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LEVI-FLAT SUBMANIFOLDS AND HOLOMORPHIC EXTENSION OF FOLIATIONS

C. REA (*)

§ 1. Introduction.

a) It is well known that the behaviour of the Levi-form of a real hypersurface Y of some complex manifold X can give a lot of information about the surface, the domain that can be bounded by it and the manifold X.

The problem of translating this theory to lower dimensional manifolds is still unsolved. (For extension of Levi-convexity see [4]).

In this paper we are involved with Levi-fiathess, i. e. with the case that the Levi-from of the functions which define Y vanishes on all complex vectors tangent to Y. For an hypersurface this is equivalent to require that Y is a union of complex hypersurfaces (not necessarily locally closed). The corresponding condition is not equivalent to Levi flatness in the lower dimensional case. We are obliged to ask something more: precisely that at each point y of Y, the dimension of the complex tangent space to Y (i. e. the Cauchy-Riemann dimension of Y at y) is minimal.

Submanifolds with this property are called *almost Levi-flat* manifolds because of their strong analogy with almost complex manifolds, or C. R. manifolds. Levi-flat and almost Levi-flat submanifolds can be also intrinsically defined and studied looking at them as C^∞ manifolds with a suitable structure that gives all the information which X gave. These are respectively semiholomorphically foliated manifolds and manifolds with semiholomorphic structure. The last one can be given associating to each point of Y a subspace of suitable dimension belonging to the complexified cotangent space. The vanishing property of the Levi form that we have for the *imbedded case*

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corresponds here to the Frobenius-Nirenberg integrability condition (see [3]) and means geometrically that the spaces we have chosen are tangent to complex manifolds lying on Y, i. e. our semiholomorphic structure is actually a semiholomorphic foliation.

b) Suppose now that Y is a Levi-flat submanifold of X. It can be easily seen that the complex submanifolds which foliate Y are locally the level sets of some C^∞ vector valued function $X \to \mathbb{C}^k$. In § 5 we prove that this function can be chosen to be holomorphic if Y is C^ω. The geometrical meaning of this fact is that the semiholomorphic foliation of Y can be holomorphically extended to some neighbourhood of Y. In § 6 we prove with a counterexample the necessity of the C^ω assumption.

Finally we remark that all this study, being purely local, could be done replacing the manifold X with an open set of $\mathbb{C}^n$ without any loss of generality but also without gain of simplicity because of the essential use that we make of local coordinates.

§ 2. Semiholomorphic structure and almost Levi-flatness.

a) Let Y be a C^∞ or C^ω manifold, its *complexified cotangent space* $T_y^{*\mathbb{C}}\, Y$ at y is the complex linear space of all $\mathbb{R}$-linear functions $T_y^{\mathbb{R}}\, Y \to \mathbb{C}$, where $T_y^{\mathbb{R}}\, Y$ is the usual tangent space of Y at y.

The differentials of the coordinates span $T_y^{*\mathbb{C}}\, Y$ over $\mathbb{C}$.

A C^∞ or C^ω *distribution of complex cotangent subspaces* of Y is a choice $y \mapsto \Omega(y)$, where $\Omega(y)$ is a complex subspace of $T_y^{*\mathbb{C}}\, Y$ of constant dimension and depends C^∞ or C^ω on y (i. e. there is a set of C^∞ or C^ω complex forms on a neighbourhood U of each $y \in Y$, which are a basis of $\Omega(z)$, for each $z \in U$).

DEFINITION. A C^∞ or C^ω *semiholomorphic structure* on Y is a C^∞ or C^ω distribution $y \mapsto \Omega(y)$ such that

$$\Omega(y) + \bar{\Omega}(y) = T_y^{*\mathbb{C}}\, Y. \tag{2.0}$$

The integer $\operatorname{cod}_{\mathbb{R}} \Omega \overset{\text{def}}{=} \dim_{\mathbb{C}} [\Omega(y) \cap \bar{\Omega}(y)] = 2 \dim_{\mathbb{C}} \Omega(y) - \dim_{\mathbb{R}} Y$ is the *codimension* of the semiholomorphic structure.

b) Suppose now that Y is a real submanifold of a complex manifold X, locally given by the equations $\Phi_1 = \ldots = \Phi_k = 0$, with $d\Phi_1 \wedge \ldots \wedge d\Phi_k \neq 0$

on Y (1), and consider, in the holomorphic tangent space $T_y X = \Sigma_\alpha \mathbb{C}(\partial/\partial z^\alpha)_y$ of X at y, the subspace

$$(2.1) \qquad T_y Y \equiv \{v \in T_y X,\ v\,\Phi_1 = \ldots = v\,\Phi_k = 0\}.$$

The complex dimension $\dim_{RC} Y_y$ of $T_y Y$ is called *Cauchy-Riemann dimension* of Y at y and is equal to $\dim_{\mathbb{C}} X - \mathrm{rk}\,(\partial\Phi_1, \ldots, \partial\Phi_k)$. Hence we have

$$(2.2) \qquad \dim_{CR} Y_y \geq \dim_{\mathbb{C}} X - \mathrm{cod}_{\mathbb{R}}\, Y.$$

DEFINITION. The submanifold Y is said to be *almost Levi-flat* if $\dim_{CR} Y_y = \dim_{\mathbb{C}} X - \mathrm{cod}_{\mathbb{R}}\, Y$ for each $y \in Y$. This is equivalent to the condition

$$(2.3) \qquad \partial\Phi_1 \wedge \ldots \wedge \partial\Phi_k \neq 0, \text{ on } Y.$$

REMARK 2.1. *Every real hypersurface is almost Levi-flat.*

LEMMA 2.1. *Y is almost Levi-flat if and only if it has no tangent complex submanifold, Z, with $0 < \mathrm{cod}_{\mathbb{C}} Z < \mathrm{cod}_{\mathbb{R}} Y$* (2).

PROOF. If such a manifold Z exists and is tangent to Y at the point y, then each vector $\Sigma\, v^\alpha \dfrac{\partial}{\partial z^\alpha}$ which is tangent to Z at y belongs also to $T_y Y$, hence $\dim_{CR} Y_y = \dim_{\mathbb{C}} T_y Y \geq \dim_{\mathbb{C}} T_y Z > \dim_{\mathbb{C}} X - \mathrm{cod}_{\mathbb{R}}\, Y$ and Y can't be Levi-flat. Conversely, suppose that $\dim_{CR} Y_y = s > \dim_{\mathbb{C}} X - \mathrm{cod}_{\mathbb{R}}\, Y$ for some $y \in Y$, and take a basis $u_1, \ldots, u_s$ of $T_y Y$. Set $m = \dim_{\mathbb{C}} X$ and consider the map $(\zeta^1, \ldots, \zeta^s) \longmapsto (y^1 + \Sigma_h \zeta^h u_h^1, \ldots, y^m + \Sigma_h \zeta^h u_h^m)$ of $D_\varepsilon \equiv \equiv \{\zeta \in \mathbb{C}^s, |\zeta| < \varepsilon\}$ into X. For sufficiently small ε this map represents a complex submanifold which is tangent to Y at y. Q. E. D.

LEMMA 2.2. *Let h be an holomorphic function defined in some open subset U of X, and Y an almost Levi-flat submanifold of X which meets U. If h vanishes on $U \cap Y$ the it vanishes identically.*

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(1) We shall always implicitely suppose that this independence condition is satisfied by the functions definining our submanifold.

(2) We mean that Z is tangent to Y at $y \in Z \cap Y$ if $T_y^{\mathbb{R}} Z \subset T_y^{\mathbb{R}} Y$.

PROOF. We can suppose $\text{cod}_{\mathbb{R}} Y > 1$ (otherwise the statement would be trivial) and prove that if all derivatives of order s vanish identically on Y, then those of order $s+1$ vanish too; the lemma will follow by induction starting from $s=0$. Assume that g is some derivative order s of h vanishing on Y, and that we have $(\partial g/\partial z^{\alpha})_y \neq 0$ for some α and $y \in Y$. Then the analytical space $S \equiv \{g = 0\}$ is a complex hypersurface in some neighbourhood $U' \subset U$ of y and contains the almost Levi-flat manifold $U' \cap Y$.

Since $U' \cap Y$ is a submanifold of $U' \cap S$, we can apply (2.2) to the couple $U' \cap S$, $U' \cap Y$, so we have $\dim_{\mathbb{C}} T_y Y \geq \dim_{\mathbb{C}} S - (\dim_{\mathbb{R}} S - \dim_{\mathbb{R}} Y) = \dim_{\mathbb{C}} X - \text{cod}_{\mathbb{R}} Y + 1$. Thus Y can't be almost Levi-flat. That is absurd.

Q. E. D.

Suppose we have any semiholomorphic structure Ω' on Y. Ω' is said to be *admissible* (with respect to the complex structure of $X \supset Y$) if, for each $\omega' \in \Omega'(y)$, there exists $\omega \in T_y^* X$, whose restriction ω_Y to the tangent vectors of Y is equal to ω'. $T_y^* X$ represents here the complex space spanned by the differentials of holomorphic coordinates of X.

PROPOSITION. 2.1. *The submanifold Y is almost Levi-flat if and only if it carries an admissible semiholomorphic structure Ω with* $\dim_{\mathbb{C}} \Omega(y) = \dim_{\mathbb{C}} X$. *In that case this structure is unique and given by*

$$\Omega(y) \equiv \{\omega' \in T_y^{*\mathbb{C}} Y \mid \exists\, \omega \in T_y^* X, \text{ with } \omega_Y = \omega'\}. \tag{2.4}$$

PROOF. We have obviously $\Omega(y) + \overline{\Omega}(y) = T_y^{*\mathbb{C}} Y$, at each $y \in Y$, and

$$\dim_{\mathbb{C}} \Omega(y) = \dim_{\mathbb{R}} Y - \dim_{CR} Y_y. \tag{2.5}$$

Hence it must be

$$\dim_{\mathbb{C}} \Omega(y) \leq \dim_{\mathbb{C}} X, \tag{2.6}$$

with sign « = » *if and only if* $\dim_{CR} Y_y = \dim_{\mathbb{C}} X - \text{cod}_{\mathbb{R}} Y$.

Since $\Omega'(y) \subset \Omega(y)$, if $\dim_{\mathbb{C}} \Omega'(y) = \dim_{\mathbb{C}} X$, we have the sign « = » in (2.6), and $\Omega'(y) = \Omega(y)$, for each $y \in Y$, and Y is almost Levi-flat. Conversely, if Y is almost Levi-flat, then we have the sign « = » in (2.6). Hence $y \longmapsto \Omega(y)$ is a distribution of constant dimension and since $\Omega(y) + \overline{\Omega}(y) = T_y^* Y$ it induces an admissible semiholomorphic structure on Y.

Q. E. D.

DEFINITION. The structure $y \longmapsto \Omega(y)$ given by (2.4) is the *induced semiholomorphic structure* on the Levi-flat submanifold Y of X.

c) Every real vector tangent to X at x can be written, in a unique way in the form $u+\bar{u}$, with $u \in T_x X$, and the map

$$(2.7) \qquad \alpha : T_x X \longrightarrow T_x^{\mathbb{R}} X, \quad u \longmapsto u+\bar{u}$$

is an $\mathbb{R}$-isomorphism. The space $T_y^{\mathbb{R}} Y$ is formed by the vetors $u+\bar{u}$ such that $u\,\Phi_j+\bar{u}\,\Phi_j=0$, for $j=1, \ldots, k$.

Hence the restrictions $\partial^Y \Phi_j$ to $T_y^{\mathbb{R}}$ of $\partial \Phi_j$ satisfy the equations $\partial^Y \Phi_j=-\bar{\partial}^Y \Phi_j$, therefore, since they are all in $\Omega(y)$, they are also in $\Lambda(y) \overset{\text{def}}{=} \Omega(y) \cap \bar{\Omega}(y)$.

We want to prove that, in the almost Levi-flat case, we have the formula

$$(2.8) \qquad \Lambda(y)=\mathbb{C}\,(\partial^Y \Phi_1)_y \oplus \ldots \oplus \mathbb{C}\,(\partial^Y \Phi_k)_y .$$

We get from (2.0) and prop. 2.1 $\dim_{\mathbb{C}} \Lambda(y)=2 \dim_{\mathbb{C}} \Omega(y)-\dim_{\mathbb{R}} Y= 2 \dim_{\mathbb{C}} X-\dim_{\mathbb{R}} Y=\operatorname{cod}_{\mathbb{R}} Y=k$.

Now formula (2.8) will follow from the independence of the $\partial^Y \Phi_j' s$ over $\mathbb{C}$.

For each $u \in T_y X$ we have $\langle \partial \Phi_j, u+\bar{u} \rangle=u\,\Phi_j=\langle \partial \Phi_j, u \rangle$, hence $\alpha \circ \partial \Phi_j=\partial \Phi_j$. Moreover the kernel of the linear map $\partial \Phi : T_y^{\mathbb{R}} X \longrightarrow \mathbb{C}^k$ given by $u+\bar{u} \longmapsto (u\,\Phi_1, \ldots, u\Phi_k)$ is the image by α of $T_y Y$ [see (2.1)], so its real dimension must be equal to $2 \dim_{CR} Y_y=\dim_{\mathbb{R}} X-2k$. For the restriction $\partial^Y \Phi$ of $\partial \Phi$ to $T_y^{\mathbb{R}} Y$, we have $\ker \partial^Y \Phi \subset \ker \partial \Phi$, hence $\dim_{\mathbb{R}} \mathcal{Im}\, \partial^Y \Phi=\dim_{\mathbb{R}} Y-\dim_{\mathbb{R}} \ker \partial^Y \Phi \geq \dim_{\mathbb{R}} Y-\dim_{\mathbb{R}} X+2k= -\operatorname{cod}_{\mathbb{R}} Y+2k=k$.

But $\partial^Y \Phi$ commutes with multiplication by i, thus it extends uniquely to a $\mathbb{C}$-linear map $T_y^{\mathbb{C}} Y=T_y^{\mathbb{R}} Y+i\,T_y^{\mathbb{R}} Y \longrightarrow \mathbb{C}^k$ whose image has complex dimension k.

So the formula (2.8) is proved.

REMARK 2.2. For any $\chi=\Sigma_\alpha a_\alpha\, dz^\alpha$ of class C^∞, we have that $d\chi= \Sigma_\alpha b_\alpha \wedge \chi^\alpha$, with $\chi^\alpha \in T^* X$ and that the restriction to Y commutes with the operator d and the exterior product.

Hence, by (2.4), for each complex C^∞ form ω on Y, such that $\omega(y) \in \Omega(y)$, $\forall y \in Y$, we have $d^Y \omega=\Sigma_\alpha a_\alpha \wedge \omega^\alpha$ with $\omega^\alpha(y) \in \Omega(y)$, for each $y \in Y$.

§ 3. Semiholomorphic foliations.

a) A k-codimensional *foliation* of class C^∞ on a differentiable manifold X of dimension N is given by an atlas of C^∞ coordinates

$(x^1, \dots, x^{N-k}, t^1, \dots, t^k)$ such that, for every change

$(x, t) \longmapsto (x', t')$, the t''s do not depend on the x's. The family of the sets $t = \text{const}$, contained in coordinate patches, form a basis of a new topology (finer) on X whose connected components are called *leaves* of the foliation.

Two foliations which have the same leaves are identical. The leaves are $N - k$ dimensional submanifolds of X (not necesserely locally closed) on which the x's form a coordinate system. The coordinates (x, t) are called *admissible.*

Definition. A submanifold Y of X is a *subfoliation* if it is a union of leaves of X.

Remark 3.1. *If Y is a submanifold and, for each $y \in Y$, the tangent space at y to the leaf through y is contained in the tangent space at y of Y, then Y is a subfoliation.*

If the manifold X is complex and the (x, t) are holomorphic coordinates, then the structure above is called an *holomorphic foliation.*

Remark 3.2. *Two holomorphic foliations which coincide on some open set are identical.*

Now we shall describe a k-codimensional foliation on a $2n + k$ dimensional real manifold Y whose leaves are complex manifolds of (complex) dimension n. This is given by an atlas of coordinates $(y, t) = (y^1, \dots, y^{2n}, t^1, \dots, t^k)$, such that, if we set $z^\alpha = y^\alpha + iy^{\alpha+n}$, for $\alpha = 1, \dots, n$, they change with the rule

$$\left\{\begin{aligned} z' &= z'(z, t) \\ t' &= t'(t), \end{aligned}\right. \tag{3.1}$$

z''s holomorphic in the z variables.

The manifold X has no complex structure but each leaf is a complex manifold with coordinates $(z^1, \dots, z^n)$. We call such a structure *semiholomorphic foliation of* (real) *codimension* k.

b) Observe that a k-codimensional semiholomorphic foliation on Y determines a k-codimensional semiholomorphic structure by setting

$$\Omega(y) = \mathbb{C}\, dz^1 \oplus \dots \oplus \mathbb{C}\, dz^n \oplus \mathbb{C}\, dt^1 \oplus \dots \oplus \mathbb{C}\, dt^k, \tag{3.2}$$

and we have

$$\Lambda(y) \overset{\text{def}}{=} \Omega(y) \cap \bar{\Omega}(y) = \mathbb{C}\, dt^1 \oplus \dots \oplus \mathbb{C}\, dt^k. \tag{3.3}$$

Let Ω be the vector-space of C^∞ (or C^ω) forms ω on Y such that $\omega(y) \in \Omega(y)$, for each $y \in Y$, $d\Omega$ for the space of differentials of forms in Ω and $\mathcal{I}\Omega$ be the ideal generated by Ω in the exterior algebra of forms of all degrees. The elements of $\mathcal{I}\Omega$ are those of the type $\sum_1^N \alpha^j \wedge \omega^j$, where the α^j's are any complex forms, N any integer and $\omega^j \in \Omega$. Take now any

$$\omega = \sum_1^n{}_\alpha\, dz^\alpha + \sum_1^k{}_j\, b_j\, dt^j \in \Omega \text{ and } \chi_k = \sum_1^k{}_j\, c_j\, dt^j \in \Lambda.$$

We have

$$d\omega = \sum_1^n{}_\alpha\, da_\alpha \wedge dz^\alpha + \sum_1^k{}_j\, db^j \wedge dt^j \in \mathcal{I}\Omega \text{ and } d\chi = \sum_1^k{}_j\, dc_j \wedge dt^j \in \mathcal{I}\Lambda.$$

Hence we can state the following

REMARK 3.3. *Every semiholomorphic structure which comes from any semiholomorphic foliation, satisfies the conditions*

$$\left.\begin{aligned} &d\Omega \subset \mathcal{I}\Omega && (3.4)\\ &d\Lambda \subset \mathcal{I}\Lambda && (3.5)\end{aligned}\right\} \text{ (integrability conditions)}$$

Conversely, by the well known Frobenius-Nirenberg theorem ([3], p. 3, th. 1), *for any semiholomorphic structure which satisfies* (3.4) *and* (3.5) *there exists a unique semiholomorphic foliation of the same codimension with the properties* (3.1), (3.2), (3.3).

Observe that if the codimension vanishes, then semiholomorphic structure means almost complex structure, and a semiholomorphic foliation is a complex stucture, the condition (3.5) is unuseful because $\Lambda = 0$ and the theorem reduces to Newlander-Nirenberg theorem [2].

§ 4. Levi-flat submanifolds.

DEFINITION 4. Let Y be any locally closed C^∞ submanifold of X. Y is said to be *Levi-flat* if each $y \in Y$ is contained in some complex submanifold

$Z \subset Y$ of X, with $\operatorname{cod}_{\mathbb{C}} Z = \operatorname{cod}_{\mathbb{R}} Y$, and there does not exists any complex submanifold of greater dimension which is tangent to Y at y (3).

Here a complex submanifold is, by definition, the image of any complex manifold M by some holomorphic, injective map $j: M \to X$ of maximal rank. An open submanifold of that is the image by j of some open set of M.

PROPOSITION 4.1. *If Y is Levi-flat then it is almost Levi-flat and the manifold Z of definition 4. can be chosen uniquely in such a way that it is connected and each other submanifold with the same properties is an open submanifold of Z.*

PROOF. The first part of the statement follows directly from lemma 2.1.

Take now a real vector $u + \bar{u}$ tangent to Z at any $z \in Z$. Since Z is a complex submanifold, the real vector $iu - \bar{iu}$ is also tangent to Z, in particular to Y. So we have $u\Phi_j + \bar{u}\Phi_j = iu\Phi_j - \bar{iu}\Phi_j = 0$, i. e. $u\Phi_j = 0$ for each $u + \bar{u} \in T_z^{\mathbb{R}} Z$ and $j = 1, \dots, k$.

But the space $\mathcal{D}(z)$ of real vectors $u + \bar{u}$ for which that is true contains $T_y^{\mathbb{R}} Z$ and is actually the image of $T_z Y$ by α [see (2.7)], hence

$$\dim_{\mathbb{R}} \mathcal{D}(z) = 2 \dim_{CR} Y_y = 2 \dim_{\mathbb{C}} X - 2 \operatorname{cod}_{\mathbb{R}} Y = 2 (\dim_{\mathbb{C}} X - \operatorname{cod}_{\mathbb{C}} Z) =$$

$$= \dim_{\mathbb{R}} Z = \dim T_y^{\mathbb{R}} Z .$$

So we get

$$T_z^{\mathbb{R}} Z = \mathcal{D}(z) \stackrel{\text{def}}{=} \{u + \bar{u} \in T_z^{\mathbb{R}} X, u\Phi_1 = \dots = \Phi_k = 0\}. \tag{4.1}$$

Therefore the manifolds Z of definition 4.1 are all integral manifolds of the real distribution $y \mapsto \mathcal{D}(y)$, then the proposition follows from the classical Frobenius Theorem ([1], 2.11. 13, p. 118)

PROPOSITION 4.2. *The submanifold Y is Levi-flat if and only if the real distribution $y \mapsto \mathcal{D}(y) = \alpha\, T_y$ has costant dimension equal to $\dim_{\mathbb{R}} X - 2 \operatorname{cod}_{\mathbb{R}} Y$ and is totally inlegrable.*

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(3) The last condition is « *almost Levi-flatness* » and is trivially satisfied by all hypersurfaces.

PROOF. The condition on the dimension of $\mathcal{D}(y)$ is trivially equivalent to the almost Levi-flatness of Y. On the other hand, the Levi-flatness of Y implies, by the final argument of the last proof, the total integrability of $y \longmapsto \mathcal{D}(y)$. It remains to be proved that all integral manifolds of $y \longmapsto \mathcal{D}(y)$ are complex submanifolds of X.

Take an open set $\Omega \subset \mathbb{R}^{2n}$ $(2n = \dim_{\mathbb{R}} X - 2\,\mathrm{cod}_{\mathbb{R}} Y)$ and an injective C^∞ map $j : \Omega \longrightarrow X$ of maximal rank such that $\mathcal{D}(jp)$ is the tangent space to $j\,\Omega$ at each point jp, i.e. an integral submanifold of class C^∞ of our distribution. *Claim* that Ω admits complex coordinates such that j is holomorphic. Actually the map $(u + \bar{u}) + i\,(v + \bar{v}) \longmapsto (u + iv) + (\overline{u + iv})$ of the complexified space $\mathcal{D}(y) + i\,\mathcal{D}(y)$ onto $T_y Y \oplus \overline{T}_y Y$ is a $\mathbb{C}$-linear isomorphism (4), so we can identify these two spaces, and j induces a $\mathbb{C}$-linear isomorphism $T_p^{\mathbb{C}}\,\Omega \longrightarrow T_{jp} Y \oplus \overline{T}_{jp} Y$, for each $p \in \Omega$. Moreover for any two complex vector fields u, v on X, such that $u(y),\ v(y) \ni T_y Y,\ \forall y \in j\,\Omega$, their Lie product $[u, v]_q$ is still in $T_y Y$. Let us consider the complex distribution $p \longmapsto j^{-1}\,T_{jp}\,Y \subset T_p^{\mathbb{C}}\,\Omega$. We have $T_p^{\mathbb{C}}\,\Omega = j^{-1}\,T_{jp}\,Y \oplus \overline{j^{-1}\,T_{jp}\,Y}$, and, for each couple a, b of complex vector fields on Ω, with $a(p),\ b(p) \in j^{-1}\,T_{jp}\,Y,\ \forall p \in \Omega$ we have $j\,[a, b]_p = [ja, jb]_{jp} \in T_{jp}\,Y$, hence $[a, b]_p \in j^{-1}\,T_{jp}\,Y$. By the Newlan-der-Nirenberg theorem [2], this implies that the distribution $p \longmapsto j^{-1}\,T_{jp}\,Y$ determines a complex structure on Ω, i.e. there are complex coordinates $\zeta^1, \dots \zeta^n$ at each point of Ω such that

$$T_{jp}\,Y = j\left[\mathbb{C}\left(\frac{\partial}{\partial \zeta^1}\right)_p \oplus \dots \oplus \mathbb{C}\left(\frac{\partial}{\partial \zeta^n}\right)_p\right].$$

Now write j in the form

$$j : (\zeta^1, \dots, \zeta^n) \longmapsto z_1(\zeta), \dots, z^m(\zeta). \tag{4.2}$$

If $\partial z^\alpha / \partial \bar{\zeta}^h$ were different from 0 at some point of Ω for some α and h, we would have at this point that

$$j\,\frac{\partial}{\partial \bar{\zeta}^h} = \Sigma_\beta\,(\partial z^\beta / \partial \bar{\zeta}^h)\,\frac{\partial}{\partial z^\beta} + \Sigma_\gamma\,(\partial \bar{z}^\gamma / \partial \bar{\zeta}^h)\,\frac{\partial}{\partial \bar{z}^\gamma}$$

has some non zero component in $T_{jp}\,Y$, so $j\,\dfrac{\partial}{\partial \zeta^h}$ has some non zero component in $\overline{T}_{jp}\,Y$ while $j\,\dfrac{\partial}{\partial \zeta^h} \in T_{jp}\,Y = 0$. Hence the functions (4.2) have to be holomorphic. Q. E. D.

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(4) Its inverse is $u + \bar{v} \longmapsto 1/2\,\{(u + v) + \overline{(u + v)} + i\,[iv - iu) + \overline{(iv - iu)}]\}$.

Proposition 4.3. *The almost Levi-flat submanifold Y of X is Levi-flat if and only if one of the following conditions is fulfilled*:

(i) *its induced semiholomorphic structure is a foliation,*

(ii) $$(\partial_\alpha \bar{\partial}_\beta \Phi_j)_y u_\alpha \bar{u}^\beta = 0, \ \forall y \in Y, \ \forall u \in T_y Y,$$

(iii) *there exist locally, real C^∞ function $f_j, j = 1, \dots, k = \operatorname{cod}_{\mathbb{R}} Y$, which satisfy the tangential differential equation*

$$\partial f_j \wedge \partial \Phi_{-1} \wedge \dots \wedge \partial \Phi_{-k} = 0, \quad on \ Y; \tag{4.3}$$

with the non triviality condition

$$df_1 \wedge \dots \wedge df_k \wedge d\Phi_1 \wedge \dots \wedge d\Phi_k \neq 0, \ on \ Y; \tag{4.4}$$

moreover, if Y is C^ω, the functions f_j can be chosen C^ω.

Proof. Note first that, if $\varphi_1, \dots, \varphi_k$ are C^∞ functions on Y and $f_1, \dots, f_k$ are C^∞ extensions to a neighbourhood, then

$$d^Y \varphi_1 \wedge \dots \wedge d^Y \varphi_k = 0 \Longleftrightarrow df_1 \wedge \dots \wedge df_k \wedge d\Phi_1 \wedge \dots \wedge d\Phi_k = 0. \tag{4.5}$$

We prove now the equivalence between Levi-flatness and (iii).
Since Y is almost Levi flat, we have $\dim_{\mathbb{R}} \mathcal{D}(y) = \dim_{\mathbb{R}} X - 2 \operatorname{cod}_{\mathbb{R}} Y$. Hence the codimension of $\mathcal{D}(y)$ in $T_y^{\mathbb{R}} Y$ is given by $\dim_{\mathbb{R}} Y - \dim_{\mathbb{R}} X + 2 \operatorname{cod}_{\mathbb{R}} Y = \operatorname{cod}_{\mathbb{R}} Y = k$.

If Y is Levi-flat, by the total integrability of the distribution $\mathcal{D}$ (prop. 4.2), we can choose, in a suitable neighbourhood $V \subset Y$ of each point of Y, real C^∞ functions $\varphi_1, \dots, \varphi_k$ such that we have

$$\mathcal{D}(y) = \{u + \bar{u} \in T_y^{\mathbb{R}} Y, (u + \bar{u}) \varphi_j = 0, \ j = 1, \dots, k\},$$

and $d^Y \varphi_1 \wedge \dots \wedge d^Y \varphi_k \neq 0$. So, for each fixed $y \in V$, the set $\varphi_j = \varphi_j(y)$ is an integral manifold of $\mathcal{D}$ in V. We take arbitrary C^∞ extensions $f_1, \dots, f_k$ of $\varphi_1, \dots, \varphi_k$ to X.

If Y is C^ω then the functions φ_j and f_j can be chosen C^ω Thus condition (4.4) is a consequence of (4.5). Take now some $u \in T_y X$, with $y \in Y$. We have $u\Phi_j = 0, \ j = 1, \dots, k \Longleftrightarrow u + \bar{u} \in \mathcal{D}(y), \ iu - i\bar{u} \in \mathcal{D}(y) \Longleftrightarrow$ $\Longleftrightarrow (u + \bar{u}) f_j = i(u - \bar{u}) f_j = 0, \ j = 1, \dots, k \Longleftrightarrow uf_j = 0, \ j = 1, \dots, k$.

This proves (4.3) and the necessity of (iii) is proved. Conversely, take functions $f_1, \dots, f_k$ satisfying (4.3) and (4.4), and their restrictions $\varphi_1, \dots \varphi_k$ to Y. From (4.4) and (4.5) we get $d^Y \varphi_1 \wedge \dots \wedge d^Y \varphi_k \neq 0$. Hence the space $\mathcal{G}(y)$ of real vectors of $T_y^{\mathbb{R}} Y$ which vanish on $\varphi_1, \dots \varphi_k$ has (real) dimension equal to $\dim_{\mathbb{R}} Y - k = \dim_{\mathbb{R}} \mathcal{D}(y)$.

But from (4.3) we get that $u\,\Phi_j = 0$, $j = 1, \dots, k$ implies $uf_j = 0$ $j = 1, \dots, k$. Therefore, for every $u + \bar{u} \in \mathcal{D}(y)$, we have $(u + \bar{u}) f_j = 0$, $j = 1, \dots, k$. But $\mathcal{D}(y) \subset T_y^{\mathbb{R}} Y$, so the last equation can be written $(u + \bar{u})\,\varphi_j = 0$, $j = 1, \dots, k$.

Hence $\mathcal{D}(g) \subset \mathcal{G}(y)$. Thus, by dimensional reasons $\mathcal{D}(y) = \mathcal{G}(y)$, $\forall y \in Y$. But the distribution $\mathcal{G}(y)$ is totally integrable by construction, so the sufficiency of (iii) follows from prop. 4.2. We want to prove now the equivalence between Levi-fiatnees and condition (ii). Take two C^∞ vector fields $u + \bar{u}$, $v + \bar{v}$ such that $u(y) + \bar{u}(y)$ and $v(y) + \bar{v}(y)$ are in $\mathcal{D}(y)$ for each $y \in Y$, that is $u\Phi_j = v\Phi_j = 0$. For every $g : X \to \mathbb{C}$ of class C^1, vanishing on Y, we have trivially $ug = vg = 0$.

Therefore $[u, v]\,\Phi_j = u\,(v\,\Phi_j) - v\,(u\Phi_j) = 0$, which implies $[u, v]_y + \overline{[u, v]} \in \mathcal{D}(y)$, $\forall y \in Y$. Now, by a simple calculation, we get $[u + \bar{u}, v + \bar{v}] = [u, v] + w(u, v) + \overline{[u, v]} + \overline{w(u, v)}$, with $w(u, v) = (\bar{u}^\alpha\, \bar{\partial}_\alpha\, v^\beta - \bar{v}^\alpha\, \bar{\partial}_\alpha\, u^\beta)\, \partial/\partial z^\beta$. Hence

$$(4.6)\quad [u + \bar{u}, v + \bar{v}]_y \in \mathcal{D}(y),\ \forall y \in Y \Longleftrightarrow w(u, v)\,\Phi_j = 0, j = 1, \dots, k, \text{ on } Y.$$

Thus, by the classical Frobenius theorem and proposition 4.2, we have to prove that the condition at the right side of (4.6) is equivalent to (ii). But

$$w(u, v)\,\Phi_j = \bar{u}^\alpha\,(\bar{\partial}_\alpha\, v^\beta)\,\partial_\beta\,\Phi_j - \bar{v}^\alpha\,(\bar{\partial}_\alpha\, u^\beta)\,\partial_\beta\,\Phi_j =$$

$$= \bar{u}\,(v\,\Phi_j) - \bar{v}\,(u\,\Phi_j) - (\bar{\partial}_\alpha\,\partial_\beta\,\Phi_j)_y\,\bar{u}^\alpha\, v^\beta + (\bar{\partial}_\alpha\,\partial_\beta\,\Phi_j)_y\,\bar{v}^\alpha\, u^\beta.$$

Now, since $v\Phi_j$ and $u\Phi_j$, vanish on Y, the first two terms vanish too, so we get $w(u, v)\,\Phi_j = 2i\,\mathcal{I}m\,[\partial_\alpha\,\bar{\partial}_\beta\,\Phi_j)_y\, u_\alpha\, \bar{v}^\beta]$.

The form $(u, v) \longmapsto (\partial_\alpha\,\bar{\partial}_\beta\,\Phi_j)_y\, u^\alpha\, \bar{v}^\beta$ on the complex space $T_y Y$ is $\mathbb{C}$-linear respect to u and $\mathbb{C}$-antilinear with respect to v, hence it can't be real without vanishing identically. So we have proved that (ii) is equivalent to Levi-flatness. We shall fiinally prove the equivalence between (i) and Levi-flatness. The (real) codimension of $\mathcal{D}(y)$ in $T_y^{\mathbb{R}} X$ is $2k$ and the real forms $d\Phi_1, \dots, d\Phi_k, i(\partial\Phi_1 - \bar{\partial}\Phi_1), \dots, i(\partial\Phi_k - \bar{\partial}\Phi_k)$ are independent and vanish on $\mathcal{D}(y)$. Hence they are a basis of the subspace of $\omega \in T_y^{*\mathbb{R}} X$ such that $\langle \omega, u + \bar{u} \rangle = 0$, $\forall\, u + \bar{u} \in \mathcal{D}(y)$.

So proposition 4.2. and the classical Frobenius theorem imply that Levi-flatness is equivalent to the fact that the differential of each form of this kind is of the type $\sum_{h=1}^{k} p^h \wedge d\Phi_h + i \sum_{h=1}^{k} q^h \wedge (\partial\Phi_h - \bar{\partial}\Phi_h)$ where p^h and q^h are real 1-forms. In other words Levi-flatness can be expressed by the condition

$$\bar{\partial}\partial\Phi_j = i \sum_{h=1}^{k} p_j^h \wedge d\Phi_h + \sum_{h=1}^{k} q_j^h \wedge (\partial\Phi_h - \bar{\partial}\Phi_h)$$

with p_j^h, q_j^h real 1-forms. We recall the relation $\partial^Y \Phi_j = - \bar{\partial}^Y \Phi_j$ between the restrictions to $T_y^{\mathbb{R}} Y$ of the forms $\partial \Phi_j$ and $\bar{\partial} \Phi_j$, so the flatness of Y implies $(\partial \bar{\partial} \Phi_j)_Y = 2i \sum_{1}^{k}{}_h \, q_j^h \wedge \partial^Y \Phi_h$ and, by (2.8), $d^Y \Lambda \subset \mathcal{I} \Lambda$.

On the other hand the remark 2.2 says precisely that the first integrability condition $d^Y \Omega \subset \mathcal{I} \Omega$ is identically verified for induced semiholomorphic structures.

Conversely, if the induced semiholomorphic structure is a foliation and Z is the leaf through the point y, by (3.4) we get

$$T_y^{\mathbb{R}} Z \equiv \{u + \bar{u} \in T_y^{\mathbb{R}} Y ;\ \langle \omega, u + \bar{u} \rangle = 0,\ \forall\, \omega \in \Lambda\}.$$

Now we apply (2.8) and obtain $T_y^{\mathbb{R}} Z \equiv \{u + \bar{u} \in T_y^{\mathbb{R}} Y, \langle \partial \Phi_j, u + \bar{u} \rangle = 0,$ $j = 1, \dots, k\}$. But $\langle \partial \Phi_j, u + \bar{u} \rangle = u \Phi_j$, and $\{u \Phi_j = 0\} \Longrightarrow u + \bar{u} \in \mathcal{D}(y) \subset T_y^{\mathbb{R}} Y$, for each each $u \in T_y X$. Hence the distribution $y \longmapsto \mathcal{D}(y) = T_y^{\mathbb{R}} Z$ is completely integrable and by prop. 4.2 the condition (i) implies Levi-flatness.

Q. E. D.

§ 5. Extension property of semiholomorphic-foliations.

Assume that the complex manifold X has some holomorphic foliation of (complex) codimension k. Every almost Levi-flat subfoliation Y of X of (real) codimension k is obviously Levi-flat.

We want now to investigate whether the semiholomorphic foliation induced on some Levi-flat submanifold Y can be continued by any holomorphic foliation on a neighbourhood V of Y in such a way that Y becomes a subfoliation of V. The answer to this question is given by the following.

THEOREM 5.1. *The holomorphic extension of the semiholomorphic foliation induced on a Levi-flat submanifold Y is unique and exists if Y is real analytic.*

In § 6 we shall give an example of a C^∞ Levi-flat hypersurface in $\mathbb{C}^2$ (or $\mathbb{C}^n$) whose induced semiholomorphic foliation can't be extended. Nevertheless there are trivial examples of Levi-flat submanifolds which are not analytic in any point and have the global extension property: take some C^∞ function of $f\colon \mathbb{R} \to \mathbb{R}$ which isn't analytic in any point and consider the Levi-flat hypersurface Y of $\mathbb{C}^2$ given by $y_2 = f(x_2)$. Its leaves are the complex lines whose equation is $z_2 = x_2 + i f(x_2)$, for fixed x_2, and the foliation extends to the trivial foliation of $\mathbb{C}^2$ with leaves $z_2 = \text{const}$.

LEMMA 5.1. *Let Y be a Levi-flat submanifold of X and $(z, t) = (z^1, \dots, z^n, t^1, \dots, t^k)$ complex coordinates on some open set U which meets Y. These coordinates are admissible for an holomorphic foliation of U which extends the induced semi-holomorhic foliation of Y if and only if they satisfy the condition*

$$(5.1) \qquad \partial \Phi_j / \partial z^\alpha = 0, \quad on \quad U \cap Y, \ \forall j \leq k, \ \alpha \leq n.$$

Moreover each other coordinate system (z', t') having this property changes with the rule $z' = z'(z, t)$, $t' = t'(t)$.

PROOF. Suppose first (z, t) are admissible coordinates for such an holomorphic foliation on U. For each $(\tilde{z}. \tilde{t}) \in Y$, the map

$$z \longmapsto (\tilde{z} + z, \tilde{t}), \qquad z \in \mathbb{C}^n, \ |z| \ small$$

is an open submanifold of the leaf through $(\tilde{z}, \tilde{t})$ of the holomorphic foliation, hence, for $|z|$sufficently small it lies in Y. Thus $\Phi_j(\tilde{z} + z, \tilde{t}) = 0$, for $|z|$ small; and (5.1) follows directly.

Conversely, assume that (5.1) is fulfilled by the coordinate system (z, t) and consider the trivial k-codimensional holomorphic foliation on U induced by these coordinates. The leaves are the (connected components of the) sets $t = \text{const}$, hence their real tangent space is given by the vectors $u + \bar{u}$ where u is of the form $\sum_{1}^{n}{}_\alpha u^\alpha \dfrac{\partial}{\partial z^\alpha}$. Since $(u + \bar{u}) \Phi_j$ vanishes on $U \cap Y$ by (5.1), those vectors are tangent to Y. Thus, by remark 3.1, $U \cap Y$ has to be a subfoliation of U.

Finally, since $\partial \Phi_j / \partial z^\alpha = 0$ on $U \cap Y$, we have

$$(5.2) \qquad 0 = \frac{\partial \Phi_j}{\partial z^\alpha} = \sum_{1}^{k}{}_{h'} \frac{\partial t^{h'}}{\partial z^\alpha} \frac{\partial \Phi_j}{\partial t^{h'}}, \quad on \quad U \cap Y.$$

But

$$\partial \Phi_1 \wedge \dots \wedge \partial \Phi_k = \det(\partial \Phi_j / \partial t^{h'}) \, dt^{1'} \wedge \dots \wedge dt^{k'},$$

hence by almost Levi-flatness condition (2.3), det $(\partial \Phi_j / \partial t^{h'})$ can't vanish on $U \cap Y$ so that (5.2) implies $\partial t^{h'} / \partial z_\alpha = 0$ on $U \cap Y$. Thus, by lemma 2.2, the holomorphic functions $\partial t^{h'} / \partial z^\alpha$ vanish identically. Q. E. D.

Observe now that if the submanifold Y has a covering of open sets U of X such that $U \cap Y$ is a subfoliation of U, then Y is a subfoliation of X. From this remark and the lemma we get immediately the following

COROLLARY 5.1. *The induced semiholomorphic foliation on the Levi-flat submanifold* $Y \equiv \{\Phi_1 = \dots = \Phi_k = 0\}$ *(with* $d\Phi_1 \wedge \dots \wedge d\Phi_k \neq 0$ *on* Y*) can be holomorhically extended to some neighbourhood of* Y*, if and only if, at each point of* Y *there are complex coordinates* $(z^1, \dots z_n, t^1, \dots, t^k)$ *such that* $\partial\Phi_j/\partial z^\alpha$ *vanish on* Y, $\forall j \leq k$, $\alpha \leq n$. *Such* (z, t)*'s are the admissible coordinates of the holomorphic foliation.*

PROPOSITION. 5.1. *Let* $Y \equiv \{\Phi_1 = \dots = \Phi_k = 0\}$ *be any real almost Levi-flat submanifold of the complex manifold* X. Y *is Levi-flat and its induced semiholomorphic foliation can be extended to an holomorphic foliation of some neighbourhood if and only if each point* $y \in Y$ *has some neighbourhood* U *where there are defined holomorphic functions* $h_1, \dots, h_k$ *satisfyng the following conditions*

$$\text{(i)} \qquad \partial h_j \wedge \partial\Phi_1 \wedge \dots \wedge \partial\Phi_k = 0, \qquad (j = 1, \dots, k)$$

$$\text{(ii)} \qquad \partial h_1 \wedge \dots \wedge \partial h_k \neq 0$$

at each point of $Y \cap U$.

PROOF. Suppose that Y is a subfoliation of some holomorphically foliated open subset V of X. For each leaf $Z \subset Y$ we have $\dim_{\mathbb{C}} Z = \dim_{C.R} Y = \dim_{\mathbb{C}} X - \mathrm{cod}_{\mathbb{R}} Y$. Thus the foliation V has complex codimension k. Set $n = \dim_{\mathbb{C}} Z$ and take admissible holomorphic coordinates $(z^1, \dots, z^n, t^1, \dots, t^k)$ in a neighbourhood U of an arbitrary point $y \in U$. We can now set $h_j = t^j$ and observe that (ii) is satisfied.

By lemma 5.1. we have $\partial\Phi_1 \wedge \dots \wedge \partial\Phi_k = a\, \partial t^1 \wedge \dots \wedge \partial t^k$ with $a \in \mathbb{C}$. Hence (i) is also satisfied. Conversely, if (ii) is fulfilled, we can take complex coordinates $(z^1, \dots, z^n, t^1, \dots, t^k)$ on a neighbourhood U of each $y \in Y$, such that $h_j = t^j$. Now, by the Rouché theorem, conditions (i), (ii) and almost Levi-fiatness (2.3) imply that $\partial h_j (= \partial t^j)$ and $\partial\Phi_j$ are basis of the same complex vector space in each point of $U \cap Y$. Hence $\partial\Phi_j/\partial z^\alpha$ vanishes on $U \cap Y$ for each $j \leq k$, and $\alpha \leq n$; thus the proposition follows from corollary 5.1

PROOF OF THEOREM 5.1. Let U and U' be admissible coordinate patches for two holomorphic extensions $\mathcal{F}$ and $\mathcal{F}'$ of the induced semiholomorphic foliation of Y with $U \cap U' \cap Y \neq \emptyset$. Both coordinate systems satisfy (5.1), thus, by lemma 5.1, they change with the law (3.1), hence they induce the same foliation on $U \cap U'$. By remark 3.2 $\mathcal{F}$ coincides with $\mathcal{F}'$ and the unicity is proved. We prove now the existence using prop. 5.1. Suppose that Y is C^ω and take the C^ω functions $f_1, \dots, f_k$ given by prop. 4.3. (iii), on some neighbourhood of an arbitrary point y of Y and their restrictions $\varphi_1, \dots, \varphi_k$ to Y. From a result of Tomassini ([6]) (5) condition (4.3) implies that $\varphi_1, \dots, \varphi_k$ can be extended to holomorphic functions on some neighbourhood of y. Now we observe that condition (4.3) involves only the traces φ_j of our functions (i. e. is purely tangential) hence is fulfilled by the functions h_j too. So the functions h_j satisfy the condition (i) of prop. 5.1.

Finally, take the real and complex part ξ_j, η_j of h_j; η_j vanishes identically on Y, so applyng (4.5) to the single function η_j we get $d\eta_j \wedge d\Phi_1 \wedge \dots \wedge d\Phi_k = 0$ on Y. Hence

$$\partial h_j \wedge d\Phi_1 \wedge \dots \wedge d\Phi_k = dh_j \wedge d\Phi_1 \wedge \dots \wedge d\Phi_k = d\xi_j \wedge d\Phi_1 \wedge \dots \wedge d\Phi_k \text{ on } Y.$$

Thus

$$(5.3) \qquad \partial h_1 \wedge \dots \wedge \partial h_k \wedge d\Phi_1 \wedge \dots \wedge d\Phi_k = d\xi_j \wedge \dots \wedge d\xi_k \wedge d\Phi_1 \wedge \dots \wedge d\Phi_k, \ \textit{on } Y.$$

Now we recall that f_j and h_j have the same trace φ_j on Y, from (4.4) and (4.5) we obtain $d^Y \varphi_1 \wedge \dots \wedge d^Y \varphi_k \neq 0$ on Y. We can apply again (4.5) to the functions $\xi_1, \dots, \xi_k$ and obtain that the right hand side of (5.3) cant't vanish on Y. This implies that $h_1, \dots, h_k$ verify the condition (ii) of prop. 5.1. Thus the existence is proved too. Q. E. D.

§ 6. Counterexample.

We have to construct a Levi-flat submanifold Y whiah doesn't have the extension property. Our Y will be a surface $\Phi = 0$ in some open set $\Omega \subset \mathbb{C}^2$, but the same construction can give a submanifold of any codimension in $\mathbb{C}^n$. Proposition 5.1 shows that is sufficient to find Φ, with $d\Phi \neq 0$ on $Y \equiv \{\Phi = 0\}$, and some $p \in Y$ in such a way that

(6.1) *h holomorphic on any neighbourhood of p, $\partial \Phi \wedge \partial h = 0 \implies h$ constant.*

Take the disk $D \equiv \{z_1 \in \mathbb{C}, |z_1| < 1\}$ and the interval $I \equiv \{t \in \mathbb{R}, |t| < 1\}$ and consider the map

$$\psi : D \times I \ni (z_1, t) \longmapsto (z_1, a(t) z_1 + t) \in \mathbb{C}^2$$

where $a \in C^\infty(\mathbb{R})$ is a real valued function with $|a'| < \frac{1}{2}$ for $|t| < 1$, so $a'(t) x_1 + 1 \neq 0$, $\forall (z_1, t) \in D \times I$.

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(5) For hypersurfaces see Severi [5].

The first three rows of the jacobian matrix of ψ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a' x_1 + 1 \end{pmatrix}$$

hence ψ is of maximal rank.

Moreover

$$\psi(z_1, t) = \psi(z_1', t') \Longleftrightarrow z_1 = z_1', \; a(t) z_1 + t = a(t') z_1 + t'.$$

Hence, if (z_1, t) and (z_1', t') were distinct, by Rolle's theorem we would get $a'(\tau) z_1 + 1 = 0$ for suitable $|\tau| < 1$. This is impossible because $|a'(\tau)| < 1/2$ and $|z_1| < 1$. Therefore ψ must be injective.

Thus there exist a C^∞ function $\Phi ; \mathbb{C}^2 \to \mathbb{R}$ and an open set $\Omega \subset \mathbb{C}^2$ such that $Y \overset{\text{def}}{=} \psi(D \times I) = \{\Phi = 0\} \cap \Omega$, and $d\Phi \neq 0$ on Y.

For each $\psi(z_1, t) \in Y$, the complex line

$$\{\zeta \in \mathbb{C}, |\zeta + z_1| < 1\} \ni \zeta \mapsto [z_1 + \zeta, a(t)(z_1 + \zeta) + t]$$

lies on Y, hence Y is Levi-flat, by definition. We choose now the function a in such a way that its zero set is the point $t = 0$ and a sequence $t_n \to 0$ composed by infinitely many distinct points (6)

We have

$$\Phi_{z_1}[z_1, a(t) z_1 + t] + a(t) \Phi_{z_2}[z_1, a(t) z_1 + t] \equiv 0. \tag{6.2}$$

Suppose now h holomorhic on some neighbourhood of the origin and $\partial h \wedge \partial \Phi = 0$ on Y, so $h_{z_1} \Phi_{z_2} - h_{z_2} \Phi_{z_1} = 0$ on Y; from (6.2) we obtain

$$\Phi_{z_2} \cdot (a h_{z_2} + h_{z_1}) = 0 \quad on \quad Y. \tag{6.3}$$

But $\Phi_{z_1}(z_1, t_n) = 0$ by (6.2) and since $d\Phi \neq 0$, we have $\Phi_{z_2}(z_1, t_n) \neq 0$. Thus (6.3) gives $h_{z_1}(z_1, t_n) = 0$ for each $z_1 \in D$ and $n \in \mathbb{N}$.

We fix now $z_1 \in D$ arbitrarely and take the function of one variable $\gamma(\zeta) = h_{z_1}(z_1, \zeta)$ holomorphic on some neighbourhood of $\zeta = 0$. Since $\gamma(t_n) = 0$ and $t_n \to 0$ has infinite distinct points, $\gamma(\zeta) \equiv 0$. Hence $h_{z_1} \equiv 0$. Now

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(6) For instance, take $t_n = 1/n$ and divide the function $\alpha(t) = e^{-\frac{1}{t^2}} \sin(\pi/t)$, for $t \leq 0$; $\alpha(t) = 0$, for $t = 0$, by the positive number $2 \max_{|t| \leq 1} |\alpha'(t)|$.

the set

$$A \equiv \{(z_1, a(t) z_1 + t] \in \mathbb{C}^2, t \neq t_n, t \neq 0\}$$

is an open (dense) subset of Y. Since we have seen that $\Phi_{z_2} | (Y - A) \neq 0$, there exists an open subset B of A such that $\Phi_{z_2} \neq 0$ at each point of B; B is a real hypersurface of $\mathbb{C}^2$. At each point $[z_1, a(t) z_1 + t]$ of B, we have by (6.3)

$$a(t) h_{z_2}[z_1, a(t) z_1 + t] = 0.$$

Since $t \neq t_n$, $t \neq 0$, $a(t)$ is different from 0 and h_{z_2} vanishes identically on the real hypersurface B and hence everywhere. Thus $h_{z_1} \equiv h_{z_2} \equiv 0$ and h must be constant. So (6.1) is proved.

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