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# A REMARK ON HARMONIC ANALYSIS OF STRONGLY ALMOST-PERIODIC GROUPS OF LINEAR OPERATORS

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## 1. Introduction.

Let us consider a Banach space  $\mathcal{X}$ , and then take a one parameter group of linear operators  $G(t)$ ,  $-\infty < t < \infty \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{X})$ , which is strongly almost-periodic; this means that for any  $x \in \mathcal{X}$ , the  $\mathcal{X}$ -valued function  $y(t) = G(t)x$  is (Bochner)-almost-periodic (see [2]).

It is a well-known result (see for example [1]), that for any  $\mathcal{X}$ -valued almost-periodic function  $f(t)$ , the mean value

$$(1.1) \quad \mathcal{M}(e^{-i\lambda t} f(t)) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{-i\lambda \sigma} f(\sigma) d\sigma$$

exists for any real number  $\lambda$ .

Furthermore, this mean-value equals  $\theta$  for all  $\lambda$  with the possible exception of a set  $(\lambda_n)_{n=1}^{\infty}$  which is finite or countable, and is denoted by  $\sigma(f)$ .

A natural problem is the following <sup>(2)</sup>:

Is there any strongly almost-periodic one-parameter group  $G(t)$ , with the property that

$$(1.2) \quad \bigcup_{x \in \mathcal{X}} \sigma(G(t)x) = \text{real line?}$$

Answering to a letter of us, professor S. Bochner indicated a solution; this will be explained here with some more details.

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<sup>(2)</sup> It arose in connection with our paper [3].

2. We shall remember here the definition of the space  $l^2[-\infty, \infty]$ .

It consists of all complex-valued functions  $a(\lambda)$ , defined for  $-\infty < \lambda < \infty$ , having the property that

$$(2.1) \quad \sum_{-\infty < \lambda < \infty} |a(\lambda)|^2 < \infty$$

In fact, (2.1) means, by definition, that for a certain constant  $c > 0$  we have

$$(2.2) \quad \sum_{i=1}^n |a(\lambda_i)|^2 < c$$

whenever arbitrary real  $\lambda_j$  are chosen (and for any  $n = 1, 2, 3, \dots$ ).

Let us remark that if  $a(\lambda) \in l^2[-\infty, \infty]$  there exists a sequence  $(\lambda_n)_{n=1}^{\infty}$  depending on  $a(\lambda)$ , such that  $a(\lambda) = 0$  if  $\lambda \neq \lambda_j$ ,  $\forall j = 1, 2, \dots$

This follows because, if we put  $\mathcal{C}_j = \left\{ \lambda \in \mathbb{R}^1, |a(\lambda)| > \frac{1}{j} \right\}$  we see that every  $\mathcal{C}_j$  is a finite set; hence  $\bigcup_{j=1}^{\infty} \mathcal{C}_j = \mathcal{C}$  is a countable set, and if  $a(\lambda) \neq 0$  then  $\lambda \in \mathcal{C}$ .

Let us denote the set  $\{\lambda; a(\lambda) \neq 0\}$  by  $Sp a(\cdot)$ ; so  $Sp a(\cdot) \subset \mathcal{C}$  is a finite or countable set,  $(\lambda_n)_{n=1}^{\infty}$ , and we take a fixed ordering of it.

It can be proved that  $l^2[-\infty, \infty]$  is a linear space on the complex field. We can introduce a scalar product on this space; if  $a(\lambda), b(\lambda) \in l^2[-\infty, \infty]$ , and  $(\lambda_n)_{n=1}^{\infty} = Sp a(\cdot), (\mu_n)_{n=1}^{\infty} = Sp b(\cdot)$ , then by definition  $(a(\lambda), b(\lambda))_{l^2[-\infty, \infty]} = \sum_{j=1}^{\infty} a(\lambda_j) \bar{b}(\mu_j)$ ; this sum becomes finite if one of  $Sp a(\cdot)$  or  $Sp b(\cdot)$  is finite.

It can be proved in the usual manner that  $l^2[-\infty, \infty]$  is a (complete) Hilbert space.

Let us consider now, for any real number  $t$ , the map of  $l^2[-\infty, \infty]$  into itself which is defined by

$$(2.3) \quad a(\lambda) \rightarrow e^{it\lambda} a(\lambda)$$

We shall denote this map by  $G_t$ ; we see that  $G_{t_1+t_2} = G_{t_1} G_{t_2}, G_0 = I$  for any pair  $t_1, t_2$  of real numbers; here  $I$  is the identity operator in  $l^2$ .

Furthermore, if  $(\lambda_n)_{n=1}^{\infty} = Sp a(\cdot)$ , we have

$$(2.4) \quad \|G_t a(\cdot)\|_{l^2}^2 = \sum_{j=1}^{\infty} |e^{it\lambda_n} a(\lambda_n)|^2 = \sum_{j=1}^{\infty} |a(\lambda_n)|^2 = \|a(\cdot)\|_{l^2}^2$$

so  $G_t$  is an isometric map of  $l^2$ ,  $\forall$  real  $t$ .

3. In this part of the paper we prove the following

**THEOREM.** *The one-parameter group  $G_t$  is strongly almost periodic in  $l^2$*

We need for the proof several Lemmas.

Consider, for a given  $\lambda_0 \in (-\infty, \infty)$ , the function  $\varphi_{\lambda_0}(\lambda)$  which equals 1 for  $\lambda = \lambda_0$ , and equals 0 for  $\lambda \neq \lambda_0$ . Obviously  $\varphi_{\lambda_0}(\lambda) \in l^2$ , and  $Sp \varphi(\cdot) = \{\lambda_0\}$ . Now, we have

**LEMMA 1.** *Let  $a(\lambda)$  be given in  $l^2$ , and  $(\lambda_n)_1^\infty = Sp a(\cdot)$ . Then we have*

$$(3.1) \quad a(\lambda) = \sum_{j=1}^{\infty} a(\lambda_j) \varphi_{\lambda_j}(\lambda), \text{ the convergence being in } l^2[-\infty, \infty]$$

Let us put in fact  $b_N(\lambda) = a(\lambda) - \sum_{n=1}^N a(\lambda_n) \varphi_{\lambda_n}(\lambda)$ .

It can be seen without difficulty that  $Sp b_N(\cdot) = (\lambda_{N+1}, \lambda_{N+2}, \dots)$ . Hence

$$\|b_N(\lambda)\|_{l^2}^2 = \sum_{j=1}^{\infty} |b_N(\lambda_{N+j})|^2; \text{ but } b_N(\lambda_{N+j}) = a(\lambda_{N+j}); \text{ hence}$$

$$\sum_{j=1}^{\infty} |b_N(\lambda_{N+j})|^2 = \sum_{j=1}^{\infty} |a(\lambda_{N+j})|^2 = \sum_{k=N+1}^{\infty} |a(\lambda_k)|^2$$

This last expression tends to 0 as  $N \rightarrow \infty$ , because  $a(\lambda) \in l^2$ . This proves Lemma.

Then we remark the trivial fact that

$$(3.2) \quad Sp(e^{it\lambda} a(\lambda)) = Sp(a(\lambda)) \text{ for any real } t.$$

Applying Lemma 1 we obtain that for any real  $t$  we have

$$(3.3) \quad e^{it\lambda} a(\lambda) = \sum_{n=1}^{\infty} e^{it\lambda_n} a(\lambda_n) \varphi_{\lambda_n}(\lambda)$$

the convergence being in  $l^2[-\infty, \infty]$ .

Also we have the simple

**LEMMA 2.** *Any function  $-\infty < t < \infty \rightarrow l^2[-\infty, \infty]$  which is given by*

$$(3.4) \quad h_n(t) = e^{it\lambda_n} a(\lambda_n) \varphi_{\lambda_n}(\cdot) \text{ is } l^2\text{-almost-periodic.}$$

This is a particular case of the fact that if  $\mathcal{X}$  is a Banach space,  $x \in \mathcal{X}$ , and  $\alpha$  is a real number, the function  $-\infty < t < \infty \rightarrow \mathcal{X}$ , given by  $e^{iat} x$

is  $\mathcal{X}$  almost-periodic (in fact it is  $\mathcal{X}$ -periodic). In our case  $\alpha = \lambda_n$ ,  $x = a(\lambda_n) \varphi_{\lambda_n}(\lambda)$ ,  $\mathcal{X} = l^2$ .

It follows from (3.3) that  $e^{it\lambda} a(\lambda)$  is  $l^2$ -almost-periodic, if we prove that the convergence in (3.3) is uniform with respect to  $t \in (-\infty, \infty)$ . This is done in

LEMMA 3. *The series  $\sum_{n=1}^{\infty} e^{it\lambda_n} a(\lambda_n) \varphi_{\lambda_n}(\lambda)$  is convergent to  $e^{it\lambda} a(\lambda)$  in  $l^2$ -norm, uniformly for  $-\infty < t < \infty$ .*

Let us consider in fact the difference

$$(3.5) \quad g_N(\lambda, t) = e^{it\lambda} a(\lambda) - \sum_{n=1}^N e^{it\lambda_n} a(\lambda_n) \varphi_{\lambda_n}(\lambda).$$

We see that

$$g_N(\lambda, t) = 0 \text{ if } \lambda \notin Sp a(\cdot) \text{ or if } \lambda \in [\lambda_1, \lambda_2, \dots, \lambda_N];$$

moreover

$$g_N(\lambda, t) = e^{it\lambda_{N+j}} a(\lambda_{N+j}) \text{ for } \lambda \in [\lambda_{N+1}, \lambda_{N+2}, \dots].$$

Consequently  $Sp g_N(\cdot, t) = (\lambda_{N+1}, \lambda_{N+2}, \dots)$  and

$$(3.6) \quad \|g_N(\lambda, t)\|_{l^2}^2 = \sum_{j=1}^{\infty} |e^{it\lambda_{N+j}} a(\lambda_{N+j})|^2 = \sum_{k=N+1}^{\infty} |a(\lambda_k)|^2$$

which tends to 0 as  $N \rightarrow \infty$ , obviously uniformly with respect to  $t \in (-\infty, \infty)$ . This proves the Theorem.

Let us consider  $a(\lambda) = \varphi_{\lambda_0}(\lambda)$  for any fixed  $\lambda_0 \in (-\infty, \infty)$ .

Then

$$G_t \varphi_{\lambda_0}(\lambda) = e^{it\lambda} \varphi_{\lambda_0}(\lambda) = e^{i\lambda_0 t} \varphi_{\lambda_0}(\lambda) \text{ as easily seen.}$$

This is a  $l^2$ -valued periodic function and  $\sigma(e^{i\lambda_0 t} \varphi_{\lambda_0}(\cdot)) = \{\lambda_0\}$ . Hence  $\bigcup_{\lambda_0 \in \mathbb{R}^1} \sigma(G_t \varphi_{\lambda_0}(\cdot)) = \text{real line } \mathbb{R}^1$  and this solves the problem in the Introduction.

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