *Then we have*

$$\int_{T_0+T/10}^{T_0+T} \sum_{\chi \bmod q} \left| \sum_{n=1}^{\infty} \frac{a_n \chi(n) \exp\left(-(n/X)^k\right)}{n^s} \right|^2 dt \ll \varphi(q)\, T(\log (qT))^4 \left(1 + \left(\frac{X}{qT}\right)^k\right).$$

PROOF. First break off the series at $n = [X(\log X)^3]$. Then consider a k-ple integral of the type

$$\iint \ldots \int_{T_0+u}^{T_0+T+u} \sum_{\chi \bmod q} |\ldots|^2 \, dt\, du \ldots$$

with suitable upper and lower limits. This leads to the lemma.

LEMMA 3. *Let* $s = l + it$ *where* l *is a real number satisfying* $2l \geqslant 6$. *Then, if* $2 \leqslant X \leqslant (qT)^{100}$, *we have*

$$\int_{T_0+T/10}^{T+T_0} \sum_{\chi \bmod q} \left| \sum_{n \geqslant X} \frac{a_n \chi(n)}{n^s} \right|^2 dt \ll \varphi(q)\, T(\log (qT))^4 \left(1 + \left(\frac{X}{qT}\right)^2\right) X^{1-2l}.$$

PROOF. Consider

$$\sum_{\chi} \int\limits_{T/100}^{T/10} du \int\limits_{T_0+u}^{T_0+T+u} \left| \sum_{n \geq X} \frac{a_n \chi(n)}{n^s} \right|^2 dt$$

$$\ll \varphi(q) T^2 \sum_{n \geq X} \frac{|a_n|^2}{n^{2l}} + \varphi(q) \sum_{\substack{m,n \geq X \\ m \equiv n(q),\, m \neq n}} \frac{|a_m a_n|}{(mn)^l} \left(\frac{m+n}{m-n}\right)^2$$

$$\ll \varphi(q) T^2 X^{1-2l} (\log(qT))^4 + \varphi(q) \left(\left(\sum \frac{|a_m|^2}{m^{2l-4}(m-n)^2} \right)^{\frac{1}{2}} \left(\sum \frac{|a_n|^2}{n^{2l}(m-n)^2} \right)^{\frac{1}{2}} + \ldots \right)$$

$$\ll \varphi(q) T^2 X^{1-2l} (\log(qT))^4 + \varphi(q)(\log(qT))^4 q^{-2} X^{3-2l},$$

and this completes the proof of the lemma.

Next in Lemma 3 of the paper we move the line of integration to $u = \frac{1}{4}$ instead of $u - (\log(qT))^{-1}$. We estimate separately the sums

$$\sum_{\chi \bmod q}^{*} \int\limits_{\frac{1}{2}T \leq |t| \leq T} \ldots \qquad \text{and} \qquad \sum_{\chi \bmod q}^{*} \int\limits_{|t| \leq q} \ldots .$$

This leads to the theorem of Montgomery.

Further details of proof will appear elsewhere.

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