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CAUCHY-SZEGÖ INTEGRALS FOR SYSTEMS OF HARMONIC FUNCTIONS

ADAM KORÁNYI [1] and STEPHEN VÁGI

Introduction. In the theory of one complex variable it is well known that the H^p space of the disc or the half-plane can be identified with a subspace of L^p of the circle or the line respectively. It follows from the classical theorem of Marcel Riesz on the conjugate function that for each p, $1 < p < \infty$ the Cauchy-Szegö integral determines a continuous projection of L^p onto this subspace. The same result holds also for the unit ball in $\mathbf{C}^n$ and for the generalized half-plane equivalent with it [4]; in these cases, however, it cannot be deduced from an assertion about the conjugate function, it is a stronger result.

In the present paper we want to establish the above result for the *harmonic* H^p spaces of the real $n+1$ dimensional $(n > 1)$ half-space, and the unit ball in $\mathbf{R}^n$ $(n > 2)$. For the half-space this is extremely easy; the problem can be formulated in terms of the Riesz transforms and is completely solved by appealing to known theorems about these (Theorem 1.1). This method can be adapted to the case of the ball, however, it loses much of its simplicity. It also has the disadvantage that additional calculations are needed to obtain an explicit formula for the reproducing kernel even in the case of the plane. For these reasons we chose to proceed differently, our approach brings the reproducing kernel itself into the foreground rather than the Riesz transform; this is conceptually simpler. The remarks following Theorem 1.1 point out the complications that arise for the ball and give some motivation for the method we actually use. The rest of the paper

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is taken up by the proof of our main theorem, Theorem 2.1. The proof consists in showing that the results of [4] about singular integrals on homogeneous spaces apply in this case.

1. **The Half space.** We denote the points of $\mathbf{R}^{n+1}$ by $(x_1, \dots, x_n, y)$ and consider the half-space $\mathbf{H}_{n+1}$ determined by the inequality $y > 0$. For a function f on $\mathbf{H}_{n+1}$ and fixed $y > 0$ we denote by f_y the function $f(x_1, \dots, x_n, y)$ regarded as a function on $\mathbf{R}^n$. A function $f: \mathbf{H}_{n+1} \longrightarrow \mathbf{R}^{n+1}$ is said to belong to $H^p(\mathbf{H}_{n+1})$ if f is the gradient of some harmonic function on $\mathbf{H}_{n+1}$ and if $f = \sup_{y>0} \|f_y\|_{L^p(\mathbf{R}^n, \mathbf{R}^{n+1})}$ is finite [5]. It is well known that, by elementary properties of the Poisson kernel, $\lim_{y \to 0} f_y$ exists in $L^p(\mathbf{R}^n, \mathbf{R}^{n+1})$ for every $f \in H^p(\mathbf{H}_{n+1})$ $(1 < p < \infty)$, and $f \longrightarrow \lim_{y \to 0} f_y$ is an isometric imbedding of $H^p(\mathbf{H}_{n+1})$ into $L^p(\mathbf{R}^n, \mathbf{R}^{n+1})$. We denote the image of this imbedding by $H^p(\mathbf{R}^n)$.

It is also well known [3] that if the component f_{n+1} of $f \in H^p(\mathbf{R}^n)$ is given, then the other components f_j are uniquely determined by it and the operators R_j defined by $f_j = R_j f_{n+1}$ $(1 \leq j \leq k)$ are continuous in $L^p(R^n, R)$ for all $1 < p < \infty$. Furthermore, given any function $g \in L^p(\mathbf{R}^n, \mathbf{R})$, the $(n+1)$-tuple $(R_1 g, \dots, R_n g, g)$ is an element of $H^p(\mathbf{R}^n)$.

We shall use the notations $R_{n+1} = I$ (the identity operator); $R_j^* = -R_j$ $(1 \leq j \leq n)$, $R_{n+1}^* = R_{n+1}$. (Of course, identifying $(L^p)'$ with $L^{p'}$, R_j^* is also the transpose).

THEOREM 1.1. The operator $P: L^p(\mathbf{R}^n, \mathbf{R}^{n+1}) \longrightarrow L^p(\mathbf{R}^n, \mathbf{R}^{n+1})$ defined by

$$(Pf)_j = \frac{1}{2} \sum_{k=1}^{n+1} R_j R_k^* f_k \qquad (1 \leq j \leq n+1)$$

is a continuous projection onto $H^p(\mathbf{R}^n)$ for every $1 < p < \infty$. If $p = 2$, it is the orthogonal projection onto $H^2(\mathbf{R}^n)$.

PROOF. Consider the case $p = 2$ first. Let $g \in L^2(\mathbf{R}^n)$. Since [3] the Fourier transform $(R_k g)^\wedge$ of $R_k g$ $(1 \leq k \leq n)$ satisfies

$$(1.1) \qquad (R_k g)^\wedge = \frac{x_k}{|x|} \hat{g},$$

the mapping

$$g \longmapsto (R_1 g, R_2 g, \dots, R_n g)$$

is an isometry, and therefore also the map

$$S : g \longmapsto \frac{1}{\sqrt{2}} (R_1 g, R_2 g, \dots , R_{n+1} g)$$

is an isometry whose range is $H^2(\mathbf{R}^n)$. By general elementary facts from operator theory SS^* is the orthogonal projection onto the range of S. Clearly, $P = SS^*$. Note that S is defined also on L^p $(1 < p < \infty)$. For $f \in L^p(\mathbf{R}^n, \mathbf{R}^{n+1})$ we have by definition $(Pf)_j = R_j g$ with $g = \frac{1}{\sqrt{2}} S^* f$. This shows that $Pf \in$ $\in H^p(\mathbf{R}^n)$. If $f \in H^p(\mathbf{R}^n)$, we know that $f_k = R_k f_{n+1}$ $(1 \leq k \leq n+1)$, hence,

$$(Pf)_j = \frac{1}{2} \sum_{k=1}^{n+1} R_j R_k^* R_k f_{n+1} = R_j f_{n+1} = f_j$$

follows from the known relation $\sum_{k=1}^{n} R_k^2 = -I$ (which is an immediate consequence of (1.1)). This concludes the proof.

REMARK 1. The decisive fact which made the preceding proof so straightforward is that for $p = 2$, $g \longmapsto (R_1 g, \dots , R_n g)$, and therefore S, are isometries. For the unit ball in $\mathbf{R}^n$ one also can define a Riesz transform [4] which is a continuous operator on L^p, $1 < p < \infty$. However, in this case *the analogue of S fails to be an isometry*; this fact causes the complications mentioned in the introduction. Our next remarks sketch an alternative method of proving Theorem 1.1, which is more complicated than the one used above, but which works the same way in the case of the ball, and which, in addition, gives explicit formulas for the reproducing kernel.

REMARK 2. The function $(Pf)^{\sim}$ corresponding to Pf in $H^p(\mathbf{H}_{n+1})$ is the Poisson integral of Pf. One can find an integral formula giving this function directly in terms of f in direct analogy to the classical Cauchy-Szegö integral. In fact, since each R_j $(1 \leq j \leq n)$ and the Poisson integral are convolution operators, this amounts simply to finding $M_{y;jk} = \frac{1}{2} R_j R_k^* P_y$ (where P_y denotes the Poisson kernel of $\mathbf{H}_{n+1}$). We will then have

$$(Pf)_{y;j} = \sum_{k=1}^{n+1} M_{y;jk} * f_k$$

and the reproducing kernel M of $H^2(\mathbf{H}_{n+1})$ (the Szegö kernel in the sense of Section 2) will be given by

$$M_{jk}(x, y\,; x', y') = M_{y+y'\,;jk}(x - x').$$

(We used the abbreviation $x = (x_1, \dots, x_n)$.) Using the characterization of the operators R_j by the Fourier transform (1.1) one finds easily the following explicit formulas:

$$M_{y\,;n+1,n+1}(x) = \frac{1}{2}P_y(x) = \frac{1}{|S^n|}\frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}$$

$$M_{y\,;n+1,j}(x) = -M_{y\,;j,n+1}(x) = \frac{1}{|S^n|}\frac{x_j}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \qquad (j \neq n+1)$$

$$M_{y\,;jj}(x) = -\frac{n+1}{|S^n|}\frac{x_j^2}{|x|^{n+2}}\int_{y/|x|}^{\infty}\frac{dt}{(1+t^2)^{\frac{n+3}{2}}} + \frac{1}{|S^n|}\frac{1}{|x|^n}\int_{y/|x|}^{\infty}\frac{dt}{(1+t^2)^{\frac{n+1}{2}}} \qquad (j \neq n+1)$$

$$M_{y\,;jk}(x) = -\frac{n+1}{|S^n|}\frac{x_j x_k}{|x|^{n+2}}\int_{y/|x|}^{\infty}\frac{dt}{(1+t^2)^{\frac{n+3}{2}}} \qquad \begin{pmatrix} j \neq k \\ j, k \neq n+1 \end{pmatrix}$$

It is also easy to see that setting $y = 0$ in these formulas we get a matrix-valued singular integral kernel which satisfies the classical conditions of Calderón and Zygmund [1]. Applying this singular integral operator to any $f \in L^p(\mathbf{R}^n, \mathbf{R}^{n+1})$ $(1 < p < \infty)$, we get $Pf - \left(\frac{1}{2n}f_1, \dots, \frac{1}{2n}f_n, \frac{1}{2}f_{n+1}\right)$. If we knew how to obtain the above expressions for M *without* using Theorem 1.1, the arguments just sketched would yield a new proof of that theorem. Now it turns out that a general principle about Hilbert spaces with reproducing kernels stated in Lemma 2.1 below, allows us to calculate M directly. The reader can easily carry out these calculations by setting $\mathfrak{H} = L^2(\mathbf{R}^n)$, $H = H^2(\mathbf{H}_{n+1})$, and, for $\Phi \in \mathfrak{H}$, $\widehat{\Phi} = \frac{1}{\sqrt{2}}(Q_y^1 * \Phi, Q_y^2 * \Phi, \dots, Q_y^n * \Phi, P_y * \Phi)$, in Lemma 2.1 ($P_y$ and Q_y^j denote the Poisson kernel and the j-th conjugate Poisson kernel, respectively).

2. The Unit Ball. Let U^n denote the unit ball in $\mathbf{R}^n$ $(n \geq 3)$, its boundary is S^{n-1}. (We shall now work in $\mathbf{R}^n$ instead of $\mathbf{R}^{n+1}$, the reason being that we want to follow the notations of [4] just as in Section 1 we followed [3].) We shall denote the points of U^n by $\xi = (\xi_1, \dots, \xi_n)$, $\eta, \dots$ and the points of S^{n-1} by $x, y, \dots$. The measure induced on S^{n-1} by the Euclidean structure of $\mathbf{R}^n$ will be denoted by dx; $L^p(S^{n-1}, \mathbf{R})$, $L^p(S^{n-1}, \mathbf{R}^n)$ will be L^p spaces with respect to this measure.

Given a function f on U^n and a number $0 < r < 1$ we denote by f_r the function defined on S^{n-1} by $f_r(x) = f(rx)$. A function $f: U^n \longrightarrow \mathbf{R}^n$ is said to belong to $H^p(U^n)$ if it is the gradient of some harmonic function on U^n and if $\|f\| = \sup\limits_{0<r<1} \|f_r\|_{L^p(S^{n-1}, \mathbf{R}^n)}$ is finite. From the properties of the Poisson kernel it follows that $\lim\limits_{r \to 1} f_r$ exists in $L^p(S^{n-1}, \mathbf{R}^n)$ for every $f \in H^p(U^n)$ $(1 < p < \infty)$, and $f \longrightarrow \lim\limits_{r \to 1} f_r$ is an isometric imbedding of $H^p(U^n)$ into $L^p(S^{n-1}, \mathbf{R}^n)$. The image of this imbedding will be denoted by $H^p(S^{n-1})$. For general information about these matters, see [2].

We want to determine the reproducing kernel, to be called the Szegö kernel, of $H^2(U^n)$. We shall do this by applying a general result about reproducing kernels. If H_1 and H_2 are Hilbert spaces, $L(H_1, H_2)$ denotes the space of all continuous linear maps of H_1 into H_2. If $T \in L(H_1, H_2)$, T^* denotes the adjoint of T.

Lemma 2.1. Let E be a set, V a Hilbert space, H a Hilbert space of functions mapping E into V, $\mathfrak{h}$ another Hilbert space.

Let there be given for every $z \in E$ a $k_z \in L(\mathfrak{h}, V)$ such that the linear map $\mathfrak{h} \ni \Phi \longmapsto \widehat{\Phi} \in H$ defined by

$$\widehat{\Phi}(z) = k_z \Phi$$

is a Hilbert space isomorphism. Then $K(z, w) = k_z k_w^*: E \times E \longrightarrow L(V, V)$ is a reproducing kernel for H. That is to say, defining $K_w: V \longrightarrow H$ by $(K_w v)(z) = K(z, w) v$, we have

$$K_w^* f = f(w)$$

for all $f \in H$ and $w \in E$.

Proof. Let $f \in H$, $w \in E$. $f = \widehat{\Phi}$ with some $\Phi \in \mathfrak{h}$. For all $v \in V$ we have $K(z, w) v = k_z(k_w^* v) = (k_w^* v)^\wedge(z)$, and, therefore, with obvious notation,

$$(K_w^* f, v)_V = (f, K_w v)_H = (\widehat{\Phi}, (k_w^* v)^\wedge)_H = (\Phi, k_w^* v)_{\mathfrak{h}}$$

$$= (k_w \Phi, v)_V = (\widehat{\Phi}(w), v)_V = (f(w), v)_V$$

and, consequently, $K_w^* f = f(w)$.

REMARK 1. If H is separable and $\{\psi_\nu\}_{\nu=1}^{\infty}$ is an orthonormal basis, and if we take $\mathfrak{H} = l^2$ with basis $\{e_\nu\}_{\nu=1}^{\infty}$, H has a reproducing kernel if and only if k_z defined by $k_z e_\nu = \psi_\nu(z)$ induces an isomorphism. The kernel in this case is of the form

$$K(z, w) = \sum_\nu \psi_\nu^*(w)\, \psi_\nu(z)$$

where $\psi_\nu^*(w)$ denotes the linear functional $v \mapsto (v, \psi_\nu(w))$ on V. More explicitly,

$$K(z, w)\, v = \sum_\nu (v, \psi_\nu(w))\, \psi_\nu(z) \qquad \text{for } v \in V$$

REMARK 2. If V is finite dimensional and $\{e_j\}$ is an orthogonal basis of V, then it is clear that H has a reproducing kernel if and only if for $f = \sum f_j e_j \in H$ the map $f \to f_j(z)$ is a continuous linear functional for all j and every $z \in E$.

We now proceed to determine the reproducing kernel of $H^2(U^n)$. We write the vectors of R^n as columns and denote by ${}^t v$ the transpose of v. For $\xi \in U^n$ we shall use the standing notation $\xi' = \xi/|\xi|$.

LEMMA 2.2. The Szegö kernel M of $H^2(U^n)$ is given by

$$M(\xi, \eta) = \frac{1}{|S^{n-1}|} \int_0^1 \left\{ \frac{I - 2\varrho\xi \cdot {}^t\eta}{|\varrho\xi - \eta|^n} + n\varrho \frac{(\eta - \varrho\,|\eta|^2\,\xi) \cdot {}^t(\xi - \varrho\,|\xi|^2\,\eta)}{|\varrho\xi - \eta|^{n+2}} \right\} d\varrho.$$

PROOF. We construct an orthonormal system in $H^2(U^n)$. Let S_k be a spherical harmonic of order k, so $u(\xi) = |\xi|^k S_k(\xi')$ is a harmonic polynomial of degree k on $\mathbf{R}^n$. ∇u is homogeneous of degree $k-1$, so we have

$$\int_{U^n} |\nabla u|^2 = \int_0^1 dr\, r^{n+2k-3} \int_{S^{n-1}} |\nabla u|^2 = \frac{1}{n + 2k - 2} \|\nabla u\|_{H^2}^2.$$

As in [4], from the divergence theorem and from the relation $\operatorname{div}(u \cdot \nabla u) = \nabla u + |\nabla u|^2$ we have $\int_{U^n} |\nabla u|^2 = k \int_{S^{n-1}} |S_k|^2$. These arguments also show that if $v(\xi) = |\xi|^l S_l(\xi')$ and $k \neq l$, then ∇u and ∇v are orthogonal in $H^2(U^n)$.

Now let $\{S_{kl}\}$, where S_{kl} is a spherical harmonic of degree k, be a complete orthonormal system in $L^2(S^{n-1}, \mathbf{R})$. It follows from the above that the system $\Phi_{kl}(\xi) = [k(n + 2k - 2)]^{-1/2}\, \nabla(|\xi|^k S_{kl}(\xi'))$ is orthonormal in

$H^2(U^n)$. We will show now that it is also complete. Since $\lim_{r\to 1} f_r = f$ for $f \in H^2(U^n)$, it follows that the functions of the form $f = \nabla\Phi$ with Φ harmonic in a ball of radius > 1 are dense in $H^2(U^n)$. It is enough to show that such an f can be represented as a series $\Sigma b_{kl}\,\Phi_{kl}$ convergent in $H^2(U^n)$. We have an expansion $\Phi|_{S^{n-1}} = \Sigma\, a_{kl}\, S_{kl}$ in $L^2(S^{n-1}, \mathbf{R})$; since Φ is harmonic in a larger ball, it follows that $\Sigma\, |a_k|^2 r^{2k} < \infty$. This implies $\Sigma k(n + 2k - 2)\,|a_{kl}|^2 < \infty$ since the number of different S_{kl}'s grows only polynomially with k. Hence, the series $\Sigma\, a_{kl}\,[k(n + 2k - 2)]^{1/2}\,\Phi_{kl}$ converges to f in $H^2(U^n)$, as was to be shown.

It follows that $M(\xi;\eta) = \sum_{kl} \Phi_{kl}(\xi)\cdot {}^t\Phi_{kl}(\eta)$. By classical formulas for spherical harmonics (for details cf. (4, Lemma 9.1]), we know that

$$\sum_{k=1}^{\infty} \frac{1}{k(n+2k-2)}\,|\xi|^k\,|\eta|^k \sum_l S_{kl}(\xi')\,S_{kl}(\eta') =$$

$$= \frac{1}{(n-2)\,|S^{n-1}|}\int_0^1 \left(\frac{1}{(1 - 2\varrho\xi\cdot\eta + \varrho^2\,|\xi|^2\,|\eta|^2)^{\frac{n-2}{2}}} - 1\right)\frac{d\varrho}{\varrho}\,.$$

Computing the gradient with respect to both ξ and η of this, we obtain our formula for M.

In the next lemmas we shall need some simple inequalities which we now list for easier reference. Throughout this paper we denote by p the point $(0, \ldots, 0, 1)$ of S^{n-1}.

For all small $A > 0$, $0 \le \varrho \le 1$ and all $x, y \in S^{n-1}$ such that $|y - p| \le A/2$, $|x - p| \ge A$ we have

$$0 < c \le \left|\frac{\varrho x - y}{\varrho x - p}\right| \le C < \infty \tag{2.1}$$

with some constants c, C.

For any positive integer m and x, y as above, the identity $a^m - b^m = (a - b)(a^{m-1} + \ldots + b^{m-1})$ and (2.1) give

$$\frac{1}{|\varrho x - y|^m} - \frac{1}{|\varrho x - p|^m} \le C\,\frac{|y - p|}{|\varrho x - p|^{m+1}}\,. \tag{2.2}$$

For all $0 \le \varrho \le 1$ and $x, y \in S^{n-1}$,

$$|\varrho x - y| \ge 1 - \varrho, \tag{2.3}$$

$$|\varrho x - y| \ge \frac{1}{2}\,|x - y|\,. \tag{2.4}$$

We also note that, obviously, $|\varrho x - y| = |x - \varrho y|$.

As $r \to 1$ from below, we have for any $m > 1$,

$$\int_0^r \frac{d\varrho}{|\varrho x - p|^m} = O\left(\frac{1}{(1-r)^{m-1}}\right), \tag{2.5}$$

as one sees immediately from (2.3).

For any $m > 1$, as $|x - p| \to 0$ we have

$$\int_0^1 \frac{d\varrho}{|\varrho x - p|^m} = O\left(\frac{1}{|x-p|^{m-1}}\right). \tag{2.6}$$

This follows by writing $\int_0^1 = \int_0^r + \int_r^1$ with $r = 1 - |x - p|$, using (2.5) for the first integral, (2.4) and the mean value theorem for the second.

For every $\beta \in \mathbf{R}$, as $r \to 0$,

$$\int_{|x-p|>r} \frac{dx}{|x-p|^{n-\beta-1}} = O(r^\beta). \tag{2.7}$$

This follows easily by directly computing dx in the neighborhood of p; for more details cf. [4, § 9].

Lemma 2.3. Let $1 < p < \infty$. For all $\varepsilon > 0$ let the operator T^ε be defined on $L^p(S^{n-1}, \mathbf{R}^n)$ by

$$(T^\varepsilon f)(x) = \int_{|y-p|>\varepsilon} M(x,y) f(y)\, dy.$$

Then the operators T^ε are uniformly bounded in norm, the limit $Tf = \lim_{\varepsilon \to 0} T^\varepsilon f$ exists in $L^p(S^{n-1}, \mathbf{R}^n)$ for all f, T is a bounded operator on L^p and preserves all Lipschitz classes Λ^β $(0 < \beta < 1)$.

Proof. We have to check the conditions of [4, Theorem 3.1]. We regard S^{n-1} as the homogeneous space $SO(n)/SO(n-1)$ with $SO(n-1)$ the isotropy group at p. σ_1 and σ_2 are now both the identity representation of $SO(n)$. We use Remark 4 after Theorem 2.2 in [4] to write out the conditions of Theorem 3.1 of [4] in terms of integrals on S^{n-1}. We note that $M(x,y) = {}^tM(y,x)$, and, therefore, in conditions (a) and (c) it is enough to prove one inequality each. So it is enough if we prove the following:

For all small $A > 0$ and all $|y - p| < A/2$

$$\int_{|x-p|>A} |M(x, y) - M(x, p)|\, dx \leq C \frac{|y-p|}{A}. \tag{2.8}$$

For all $r > 0$ and for some (hence any) $\beta > 0$,

$$\int_{|x-p|<r} |x - p|^{\beta} |M(x, p)|\, dx \leq Cr^{\beta}. \tag{2.9}$$

Finally, for all small $r, s > 0$,

$$\int_{r<|x-p|<r(1+s)} M(x, p)\, dx \leq Crs. \tag{2.10}$$

We write $M = M^{(1)} - 2M^{(2)} + nM^{(3)}$ with

$$M^{(1)}(x, y) = \int_0^1 \frac{d\sigma}{|\varrho x - y|^n} I,$$

$$M^{(2)}(x, y) = \int_0^1 \frac{x \cdot {}^t y}{|\varrho x - y|^n} \varrho\, d\varrho$$

$$M^{(3)}(x, y) = \int_0^1 \frac{(y - \varrho x) \cdot {}^t (x - \varrho y)}{|\varrho x - y|^{n+2}} \varrho\, d\varrho$$

and prove (2.8) for each $M^{(\varrho)}$ separately. For $M^{(1)}$ (2.8) is immediate from (2.2), (2.6) and (2.7). For $M^{(2)}$ we have

$$|M^{(2)}(x, y) - M^{(2)}(x, p)| \leq$$

$$\leq \int_0^1 \frac{|x \cdot {}^t (y - p)|}{|\varrho x - y|^n} \varrho\, d\varrho + \int_0^1 \left| \frac{1}{|\varrho x - y|^n} - \frac{1}{|\varrho x - p|^n} \right| |x \cdot {}^t p|\, \varrho\, d\varrho$$

$$\leq |y - p| \int_0^1 \frac{d\varrho}{|\varrho x - y|^n} + \int_0^1 \left| \frac{1}{|\varrho x - y|^n} - \frac{1}{|\varrho x - p|^n} \right| d\varrho$$

and we get (2.8) in the same way as for $M^{(1)}$. For $M^{(3)}$ we proceed similarly, using (2.2) with $m = n + 2$ and using (2.1) along the way.

To check (2.9) we note that, clearly,

$$| M(x, p) | \leq C \int_0^1 \frac{d\varrho}{| \varrho x - p |^n}$$

so (2.9) follows at once from (2.6) and (2.7).

To check (2.10) it seems most convenient to consider the matrix entries $M_{jk}(x, p)$ of $M(x, p)$ separately. The off-diagonal entries of $M^{(1)}(x, p)$ are zero. In $M^{(2)}(x, p)$ the off-diagonal terms are zero, except in the last column; even there, $M^{(2)}{}_{jn}(x, p)$ is an odd function of x_j. Since in (2.10) we integrate it on a set symmetric with respect to x_j, the integral is 0. Similar considerations show that $M^{(3)}$ also contributes only to the diagonal terms in the integral of (2.10).

Next we notice that

$$M_{nn}(x, p) = \int_0^1 \left(\frac{1 - 2\varrho x_n}{| \varrho x - p |^n} + n\varrho \frac{(1 - \varrho x_n)(x_n - \varrho)}{| \varrho x - p |^{n+2}} \right) d\varrho =$$

$$= - x_n \int_0^1 \frac{\varrho \, d\varrho}{| \varrho x - p |^n} + \int_0^1 (1 - \varrho x_n) \frac{\partial}{\partial \varrho} \left(\frac{\varrho}{| \varrho x - p |^n} \right) d\varrho,$$

and after integrating by parts we find

$$M_{nn}(x, p) = \frac{1 - x_n}{| x - p |^n} = \frac{1}{2 | x - p |^{n-2}}. \tag{2.11}$$

It follows easily that (2.10) holds for $M_{nn}(x, p)$ (details of this step are also in [4, proof of Lemma 9.3]).

Now let $j \neq n$. Then

$$M_{jj}(x, p) = \int_0^1 \frac{d\varrho}{| \varrho x - p |^n} - n x_j^2 \int_0^1 \frac{\varrho^2}{| \varrho x - p |^{n+2}} d\varrho. \tag{2.12}$$

By elementary algebra:

$$\int_0^1 \frac{\varrho^2}{| \varrho x - p |^{n+2}} d\varrho = \int_0^1 \frac{(\varrho - x_n)^2}{| \varrho x - p |^{n+2}} d\varrho + x_n \int_0^1 \frac{2(\varrho - x_n)}{| \varrho x - p |^{n+2}} d\varrho + x_n^2 \int_0^1 \frac{d\varrho}{| \varrho x - p |^{n+2}} \tag{2.13}$$

$$(2.14) \qquad \int_0^1 \frac{d\varrho}{|\varrho x - p|^n} = \int_0^1 \frac{1 - x_n^2}{|\varrho x - p|^{n+2}}\, d\varrho + \int_0^1 \frac{(\varrho - x_n)^2}{|\varrho x - p|^{n+2}}\, d\varrho$$

and, by integration by parts,

$$(2.15) \qquad \int_0^1 \frac{(\varrho - x_n)^2}{|\varrho x - p|^{n+2}}\, d\varrho = -\frac{1}{n}\,\frac{\varrho - x_n}{|\varrho x - p|^n}\Bigg|_0^1 + \frac{1}{n}\int_0^1 \frac{d\varrho}{|\varrho x - p|^n}\,.$$

Using (2.13), (2.14), (2.15) to express the second integral in (2.12) by means of the first one we find

$$(2.16) \qquad M_{jj}(x, p) = \left(1 - x_j^2 - (n-1)\frac{x_j^2\, x_n^2}{1 - x_n^2}\right)\int_0^1 \frac{d\varrho}{|\varrho x - p|^n} + $$

$$+ x_j^2\left(\frac{1 + 2x_n}{(1 + x_n)\,|x - p|^n} - \frac{x_n}{1 - x_n^2}\right).$$

Since the domain of integration in (2.10) is symmetric, each $M_{jj}(x, p)$ $(1 \leq j \leq n - 1)$ has the same integral. Therefore, adding and using $\sum\limits_1^{n-1} x_j^2 = 1 - x_n^2$, we have

$$(2.17) \qquad \int\limits_{r < |x-p| < r(1+s)} M_{jj}(x, p)\, dx =$$

$$= \frac{1}{(n-1)\,|S^{n-1}|}\int\limits_{r<|x-p|<r(1+s)} \Bigg\{(n-2)(1 - x_n^2)\int_0^1 \frac{d\varrho}{|\varrho x - p|^n} +$$

$$+ \frac{1 + 2x_n}{2}\,\frac{1}{|x - p|^{n-2}} - x_n\Bigg\}\, dx.$$

Since $1 - x_n^2 = \dfrac{1 + x_n}{2}\,|x - p|^2 \leq |x - p|^2$, it follows that the integrand is $O\left(\dfrac{1}{|x - p|^{n-2}}\right)$, and (2.10) follows in the same way as for M_{nn}.

LEMMA 2.4. (i) For all $0 < r < 1$, $x \in S^{n-1}$,

$$\int\limits_{S^{n-1}} M(rx, y)\, dy = I$$

and, (ii) for all $x \in S^{n-1}$,

$$\lim_{\varepsilon \to 0} \int\limits_{|y-x|>\varepsilon} M(x, y)\, dy = \begin{pmatrix} 1 - \frac{1}{2(n-1)} & & & O \\ & \ddots & & \\ & & 1 - \frac{1}{2(n-1)} & \\ O & & & \frac{1}{2} \end{pmatrix}.$$

Proof. (i) follows from the fact that every constant vector is contained in $H^2(U^n)$, and is reproduced by M.

The left side of (ii) is the transpose of

$$\lim_{\varepsilon \to 0} \int\limits_{|x-p|>\varepsilon} M(x, p)\, dx,$$

as one readily sees by using group invariance. Integrals of this type were considered in the proof of (2.10) in Lemma 2.3; the arguments there show that the integral of all off-diagonal terms is zero.

To compute the integral of $M_{nn}(x, p)$ we use (2.11), the identity $|x-p|^2 = 2(1-x_n)$ and the fact that the surface element on S^{n-1} for sets that are described by a condition on x_n alone (i.e., for sets that are $O(n-1)$-invariant) is

$$|S^{n-2}|(1-x_n^2)^{\frac{n-3}{2}}\, dx_n. \tag{2.18}$$

We find

$$\int\limits_{S^{n-1}} M_{nn}(x, p)\, dx = \frac{|S^{n-2}|}{2^{n/2}|S^{n-1}|} \int_{-1}^{1} \frac{(1-x_n^2)^{\frac{n-3}{2}}}{(1-x_n)^{\frac{n-2}{2}}}\, dx_n$$

which, after the change of variable $x_n = 2t - 1$ becomes an Euler B-integral and is found to equal $1/2$. To find the integral of $M_{jj}(x, p)$ $(j \neq n)$, we use (2.17) and again (2.18). By an argument similar to the one used for $M_{nn}(x, p)$ we find that the integral of the rational part is $\frac{1}{2(n-1)} + \frac{1}{n(n-1)}$. The remaining term leads to the integral

$$\frac{(n-2)|S^{n-2}|}{(n-1)|S^{n-1}|} \int_{-1}^{1} \int_{0}^{1} \frac{(1-x_n^2)^{\frac{n-1}{2}}}{[(\varrho - x_n)^2 + 1 - x_n^2]^{n/2}}\, d\varrho\, dx_n.$$

Introducing new variables, φ, θ by

$$\cos\varphi = x_n$$

$$\tan\theta = \frac{\varrho - x_n}{(1 - x_n^2)^{1/2}}$$

and interchanging the order of integration this is found to be equal to

$$2\frac{n-2}{n-1}\frac{|S^{n-2}|}{|S^{n-1}|}\int_0^{\pi/2}\cos^n\theta\, d\theta = \frac{n-2}{n}.$$

Adding this gives $1 - \frac{1}{2(n-1)}$, finishing the proof.

LEMMA 2.5. Let $1 < p < \infty$. For all $f \in L^p(S^{n-1}, \mathbf{R}^n)$

$$\lim_{r\to 1}\int_{|y-p|>1-r}[M(rx, y) - M(x, y)]f(y)\, dy = \frac{1}{2(n-1)}f_{\tan}(x) + \frac{1}{2}f_{\mathrm{rad}}(x) \tag{2.19}$$

in the L^p sense, where $f_{\mathrm{rad}}(x) = (f(x)\cdot x)\, x$, $f_{\tan}(x) = f(x) - f_{\mathrm{rad}}(x)$.

PROOF. Making an obvious change of variable in the integral defining M (Lemma 2.2) we find

$$M(rx, y) = \frac{1}{|S^{n-1}|}\frac{1}{r}\int_0^1\left(\frac{I - \varrho x\cdot {}^t y}{|\varrho x - y|^n} + n\varrho\frac{(y - \varrho x)\cdot {}^t(x - \varrho y)}{|\varrho x - y|^{n+2}}\right)d\varrho.$$

By (2.5) this implies

$$|M(rx, p)| \leq \frac{C}{(1-r)^{n-1}}. \tag{2.20}$$

Similarly, it is easy to see that

$$|M(rx, p) - M(x, p)| \leq C\frac{1-r}{|x-p|^n}. \tag{2.21}$$

We now have to work in the group $SO(n)$. For $g \in SO(n)$ define $\tilde{f}(g) = f(gp)$. Define $K(g) = M(gp, p)$. $|gp - p| = |g|$ defines a gauge in the sense of [4] on $SO(n)$. For $0 < r < 1$ let K^{1-r} be defined by

$$K^{1-r}(g) = \begin{cases} 0 & \text{for } |g| < 1-r \\ K(g) & \text{for } |g| \geq 1-r \end{cases}$$

and K_r by $K_r(g) = K(rgp, p)$. dh or the like denotes Haar measure on $SO(n)$. It is normalized by $\int_{SO(n)} 1\, dh = |S^{n-1}|$. If we set $x = gp$ and $y = hp$, then the integral in (2.19) becomes

$$\int_{SO(n)} [hK_r(h^{-1}g)h^{-1} - hK^{1-r}(h^{-1}g)h^{-1}]\tilde{f}(h)\,dh =$$

$$= \int_{SO(n)} gl^{-1}(K_r(l) - K^{1-r}(l))\, lg^{-1}\tilde{f}(gl^{-1})\,dl =$$

$$g \int_{|l|<1-r} l^{-1} K_r(l)\, l\,[\tilde{f}(gl^{-1}) - f(g)]\,dlg^{-1}$$

$$+ g \int_{|l|<1-r} l^{-1} K_r(l)\, l\tilde{f}(g)\,dlg^{-1} +$$

$$+ g \int_{|l|\geq 1-r} l^{-1}(K_r(l) - K^{1-r}(l))\, l\,[\tilde{f}(gl^{-1}) - f(g)]\,dlg^{-1} +$$

$$+ g \int_{|l|\geq 1-r} l^{-1}(K_r(l) - K^{1-r}(l))\, l\tilde{f}(g)\,dlg^{-1} =$$

$$= J_r^1(g) + I_r^1(g) + J_r^2(g) + I_r^2(g).$$

By Lemma 2.4, for $r \to 0$

$$I_r^1(g) + I_r^2(g) \to \tilde{f}(g) - g \begin{pmatrix} \frac{2n-3}{2(n-1)} & & \\ & \ddots & \\ & & \frac{1}{2} \end{pmatrix} g^{-1}\tilde{f}(g)$$

$$= \frac{1}{2(n-1)} f_{\tan}(x) + \frac{1}{2} f_{\text{rad}}(x).$$

By Minkowski's integral inequality,

$$\| J_r^1 \|_p \leq \int\limits_{|l| < -r} | K_r(l) | \, \| R_{l^{-1}} \tilde{f} - \tilde{f} \|_p \, dl$$

where $R_{l^{-1}}$ is right translation by l^{-1}. By (2.20) we therefore have

$$\| J_r^1 \|_p \leq C \sup_{|l| \leq 1-r} \| R_{l^{-1}} \tilde{f} - \tilde{f} \|_p ,$$

and this tends to zero for $r \rightarrow 1$.

For J_r^2, again by Minkowski's inequality, we have

$$\| J_r^2 \|_p \leq \Big(\int\limits_{1-r < |l| < \eta} + \int\limits_{|l| \geq \eta} \Big) | K_r(l) - K(l) | \, \| R_{l^{-1}} \tilde{f} - \tilde{f} \|_p \, dl = L_r' + L_r'' .$$

Using (2.21) we have

$$L_r' \leq \sup_{|l| < \eta} \| R_{l^{-1}} \tilde{f} - \tilde{f} \|_p \int\limits_{1-r < |x-p| < \eta} \frac{1-r}{|x-p|^n} \, dx.$$

The last integral is bounded [4, § 9], hence, by choosing η small, L_r' can be made arbitrarily small.

Again by (2.21) $|K_r(l) - K(l)| \rightarrow 0$ uniformly for $|l| > \eta$, as $r \rightarrow 0$. Therefore, $L_r'' \rightarrow 0$. This concludes the proof of the lemma.

THEOREM 2.1. Let $1 < p < \infty$. For all $f \in L^p(S^{n-1}, \mathbf{R}^n)$ the limit

$$(Pf)(x) = \frac{1}{2} f_{\text{rad}}(x) + \frac{1}{2(n-1)} f_{\tan}(x) + \lim_{\varepsilon \to 0} \int\limits_{|y-x| > \varepsilon} M(x, y) f(y) \, dy$$

exists in $L^p(S^{n-1}, \mathbf{R}^n)$; P is a bounded projection onto $H^p(S^{n-1})$, if $p = 2$ it is the orthogonal projection. Pf is the boundary function of the $H^p(U^n)$-function $g(\xi) = \int\limits_{S^{n-1}} M(\xi, y) f(y) \, dy$. P maps each of the Lipschitz classes $\Lambda^\beta(S^{n-1}, \mathbf{R}^n)$ $(0 < \beta < 1)$ into themselves.

PROOF. The limit exists and P is bounded and P preserves Λ^β by Lemma 2.3. By Lemma 2.5, $\lim g_r = Pf$ in $L^p(S^{n-1}, \mathbf{R}^n)$. Hence, also the L^p-norm of g_r is bounded, which shows that $g \in H^p(U^n)$ and $Pf \in H^p(S^{n-1})$.

If $p = 2$, it follows from the reproducing property of M that P is a projection onto $H^2(S^{n-1})$; from ${}^tM(x, y) = M(y, x)$ it follows that P is self-adjoint, so an orthogonal projection.

For any $1 < p < \infty$, $H^p \cap H^2$ is dense in H^p. Therefore, $P^2 = P$ and the range of P is dense in H^p. It follows that P is a projection onto H^p. Finally we have the obvious

COROLLARY 2.1. For $1 < p < \infty$, M is the reproducing kernel for $H^p(U^n)$.

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