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INVARIANT MEANS ON VECTOR-VALUED FUNCTIONS II

by T. HUSAIN and JAMES C. S. WONG

1. Introduction.

In this paper, we continue with the study of invariant means on vector-valued functions initiated in [6]. For brevity, notations and terminologies in [6] will be used without further explanations. We shall mainly be concerned with spaces of functions defined on a locally compact group. For general terms in harmonic analysis and topological vector spaces, we follow Hewitt and Ross [5] and Robertson and Robertson [8] respectively.

Recall that if S is a semigroup, E a separated locally convex space which is quasi-barrelled (i. e. strongly bounded subsets of E^* are equicontinuous) and $l_\infty(S, E^*)$ the linear space of all functions $f: S \longrightarrow E^*$ such that $f(S)$ is strongly bounded in E^*, then a mean on a linear subspace X of $l_\infty(S, E^*)$ is defined to be a linear mapping $M: X \longrightarrow E^*$ such that $M(f)$ belongs to the weak* closed convex hull of $\{f(s): s \in S\}$ in E^* for any $f \in X$. (See [6] for details). Now if G is a locally compact group and $L_\infty(G)$ the Banach space of all bounded Haar measurable functions on G with essential supremum norm (we fix a left Haar measure λ and identify functions which are equal λ-locally almost everywhere), then the above definition of a mean does not make sense because the set $\{f(s): s \in G\}$ now depends on the function f chosen from its equivalence class in $L_\infty(G)$. To overcome this, we replace in the above definition the set $\{f(s): s \in G\}$ by the intersection of the sets $\{g(s): s \in G\}$ where g runs over all functions in the same equivalence class of f. The precise definition of a mean on vector-valued functions will be given in the next section. Many results in [6] and [11] are then extended.

The theory we are going to develop is quite general. It covers the situations when E is a barrelled space (a fortiori if E is a Banach space)

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or a metrisable locally convex space. However, there are quasi-barrelled spaces which are neither metrisable nor barrelled (see [8]).

2. Basic definitions and lemmas.

Let G be a locally compact group with a fixed left Haar measure λ, E a separated locally convex space (with continuous dual E^*) which is quasi-barrelled (i. e. strongly bounded subsets in E^* are equi continuous) and $l_\infty(G, E^*)$ the linear space of all functions $f: G \to E^*$ such that $f(G)$ is strongly bounded. For each bounded subset A of E, define a semi-norm q_A on $l_\infty(G, E^*)$ by

$$q_A(f) = \sup_{s \in G} p_A(f(s)) = \sup_{s \in G} \sup_{x \in A} |f(s)x|$$

$f \in l_\infty(G, E^*)$. (Here p_A is the semi-norm on E^* defined by $p_A(x^*) = \sup_{x \in A} |x^*(x)|$, $x^* \in E^*$). Then as in [6], $l_\infty(G, E^*)$ becomes a separated locally convex space.

Let $BM(G, E^*)$ be the linear subspace of all functions $f \in l_\infty(G, E^*)$ such that $f(\cdot)x$ is λ-measurable for each $x \in E$ and let $N(G, E^*)$ be the closed linear subspace of all functions $f \in BM(G, E^*)$ such that $f(\cdot)x$ is locally null for each $x \in E$. In otherwords, for each $x \in E$, there is a locally null set N depending on x and f such that $f(s)x = 0$ for any $s \notin N$. Let $L_\infty(G, E^*) = BM(G, E^*)/N(G, E^*)$ be the quotient linear space, then $L_\infty(G, E^*)$ is a separated locally convex space (See Robertson and Robertson [8]) with quotient semi-norms

$$\bar{q}_A(\bar{f}) = \inf\{q_A(g) : g \in BM(G, E^*),\ g \backsim f\}$$

where A is any bounded subset of E. Here $\backsim$ is the equivalence relation on $BM(G, E^*)$ defined by $f \backsim g$ iff $f - g \in N(G, E^*)$ and $\bar{f}$ denotes the equivalence class to which f belongs. It is then straight forward to verify that the usual left translation operator l_a on $BM(G, E^*) \subset l_\infty(G, E^*)$ induces a left translation operator $\bar{l}_a$ on $L_\infty(G, E^*)$ such that $\bar{l}_a(\bar{f}) = \overline{l_a f}$ for any $\bar{f} \in L_\infty(G, E^*)$ and that $L_\infty(G, E^*)$ is left translation invariant. Moreover $\bar{q}_A(\bar{l}_a \bar{f}) \leq \bar{q}_A(\bar{f})$ for any $a \in G$, $f \in L_\infty(G, E^*)$ and A bounded subset of E.

Very often, we use f to denote also its equivalence class. If will be clear from the context whether we mean the function or its equivalence class in $L_\infty(G, E^*)$.

DEFINITION 2.1. Let X be a linear subspace of $L_\infty(G, E^*)$ a mapping $M: X \rightarrow E^*$ is called a *mean* on X if

(a) M is linear

(b) $M(f) \in \bigcap \{K_g : g \in BM(G, E^*) \text{ and } g \sim f\}$ for each $f \in X$, where $K_g = w^*\, CL\, CO\, \{g(s) : s \in G\}$ is the weak* closure of the convex hull of $\{g(s) : s \in G\}$ in E^*.

If in addition, X is invariant under left translation, then M is called *left invariant* on X iff $M(l_a f) = M(f)$ for any $a \in G$ and $f \in X$.

REMARK 2.2. It is obvious that for *discrete* groups, this definition of a mean agrees with that given in [6]. The next theorem shows that when E is the space of *real numbers* and X contains the *constants*, our definition coincides with that used in Greenleaf [4, § 2].

THEOREM 2.3. Let E be the space of real numbers and X contain the constants, then M is a mean on X iff

(a) M is linear

(b) $\text{ess inf} f \leq M(f) \leq \text{ess sup} f$ for any $f \in X$.

Proof: Suppose M is a mean on X, then clearly, $M(1) = 1$ and $|M(f)| \leq \|g\|_u = \sup_{s \in G} |g(s)|$ for any $g \sim f$ in $L_\infty(G)$. It follows that $|M(f)| \leq \|f\|_\infty = \inf\{\|g\|_u : g \sim f\}$. Therefore $\|M\| = M(1) = 1$. Let $f \in X$ and $\alpha = \text{ess inf} f$. Put $g = f - \alpha \in X$, then $g \geq 0$ locally almost everywhere on G. But $M(f) - \alpha = M(f - \alpha) = M(g) \leq \|g\|_\infty = \text{ess sup}\, g$ (since $g \geq 0$ locally almost everywhere) $= \text{ess sup}\,(f - \alpha) = \text{ess sup} f - \alpha$. Therefore $M(f) \leq \text{ess sup} f$. Replacing f by $-f$, we get $M(f) \geq \text{ess inf} f$.

Conversely if M is linear and $\text{ess inf} f \leq M(f) \leq \text{ess sup} f$ for any $f \in X$, we claim that $M(f) \in K_g$ for any $g \sim f$. Otherwise for some $g \sim f$, $M(f) \notin K_g = [\inf g, \sup g]$ (g is real-valued). Then either $M(f) > \sup g$ or $M(f) < \inf g$. In the former case $M(f) > \sup g \geq \text{ess sup}\, g = \text{ess sup} f$ and in the latter, $M(f) < \inf g \leq \text{ess inf}\, g = \text{ess inf} f$, both leading to a contradiction. This completes the proof.

As in [6], we can define the weak* operator topology on $\Pi\{E^* : f \in X\}$ as the product of the weak* topologies. Many results in [6] concerning the set of means (for the discrete case) can be carried over. In particular, we have the following lemma. The proof is the same as in the discrete case (see [6], Lemmas 3.4 and 3.5). We omit the details.

LEMMA 2.4. (*a*) $p_A(M(f)) \leq \overline{q}_A(f)$ for any mean M on X, $f \in X$ and A bounded subset of E. Here p_A is defined by $p_A(x^*) = \sup_{x \in A} |x^*(x)|$, $x^* \in E^*$ as in [6].

(*b*) The set of means on X is compact convex in the weak* operator topology.

LEMMA 2.5. Let $(\ ,\ )$ denote the defining bilinear functional of the pair $L_\infty(G)$ and $L_1(G)$. Each $\Phi \in L_1(G)$ can be regarded as a linear mapping from X into E^* such that $\Phi(f)x = \int f(s)\,x\,\Phi(s)\,ds = (f(\cdot)x, \Phi)$ for any $f \in X$ and $x \in E$. In particular, if $\Phi \in P(G) = \{\Phi \in L_1(G) : \Phi \geq 0 \text{ and } \|\Phi\|_1 = 1\}$, then Φ is a mean on X.

Proof: For each $x \in E$, $f(\cdot)x$ is a bounded measurable scalar-valued function. Hence $\int f(s)(x)\,\Phi(s)\,ds$ is finite for $\Phi \in L_1(G)$. If $f \backsim g$ then $f(\cdot)x = g(\cdot)x$ locally almost everywhere. Thus $\Phi : X \longrightarrow E^*$ in well defined if we can show that $\Phi(f) \in E^*$. Now linearity of $\Phi(f)x$ in x is clear. Also $|\Phi(f)x| \leq \sup_{s \in G} |f(s)x| \cdot \|\Phi\|_1$, since the set $\{f(s) : s \in G\}$ is strongly bounded, it is equi-continuous (E is quasi-barrelled). Therefore $p(x) = \sup_{s \in G} |f(s)x|$ is a continuous semi-norm on E (See [8, Proposition 3, p. 48]). Now $\Phi(f)$ is dominated by a scalar multiple of p. Hence $\Phi(f) \in E^*$.

Evidently, $\Phi : X \longrightarrow E^*$ is linear. If $\Phi \in P(G)$, we claim that $\Phi(f) \in \bigcap \{K_g : g \backsim f\}$. Otherwise for some $g \backsim f$, $\Phi(f) \notin K_g$. An application of Hahn-Banach Theorem shows that there exist some $x \in E$ and some real number α such that $\sup\{g(s)x : s \in G\} \leq \alpha < \Phi(f)x = \Phi(g)x = \int g(s)\,x\,\Phi(s)\,ds \leq \sup\{g(s)x : s \in G\}$ (Because $\Phi \in P(G)$). This is a contradiction. Hence Φ is a mean on X if $\Phi \in P(G)$.

REMARK 2.6. (*a*) The assumption that E is quasi-barrelled is needed to show that $\Phi(f) \in E^*$ in Lemma 2.5. Such arguments have been used before in [6, § 5] and will be used quite often again in later discussions.

(*b*) Lemma 2.5 shows that $P(G)$ is a (convex) subset of the set of the means on X. Let $K(X)$ denote the weak* operator closure of $P(G)$, then $K(X)$ is again a subset of the set of means on X. It is not clear whether $K(X)$ contains all the means. (See [6, § 4]).

DEFINITION 2.7. We shall use the notation $\langle f, g \rangle$ for $\Phi(f)$, $f \in L_\infty(G, E^*)$ $g \in L_1(G)$. It is clear that $\langle f, g \rangle$ is bilinear. Moreover $\langle f, g \rangle = 0$ for all

$g \in L_1(G)$ iff $f = 0$ in $L_\infty(G, E^*)$ (i.e. $f(\cdot)x = 0$ locally almost everywhere for each $x \in E$) because $\langle f, g\rangle x = (f(\cdot)x, g)$. Thus to show that $f = f_1$ in $L_\infty(G, E^*)$, it is enough to show that $\langle f, g\rangle = \langle f_1, g\rangle\ \forall g \in L_1(G)$, as in the real case.

3. Convolutions and topological invariant means.

DEFINITION 3.1. Let $\Phi \in L_1(G)$ and $f \in L_\infty(G, E^*)$, define a mapping $\Phi * f\colon G \to G^*$ by $\Phi * (f(s))\, x = (\Phi * f(\cdot)\, x)(s) = \int \Phi(t) f(t^{-1} s)\, x dt$, $s \in G$ and $x \in E$. Using the same arguments as in Lemma 2.5, we can easily show that $\Phi * f\colon G \to E^*$ is well-defined. In fact we have the following lemma.

LEMMA 3.2. If $\Phi \in L_1(G)$ and $f \in L_\infty(G, E^*)$, then $\Phi * f \in L_\infty(G, E^*)$. Moreover $\overline{q}_A(\Phi * f) \leq \overline{q}_A(f) \cdot \|\Phi\|_1$ for any bounded subset A of E.

Proof: For any $g \backsim f$, $s \in G$, $x \in E$, we have $|(\Phi * f(s))\, x| = \left| \int \Phi(t) f(t^{-1} s)\, x dt \right| \leq \sup_{s \in G} |g(s)\, x| \cdot \|\Phi\|_1$. Hence for any $A \subset E$ bounded, and $s \in G$, $p_A(\Phi * f(s)) \leq q_A(g) \cdot \|\Phi\|_1$, $g \backsim f$. Taking infimum over $g \backsim f$ and supremum over $s \in G$, we have $\overline{q}_A(\Phi * f) \leq q_A(\Phi * f) \leq \overline{q}_A(f) \cdot \|\Phi\|_1$. Therefore $\Phi * f \in L_\infty(G, E^*)$ (measurability of $(\Phi * f)(\cdot)\, x$ is clear since $(\Phi * f)(\cdot)\, x = \Phi * f(\cdot)\, x$ is even continuous).

REMARK 3.3. We can also define $f * \Phi^\sim\colon G \to E^*$ by $(f * \Phi^\sim)(s)\, x = (f(\cdot)\, x * \Phi^\sim)(s) = \int f(t)\, x\, \Phi^\sim(t^{-1} s)\, dt$, $s \in G$, $x \in E$. By similar arguments as above, we can prove that $f * \Phi^\sim \in L_\infty(G, E^*)$ and $\overline{q}_A(f * \Phi^\sim) \leq \overline{q}_A(f) \cdot \|\Phi\|_1$ for any $A \subset E$ bounded.

DEFINITION 3.4. Following the notations in Wong [11] for real-valued functions, we define for each $\Phi \in L_1(G)$ the operators $l_\Phi, r_\Phi\colon L_\infty(G, E^*) \to L_\infty(G, E^*)$ by $l_\Phi(f) = \frac{1}{\Delta} \Phi^\sim * f$ and $r_\Phi(f) = f * \Phi^\sim$, $f \in L_\infty(G, E^*)$.

The next lemma is a generalisation of [11, Lemma 3.1 (c)].

LEMMA 3.5. If $f \in L_\infty(G, E^*)$, Φ, $g \in L_1(G)$, then $\langle \frac{1}{\Delta} \Phi^\sim * f, g\rangle = \langle f, \Phi * g\rangle = \langle f * g^\sim, \Phi\rangle$.

Proof: For each $x \in E$, we have by [11, Lemma 3.1] (*e*)],

$$\left(\frac{1}{\Delta} \Phi^{\sim} * f(\cdot)x, g\right) = (f(\cdot)x, \Phi * g) = (f(\cdot)x * g^{\sim}, \Phi).$$

Hence

$$\langle \frac{1}{\Delta} \Phi^{\sim} * f, g \rangle = \langle f, \Phi * g \rangle = \langle f * g^{\sim}, \Phi \rangle.$$

DEFINITION 3.6. Let X be a linear subspace of $L_\infty(G, E^*)$ which is topological left invariant (i.e. $l_\Phi f \in X$ for any $\Phi \in P(G)$ and $f \in X$). A mean M on X is called *topological left invaviant* iff $M(l_\Phi f) = M(f)$ for any $f \in X$, $\Phi \in P(G)$.

Obviously our definition agrees with that given in [11] for real-vàlued functions.

An immediate consequence of this definition is the following lemma which extends [4, Proposition 2.1.3].

LEMMA 3.7. Let X be a left invariant and topological left invariant linear subspace of $L_\infty(G, E^*)$, then any topological left invariant mean on X is also left invariant.

Proof: Let $a \in G$, $\Phi \in P(G)$ and $f \in L_\infty(G, E^*)$, then $(\Phi * l_a f)(\cdot)x = \Phi * (l_a f(\cdot)x) = \Phi * l_a(f(\cdot)x) = \Delta(a)(r_a \Phi * f(\cdot)x)$ (by [5, Theorem 20.11 (iii)]) $= (\Delta(a) r_a \Phi * f)(\cdot)x$ for any $x \in E$. Hence $\Phi * l_a f = \Delta(a) r_a \Phi * f$. Since $\Delta(a) r_a \Phi \in P(G)$ again, we must have $M(l_a f) = M(\Phi * l_a f) = M(f)$ for any $f \in X$, and any topological left invariant mean M on X.

Some other results in Greenleaf [4] can be extended in the same way, we shall not attempt to enumerate them here.

4. Arens product and localisation theorem.

In order to define an Arens product in the set of means which extends the real case [11, § 4] and parallels the discrete case [6, § 3.6], we need the concept of a lifting. For details, the reader is referred 'to Tulcea and Tulcea [9], [10]. We present here a brief description of a special case which is required in our discussions.

DEFINITION 4.1. *A linear lifting* is a map $\varrho: BM(G) \rightarrow BM(G)$ such that

(1) $\varrho(f) \backsim f$

(2) $f \backsim g$ implies $\varrho(f) = \varrho(g)$

(3) $f \geq 0$ implies $\varrho(f) \geq 0$

(4) $\varrho(1) = 1$

(5) $\varrho(\alpha f + \beta g) = \alpha \varrho(f) + \beta \varrho(g)$, α, β scalars.

REMARK 4.2. If ϱ is a linear lifting, then $f \geq 0$ locally almost everywhere implies $\varrho(f) \geq 0$. For $f \geq 0$ locally almost everywhere implies $\varrho(f) \geq 0$. For $f \geq 0$ locally almost everywhere implies $f \backsim g \geq 0$ for some g. Hence $\varrho(f) = \varrho(g) \geq 0$ by (2) and (3).

Also from (3), (4) and (5), it follows that $\| \varrho(f) \|_u \leq \| f \|_u$ for any $f \in BM(G)$ (Tulcea and Tulcea [9, § 2]).

THEOREM 4.3. (Tulcea and Tulcea), For any locally compact group G there always exists a linear lifting ϱ on $BM(G)$ such that

(1) ϱ commutes with left translations (i. e. $\varrho(l_a f) = l_a(\varrho(f))$ for any $a \in G$, $f \in BM(G)$.

(2) $\varrho(f) = f$ for any continuous function $f \in BM(G)$.

Proof: See Tulcea and Tulcea [9].

LEMMA 4.3. Let X be a topological left invariant linear subspace of $L_\infty(G, E^*)$, M a mean on X and ϱ a linear lifting on $BM(G)$ which commutes with left translations. For each $f \in X$, $x \in E$, the mapping $\Phi \rightarrow M\left(\frac{1}{\Delta} \Phi^{\sim} * f\right)x$ is a continuous linear functional on $L_1(G)$. Let $g_{x,f}^M \in L_\infty(G)$ be such that $(g_{x,f}^M, \Phi) = M\left(\frac{1}{\Delta} \Phi^{\sim} * f\right)x$, $\Phi \in L_1(G)$ and define $M_L(f) : G \rightarrow E^*$ by $(M_L(f)(s))x = \varrho(g_{x,f}^M)(s)$, $s \in G$, $x \in E$. Then $M_L(f) \in L_\infty(G, E^*)$. Moreover $M_L(f)$ is linear in f and is independent of ϱ.

PROOF: Since M is a mean, we have $\left| M\left(\frac{1}{\Delta} \Phi^{\sim} * f\right)x \right| \leq \sup_{s \in G} \left| \frac{1}{\Delta} \Phi^{\sim} * f(s) x \right|$ $\leq \sup_{s \in G} |f(s)x| \cdot \| \Phi \|_1$. Hence $\Phi \rightarrow M\left(\frac{1}{\Delta} \Phi^{\sim} * f\right)x$ is indeed a continuous linear functional on $L_1(G)$ (Linearity is obvious). Let $g_{x,f}^M \in L_\infty(G)$ be defined by $g_{x,f}^M, \Phi) = M\left(\frac{1}{\Delta} \Phi^{\sim} * f\right)x$, $\Phi \in L_1(G)$. Define $M_L(f) : G \rightarrow E^*$ by $M_L(f)(s)x = \varrho(g_{x,f}^M)(s)$, $s \in G$ and $x \in E$. Since $g_{x,f}^M \backsim \hat{g}_{x,f}^M$ implies $\varrho(g_{x,f}^M) =$

$= \varrho\,(\hat{g}^M_{x,f})$. The right hand side is independent of the choice of $g^M_{x,f}$ from its equivalence class in $L_\infty(G)$. We first show that $M_L(f)(s)x$ is linear in x. From the definition of $g^M_{x,f}$, it is clear that $g^M_{x+y,f} = g^M_{x,f} + g^M_{y,f}$ locally almost everywhere. Hence $\varrho\,(g^M_{x+y,f}) = \varrho\,(g^M_{x,f}) + \varrho\,(g^M_{y,f})$ and therefore $M_L(f)(s)(x+y) = M_L(f)(s)x + M_L(f)(s)y$ for any $s \in G$, $x, y \in E$. Similarly $M_L(f)(s)(\alpha x) = \alpha M_L(f)(s)x$ for any $s \in G$, $x \in E$, α real. Thus $M_L(f)(s)x$ is linear in x. Also $|M_L(f)(s)x| = |\varrho\,(g^M_{x,f})(s)| = |\varrho\,(\hat{g}^M_{x,f})(s)| \leq \leq \|\varrho\,(\hat{g}^M_{x,f})\|_u \leq \|\varrho\,(\hat{g}^M_{x,f})\|_u \leq \|\hat{g}^M_{x,f}\|_u$ for any $\hat{g}^M_{x,f} \backsim g^M_{x,f}$ in $L_\infty(G)$. Hence

$$|M_L(f)(s)x| \leq \|g^M_{x,f}\|_\infty$$

$$= \sup_{\|\Phi\|_1 \leq 1} \left| M\left(\frac{1}{\Delta}\,\Phi^\sim * f\right)x \right|$$

$$\leq \inf_{g \sim f} \sup_{s \in G} |g(s)x|.$$

In particular, $\|M_L(f)(s)x| \leq \sup_{s \in G} |f(s)x|$ which is a continuous semi-norm on E. Hence $M_L(f)(s) \in E^*$ for any $s \in G$. Since $M_L(f)(\cdot)x = \varrho\,(g^M_{x,f}) \in BM(G)$, $M_L(f) \in L_\infty(G, E^*)$. In fact for any $A \subset E$ bounded, $q_A(M_L(f)) = = \sup_{s \in G} \sup_{x \in A} |M_L(f)(s)x| \leq \sup_{s \in G} \sup_{x \in A} |g(s)x| = q_A(g)\ \forall\, g \backsim f$. Therefore, $\overline{q}_A(M_L(f)) \leq \overline{q}_A(f)$.

If now ϱ' is another linear lifting of $BM(G)$, then $\varrho'(g^M_{x,f}) \backsim g^M_{x,f} \backsim \varrho\,(g^M_{x,f})$. Hence for each $x \in E$, $M_L(f)(\cdot)x = \varrho\,(g^M_{x,f}) \backsim \varrho'(g^M_{x,f}) = M'_L(f)(\cdot)x$ where $M'_L(f)$ arises from the linear lifting ϱ'. In order words, $M_L(f)$ and $M'_L(f)$ both belong to the some equivalence class in $L_\infty(G, E^*)$. Thus, $M_L(f)$ is independent of ϱ. It is clear from the definition that $M_L(f)$ is also independent of the function f chosen from its equivalence class in $L_\infty(G, E^*)$, while the linearity of $M_L(f)$ in f can be handled exactly as we have done for x.

DEFINITION 4.4. $M_L(f)$ is called the *topological left introversion* of f by M. X is called topological left introverted if $M_L(X) \subset X$. In this case, if M and N are means on X, we can define the Arens product $M \odot N$ by $(M \odot N)(f) = M(N_L(f))$, $f \in X$.

REMARK 4.5. For real-valued function, our definitions of topological left introversion and Arens product agree with those given in Wong [11] becouse $(M_L(f), \Phi) = (\varrho\,(g^M_f), \Phi) = (g^M_f, \Phi) = M\left(\frac{1}{\Delta}\,\Phi^\sim * f\right)$ for any $\Phi \in L_1(G)$

since $\varrho(g_f^M) \backsim g_f^M$. $\left(\text{Here } g_f^M \text{ is defined by } (g_f^M, \Phi) = M\left(\frac{1}{\Delta}\,\Phi^{\sim} * f\right),\ \Phi \in L_1(G)\right.$. We can forget about the «x» since we are dealing with real scalars).

LEMMA 4.6. Let X be a topological left introverted and topological left invariant linear subspace of $L_\infty(G, E^*)$, $\Phi \in L_1(G)$, $f \in X$, and M, N and P means on X. Then

(1) $\langle M_L(f), \Phi \rangle = M\left(\frac{1}{\Delta}\,\Phi^{\sim} * f\right)$

(2) $M \odot N$ is also a mean on X.

(3) $\Phi * M_L(f) = M_L(\Phi * f)$

(4) $M_L \circ N_L = (M \odot N)_L$

(5) $(M \odot N) \odot P = M \odot (N \odot P)$

(6) For fixed N the map $M \longrightarrow M \odot N$ is affine and weak* operator continuous from the set of means on X into itself.

Proof: (1) $\langle M_L(f), \Phi \rangle x = (M_L(f)(\cdot)x, \Phi) = (\varrho(g^M_{x,f}), \Phi) = (g^M_{x,f}, \Phi) =$ $= M\left(\frac{1}{\Delta}\,\Phi^{\sim} * f\right)x$ for any $x \in E$. This proves (1).

(2) Obviously, $M \odot N : X \longrightarrow E^*$ is linear. For each $s \in G$, $\Phi \in P(G)$, we claim that $(\Phi * f)(s) \in K_g$ for any $g \backsim f$. Otherwise by Hahn-Banach Theorem, there exist some $x \in E$ and some real α such that

$$(\Phi * f)(s)\,x > \alpha \geq \sup\{g(s)\,x : s \in G\}$$

for some fixed $g \backsim f$. But $(\Phi * f)(s)\,x = \int \Phi(t) f(t^{-1}s)\,x\,dt = \int \Phi(t)\, g(t^{-1}s)\,x\,dt$ $\leq \sup\{g(t^{-1}s)\,x : t \in G\} \leq \sup\{g(s)\,x : s \in G\} \leq \alpha$ (since $\Phi \geq 0$, $\|\Phi\|_1 = 1$) contradiction.

Next, we want to show that for any $s \in G$ $N_L(f)(s) \in K_g$ for any $g \backsim f$. Since N is a mean, $N\left(\frac{1}{\Delta}\,\Phi^{\sim} * f\right) \in w^*\,CL\,CO\left\{\frac{1}{\Delta}\,\Phi^{\sim} * f(t) : t \in G\right\} \subset K_g$ for any $\Phi \in P(G)$, $g \backsim f$ (By what we have just proved). Therefore for any $x \in E$, $\Phi \in P(G)$, $g \backsim f$, we have $(g^N_{x,f}, \Phi) = N\left(\frac{1}{\Delta}\,\Phi^{\sim} * f\right)x \in CL\,CO\,\{g(s)\,x :$ $s \in G\}$ (closure now taken in the usual topology of the reals) $= [\alpha, \beta]$ where $\alpha = \inf\{g(s)\,x : s \in G\}$ and $\beta = \sup\{g(s)\,x : s \in G\}$ (α, β both depend on x and g but not Φ). This is true for any $\Phi \in P(G)$. It follows that $g^N_{x,f} \leq \beta$ locally almost everywhere. Hence $\varrho(g^N_{x,f}) \leq \beta$ (everywhere). Another application of Hahn Banach Theorem shows that $N_L(f)(s) \in K_g$ for any $g \backsim f$. Since M is a mean, $(M \odot N)(f) = M(N_L(f)) \in w^*\, CL\, CO\, \{N_L(f)(s) : s \in G\} \subset K_g$, $g \backsim f$ or $M \odot N$ is also a mean.

(3) Since $(g^M_{x, \psi * f}, \Phi) = M\left(\frac{1}{\Delta}\, \Phi^\sim * (\psi * f)\right) x = M\left(\frac{1}{\Delta}\left(\frac{1}{\Delta}\, \psi^\sim * \Phi\right)^\sim * f\right) x$ $= \left(g^M_{x,f}, \frac{1}{\Delta}\, \Phi^\sim * \Phi\right) = (\psi * g^M_{x,f}, \Phi)$ (See Wong [11, Lemma 3.1 (c)]). Therefore $g^M_{x, \psi * f} \backsim \psi * g^M_{x,f}$. Hence $(\Phi * M_L(f))(\cdot)\, x = \Phi * M_L(f)(\cdot)\, x$ $= \Phi * \varrho(g^M_{x,f}) = \varrho(\Phi * g^M_{x,f})$ (By (1) above) $= \varrho(g^M_{x, \Phi * f}) = M_L(\Phi * f)(\cdot)\, x$. Thus $\Phi * M_L(f) = M_L(\Phi * f)$ for any $\Phi \in L_1(G)$, $f \in X$, and M any mean on X.

(4) Let $f_1 = N_L(f)$, $f \in X$. We have

$$\begin{aligned}(g^M_{x,f_1}, \Phi) &= M\left(\frac{1}{\Delta}\, \Phi^\sim * f_1\right) x \\ &= M\left(\frac{1}{\Delta}\, \Phi^\sim * N_L(f)\right) x \\ &= \left(M\left(N_L\left(\frac{1}{\Delta}\, \Phi^\sim * f\right)\right)\right) x \qquad \text{(By (3) above)} \\ &= (M \odot N)\left(\frac{1}{\Delta}\, \Phi^\sim * f\right) x \\ &= (g^{M \odot N}_{x,f}, \Phi), \quad \Phi \in L_1(G).\end{aligned}$$

Therefore $g^M_{x, N_L(f)} \backsim g^{M \odot N}_{x,f}$.

Now $\{(M_L \circ N_L)(f)\}(\cdot)\, x = M_L(f_1)(\cdot)\, x = \varrho(g^M_{x, N_L(f)}) = \varrho(g^{M \odot N}_{x,f})$ $= (M \odot N)_L(f)(\cdot)\, x$. Consequently, $M_L \circ N_L = (M \odot N)_L$.

(5) This follows immediately from (4) above as in the real case [11, § 4] and the discrete case) [6, § 3].

(6) This can be proved exactly as in the discrete case [6, § 3].

THEOREM 4.7. Let X be a topological left introverted and topological left invariant linear subspace of $L_\infty(G, E^*)$ and $K(X)$ the weak* operator closure of $P(G)$. Then there is a topological left invariant mean M on X (in $K(X)$) iff for each $f \in X$, there is a mean M_f on X (in $K(X)$) such that $M_f(l_\Phi f) = M_f(f)$ for any $\Phi \in P(G)$.

Proof: The proof is similar to that of [11, Theorem 5.2]. We only have to replace the bilinear functional (f, g), $f \in L_\infty(G)$, $g \in L_1(G)$ by the bilinear mapping $\langle f, g \rangle$, $f \in L_\infty(G, E^*)$, $g \in L_1(G)$ in the proof of [11, Theorem 5.2]. (Recall that $\langle f, g \rangle = 0$ $\forall g \in L_1(G)$ implies $f = 0$ in $L_\infty(G, E^*)$. See also Definition 2.7).

For the assertion about neans in $K(X)$, one follows the argument in the discrete case [6, Theorem 4.3] to show that $K(X)$ is closed under Arens product (Observe that $\Phi \odot \psi = \Phi * \psi$ $\forall \Phi, \psi \in P(G)$ and $\Phi \odot N = N \circ l_\Phi$ for any $\Phi \in P(G)$ and any mean N on X) and the rest is immediate.

5. The weak** topology and topological right Stationarity.

DEFINITION 5.1. Consider the product $\Pi\{L_\infty(G) : x \in E\}$. The product of the weak* topologies of $L_\infty(G) = L_1(G)^*$ is called the weak* topology.

Notice that the space $L_\infty(G, E^*)$ can be embedded in the product $\Pi\{L_\infty(G) : x \in E\}$ if we associate the element $(f(\cdot)x)_{x \in E}$ of the product with the function $f \in L_\infty(G, E^*)$. This mapping is well-defined and one to-one because $f \backsim g$ iff $f(\cdot)x = g(\cdot)x$ locally almost everywhere for each $x \in E$.

If x is a linear subspace of $L_\infty(G, E^*)$, we shall identify X with its image under the natural embedding defined above. Thus $f_a \longrightarrow f$ in w^{**} topology of X iff for each $x \in E$, $f_\alpha(\cdot)x \longrightarrow f(\cdot)x$ in weak* topology of $L_\infty(G)$. Alternatively, this is the case iff $\langle f_\alpha, \Phi \rangle \longrightarrow \langle f, \Phi \rangle$ in weak* topology of E^* for any $\Phi \in L_1(G)$. (Hence the name weak** topology).

DEFINITION 5.2. Let X be a topological left introverted and topological left invariant linear subspace of $L_\infty(G, E^*)$. X is called *topological right stationary* if for each $f \in X$, $Z_R(f) =$ weak** closure of $\{r_\Phi f : \Phi \in P(G)\}$ contains a constant function (A function $f \in BM(G, E^*)$ is a constant function if $f(G)$ is a singleton. $f \in L_\infty(G, E^*)$ is a constant function if the equivalence class f contains a constant function in $BM(G, E^*)$).

This definition obviously agrees with the one used in Wong [11] for real-valued functions. As in [6], we make no distinction between the element $x^* \in E^*$ and the constant function on G which is identically equal to x^*.

THEOREM 5.3. Let X be a topological left introverted and topological left invariant linear subspace of $L_\infty(G, E^*)$, then there is a topological left invariant mean in $K(X)$ iff A is topological right stationary. In this case, if $f \in X$, then $x^* \in Z_R(f)$ iff there is a topological left invariant mean M in $K(X)$ such that $M(f) = x^*$.

Proof: Combine the arguments used in the real case [11, Theorem 5.4] and the discrete case [6, Theorem 4.5]. We omit the details.

6. Relation Between the means on vector-valued and scalar-valued functions.

Let X be a topological left invariant linear subspace of $L_\infty(G, E^*)$, $f \in X$ and $x \in E$, then $f(\cdot)\, x$ is bounded measurable and defines an element of $L_\infty(G)$ which is independent of the choice of f from its equivalence class in $L_\infty(G, E^*)$. In general the functions of the form $f(\cdot)\, x$, $f \in X$, $x \in E$ need not form a linear space nor should it contain the constant.

In analogy with [11, Theorem 5.1], we have the following theorem.

THEOREM 6.1. Let m be a mean on $L_\infty(G)$ and define $M: X \to E^*$ by $M(f)\, x = m(f(\cdot)\, x)$, $f \in X$, $x \in E$. Then M is a mean in $K(X)$. Moreover, M is topological left invariant of f iff m is topological left invariant on $f(\cdot)\, x$ for any $x \in E$.

Conversely, any mean in $K(X)$ is of this form.

Proof: Obviously $M(f)\, x$ is linear in x and depends only on the equivalence class f. Also $|M(f)\, x| = |m(f(\cdot)\, x)| \leq \|f(\cdot)\, x\|_\infty \leq \|f(\cdot)\, x\|_u = \sup_{s \in G} |f(s)\, x|$ wich is a continuous semi-norm on E. Hence $M: X \to E^*$ is well-defined and is clearly linear in f. Now take any $g \backsim f$ aud suppose $M(f) \notin K_g$. By Hahn-Banach Theorem again, there exist some $x \in E$ and some real α such that

$$\begin{aligned} m(g(\cdot)\, x) &= m(f(\cdot)\, x) \\ &= M(f)\, x \\ &> \alpha \\ &\geq \sup\{g(s)\, x : s \in G\} \\ &\geq \text{ess sup}\{g(s)\, x : s \in G\} \\ &\geq m(g(\cdot)\, x) \end{aligned}$$

which is a contradiction. Therefore $M(f) \in \bigcap \{K_g : g \backsim f\}$ consequently, M is a mean on X. Since $(\Phi * f)(\cdot)\,x = \Phi * f(\cdot)\,x$, it is clear that M is topological left invariant on f iff m is topological left invariant on $f(\cdot)\,x$ for any $x \in E$. Also one can use the same arguments as in the proof of [6, Theorem 5.1] to show that $M \in K(X)$ and that any mean $M \in K(X)$ is « induced » by some mean in on $L_\infty(G)$ by the equations $M(f)x = m(f(\cdot)x)$, $x \in E$ (In general m is not unique, see [6, Remark 5.2] also).

7. Comments on generalisations.

The theory we have developed in this paper and in [6] is by no means the only possible one. We can also consider functions defined on a semigroup or locally compact group with values in a general separated locally convex space E (instead of the continuous dual of a quasi-barrelled space). However, as first observed by Dixmier in [2, § 3] it is necessary to restrict ourselves to those functions such that the closures of the convex hulls of their images are weakly compact in E (in order to have an interesting theory). In defining a mean for these functions, we use $CL\,CO\,\{f(s) : s \in S\}$ for the semigroup case and $\bigcap_{g \sim f} CL\,CO\,\{g(s) : s \in G\}$ for the locally compact group case, where $g \backsim f$ means $x^* g(\cdot) = x^* f(\cdot)$ locally almost everywhere for each $x^* \in E^*$. Of course, in the latter case, we identify functions f and g such that $f \backsim g$ and considere only functions f such that $x^* f(\cdot) \in BM(G)$ for each $x^* \in E^*$. (Note that the closure of a convex set in E is the same in any topology of the pair (E, E^*)).

Many results we have in this paper and [6] can be carried over smoothly. For example, the compactness of the set of means on X in the product space $\Pi\,\{E : f \in X\}$ where each E is endowed with the weak topology.

Notice that if E is a quasi-barrelled space and $f \in l_\infty(S, E^*)$, then the (weak) closure of the convex hull of the range of f is weakly compact in E_1 where E_1 is the separated locally convex space E^* with the weak* topology. (Since $f(S)$ is strongly bounded, the same is true for its convex hull. Hence both are equicontinuous and in particular, the weak* closure of the convex hull of $f(S)$ is weak* compact). In this sense, the theory of [6] can also be developed by the new approach mentioned just above. Similar considerations apply to the locally compact group case.

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