Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

Oldřich Kowalski

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 26, nº 1 (1972), p. 233-245

<http://www.numdam.org/item?id=ASNSP_1972_3_26_1_233_0>

PROPERTIES OF HYPERSURFACES WHICH ARE CHARACTERISTIC FOR SPACES OF CONSTANT CURVATURE

OLDŘICH KOWALSKI, Praha

If is well-known that the spaces of constant curvature are characterized among all Riemannian spaces by the following property: *for any* $(n-1)$-*dimensional linear element* E_{n-1} *of a Riemann space* N ($\dim N = n \geq 3$) *there is a totally geodesic hypersurface* $M \subset N$ *which is tangent to* E_{n-1}. (Cf. [1]). The purpose of this Note is to present a number of theorems of the above type; only the requirement that our hypersurfaces should be totally geodesic will be replaced be another geometrical or analytical postulates. Umbilical points of hypersurfaces and so called « normal Bianchi identity » play the leading part here.

Throughout the paper we shall keep all the notations and conventions of the famous book by Kobayashi and Nomizu ([2], [3]).

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Let N be a Riemannian manifold of dimension $n \geq 3$, g the corresponding Riemann metric on N and $M \subset N$ a hypersurface. Because all the postulates put on hypersurfaces will be purely local, we can suppose (if not otherwise stated) that M is a « small » hypersurface, diffeomorphic with an open region in $\mathbb{R}^{n-1}$.

Denote by ∇, ∇' the covariant differentation on N, M respectively. Let ξ be a field of unit normal vectors and X, Y tangent vector fields on M. *The formulas of Gauss and Weingarten* are given by

$$\nabla_X Y = \nabla'_X Y + h(X, Y)\,\xi, \tag{1}$$

$$\nabla_X \xi = -AX. \tag{2}$$

Pervenuto alla Redazione il 13 Febbraio 1971.

Here $h(X, Y)$ is *the second fundamental form* and A is a symmetric transformation on each tangent space $T_x(M)$, $x \in M$. Moreover, we have $g(AX, Y) = g(X, AY) = h(X, Y)$.

A point $x \in M$ is called *an umbilical point* if $A = \lambda \cdot I$ on $T_x(M)$, where λ is a scalar and I is an identity transformation.

Definition 1. A hypersurface $M \subset N$ is called *a U-sphere* if $A = \lambda \cdot I$ on the tangent bundle $T(M)$, where λ is a constant, $\lambda \neq 0$.

Denote by R, R' the Riemann curvature tensors of N, M respectively. At any point $x \in M$ we have

$$R(X, Y)\xi = (\nabla'_Y A)(X) - (\nabla'_X A)(Y), \tag{3}$$

$$\operatorname{proj}_{T(M)} R(X, Y) Z = R'(X, Y) Z + g(AX, Z) AY - g(AX, Z) AX, \tag{4}$$

where $X, Y, Z \in T_x(M)$ and ξ is a normal vector to M at x. (Cf. [3]).

We introduce new tensors $\Delta(X, Y)$, $B(X, Y)Z$ on M by

$$\Delta(X, Y) = (\nabla'_Y A)(X) - (\Delta'_X A)(Y), \tag{5}$$

$$B(X, Y) Z = g(AX, Z) AY - g(AY, Z) AX. \tag{6}$$

Let us remind *the first Bianchi identity* $\sigma(R'(X, Y)Z) = 0$ and *the second Bianchi identity* $\sigma((\nabla'_X R')(Y, Z)) = 0$, where X, Y, Z are tangent vector fields on M and σ denotes the cyclic sum with respect to X, Y, Z.

The tensor $B(X, Y)Z$ defined by (6) also satisfies the first Bianchi identity but not the second one, in general.

Definition 2. We say that a hypersurface $M \subset N$ satisfies *the normal Bianchi identity* if $\sigma((\nabla'_X B)(Y, Z)) = 0$ for any vector fields X, Y, Z on M.

Remark. This definition is independent of the choice of a normal unit vector field ξ.

A routine calculation leads to the following

Proposition 1. *The normal Bianchi identity holds on a hypersurface* $M \subset N$ *if and only if*

$$\Delta(X, Y) \wedge AZ + \Delta(Y, Z) \wedge AX + \Delta(Z, X) \wedge AY = 0 \tag{7}$$

in the vector bundle $\Lambda^2 T(M)$.

If N is a Riemann manifold with the constant curvature C, then

$$R(X, Y)Z = C\{g(Y, Z)X - g(X, Z)Y\}. \tag{8}$$

Hence and taking into account (3) - (6) we obtain

$$\Delta(X, Y) = (\nabla'_X A)(Y) - (\nabla'_Y)(X) = 0 \quad (\textit{Equation of Codazzi}), \tag{9}$$

$$B(X, Y)Z = C\{g(Y, Z)X - g(X, Z)Y\} - R'(X, Y)Z \tag{10}$$

(*Equation of Gauss*).

Finally, (7) and (9) imply

PROPOSITION 2. *If N is a space of constant curvature then any hypersurface $M \subset N$ satisfies the Codazzi equation $\Delta(X, Y) = 0$ and also the normal Bianchi identity.*

In order to prove a converse we shall remind some well-known definitions.

For each plane p in the tangent space $T_x(N)$, i. e., for any 2-dimensional subspace of $T_x(N)$, *the sectional curvature* $K(p)$ is defined by $K(p) = R(e_1, e_2, e_1, e_2) = g(R(e_1, e_2)e_2, e_1)$, where $\{e_1, e_2\}$ is an orthonormal basis of p. (In the following we shall put $K(e_1, e_2) = R(e_1, e_2, e_1, e_2)$ for abbreviation).

A point $x \in N$ is called *isotropic* if the sectional curvature $K(p)$ is the same for any plane $p \subset T_x(N)$.

Now we shall present a number of lemmas the statements of which are well-known.

LEMMA 1. *Suppose that there is an orthonormal basis $\{e_1, \dots, e_n\}$ in $T_x(N)$ and a constant C such that $K(e_i, e_j) = C$ for any $i, j = 1, \dots, n$, $i \neq j$. Then x is an isotropic point of N.*

LEMMA 2. *Suppose that $g(R(e_i, e_j)e_k, e_j) = 0$ for any orthonormal triplet $\{e_i, e_j, e_k\}$ of vectors of $T_x(N)$. Then x is an isotropic point of N.*

PROOF. Consider an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(N)$, and for any triplet of indices $\{i, j, k\} \subset \{1, \dots, n\}$ put $f_i = \dfrac{e_i + e_k}{\sqrt{2}}$ $f_j = e_j$, $f_k = \dfrac{e_i - e_k}{\sqrt{2}}$. The equality $g(R(f_i, f_j)f_k, f_j) = 0$ implies $K(e_i, e_j) = K(e_k, e_j)$ and we can use Lemma 1.

LEMMA 3. (Schur's lemma). *Let N be a Riemannian manifold of dimension $n \geq 3$. If all points of N are isotropic, then N is a space of costant curvature.* (Cf [2]).
Now we can derive

PROPOSITION 3. *Let N be a Riemannian manifold of dimension $n \geq 3$ with the following property: to any point $x \in N$ and any hyperplane $E_{n-1} \subset \subset T_x(N)$ there is a hypersurface $M \subset N$ such that*

a) $M \ni x$, $T_x(M) = E_{n-1}$, b) M satisfies the Codazzi equation at x. Then N is a space of constant curvature.

PROOF. Let a point $x \in N$ and an orthonormal triplet $\{e_i, e_j, e_k\}$ of vectors of $T_x(N)$ be given. Denote $E_{n-1} = \{X \in T_x(N) \mid g(X, e_k) = 0\}$. Let $M \subset N$ be a hypersurface satisfying the conditions a), b) of the Proposition with respect to E_{n-1}. According to (3) we obtain $R(e_i, e_j)\, e_k = 0$. Now we apply Lemma 3 to complete the proof.
Propositions 2 and 3 give us

THEOREM 1. *Let N be a Riemannian manifold of dimension $n \geq 3$. Then the following two statements are equivalent:*

(i) *Any hypersurface $M \subset N$ satisfies the Codazzi equation $\Delta(X, Y) = 0$.*

(ii) *N is a space of constant curvature.*

We are in a position to prove also the following

THEOREM 2. *Let N be a Riemannian manifold of dimension $n \geq 4$. Then the following two statements are equivalent:*

(i) *Any hypersurface $M \subset N$ satisfies the normal Bianchi identity.*

(ii) *N is a space of constant curvature.*

PROOF. The implication (ii) $\Longrightarrow$ (i) was stated in Proposition 2. Let us prove the converse. Let $x \in N$ be a fixed point and consider *a system of normal coordinates* $\{x^1, \dots, x^n\}$ in a neighbourhood V_x of x. In this way the neighbourhood V_x of x can be represented as an open region U_x of a coordinate space $\mathbb{R}^n(x^1, \dots, x^n)$; the point x is mapped onto the origin $0 \in \mathbb{R}^n$. The space $\mathbb{R}^n$ provided with its canonical euclidean metric is called *the osculating euclidean space of N at the point x.* (Cf. [1]). We have a canonical isomorphism of $T_x(N)$ onto $T_0(\mathbb{R}^n)$. Suppose that $\{e_i, e_j, e_k\}$ is an orthonormal triplet of vectors of $T_x(N)$ and $\{e'_i, e'_j, e'_k\}$ the corresponding orthonormal triplet of $T_0(\mathbb{R}^n)$.

Let us construct a small piece M' of a cylinder $\mathbb{S}^1 \times \mathbb{R}^{n-2} \subset \mathbb{R}^n$ with the following properties:

a) M' passes through the origin $0 \in \mathbb{R}^n$ and $M' \subset U_x$,

b) M' is normal to e'_k at 0,

c) with respect to the canonical identification

$$T_0(M') = T_0(\mathbb{S}^1) \dot{+} T_0(\mathbb{R}^{n-2}) \qquad \text{(orthogonal decomposition !)}$$

we have $e'_i, e'_j \in T_0(\mathbb{R}^{n-2})$.

Let Z be a vector of $T_x(N)$ corresponding to a vector $Z' \in T_0(\mathbb{S}^1)$.

If M is a hypersurface of N corresponding to M', then it is well-known that the second fundamental form of $M \subset N$ at x is the same as the second fundamental form of $M' \subset \mathbb{R}^n$ at 0. Hence $Ae_i = Ae_j = 0$, $AZ = \lambda \cdot Z$. Formula (7) implies $\Delta(e_i, e_j) = \mu \cdot Z$ and from (3) we obtain $R(e_i, e_j)\,e_k = \mu \cdot Z$. Hence $g(R(e_i, e_j)\,e_k, e_j) = 0$ and x is an isotropic point of N according to Lemma 2. Now the Schur's lemma completes our proof.

REMARK. If N is a Riemannian manifold of dimension 3, then the normal Bianchi identity is trivially satisfied on each surface $M \subset N$.

The rest of this paper is devoted mainly to the study umbilical points and U spheres. The following theorem of the linear algebra is well-known and it will be usefull for our further calculations:

LEMMA 4. *Let g, h be two quadratic forms in a real vector space $\mathbb{R}^n$ and let g be positively definite. Then there is a basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ such that $g = \sum_{i=1}^{n} (\xi^i)^2$, $h = \sum_{i=1}^{n} \lambda_i (\xi^i)^2$ with respect to the dual basis $\{\xi^1, \dots, \xi^n\}$* of $(\mathbb{R}\ )^*$.

CONVENTION: $N(C)$ will denote a Riemannian manifold with the constant curvature C.

PROPOSITION 4. *Let $N(C)$ be a space of dimension $n \geq 4$ and $M \subset N(C)$ a hypersurface. Then the following two statements are equivalent:*

(i) *M is a U-sphere.*

(ii) *M is a space $M(C')$, where $C' > C$.*

PROOF. Let us multiply (6) and (10) by the vector X to the right and put $Z = Y$. We obtain $g(AX, Y)^2 - g(AX, X)\,g(AY, Y) = C\{g(X, X)\,g(Y, Y) - g(X, Y)^2\} - g(R'(X, Y)\,Y, X)$, where $g(AX, Y) = h(X, Y)$ is the second fundamental form of the hypersurface $M \subset N$. Denote by K_M the sectional curvature of M. if X, Y are orthonormal vectors of $T_x(M)$, it follows

$$g(AX, Y)^2 - g(AX, X)\, g(AY, Y) = C - K_M(X, Y). \tag{11}$$

The proof of (i) $\Longrightarrow$ (ii): If M is a U-sphere, $A = \lambda . I$, $\lambda \neq 0$, we obtain $K_M(X, Y) = C + \lambda^2 =$ const., *q. e. d.*

The proof of (ii) $\Longrightarrow$ (i): Let us re-write (11) in the form $h(X, X) h(Y, Y) - h(X, Y)^2 = K_M(X, Y) - C$. If (ii) is satisfied, we obtain

$$h(X, X)\, h(Y, Y) - h(X, Y)^2 = C' - C > 0 \tag{12}$$

for any orthonormal pair $\{X, Y\}$ of $T_x(M)$.

Let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis of $T_x(M)$ such that $h(X, Y)$ assumes a diagonal form with respect to this basis (Lemma 4). Put $h(e_i, e_i) = \lambda_i$ for $i = 1, \dots, n-1$; then (12) implies $\lambda_i \lambda_j = C' - C > 0$ for $i, j = 1, \dots, n-1 \geq 3$ as required, we obtain $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = \pm \sqrt{C' - C}$ and $h = \pm \sqrt{C' - C} \cdot g$ on M. Hence follows $A = \pm \sqrt{C' - C} \cdot I$ on M; the sign depends on the orientation of M.

REMARK. If dim $N(C) = 3$, the implication (ii) $\Longrightarrow$ (i) is false as the following example shows: let M be a small piece of a sphere S^2 in the euclidean space E^3; consider a non-trivial isometric deformation of M in E^3. The deformed surface is a space of constant curvature but not a U-sphere.

Let N be a Riemannian manifold. To any point $x \in N$ and any sufficiently small number $r > 0$ the locus of all points at the distance r from x is a regular submanifold $S(x; r)$ of dimension $n - 1$, called *a metric sphere with center x and radius r.*

PROPOSITION 5. *Let $N(C)$ be a space of dimension $n \geq 3$. Then any sufficiently small metric sphere in $N(C)$ is a space form $N(C')$ with $C' > C$.*

PROOF. It suffices to cite some classical results only (cf. [5]).
For $C = 0$ the assertion is trivial.
For $C > 0$ (*elliptic* geometry) we put $k = 1/\sqrt{C}$. The following result it well-known: *The metric of the space $N(1/k^2)$ induces on a sphere of radius r the metric of a euclidean sphere of radius $k \cdot \sin(r/k)$.*

For $C < 0$ (hyperbolic geometry) we put $k = 1/\sqrt{-C}$. Let us remember the following theorem: *The metric of the space $N(-1/k^2)$ induces on a sphere of radius r the metric of a euclidean spere of radius $k \cdot \mathrm{Sh}(r/k)$.*
Hence our Proposition follows.

PROPOSITION 6. *Let $N(C)$ be a space of dimension $n \geq 3$. Then any sufficiently small metric sphere in $N(C)$ is a U-sphere.*

PROOF. If the dimension $n \geq 4$, the result follows from Propositions 4 and 5. For $n = 3$, we can identify $N(C)$ with a totally geodesic hypersurface of a space form $N'(C)$ of dimension 4. Any metric sphere in $N(C)$ can be represented as an intersection of a metric sphere in $N'(C)$ by $N(C)$. Hence our result follows.

REMARK. The converse of Proposition 6 is true in the euclidean and the elliptic case only. As for a hyperbolic space $N(-1/k^2)$, any U-sphere in N is *locally* either *a metric sphere* (for $|\lambda| > 1/k$), or *a limit hypersurface* bearing a euclidean metric (for $|\lambda| = 1/k$), or finally *an equidistant hypersurface* (for $0 < |\lambda| < 1/k$).

PROPOSITION 7. *Let N be a Riemannian space of dimension $n \geq 3$. Suppose that to any linear $(n-1)$-dimensional element E_{n-1} of N there is a U-sphere tangent to E_{n-1}. Then N is a space of constant curvature.*

PROOF. Let M be a U-spere tangent to E_{n-1}. We have $A = \lambda \cdot I$ along M, $\lambda = \text{const.}$ Hence we see that the Codazzi equation $(\nabla'_Y A)(X) - (\nabla'_X A)(Y) = 0$ holds at the base point x of E_{n-1}. Now we can apply Proposition 3.

PROPOSITION 8. *Let N be a Riemannian space of dimension n. Then to any linear $(n-1)$-dimensional element E_{n-1} of N there is a metric sphere M tangent to E_{n-1}.*

PROOF. Let x_0 be the base point of E_{n-1}. According to [2], Theorem 8.7. there is a spherical normal coordinate neighbourhood $U(x_0; \varrho)$ such that each point of $U(x_0; \varrho)$ has a normal coordinate neighbourhood containing $U(x_0; \varrho)$. Let γ be a geodesic emanating from x_0 and orthogonal to E_{n-1} at x_0. Choose a point y on γ at the distance $d < \frac{\varrho}{2}$ from x_0. Then the metric sphere $S(y; d)$ satisfies our Proposition.

THEOREM 3. *Let N be a Riemannian manifold of dimension $n \geq 3$. Then the following statements* (i) - (iii) *are equivalent*:

(i) *To any $(n-1)$-dimensional linear element E_{n-1} of N there is a U-sphere tangent to E_{n-1}.*

(ii) *Any sufficiently small metric sphere of N is a U-sphere.*

(iii) *N is a space of constant curvature.*

PROOF: (iii) $\Longrightarrow$ (ii) - Proposition 6,
(i) $\Longrightarrow$ (iii) - Proposition 7,
(ii) $\Longrightarrow$ (i) - Proposition 8.

Now, let us remind some further definitions. *The Ricci tensor field* is a covariant tensor field of degree 2 on N defined as follows: for any vectors $X, Y \in T_x(N)$ and any orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(N)$ we put $S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X) Y, e_i)$. A Riemannian manifold N is called *an Einstein manifold if* $S = \lambda \cdot g$ on N, where λ is a constant.

REMARK. From a more general point of view: whenever we speak about an Einstein manifold, we mean an Einstein monifold the signature of which in zero (cf. [4]).

It is well-known that an Einstein manifold of dimension $n \leq 3$ is a space of constant curvature. We can characterize Einstein manifolds of dimension 4 as follows:

LEMMA 5. *A 4-dimensional Riemannian manifold N is Einsteinian if and only if the following property is satisfied: For any point $x \in N$ and any orthogonal decomposition of $T_x(N)$ into two planes p, p' we have $K(p) = K(p')$.*

PROOF. 1) Let N be Einsteinian, $S = \lambda \cdot g$, and let us have an orthogonal decomposition $T_x(N) = p \dot{+} p'$. Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that $p = (e_1, e_2), p' = (e_3, e_4)$. Then $S(e_i, e_i) = \lambda \cdot g(e_i, e_i) = \lambda$ for $i = 1, \dots, 4$; on the other hand $S(e_i, e_i) = \sum_{j=1}^{4} g(R(e_j, e_i) e_i, e_j) = \sum_{j=1}^{4} K(e_i, e_j)$ $(j \neq i)$. We obtain easily $K(p) - K(p') = K(e_1, e_2) - K(e_3, e_4) = (1/2)[S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) - S(e_4, e_4)] = (1/2)(2\lambda - 2\lambda) = 0$, q. e. d.

2) Let us accept the implication $T_x(N) = p \dot{+} p' \Longrightarrow K(p) = K(p')$. On a fixed space $T_x(N)$ the Ricci tensor S is a symmetric bilinear form and g is positively definite. Thus, according to Lemma 4, there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$ in which S takes on a diagonal form. It means that $S(e_i, e_j) = 0$ for $i \neq j$. Further, $S(e_i, e_i) = \sum_{j=1}^{4} K(e_i, e_j)$ for $i = 1, \dots, 4$, and because $K(e_1, e_2) = K(e_3, e_4)$, $K(e_1, e_3) = K(e_2, e_4)$, $K(e_1, e_4) = K(e_2, e_3)$, we obtain easily $S(e_1, e_1) = \dots = S(e_4, e_4)$. Hence we

get $S = \lambda \cdot g$ on $T_x(N)$. According to [2], p. 292, the coefficient λ is independent of the point $x \in N$. Consequently, N is an Einstein space.

The following Theorem is added to complete our theory of U-spheres:

THEOREM 4. *Let N be an Einstein manifold of dimension $n \geq 3$ and $M \subset N$ a hypersurface all points of which are umbilical. Then M is a U-sphere, or a totally geodesic submanifold.*

PROOF. We suppose that $A = \mu I$ on M where μ is a real function. Then

$$\Delta(X, Y) = (\nabla'_Y A)(X) - (\nabla'_X A)(Y) = (Y\mu) X - (X\mu) Y. \tag{13}$$

Let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis of $Z_x(M)$ and e_n a unit vector of $T_x(N)$ which is normal to $T_x(M)$. According to (3) and (13) $R(e_i, e_j) e_n = (e_j \mu) e_i - (e_i, \mu) e_j$ for $i, j = 1, \dots, n-1)$ and $S(e_j, e_n) = \sum_{i=1}^{n} g(R(e_i, e_j) e_n, e_i) = \sum_{i=1, i \neq j}^{n-1} g(R(e_i, e_j) e_n, e_i) = (n-2)(e_j \mu)$. On the other hand, $S(e_j, e_n) = \lambda \cdot g(e_j, e_n) = 0$ because N is Einsteinian. Hence we obtain $e_j \mu = 0$ for any $j = 1, \dots, n-1$ or, what is the same, the differential $(d\mu)_x = 0$. Consequently, $d\mu = 0$ on M and $\mu = \text{const.}$

In the end we shall study the normal Bianchi identity at the umbilical points of a hypersurface. We start with

LEMMA 6. *Let N be a Riemannian manifold of dimension $n \geq 3$. Then to any $(n-1)$-dimensional linear element E_{n-1} of N there is a hypersurface $M \subset N$ tangent to E_{n-1} at its base point x and such that $A = I$ at x.*

PROOF. We employ the concept of an osculating euclidean space as in the proof of Theorem 2.

PROPOSITION 9. *Let N be a Riemannian manifold of dimension $n \geq 4$ and $x \in N$ a fixed point. Then the following two statements are equivalent:*

(i) *Any hypersurface $M \subset N$ for which x is an umbilical point satisfies the normal Bianchi identity at x.*

(ii) *For any two totally orthogonal planes p, p' of $T_x(N)$ we have $K(p) = K(p')$.*

PROOF. (i) $\Longrightarrow$ (ii). Let e_1, e_2, e_3, e_4 be orthonormal vectors of $T_x(N)$ such that $p = (e_1 e_2)$, $p' = (e_3 e_4)$. Put $E_{n-1} = \{X \in T_x(N) \mid g(X, e_4) = 0\}$. According to Lemma 6 there is a hypersurface $M \subset N$ such that $T_x(M) = E_{n-1}$ and x is a «unit» umbilic: $AX = X$ for any $X \in E_{n-1}$. Formula (7) takes on the form

$$\Delta(X, Y) \wedge Z + \Delta(Y, Z) \wedge X + \Delta(Z, X) \wedge Y = 0 \quad (X, Y, Z \in E_{n-1}). \tag{14}$$

Put $X = e_1$, $Y = e_2$, $Z = e_3$, $\xi = e_4$, then (3) and (14) imply

$$R(e_1, e_2) e_4 \wedge e_3 + R(e_2, e_3) e_4 \wedge e_1 + R(e_3, e_1) e_4 \wedge e_2 = 0. \tag{15}$$

According to the Cartan's lemma there are numbers a, b, c, d, f, g such that

$$\begin{aligned} R(e_1, e_2) e_4 &= a \cdot e_3 + b \cdot e_1 + c \cdot e_2 \\ R(e_2, e_3) e_4 &= b \cdot e_3 + d \cdot e_1 + f \cdot e_2 \\ R(e_3, e_1) e_4 &= c \cdot e_3 + f \cdot e_1 + g \cdot e_2 \end{aligned} \tag{16}$$

Particularly, we have $g(R(e_1, e_2) e_4, e_1) = g(R(e_2, e_3) e_4, e_3) = b$. Now, the orthonormal quadruple $\{e_1, e_2, e_3, e_4\}$ was arbitrary and it can be replaced by the quadruple $\{f_1, f_2, f_3, f_4\}$, where

$$f_1 = e_1, f_2 = \frac{e_2 + e_4}{\sqrt{2}}, f_3 = e_3, f_4 = \frac{e_2 - e_4}{\sqrt{2}}. \tag{17}$$

We get $g(R(f_1, f_2) f_4, f_1) + g(R(f_3, f_2) f_4, f_3) = 0$, whence

$$K(e_1, e_2) + K(e_3, e_2) = K(e_1, e_4) + K(e_3, e_4).$$

Similarly, we obtain

$$K(e_2, e_3) + K(e_4, e_3) = K(e_2, e_1) + K(e_4, e_1).$$

By the subtraction we get finally

$$K(p) = K(e_1, e_2) = K(e_3, e_4) = K(p').$$

(ii) $\Longrightarrow$ (i): Suppose first dim $N = 4$. Let $M \subset N$ be a hypersurface having an umbilic at x. Denote by e_4 a unit vector of $T_x(N)$ normal to $E_3 = T_x(M)$. It suffices to prove (15), or equivalently, (16) for any *orthonormal* triplet $\{e_1, e_2, e_3\}$ of E_3.

Now, each $R(e_i, e_j) e_4 \ (i, j = 1, 2, 3)$ is normal to e_4 and hence it is a linear combination of e_1, e_2, e_3. We have only to show the relation

$$g(R(e_1, e_2) e_4, e_1) + g(R(e_3, e_2) e_4, e_3) = 0$$

for any orthonormal triple $\{e_1, e_2, e_3\}$ of E_3. For this purpose, take the quadruple $\{f_1, f_2, f_3, f_4\}$ given by (17) and write up the identity $K(f_1, f_2) = K(f_3, f_4)$. Taking into account $K(e_1, e_2) = K(e_3, e_4)$, $K(e_1, e_3) = K(e_1, e_4)$, we obtain the wanted relation.

Consider now the case $\dim N \geq 5$. Then, according to the assumption (ii) and Lemma 1, x is an isotropic point of N. Formula (8) shows that (15) is satisfied trivially for any *orthonormal* quadruple $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$. This completes the proof.

Now we can formulate two theorems, which are consequences of Lemma 5, Proposition 9, Lemma 1 and Lemma 3.

THEOREM 5. *Let N be a Riemannian manifold of dimension $n = 4$. Then the following two statements are equivalent:*

(i) *Any hypersurface $M \subset N$ satisfies the normal Bianchi identity at its umbilical points.*

(ii) *N is an Einstein space.*

THEOREM 6. *Let N be a Riemannian manifold of dimension $n \geq 5$. Then the following two statements are equivalent:*

(i) *Any hypersurface $M \subset N$ satisfies the normal Bianchi identity at its umbilical points.*

(ii) *N is a space of constant curvature.*

REMARK. Theorem 6 is closely related to a theorem by T. Y. Thomas: *If $M \subset E^n, n \geq 5$, is a hypersurface the type number of which is ≥ 4 at each point, then the Codazzi equations are algebraic consequences of the Gauss equations.* (Cf. [6]).

Appendix.

In Theorem 4 we have characterized space of constant curvature as Riemannian manifolds containing « sufficiently many » U-spheres. A direct generalization of a U-sphere is *an umbilical hypersurface*, i. e., a hypersurface $M \subset N$ all points of which are umbilical. One can ask about the nature of spaces containing « sufficiently many » umbilical hypersurfaces. The answer is given by the following Theorem:

THEOREM 7. *Let N be a Riemannian manifold of dimension $n \geq 4$. Then the following two statements are equivalent:*

(i) *To any $(n-1)$ dimensional linear element E_{n-1} of N there is an umbilical hypersurface tangent to E_{n-1}.*

(ii) *N is locally conformally flat.*

PROOF. (ii) $\Longrightarrow$ (i). Remind the well-know fact that all umbilical points of a submanifold $M \subset N$ remain umbilical under conformal transformations of the metric g on N (See e. g. [7]). Thus, we can always use an auxiliary *euclidean* neighbourhood to realize a geometrical construction required by (i).

(i) $\Longrightarrow$ (ii). Let $x \in N$, $E_{n-1} \subset T_x(N)$ be given, and let $M \subset N$ be a hypersurface such that $T_x(M) = E_{n-1}$, $A = \mu \cdot I$ on M. Similarly as in the proof of Theorem 4 we obtain the relation $R(e_i, e_j)e_n = (e_j\mu)e_i - (e_i\mu)e_j$ for any $e_i, e_j \in E_{n-1}$ and a unit e_n normal to E_{n-1}. Hence $e_j\,\mu = g(R(e_i, e_j)e_n, e_i) = g(R(e_k, e_j)e_n, e_k)$ for any orthonormal quadruple $\{e_i, e_j, e_k, e_n\}$ of $T_x(N)$, $e_n \perp E_{n-1}$. Because the element $E_{n-1} \subset T_x(M)$ can be arbitrary, we obtain $g(R(e_1, e_2)e_4, e_1) = g(R(e_3, e_2)e_4, e_3)$ for *any* orthonormal quadruple $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$. Using a similar argument as in the proof of Proposition 9 (Formula (17), we obtain finally:

$$K(e_1, e_2) + K(e_3, e_4) = K(e_1, e_4) + K(e_2, e_3)$$

for any orthonormal quadruple $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$.

As was shown by R. S. KULKARNI, [8], this last condition is equivalent to the requirement that *the conformal curvature tensor C vanishes at the point x.* Consequently, the Riemannian space N is locally conformally flat.

As a consequence of Theorem 7 we can state

THEOREM 8. *Let N be a Riemannian manifold of dimension $n \geq 4$. Suppose that any sufficiently small metric sphere of N consists of umbilical points. Then N is locally conformally flat.*

PROBLEM. Does a converse of Theorem 8 hold, too?

Added in Proof, March 15, 1972: It has been pointed to me by R. S. Kulkarni that the conclusion of Theorem 8 may be strenghtened to the effect that N be a space of constant curvature.

Dr. Oldřich Kowalski
Matematický Ústav Univ. Karlovy
Malostranské nám. 25
Praha 1, Czechoslovakia

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