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# FOURIER TRANSFORMS OF HOMOGENEOUS DISTRIBUTION

CARLOS LEMOINE

## Introduction.

The purpose of the paper\* is to study the relations between the regularity of a homogeneous distribution and that of its-Fourier transform; this problem has been treated by Calderon, Zygmund and Hormander, our results are extensions of theirs.

In preparing the basis for our study we obtain a characterization of the continuous linear maps, from the space of distributions in the unit sphere into itself, that commute with rotations (Chap 1). A clear presentation of the spaces  $L_p^s$  in a compact manifold, is given in chap 2.

## CHAPTER I

### SPHERICAL HARMONICS

#### *Summary.*

- 1.1. Some notations are introduced, and some fundamental facts are recalled.
- 1.2. The expansion of a distribution on the unit sphere in a convergent series of spherical harmonics is given and also a characterization of the continuous linear maps from  $D(\Sigma)$  to  $D'(\Sigma)$  that commute with rotations. As an application we consider the operators  $J^\alpha$  defined by

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(\*) The paper contains essentially the author dissertation submitted at the University of Maryland. The work was directed by Professor Umberto Neri.

Seely and give their explicit expression in terms of spherical harmonics.  
 1.3. We make some remarks on the Fourier transform of-distributions of the form  $r^s Y_{nm}$ .

### 1.0. Notations.

$C$  will denote a constant not necessarily the same in a given statement,  $\mathbb{R}$  the real numbers,  $\mathbb{C}$  the complex numbers,  $\mathbb{E}^k$  the Euclidean  $k$ -dimensional space,  $x = (x_1, \dots, x_k)$ ,  $y = (y_1, \dots, y_k)$  arbitrary points in  $\mathbb{E}^k$ ,  $x, y$  the inner product of  $x$ , and  $y$ ,  $|x| = (\langle x, x \rangle)^{\frac{1}{2}}$ ,  $\alpha$  a multi-index i. e.  $\alpha = (a_1 \dots a_k)$  a point in  $\mathbb{E}^k$  with positive integers as co-ordinates,  $|\alpha| = a_1 + a_2 + \dots + a_k$ ,  $D^\alpha$  the operator

$$\frac{\partial^{|\alpha|}}{\partial x_1^{a_1}, \dots, \partial x_n^{a_n}}$$

If  $X$  is a  $C^\infty$  manifold,  $D(X)$  will designate the space of  $C^\infty$  functions on  $X$  with compact support provided with the Schwartz topology,  $D'(X)$  its dual with the weak topology.

### 1.1. Preliminaries.

For the proofs of the results stated here we refer to Neri [1]. Let  $\mathbb{E}^k$  be the  $k$ -dimensional Euclidean space, and let  $\Sigma_{k-1} = \{x; |x| = 1\}$  be the unit sphere. The restrictions to  $\Sigma$  of homogeneous harmonic polynomials of degree  $n$  are called spherical harmonics. The spherical harmonics of degree  $n$  form a complex vector space  $\{Q_n\}$  of dimension  $N_n^k = O(n^{k-2})$ . By  $\{Y_{nm}\}_n$  ( $m = 1, \dots, N_n^k$ ) we will denote a base of  $\{Q_n\}$  formed by restrictions of homogeneous harmonic polynomials with real coefficients and orthonormal with respect to the inner product

$$(1.1.1) \quad \langle f, g \rangle = \int_{\Sigma} f \bar{g} d\sigma$$

where  $d\sigma$  denotes the Lebesgue measure on  $\Sigma$ . Whenever  $m$  appears together with  $n$  we will assume that  $m$  runs from 1 to  $N_n^k$ .  $\bigcup_{n \geq 0} \{Y_{nm}\}_n$  is a Hilbert base of  $L_2(\Sigma)$ ; we will denote by  $\{Y_{nm}\}$  this base. If  $f \in C^\infty(\Sigma)$ , the coefficients  $a_{nm}$  of its expansion in spherical harmonics satisfy for every

$k' > 0$

$$(a_{nm}) = O(n^{-k'}).$$

Reciprocally if  $a_{nm} = O(n^{-k'})$  for every  $k'$  and  $f$  is given by

$$f = \sum a_{nm} Y_{nm}$$

$f \in C^\infty(\Sigma)$ . In this chapter we assume  $k \geq 2$ .

### 1.2. Fourier Series of Spherical Harmonics.

The following proposition is a natural extension of a well-known result of L. Schwartz.

**PROPOSITION. 1.2.1.** Let  $u \in D'(\Sigma)$  and  $\{Y_{nm}\}$  be a Hilbert basis of  $L_2(\Sigma)$ , formed by orthonormal spherical harmonics. If  $a_{nm} = \langle Y_{nm}, u \rangle$  then:

- (i) there is an integer  $N$  such that  $|a_{nm}| = O(n^{-N})$ ;
- (ii) the series

$$(1.2.1) \quad \sum a_{nm} Y_{nm}$$

converges weakly to  $u$ .

Reciprocally: If  $\{a_{nm}\}$   $n = 1, \dots, \infty, m = 1, \dots, N_n^k$ , satisfies (i), (1.2.1) is weakly convergent to a distribution  $u \in D'(\Sigma)$ .

**PROOF.** Let  $C^s(\Sigma)$  be the space of complex-valued functions with continuous derivatives up to the order  $s$ . If  $f \in C^s(\Sigma)$  let  $r^0 f$  be its extension to  $\mathbb{E}^k - \{0\}$  as a homogeneous function of degree 0.  $C^s(\Sigma)$  is a Banach space with the following norm.

$$(1.2.2) \quad \|f\| = \sum_{|\alpha| \leq s} \sup_{|x|=1} |D^\alpha(r^0 f)(x)|$$

where  $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_k^{\alpha_k}}$ .

Now since  $\Sigma$  is compact  $u$  is of finite order  $s$ , i. e., can be extended to  $C^s(\Sigma)$  as a continuous form for the norm defined above. If  $b_{nm}$  is any sequence of constants such that  $|b_{nm}| = O(n^{-[\frac{3}{2}(k-2)+s+2]})$  the series

$$(1.2.3) \quad \sum_{n,m} b_{nm} Y_{nm}$$

is convergent in  $C^s(\Sigma)$ , this follows from the estimates

$$(1.2.4) \quad |D^\alpha(r^0 Y_{nm}(x'))| \leq C_{k,|\alpha|} n^{1/2(k-2)+|\alpha|}$$

(Cfr. Neri, [1]) and the fact that the number of linearly independent spherical harmonics of degree  $n$  is  $N_n^k = 0 (n^{k-2})$ . The convergence of (1.2.3) in  $C^s(\Sigma)$  and the continuity of  $u$  imply that the series

$$(1.2.5) \quad \sum_{n, m} a_{nm} b_{nm} = \sum_{n, m} \langle u, Y_{nm} \rangle b_{nm} = \langle u, \sum_{n', m'} b_{n'm'} Y_{n'm'} \rangle$$

is convergent. Consequently, since  $\{b_{nm}\}$  is arbitrary, the estimate (i) and the weak convergence of (1.2.1) follow readily.

Reciprocally, assume that  $a_{nm} = 0 (n^N)$ . Let  $f \in D(\Sigma)$  and  $\langle f, Y_{nm} \rangle = b_{nm}$ , then  $b_{nm} = 0 (n^{-k'})$  for every  $k'$ . Hence (1.2.5) is convergent and this implies that (1.2.1) is weakly convergent. ■

**DEFINITION 1.2.2.** Given  $f \in D'(\Sigma)$  and  $\{Y_{nm}\}$  a Hilbert base of  $L_2(\Sigma)$  formed by spherical harmonics,  $\langle f, Y_{nm} \rangle = a_{nm}$  will be called the harmonic components of  $f$  with respect to the base  $Y_{nm}$ , or simply the components of  $f$ .

Let  $f \in D'(\Sigma)$  have components  $a_{nm}$ , and  $u \in D(\Sigma)$  with components  $b_{nm}$ . The condition (i) and the relations  $(b_{nm}) = 0 (n^{-k'})$  for every  $k'$ , imply that  $a_{nm} b_{nm} = 0 (n^{-k'})$  for every  $k'$ . Hence,  $a_{nm} b_{nm}$  are the components of an element  $g \in D$ . By the Closed Graph theorem it follows that

$$(1.2.6) \quad T_f : u \rightarrow g$$

is a continuous linear map from  $D(\Sigma)$  to  $D(\Sigma)$ . The relation between continuous linear mappings from  $D(\Sigma)$  to  $D'(\Sigma)$  commuting with rotations and mappings of the form (1.2.6) associated to a distribution  $f$  is as follows.

**PROPOSITION 1.2.3.** A continuous linear map  $T$  from  $D(\Sigma)$  to  $D'(\Sigma)$  that commutes with rotations, maps  $D(\Sigma)$  continuously into itself and is of the form (1.2.6). Furthermore, if  $k > 2$ ,  $f$  has the form

$$(1.2.7) \quad f = \sum_{n, m} \lambda_n Y_{nm}.$$

Reciprocally if  $k > 2$  and  $f \in D'(\Sigma)$  is of the form (1.2.7),  $T_f$  is a continuous linear map from  $D(\Sigma)$  to  $D(\Sigma)$  that commutes with rotations. If  $k = 1$ , for every  $f \in D'(\Sigma)$ ,  $T_f(u) = \frac{1}{\pi} u * f$  is a continuous linear map from  $D(\Sigma)$  to  $D(\Sigma)$  that commutes with rotations.

**PROOF.** Let us assume first that  $T$  maps  $D(\Sigma)$  into  $C^2(\Sigma)$ . By evaluating  $T$  on the coefficients of a differential form we extend  $T$  to the diffe-

rential forms, i. e.,  $T$  can be considered as a continuous linear map from the space of differential forms with coefficients in  $C^\infty$  into the space of differential forms with coefficients in  $C^2$  (on the differential forms we consider the product topology).

We claim that  $T$  commutes with the exterior differential operator. This follows immediately if  $k = 2$ . If  $k > 2$  and  $P$  is an arbitrary point we take cartesian coordinates such that  $P = (0, 0, \dots, 0, 1)$  and consider the associated spherical coordinates

$$\begin{cases} x_k = r \cos \theta_{k-1} \\ \vdots \\ x_1 = r \sin \theta_{k-1} \dots \sin \theta_1 \end{cases}$$

In a neighborhood of  $P$  in  $\Sigma$ ,  $(\theta_1, \theta_2, \dots, \theta_{k-1})$  is a valid system of coordinates. Using the commutativity of  $T$  with rotations, one readily see that

$$(dT(f))(P) = (T(df))(P)$$

This proves our claim.

$T$  clearly commutes with the Hodge operator  $*$ , hence  $T$  commutes with the Laplace-Beltrami operator  $\Delta = - * d * d$ . Consequently, the image under  $T$  of a spherical harmonics,  $Y_{nm}$  of degree  $n$  will be of the form

$$(1.2.8) \quad T(Y_{nm}) = \sum \lambda_{m'} Y_{nm'}$$

(This is readily seen if we compare  $\Delta T(Y_{nm})$  with  $T\Delta(Y_{nm})$ ).

If  $k > 2$  we will denote by  $Y_z^n(x)$  a zonal harmonics of degree  $n$  and pole  $z$  (Cfr. Neri [1] for the definition of zonal harmonics). The commutativity of  $T$  with rotations and (1.2.8) imply that

$$T(Y_z^n) = C_n Y_z^n$$

On the other hand, for an arbitrary spherical harmonic  $Y_{nm}$  of degree  $n$  we have:

$$Y_{nm}(z) = C \int_{\Sigma} Y_z^n(y') Y_{nu}(y') d\sigma.$$

From this and the continuity and linearity of  $T$  we obtain

$$T_z(Y_{nm}(z)) = C \int_{\Sigma} T_z(Y_z^n(y')) Y_{nu}(y') d\sigma = C_n Y_{nm}(z).$$

(Here we have used  $T_z$  instead of  $T$  to make clear that  $y'$  is a parameter with respect to  $T$ .) Therefore, if  $u = \sum a_{nm} Y_{nm}$ ,

$$(1.2.9) \quad T(u) = \sum C_n a_{nm} Y_{nm}.$$

Let us now show that the  $C_n$  satisfy the condition (i) of Proposition 1.2.1. If not, there are sequences of numbers  $C_k \rightarrow \infty$  and integers  $n_k \rightarrow \infty$  such that  $|C_{n_k}| > C_k n_k^{k'}$ , but then if  $u = \sum_{k'=1}^{\infty} \sum_m \frac{1}{n_k^{k'/2}} Y_{n_k m}$ ,  $u \in D(\Sigma)$  (cfr. preliminaries) and  $T(u) \notin D'(\Sigma)$ . In other words, we have proved that if  $k' > 2$  a continuous map from  $D(\Sigma)$  to  $D'(\Sigma)$  that commutes with rotations and whose range is contained in  $C^2(\Sigma)$ , is of the form (1.2.6) with  $f \in D'(\Sigma)$  of the form (1.2.7).

Now we drop the condition  $T(D) \subset C^2(\Sigma)$ . Let  $T: D(\Sigma) \rightarrow D'(\Sigma)$  be continuous and commuting with rotations. Let  $V_n$  be the inverse image under  $T$  of the subspace  $\{Q_n\}$  generated by  $\{Y_{nm}\}_n$ , and let  $T_n$  be the linear mapping that coincides with  $T$  on  $V_n$  and is zero outside.  $T_n$  commutes with rotations and its image is contained in  $D(\Sigma)$  hence  $T_n(Y_{nm}) = C_n Y_{nm} = T(Y_{nm})$ , i.e.,  $T$  is of the form (1.2.9). By the same reasoning used in the case  $T(D(\Sigma)) \subset C^2(\Sigma)$ , it follows that the  $C_n$  satisfy (i). On the other hand, if  $T$  is of the form (1.2.9), then  $T$  commutes with rotations. In fact, if  $Y_x^n(y)$  is a normalized zonal harmonic, i. e.,

$$Y_{nm}(z) = \langle Y_z^n(x), Y_{nm}(x) \rangle,$$

we may write

$$(1.2.10) \quad T(u)(z) = \sum C_n a_{nm} \langle Y_z^n(x), Y_{nm}(x) \rangle, \quad (u = \sum a_{nm} Y_{nm} \in D(\Sigma)).$$

Let  $\varrho$  be a rotation; for an arbitrary function  $u$  on  $\Sigma$  we define  $\varrho(u)(x) = u(\varrho^{-1}(x))$ . It is clear that for every  $x', y' \in \Sigma_{k-1}$ , and every  $\varrho$ ,

$$\langle \varrho(x'), \varrho(y') \rangle = \langle x', y' \rangle.$$

Hence

$$Y_{\varrho(z)}^n(\varrho(x)) = Y_z^n(x).$$

Thus

$$\begin{aligned} (\varrho(Tu))(z) &= \varrho\left(\sum C_n a_{nm} \langle Y_{\varrho(z)}^n(\varrho(x)), Y_{nm} \rangle\right) \\ &= \varrho\left(\sum C_n a_{nm} \langle Y_z^n(x), \varrho(Y_{nm}) \rangle\right) \\ &= \left(\sum C_n a_{nm} \langle Y_z^n(x), \varrho(Y_{nm}) \rangle\right) \\ &= T(\varrho u)(z). \end{aligned}$$

Finally, if  $k = 2$  and  $T$  commutes with rotations, we have shown that  $T$  commutes with  $\frac{d}{d\theta}$ , and  $\frac{d^2}{d\theta^2}$  ( $\theta = \text{angle in } \Sigma_1$ ). It follows by an elementary calculation that

$$\begin{aligned} T(\cos n\theta) &= a_n \cos n\theta + b_n \sin n\theta \\ T(\sin n\theta) &= -b_n \cos n\theta + a_n \sin n\theta \end{aligned}$$

where  $a_n$  and  $b_n$  satisfy the condition (i), i. e.,  $T(u) = u * f = T_f(u)$  where  $f = \Sigma \left( \frac{a_n}{\pi} \cos n\theta + \frac{b_n}{\pi} \sin n\theta \right)$ . Reciprocally, if  $f \in D'(\Sigma)$  it is clear that  $u \rightarrow u * f$  is invariant under rotations. The proof of the proposition is complete. ■

EXAMPLE 1.2.4. (The Laplace-Beltrami operator  $\Delta$ ). The formula

$$\Delta Y_{nm} = (-n)(n+k-2) Y_{nm}$$

shows that  $\Delta$  is an operator of the form  $T_f$  associated to the distribution.

$$f = \sum_{n,m} (-n)(n+k-2) Y_{nm}.$$

EXAMPLE 1.2.5. If  $\beta$  is a complex number  $\neq n(n+k-2)$  [ $n = 0, 1, 2, \dots$  etc.],  $(\beta + \Delta)$  is invertible as a linear map from  $D(\Sigma)$  to  $D(\Sigma)$ .

Let  $L > 0$  and  $\alpha > 0$ , then it follows from the preceding example that the operator  $\beta^{-\alpha/2}(\beta - L + \Delta)^{-1}$  is associated to the distribution

$$(1.2.11) \quad f_\beta = \sum_{n,m} \beta^{-\alpha/2} (\beta - L - (n(n+k-2)))^{-1} Y_{nm}$$

$f_\beta$  is an analytic family of distributions, i. e., for every  $u \in D(\Sigma)$

$$(1.2.12) \quad \beta \rightarrow \langle f_\beta, u \rangle$$

is analytic in the  $\beta$  plane slit along the negative real axis. Our purpose is to integrate the function

$$\beta \rightarrow \frac{1}{2\pi i} \langle f_\beta, u \rangle$$

along the path  $\text{Re } \beta = \frac{L}{2}$  traversed from top to bottom, and show that

$$u \rightarrow \frac{1}{2\pi i} \int_{\text{Re } \beta = \frac{L}{2}} \langle f_\beta, u \rangle d_i \beta$$



is a distribution. Let  $u = \sum a_{nm} Y_{nm} \in D(\Sigma)$ , then (1.2.11) implies that

$$(1.2.13) \quad \langle f_\beta, u \rangle = \sum_{n,m} \beta^{-a/2} (\beta - L - n(n+k-2))^{-1} a_{nm}.$$

Integrating (1.2.13) along the path  $\operatorname{Re} \beta = \frac{L}{2}$ , and using the residue theorem, we obtain

$$(2\pi i)^{-1} \int_{\operatorname{Re} \beta = \frac{L}{2}} \langle f_\beta, u \rangle d\beta = \sum_{n,m} (n(n+k-2) + L)^{-a/2} a_{nm},$$

i. e., the form

$$u \rightarrow \frac{1}{2\pi i} \int_{\operatorname{Re} \beta = \frac{L}{2}} \langle f_\beta, u \rangle$$

is a distribution whose expansion in spherical harmonics is

$$\sum (n(n+k-2) + L)^{-a/2} Y_{nm}.$$

The rotation invariant operator associated with this distribution on  $\Sigma$  is denoted by  $J^\alpha$  and its explicit expression in terms of spherical harmonics is

$$(1.2.14) \quad J^\alpha(Y_{nm}) = (n(n+k-2) + L)^{-a/2} Y_{nm}.$$

The operators  $J^\alpha$  were defined by Seeley (cfr. Seeley [1]). From the expression (1.2.14) it is clear that  $J^\alpha J^\beta = J^{\alpha+\beta}$  for every  $\alpha$  and  $\beta$  complex, and also that  $J^\alpha$  depends analytically on  $\alpha$ . Thus,  $\{J^\alpha\}_{\alpha \in \mathbb{C}}$  is an analytic abelian group of operators on  $D(\Sigma)$ , and  $D'(\Sigma)$ .

REMARK 1.2.5. The Proposition 1.2.1 shows that a distribution  $f = \sum a_{nm} Y_{nm}$  defines in a unique way a harmonic function  $F$  in  $|x| < 1$  given by

$$F(x) = \sum a_{nm} P_{nm}(x) = \sum r^n a_{nm} Y_{nm} \left( \frac{x}{|x|} \right), \quad (|x| = r).$$

(The convergence of the series is a consequence of the condition (i).) Given  $0 < r < 1$ , we may consider on  $\Sigma$  the function  $F_r$  given by

$$(1.2.15) \quad F_r(x') = F(rx') = \sum a_{nm} r^n Y_{nm}(x'), \quad (|x'| = 1).$$

In the sense of distributions we have

$$(1.2.16) \quad \lim_{\substack{r \rightarrow 1 \\ r < 1}} F_r = f,$$

i. e., a distribution  $f$  on  $\Sigma$  is the boundary value of a harmonic function  $F$  in  $|x| < 1$ .

Using Proposition 1.2.1 we may prove the existence of constants  $C$  and  $K$  such that for every  $1 > \varepsilon > 0$

$$(1.2.17) \quad \sup_{|x|=1-\varepsilon} |F(x)| \leq C(1 + \varepsilon^{-K})$$

and reciprocally if  $F(x)$  is harmonic in  $|x| < 1$  and satisfies (1.2.17) for some  $C$  and  $K$  there is a distribution  $f$  such that (1.2.16) holds. Since we will not use this result we omit the proof.

### 1.3. Fourier Transforms of Homogeneous Spherical Harmonics.

If  $\Phi \in \mathcal{D}(\mathbb{E}^k)$ ,  $r \geq 0$ ,  $x' \rightarrow \Phi(rx')$  ( $x' \in \Sigma$ ) is an element of  $D(\Sigma)$ . If  $f \in D'(\Sigma)$  we associate to  $f$  an analytic family of tempered distributions  $r^\lambda f$  for  $\text{Re } \lambda > -k$  given by

$$\langle r^\lambda f, \Phi \rangle = \int_0^\infty \langle f, \Phi(rx') \rangle r^{\lambda+k-1} dr.$$

$r^\lambda f$  is analytic in the whole plane with exception of the points  $\lambda = -k, -k - 1$  etc.  $r^\lambda f$  is regular at  $\lambda = -k - n^*$  ( $n^* = \text{integer} \geq 0$ ) if and only if (cfr. Gel'fand [1], p. 310)

$$(1.3.1) \quad \langle f, x_1^{\alpha_1} \dots x_k^{\alpha_k} \rangle = 0$$

for every multi-index  $\alpha$  with  $\sum \alpha_i = n^*$ . Since a spherical harmonic  $Y_{nm}$  is orthogonal to all polynomials of degree  $< n$  and also to all homogeneous polynomials of degree  $v$  such that  $v - n$  is odd, it follows that the condition (1.3.1) can be written in terms of the components  $a_{nm}$  of  $f$  in the following form

$$(1.3.1') \quad a_{nm} = 0 \quad \text{for } n = n^*, \quad n^* - 2l, \quad l = 0, 1, 2, \dots, \text{ etc.}$$

In particular,  $r^\lambda Y_{nm}(x')$  will be regular for all  $\lambda \neq -n - k, -n - k - 2, \dots$ , etc.

Let  $r = (\sum x_i^2)^{1/2}$ , and  $r^s Y_{nm} \left( -k < s < \frac{k}{2} \right)$  be the positively homogeneous function of degree  $s$  that coincides with  $Y_{nm}$  on  $\Sigma$ . Let  $\theta$  be the characteristic function of the unit ball in  $\mathbb{E}^k$ . Then,  $\theta(r^s Y_{nm}) \in L_1(\mathbb{E}^k)$  and  $(1 - \theta)(r^s Y_{nm}) \in L_2(\mathbb{E}^k)$ . The Fourier transform of  $\theta(r^s Y_{nm})$  is continuous and bounded (theorem of Riemann-Lebesgue) and  $[(1 - \theta)r^s Y_{nm}]^\wedge \in L_2(\mathbb{E}^k)$ .

It follows that the Fourier transform of  $r^s Y_{nm}$  is a homogeneous, locally integrable function that can be written in the form  $r^{-s+k} g$  ( $g \in L_2(\Sigma)$ ). To find the expansion of  $g$  in terms of spherical harmonics we recall that if  $P_{n'm'}$  is a homogeneous harmonic polynomial of degree  $n$  we have

$$(1.3.2) \quad (P_{n'm'} e^{-r^2/2})^\wedge = (-i)^{n'} (2\pi)^{k/2} P_{n'm'} e^{-r^2/2},$$

(where  $r = (\sum x_i^2)^{1/2}$ ).

From (1.3.2) it follows that the Fourier transform of a distribution orthogonal to  $P_{n'm'} e^{-r^2/2}$  is also orthogonal to  $P_{n'm'} e^{-r^2/2}$ . Thus,  $g$  must be of the form  $C Y_{nm}$ , where  $C$  is a constant. Let us evaluate  $C$ . Using the definition of the Fourier transform of a distribution and (1.3.2) we obtain

$$C \langle r^{-s-k} Y_{nm}, P_{nm} e^{-r^2/2} \rangle = (-i)^n (2\pi)^{k/2} \langle r^s Y_{nm}, P_{nm} e^{-r^2/2} \rangle$$

from which we deduce that

$$C = (-i)^n \pi^{k/2} 2^{s+k} \frac{\Gamma\left(\frac{s+n'+k}{2}\right)}{\Gamma\left(\frac{n'-s}{2}\right)}.$$

Therefore,

$$(1.3.3) \quad (r^s Y_{nm})^\wedge = (-i)^n \pi^{k/2} 2^{s+k} \frac{\Gamma\left(\frac{s+n+k}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)} Y_{nm} r^{k-s}.$$

It is clear that if we have an analytic family  $f_\lambda$  of tempered distributions in a region  $A_1$  of the complex plane  $\widehat{f}_\lambda$  is also analytic in  $A_1$ , furthermore, if  $f_\lambda$  can be extended to a larger region  $A_2$  also  $\widehat{f}_\lambda$  can be extended, and the same relation between the two families holds in  $A_2$ . Using this and our remarks on the points at which  $r^s Y_{nm}$  is regular, we conclude that (1.3.3) is valid in the whole  $s$  plane with exception of the points  $-n-k, -n-k-2, \dots$ , etc.

REMARK 1.3.1. By analytic continuation we deduce from (1.3.3)

$$(-i)^n \pi^{k/2} 2^{s+k} \frac{\Gamma\left(\frac{s+n+k}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)} Y_{nm} r^{k-s} = (-1)^l (-i)^n \Delta^l P_{nm}(D) \delta$$

for  $s = n + 2l$ . In fact, the left hand side of (1.3.3) is  $(r^{n+2l} Y_{nm})^\wedge = (r^{2l} P_{nm}(x))^\wedge$ .

CHAPTER II

THE SPACES  $L_p^s$

*Summary.*

- 2.1. Definition and basic properties.
- 2.2. Characterization of  $L_\infty^s \cap \mathfrak{E}'(\mathfrak{E}^k)$  and  $L_1^s \cap \mathfrak{E}'(\mathfrak{E}^k)$ .
- 2.3. Definition of the spaces  $L_p^s(M)$ ,  $1 \leq p \leq \infty$ ,  $-\infty < s < \infty$  where  $M$  compact manifold.

2.1. Definition of the  $L_p^s$  in  $\mathfrak{E}^k$ .

2.1.1. Let  $d(x) = d(|x|)$  be a positive infinitely differentiable function such that for  $|x| > 1$  coincides with  $|x|$ . If  $s$  a complex number we define an operator  $I^s$  on  $S'(\mathfrak{E}^k)$  by the relation (cfr. Calderón [1], p. 36)

$$(2.1.1) \quad (I^s f)^\wedge = \widehat{f} (d(x))^{-s}.$$

When  $s$  is a real number the image of  $L_p(\mathfrak{E}^k)$  by  $I^s$  denoted by  $L_p^s(\mathfrak{E}^k)$ . On  $L_p^s(\mathfrak{E}^k)$  we define a norm by  $\|f\|_{ps} = \|I^{-s}f\|_p$ ; with this norm  $L_p^s(\mathfrak{E}^k)$  is a Banach space. If  $s$  is a positive integer  $L_p^s(\mathfrak{E}^k)$  ( $1 < p < \infty$ ) is the class of functions whose distribution derivatives of order  $\leq s$  belong to  $L_p(\mathfrak{E}^k)$ . If  $s$  is a positive integer the sum of the  $L_p$  norms of the function and its derivatives of order  $\leq s$ , is equivalent to the norm defined above (cfr. Calderón [2], p. 36, and Neri [1]). Furthermore, if  $1 \leq p < \infty$  the dual of  $L_p^s(\mathfrak{E}^k)$  is  $L_{p'}^{-s}(\mathfrak{E}^k) \left( \frac{1}{p} + \frac{1}{p'} = 1 \right)$ , for every  $p$  all  $L_p^s$  are isomorphic, for every  $s$ .

REMARK 2.1.2. The space  $L_p^s(\mathfrak{E}^k)$  ( $1 < p < \infty$ ) do not depend on the particular choice of  $d(|x|)$ . In fact, if  $d_1$  and  $d_2$  have the property stated in 2.1.1, then  $\Phi = d_1^s d_2^{-s}$  satisfies the conditions:

$$(2.1.2) \quad |x|^r \left| \frac{\partial^r \Phi}{\partial x^r} \right| \leq A, \quad 0 \leq |r| < k.$$

Hence by Mihlin's theorem (cfr. Hörmander [1], p. 120) the linear mapping defined by

$$(Tf)^\wedge = \widehat{\Phi f}$$

is continuous from  $L_p$  to  $L_p$  with norm  $AC_p$ . Therefore, if  $I_1$  and  $I_2$  are the operators associated by (2.1.1) to  $d_1$  and  $d_2$ , the operator  $I_1^s I_2^{-s}$  is an isomorphism from  $L_p(\mathfrak{E}^k)$  onto  $L_p(\mathfrak{E}^k)$ .

REMARK 2.1.3. If  $v$  is real  $(d(x))^{iv}$  satisfies (2.1.2) with  $A = (1 + |v|)^k$ . Furthermore, if  $\text{Re } z > 0$ , both  $d(x)^{-z}$  and its derivative with respect to  $z$  satisfy (2.1.2) so, for  $\text{Re } z > 0$ ,  $I^z$  is an analytic family of operators from  $L_p(\mathfrak{E}^k)$  to  $L_p(\mathfrak{E}^k)$ .

Moreover, for any element  $f \in L_p(\mathfrak{E}^k)$   $I^z f$  defines an analytic family of distributions. The following simplified version of a theorem of Calderón will be useful to us (cfr. Calderón [2], p. 40).

LEMMA 2.1.4. Let  $A$  be a linear map defined in  $C_0^\infty(\mathfrak{E}^k)$  with values in  $\mathcal{S}'$ . Let  $\theta_0 = (\xi_0^1, \xi_0^2, \xi_0^3, \xi_0^4)$   $\theta_1 = (\xi_1^1, \xi_1^2, \xi_1^3, \xi_1^4)$   $\theta_t = (1 - t)\theta_0 + t\theta_1$ ,  $0 < \xi_0^i < 1$ ,  $0 \leq t \leq 1$ ,  $i = 1, 2$ . If there are two constants  $C_i (i = 0, 1)$  such that

$$(2.1.3) \quad \|Af\|_{\xi_t^1, \xi_t^3} \leq C_t \|f\|_{\xi_t^2, \xi_t^4} \quad t = 0, 1$$

then there is a logarithmically convex function  $C_t (0 \leq t \leq 1)$  such that (2.1.3) holds in the whole interval  $[0, 1]$ .

PROOF. We will use the following

LEMMA 2.1.5. Let  $\Phi(z)$  be analytic in the strip  $0 < \text{Re}(z) < 1$ , and suppose that  $\Phi(z)$  is bounded there. Let  $M_t = \sup_{-\infty < y < \infty} |\Phi(t + iy)|$ . Then  $Lg M_t$  is a convex function of  $t$  in  $[0, 1]$  (cfr. Stein [1], p. 422).

Let  $l_0$  and  $l_1$  be linear functions with real coefficients such that  $l_1(0) = \xi_0^3$ ,  $l_1(1) = \xi_1^3$ ,  $l_2(0) = -\xi_0^4$ ,  $l_2(1) = -\xi_1^4$ . If  $f \in C_0^\infty$ ,  $g \in C_0^\infty$ ,  $I^{l_1(z)} f \in L_p^s$  for every  $p (1 < p < \infty)$  and  $s$ . Using this and the continuity of  $A$  we may define  $AI^{l_1(z)} f$  and, by remark 2.1.3,  $I^{l_2(z)} AI^{l_1(z)} f$  will be an analytic family of distributions. Let

$$(2.1.4) \quad F(z, f, g, n) = \left\langle g, \frac{I^{l_2(z)} AI^{l_1(z)} f}{\exp\left(\frac{\cos nz}{n}\right)} \right\rangle \quad n > 0.$$

$F(z, f, g, n)$  has the following properties :

a) It is analytic in  $0 \leq \operatorname{Re} z \leq 1$  (for fixed  $n, f$  and  $g$ ).

b) It is bounded there; in fact, by remark (2.1.3), the numerator may be estimated by  $C|1 + \operatorname{Im} z|^{2k}$  and the denominator increases exponentially when  $\operatorname{Im} z \rightarrow \pm \infty$ .

Hence by Lemma 2.1.5 if  $z = t + iy$ ,  $LgM$  given by  $LgM(t, f, g, n) = \lg \sup_{-\infty < \operatorname{Im} z < \infty} |F(z, f, g, n)|$  is convex. On the other hand, for arbitrary  $t$ , the function  $LgM^*(t, f, g)$  given by  $LgM^*(t, f, g) = \lim_{n \rightarrow \infty} LgM(t, f, g, n) = \lg |\langle g, I^{1(t)} A I^{1(t)} f \rangle|$  is the limit of a sequence of convex functions, hence it will be convex (cfr. Bourbaki [1], p. 46).

Now  $Lg C_t$  will be the upper envelope of  $LgM^*(t, f, g)$  when  $\|f\|_{\frac{1}{\xi_0^2}} = 1$ ,

$\|g\|_{\frac{1}{\xi_0^1}} = 1$ . Hence  $\log D_t$  is convex.

## 2.2. The Spaces $L_\infty^s \cap F'$ and $L_1^s \cap E'$ .

We will use the following extension of a theorem of Calderón (cfr. Calderón [1], p. 36),

LEMMA 2.2.1. Let  $A^\alpha (\alpha > 0)$  be the operator defined by  $(A^\alpha f)^\wedge = |x|^\alpha \hat{f}$ . Then ;

$$(2.2.1) \quad I^\alpha = A^\alpha + S$$

where  $S$  is an operator which commutes with translations and maps  $L_p(E^k)$  ( $1 < p < \infty$ ) into the space  $S_p$  of functions in  $L_r$  with derivatives of all orders in  $L_r$  for  $0 \leq \frac{1}{r} \leq \frac{1}{p}$ .

PROOF. Let  $\Phi_1$  and  $\Phi_2$  be two functions in  $C_0^\infty(\mathbb{E}^k)$  such that  $d(x)^\alpha = |x|^\alpha (1 - \Phi_1) + \Phi_2$ . Let  $\Psi_1$  and  $\Psi_2$  be the inverse Fourier transforms of  $|y|^\alpha \Phi_1$  and  $\Phi_2$  respectively, and denote by  $S$  the convolution with  $\Psi_1 + \Psi_2$ .

i)  $\Psi_2$  is the Fourier transform of an element in  $S(E^k)$ , hence belongs to  $S_q$ ,  $1 \leq q \leq \infty$ . By Young's convolution theorem and differentiation under the integral sign, we deduce that  $L_p * \Psi_2 \subset S_p$ .

ii) We may write  $|x|^\alpha \tilde{\Phi}_1 = |x|^\beta \Phi_1$ , where  $0 \leq \beta < 2$  and  $\tilde{\Phi}_1 \in C_0^\infty(\mathbb{E}^k)$ . Let  $\tilde{\Psi}_1 \in \mathcal{S}(\mathbb{E}^k)$  be the inverse Fourier transform of  $\tilde{\Phi}_1$ . Let  $\Phi \in C_0^\infty\{x; |x| < 1\}$  be equal to 1 in a neighborhood of the origin. Let us assume if  $k \geq 2$ ,

$0 < \beta < 2$ , if  $k = 1$ ,  $0 < \beta \leq 2$  but  $\beta \neq 1$  then (cfr. (1.3.3))

$$(|x|^\beta)^\wedge = C|x|^{-\beta-k}$$

and

$$(2.2.3) \quad \Psi_1 = C(\Phi|x|^{-k-\beta} * \tilde{\Psi}_1) + C(1 - \Phi)|x|^{-k-\beta} * \tilde{\Psi}_1$$

The first term in the right hand side of (2.2.3) can be estimated by

$$(2.2.4) \quad |C(\Phi|x|^{-k-\beta} * \tilde{\Psi}_1)(y)| \leq C' \sum_{\substack{|z-y| < 1 \\ |\alpha| < N}} \sup |D^\alpha \tilde{\Psi}(z)| \quad (\text{for some } N)$$

because  $\Phi|x|^{-k-\beta}$  is a distribution of finite order with support in  $\{x; |x| < 1\}$ . Since  $\Psi(z)$  decreases faster than any rational function when  $|z| \rightarrow \infty$  it follows that  $C\Phi|x|^{-k-\beta} * \tilde{\Psi}_1$  belongs to  $\mathcal{S}_q$  for  $1 \leq q \leq \infty$ .

The second term in (2.2.3) is the convolution of the function  $C(1-\Phi)|x|^{-k-\beta}$ , that belongs to  $\mathcal{S}_q$  ( $1 \leq q \leq \infty$ ), with  $\tilde{\Psi}_1 \in \mathcal{S}$ , using Young's convolution theorem we conclude that  $C(1-\Phi)|x|^{-k-\beta} * \tilde{\Psi}_1 \in \mathcal{S}_q$  ( $1 \leq q \leq \infty$ ). I. e.,  $\Psi_1$  in this case  $\in \mathcal{S}_q$  ( $1 \leq q \leq \infty$ ), and by the argument in (i)  $L_p * \Psi_1 \subset \mathcal{S}_p$ .

If  $k = 1$  and  $\beta = 1$ ,  $|x| \tilde{\Phi}_1 = (\text{sgn } x)(x\tilde{\Phi}_1)$ , hence if we denote by  $Hf$  the Hilbert transform of  $f$ , we obtain

$$\Psi_1 * f = C' \left( \frac{d}{dx} \tilde{\Psi} \right) * Hf.$$

Since if  $f \in L_p(\mathbb{R}^k)$ ,  $Hf \in L_p$ , and  $\frac{d}{dx} \tilde{\Psi} \in \mathcal{S}_q$  ( $1 \leq q \leq \infty$ ) the argument in (i) proves that  $\Psi_1 * L_p \subset \mathcal{S}_p$  also in this case.

If  $\beta = 0$ , we use for  $\Psi_1$  the argument (i). The proof of the lemma is thus complete. ■

COROLLARY 2.2.2. a)  $S$  maps  $L_p^s$  into  $V = \bigcap_{\substack{0 < \frac{1}{r} \leq \frac{1}{p} \\ t < \infty}} L_r^t$

b)  $S$  maps  $\mathcal{C}'$  into  $\mathcal{S}_1$ .

PROOF. a) In fact:  $I^s V = V$  and  $\mathcal{S}_p \subset V$ , then  $S(L_p^s) = S(I^s L_p) \subset \subset I^s(\mathcal{S}_q) I^s(S(L_p))$ .

b) With the notations used in the proof of Lemma 2.2.1, we have for  $u \in \mathcal{C}'$ :  $S(u) = C|x|^{-k-\beta} * (\tilde{\Psi}_1 * u) + \Psi_2 * u$ . It is readily seen that  $\Psi_2 * u$  and  $\tilde{\Psi}_1 * u$  belong to  $\mathcal{S}$ , and from the argument used in ii) it follows that  $C|x|^{-k-\beta} (\tilde{\Psi}_1 * u)$  belongs to  $\mathcal{S}_1$ .

LEMMA 2.2.3. Let  $K$  be a compact set,  $\Omega$  a neighborhood of  $K$ , and  $u$  a distribution with  $(\text{supp } u) \subset K$ . Then  $u \in L_p^s(\mathfrak{E}^k)$ , if and only if  $I^{-s} u$ , restricted to  $\Omega$ , belongs to  $L_p(\Omega)$ .

PROOF. The definition of  $L_p^s(\mathfrak{E}^k)$  implies that the condition is necessary. Let  $s > 0$ . By Lemma 2.2.1,  $I^{-s} = A^s + S$ . By corollary 2.2.2 part b),  $S(u) \in S_1$ , hence  $I^{-s} u \in L_p$  if and only if  $A^s u \in L_p$  ( $1 \leq p \leq \infty$ ). If  $s = 2l$  is an even integer, then  $A^s u = CA^l u$  will belong to  $L_p$  if and only if its restriction to  $\Omega$  belongs to  $L_p$ . If  $s$  is not an even integer let  $\Phi \in C_0^\infty$ ,  $\Phi \equiv 1$  near zero and  $(\text{supp } \Phi) \subset \{x; |x| < \delta\}$  where  $2\delta = \text{dist}(K, C\Omega)$ . We may write

$$(2.2.5) \quad A^s u = C(\Phi |x|^{-s-k} * u) + C((1 - \Phi) |x|^{-s-k} * u)$$

The first term in the right hand side has support contained in  $\Omega$  and the second belongs to  $S_1$ . Hence  $A^s u \in L_p$  if and only if its restriction to  $\Omega$  belongs to  $L_p$ .

If  $s < 0$ , using the relation  $\langle f, g \rangle = \langle (I^{-s} f), (I^s g) \rangle$  and what we have already proved, the result follows.

COROLLARY 2.2.4. a) If  $u \in E'$  then  $u \in L_\infty^s$  if and only if for every  $p < \infty$ ,  $u \in L_p^s$  and  $\|u\|_{p^s}$  is bounded independently of  $p$ .

b) If  $u \in L_p^s(\mathfrak{E}^k) \cap E'(\mathfrak{E}^k)$ , and  $D^\alpha u \in L_p^s(\mathfrak{E}^k)$  for  $|\alpha| \leq k$ , then  $u \in L_\infty^s(\mathfrak{E}^k)$ .

PROOF. a) Let us first assume that  $\|u\|_{p^s} < N$ . Let  $\Omega$  be a bounded neighborhood of  $K$  and  $\omega$  be the characteristic function of  $\Omega$ . Then for every  $p < \infty$ ,  $\|\omega I^{-s} u\|_p \leq N$  implies that  $\text{ess sup } |\omega I^{-s} u| \leq N$ , and hence, by Lemma 2.2.3,  $u \in L_\infty^s$ . Reciprocally, let us assume that  $u \in L_\infty^s(\mathfrak{E}^k)$ , and  $\Omega$  is as above. We have shown (cfr. proof of Lemma 2.3.3) that  $(1 - \omega) I^{-s} u \in S_1$ . The convexity of the norms implies that there is a constant  $M > 0$  such that, for every  $p$ ,  $\|(I^{-s} u)(1 - \omega)\|_p \leq M$ . On the other hand, since  $\Omega$  is bounded and  $I^{-s} u \in L_\infty$ , it follows that there is a  $M_1$  such that, for all  $p$ ,  $\|(I^{-s} u)\omega\|_p \leq M_1$ . Consequently,  $u \in L_p^s$  for every  $1 \leq p < \infty$  and  $\|u\|_{p^s} \leq M + M_1$  independently of  $p$ .

b) For  $s = 0$ , the result is well-known (e. g., Hörmander [1], p. 97). With  $\Omega$  and  $\omega$  as above and  $|\alpha| \leq k$ ,  $\omega D^\alpha I^{-s} u = \omega I^{-s}(D^\alpha u) \in L_p$ , hence  $I^{-s} u \in L_\infty$  in  $\Omega$ , i.e.,  $u \in L_\infty^s$ .

COROLLARY 2.2.5. Let  $u \in E'$  and suppose that  $\Psi \in C^\infty$  vanishes on a neighborhood of  $\text{supp } u$  and  $\Psi = 1$  outside a compact set. Then, for all  $s > 0$ ,  $\Psi A^s u \in S_1$ .



**PROOF.** Consider  $\Phi \in C_0^\infty$  such that  $\Phi \equiv 1$  near the origin and  $\text{supp } \Phi \in \{|x| < \delta\}$  where  $\delta = \text{distance from } \text{supp } u \text{ to } \text{supp } \Psi$ . Then the result follows from expression (2.2.5) and the argument following it.

**COROLLARY 2.2.6.** Let  $u \in E'$  and  $s > 0$ . Then  $u \in L_p^s$ ,  $1 \leq p \leq \infty$ , if and only if  $\Lambda^s u \in L_p$ . In particular,  $L_1^s \cap E'$  and  $L_\infty^s \cap E'$  do not depend on the particular choice of  $d(x)$ .

**PROOF.** This is contained in the proof of Lemma 2.2.3.

**COROLLARY 2.2.7.** Let  $K$  be a compact set,  $\Omega$  a neighborhood of  $K$ ,  $u$  a distribution with support in  $K$ . Then  $u \in L_p^s(\mathbb{E}^k)$ ,  $1 < p < \infty$ , if and only if  $I^{-s+iv} u$  restricted to  $\Omega$  belong to  $L_p(\Omega)$ , ( $v$  real).

**PROOF.** Using the fact that  $I^{iv}$  is an isomorphism from  $L_p$  onto  $L_p$ , the proof is the same as for Lemma 2.2.3.

### 2.3. Definition of the Spaces $L_p^s$ on a Compact Manifold.

We recall that given two open sets  $\Omega_1, \Omega_2$  in  $\mathbb{E}^k$ , and  $\Psi: \Omega_1 \rightarrow \Omega_2$  a  $C^\infty$  diffeomorphism from  $\Omega_1$  onto  $\Omega_2$ , to each distribution  $u \in D'(\Omega_2)$  we associate a distribution on  $\Omega_1$  that we denote by  $u \circ \Psi$  and is given

$$(2.3.1) \quad \langle u \circ \Psi, \Phi \rangle = \langle u, (\Phi \circ \Psi^{-1}) |J| \rangle \quad \Phi \in C_0^\infty(\Omega_1)$$

and  $|J|$  denotes the Jacobian determinant of the mapping  $\Psi^{-1}$ .

**PROPOSITION 2.3.1.** With the notations above if  $u \in E'(\Omega_2)$  then  $u \in L_p^s$ ,  $1 \leq p \leq \infty$  and  $-\infty < s < +\infty$ , if and only if  $u \circ \Psi \in L_p^s$ .

**PROOF.** Let  $u \in \mathcal{C}'(\Omega_2)$ ,  $\Phi_1 \in C_0^\infty(\Omega_1)$  and  $\Phi_2 \in C_0^\infty(\Omega_2)$  be such that  $\Phi_1 = 1$  in a neighborhood of  $\text{supp } u \circ \Psi$  and  $\Phi_2 = 1$  in a neighborhood of  $\text{supp } u$ .

Let  $T_\Psi$  be the linear mapping from  $D'(\Omega_2)$  to  $D'(\Omega_1)$  defined by

$$(2.3.1) \quad T_\Psi(v) = \Phi_1((v \Phi_2) \circ \Psi).$$

Since  $T_\Psi(u) = u \circ \Psi$ , it will be enough to show that  $T_\Psi$  is continuous from  $L_p^s$ ,  $1 \leq p \leq \infty$  and  $-\infty < s < \infty$ , into  $L_p^s$ .

If  $s \geq 0$  is even, Corollary 2.2.6 shows that the image of  $L_p^s$  by  $T_\Psi$  is contained in  $L_p^s$ . The Closed Graph theorem proves the continuity. By duality it follows that our statement holds for an arbitrary even integer.

If  $n$  is an even integer, let  $C_p^n$  and  $C_p^{n+p}$  be such that

$$(2.3.2) \quad \|T_\Psi f\|_{ps} \leq C_p^s \|f\|_{ps} \quad s = n, n + 2; 1 \leq p \leq \infty.$$

This is equivalent to:

$$(2.3.3) \quad \|I^{-s} T_\Psi I^{+s} g\|_p \leq C_p^s \|g\|_p, \quad 1 \leq p \leq \infty \text{ and } s = n, n + 2.$$

Therefore by the convexity theorem of Riesz-Thorin (cfr. Zygmund [2], p. 225) there exists a constant  $C_*^s$ , independent of  $p$ , such that (2.3.3) and (2.3.2) hold for every  $p$  with  $C_p^s = C_*^s$ . Using this fact and Lemma 2.1.4 it follows that for every  $t \in [n, n + 2]$  there exist  $C_*^t$  such that, for every  $1 < p < \infty$

$$(2.3.4) \quad \|T_\Psi f\|_{pt} \leq C_*^t \|f\|_{pt}$$

By corollary 2.2.4 and the fact that in (2.3.4) the constant is independent of  $p$  it follows that the image by  $T_\Psi$  of a function in  $L_\infty^s$  belongs to  $L_\infty^s$ ; the Closed Graph theorem then implies the continuity of  $T_\Psi$  in the spaces  $L_\infty^s$ . To prove the continuity in the spaces  $L_1^s(\mathbb{E}^k)$  we note first that by (2.3.1)

$${}^t T_\Psi(\Phi) = |J| \Phi_2((\Phi_1 \Phi) \circ \Psi^{-1})$$

This formula and what we proved above shows that  ${}^t T_\Psi$  is continuous from  $L_\infty^s(\mathbb{E}^k)$  to  $L_\infty^s(\mathbb{E}^k)$ . On the other and it is clear that  $T_\Psi$  is closed and densely defined as an operator from  $L_1^{-s}(\mathbb{E}^k)$  into  $L_1^{-s}(\mathbb{E}^k)$ , thus  $T_\Psi$  is continuous from  $L_1^{-s}(\mathbb{E}^k)$  into  $L_1^{-s}(\mathbb{E}^k)$  (cfr. S. Goldberg [1], p. 57).

The preceding proposition justifies the following:

**DEFINITION.** Let  $M$  be a compact  $C^\infty$  manifold,  $\mathcal{U} = \{U_i\}$  a finite open covering of  $M$  by coordinate neighborhoods, and  $\{\Phi_i\}$  a  $C^\infty$  partition of unity such that  $\text{supp } \Phi_i \subset U_i$ . If  $1 \leq p \leq \infty$  and  $-\infty < s < \infty$  a distribution  $u$  on  $M$  is said to belong to  $L_p^s(M)$  if and only if  $\Phi_i u$  expressed in the coordinates of  $U_i$  belongs to  $L_p^s$ .

**PROPOSITION 2.3.2.** Let  $u$  be a distribution on  $\Sigma_{k-1} = \{x \in \mathbb{E}^k\}$ . Let us consider the following conditions:

- a)  $u \in L_p^s(\Sigma_{k-1})$
- b) For an arbitrary  $\Psi \in C_0^\infty(\mathbb{E}^k - \{0\})$ ,  $\Psi(r^0 u) \in L_p^s(\mathbb{E}^k)$  (cfr. 1.3 for the definition of  $r^0 u$ ).
- c) There is  $f \in L_p(\Sigma_k)$  such that  $J^s f = u$ .

Then, for  $1 \leq p \leq \infty$ ,  $-\infty < s < +\infty$  a) and b) are equivalent, for  $1 < p < \infty$  and  $-\infty < s < +\infty$  a) and b) and c) are equivalent.

PROOF. If  $u \in E'(\mathbb{E}^{k-1})$  we may extend  $u$  into a distribution  $x_k^0 u$  in  $D'(\mathbb{E}^k)$  by means of the relation

$$\langle x_k^0 u, \Phi \rangle = \int_{-\infty}^{+\infty} \langle u, \Phi(x_1, \dots, x_{k-1}, x_k) \rangle dx_k.$$

Let us prove that  $u \in L_p^s(\mathbb{E}^{k-1})$  ( $1 \leq p \leq \infty$ ,  $-\infty < s < +\infty$ ) if and only if for every  $\Phi \in C_0^\infty(\mathbb{E}^k)$ ,  $x_k^0 u \Phi \in L_p^s(\mathbb{E}^k)$ . In fact if  $s$  is even  $\geq 0$ , the result is easily obtained from Corollary 2.2.6. An argument similar to the one used in the proof 2.3.1 extends the result to the general case.

Let  $x = (x_1, x_2, \dots, x_{k-1}, 0)$  be such that  $0 < (\sum x_i^2)^{1/2} < 1$ , and let  $\Psi$  be the diffeomorphism from a neighborhood  $\Omega_1$  of  $x$ , onto a neighborhood

$\Omega_2$  of  $x^* = (x_1, \dots, x_{k-1}, (1 - \sum x_i^2)^{1/2}) \in \Sigma_{k-1}$  given by

$$\Psi: (x_1, \dots, x_k) \rightarrow \left( x_1, \dots, x_{k-1}, \sqrt{1 - \sum x_i^2} \right) e^{x_k}$$

If  $\Omega_1$  and  $\Omega_2$  are small enough,  $\Psi$  is one-to-one and onto, and send  $\Omega_1 \cap \{x; x_k = \text{constant}\}$  onto  $\Omega_2 \cap \{x; |x| = \text{constant}\}$  and the lines normal to the hyperplane  $x_k = 0$  into the normals to the sphere  $\Sigma_{k-1} = \{x; |x| = 1\}$ , hence a distribution in  $\Omega_2$  of the form  $r_0 f$  if and only if  $(r_0 f) \circ \Psi$  is of the form  $x_k^0 u$  ( $u \in D'(\Omega_1 \cap \{x; x_k = 0\})$ ).

The equivalence between a) and b) is thus a consequence of Proposition 2.3.1 and what we proved above. The equivalence between a) and c) follows from the fact that  $J^s$  is a pseudo-differential operator of order  $s$ , and is invertible (cfr. example 1.2.5 and Seeley [2]).

REMARK 2.3.3. For  $f \in L_p^s(\Sigma_{k-1})$  we shall use the norms  $\|f\|_{ps} = \|J^{-s} f\|_p$ , if  $1 < p < \infty$ . For  $p = 1$  or  $\infty$  if  $\{\Phi_i\}$  is any finite  $C^\infty$  partition of the unity, subordinate to a covering  $\{U_i\}$ , we define the  $L_p^s(\Sigma_{k-1})$  norms by  $\|f\|_{ps} = \sum \|\Phi_i f\|_{ps}$ . By Proposition 2.3.1 the norms corresponding to different coverings and different partitions are equivalent.

## CHAPTER III

## FOURIER TRANSFORMS OF HOMOGENEOUS DISTRIBUTIONS

*Summary.*

- 3.1. Structure and integral representation of homogeneous distributions.
- 3.2. Relations of regularity in  $L_2^s$  between a homogeneous distribution and its Fourier transform.
- 3.3. Relation of regularity of the homogeneous distributions and their Fourier transforms in  $L_p^s$ .
- 3.4. A counterexample.

**2.1. Structure of Homogeneous Distributions.**

Given a distribution  $\tau \in \mathcal{S}'(\mathbb{E}^k)$  and a complex number  $\lambda$  we say that  $\tau$  is homogeneous of degree  $\lambda$  if for any  $\Psi \in D(\mathbb{E}^k)$  and any  $\alpha > 0$

$$(3.1.0) \quad \left\langle \tau, \Psi\left(\frac{x}{\alpha}\right) \right\rangle = \alpha^{\lambda+k} \langle \tau, \Psi \rangle$$

**EXAMPLE 3.1.0.** If  $f \in D'(\Sigma)$  and  $r^\lambda f$  is a distribution, it is homogeneous of degree  $\lambda$  (cfr. 1.3 for the definition of  $r^\lambda f$ ). If  $P_n$  is a homogeneous polynomial of degree  $n$   $P_n(D)\delta$  is homogeneous of degree  $-n-k$ .

From now on we will assume that  $k \geq 2$ .

Our aim in this section is the following :

**THEOREM 3.1.1.** Let  $\lambda$  be a complex number and  $\tau$  a homogeneous distribution of degree  $\lambda$ . Then,

- a) If  $\lambda$  is not an integer  $\leq -k$ , there exists an  $f \in D'(\Sigma)$  such that

$$(3.1.1) \quad \tau = r^\lambda f.$$

- b) If  $\lambda = -k, -k-1, \dots$ , etc. There are  $f \in D'(\Sigma)$  and a polynomial  $P_{-\lambda-k}$  homogeneous of degree  $-\lambda-k$  such that

$$(3.1.1') \quad \tau = r^\lambda f + P_{-\lambda-k}(D)\delta \quad (\delta = \text{Dirac measure}).$$

To prove this theorem we will use some lemmas

LEMMA 3.1.2. If  $\lambda$  is a complex number such that  $\operatorname{Re} \lambda > -k$  the linear map  $L_\lambda$  from  $D'(\Sigma)$  to  $S'(\mathbb{E}^k)$  defined by

$$L_\lambda(f) = r^\lambda f$$

is continuous for the weak and strong topologies on  $D'(\Sigma)$  and  $S'(\mathbb{E}^k)$ .

PROOF. Since the continuity for the strong topologies is a consequence of the weak continuity, it suffices to prove (cfr. Horváth [2], p. 224) that for every  $\Phi' \in S(\mathbb{E}^k)$  there is a  $\tilde{\Phi} \in D(\Sigma)$  such that

$$\langle f, \tilde{\Phi} \rangle = \langle L_\lambda(f), \Phi' \rangle$$

If  $\Phi' \in S(\mathbb{E}^k)$  it is readily seen that  $\Phi(x) = \int_0^\infty \Phi'(tx) t^{\lambda+k-1} dt$  is homo-

geneous of degree  $-(\lambda+k)$  and infinitely differentiable: in  $|x| > 0$ . Let  $\tilde{\Phi}$  be the restriction to  $\Sigma$  of  $\Phi$ ; then,  $\tilde{\Phi} \in D(\Sigma)$ . On the other hand, from the continuity of  $f$  we have

$$\begin{aligned} \langle f, \tilde{\Phi} \rangle &= \left\langle f, \int_0^\infty \Phi'(tx') t^{\lambda+k-1} dt \right\rangle = \int_0^\infty \langle f, \Phi'(tx') \rangle t^{\lambda+k-1} dt \\ &= \langle r^\lambda f, \Phi' \rangle = \langle L_\lambda(f), \Phi' \rangle \end{aligned}$$

LEMMA 3.1.3. If  $\tau$  is a homogeneous distribution of degree  $\lambda$  and  $\operatorname{Re} \lambda > -k$ , or  $\lambda$  is not an integer,  $\tau$  can be expressed in the form (3.1.1).

PROOF. We recall that in  $R_+$  there is, up to constant factors, one and only one homogeneous distribution of degree  $\lambda$ , i. e.,  $r_+^\lambda$  (Gel'fand and Shilov [1], p. 80). Given  $u \in D(R_+)$  and  $k \in D(R_+)$  we define on  $\mathbb{E}^k$  an element  $k \otimes u$  of  $D(\mathbb{E}^k)$  by

$$(k \otimes u)(x) = k(|x|) u\left(\frac{x}{|x|}\right).$$

Consequently, the linear form

$$k \mapsto \langle \tau, k \otimes u \rangle \quad k \in D(R_+)$$

is a homogeneous distribution on  $R_+$  that can be written in the form

$$C_{\tau u} r^{\lambda+k-1}$$

where  $C_{\tau u}$  is a constant. If  $k(t) \in D(R_+)$  satisfies

- (i)  $\text{supp } k(t)$  is contained in  $1 < |x| < 2$  and
- (ii)  $\langle r^{\lambda+k-1}, k \rangle = 1$ , we define a distribution  $f \in D'(\Sigma)$  by

$$(3.1.2) \quad \langle f, u \rangle = \langle \tau, k(r) \otimes u(x') \rangle \quad u \in D(\Sigma).$$

We claim that, for any test-function of the form  $\Psi \otimes u$  ( $\Psi \in D(\mathbb{R}_+)$  and  $u \in D(\Sigma)$ ),

$$(3.1.3) \quad \langle r^\lambda f, \Psi \otimes u \rangle - \langle \tau, \Psi \otimes u \rangle = 0.$$

In fact, using (3.1.2), the left hand side of (3.1.3) becomes

$$(3.1.4) \quad \int_0^\infty \langle \tau, k(t) \otimes u(x') \rangle r^{\lambda+k-1} \Psi(r) dr - \langle \tau, \Psi(r) \otimes u(x') \rangle$$

$$= C_{ru} \langle r^{\lambda+k-1}, k(t) \rangle \langle r^{\lambda+k-1}, \Psi(r) \rangle - C_{ru} \langle r^{\lambda+k-1}, \Psi(r) \rangle$$

$$= 0$$

by condition (ii) for  $k$ .

We now prove that  $r^\lambda f = \tau$  also in  $U = \{x; |x| > 1/2\}$ . To do so, we only need to show that if  $\Psi \in D(U)$ ,  $\Psi$  is the limit of functions of the form

$$(3.1.5) \quad \sum_j \kappa_n^j \otimes \Phi_n^j \quad (\kappa_n^j \in D(\mathbb{R}_+), \Phi_n^j \in D(\Sigma)).$$

Using a partition of unity, we may assume that

$$\text{supp } \Psi \subset V = \left\{ x; \frac{\langle x, e_1 \rangle}{\|x\|} > 1/2 \right\} e_1 = (1, 0, \dots, 0).$$

In  $U \cap V$  we may take a system of coordinates  $(r, x'_2, \dots, x'_k)$  where  $(x'_2, \dots, x'_k)$  is a system of coordinates valid on  $\Sigma \cap V$  and  $r$  is the distance from the origin. Then  $\Psi(r, x'_2, \dots, x'_k)$  is the limit in  $D$  of functions of the form (3.1.5) (cfr. Horváth [2], p. 369).

From (3.1.3) and the preceding argument we conclude that  $r^\lambda f - \tau$  is a homogeneous distribution of degree  $\lambda$  with support at the origin. But any such distribution must be of the form  $P_n(D)(\delta)$  and hence it cannot be homogeneous of degree  $\lambda$  (cfr. examples 3.1.0) unless it be zero. ■

REMARK 3.1.4. A distribution  $\tau$  defined in an open convex cone  $\Omega$  will be called *homogeneous of degree  $\lambda$*  if (3.1.0) holds for every  $\Psi \in D(\Omega)$ .

The proof of Lemma 3.1.3 shows that, if  $\Omega \neq \mathfrak{E}^k$  is an open convex cone and  $\Omega \cap \Sigma = \Sigma^*$ , any homogeneous distribution  $\tau$  in  $\Omega$  can be written in the form  $\tau = r^\lambda f$ ; where  $f \in D'(\Sigma^*)$ .

PROOF OF THEOREM 3.1.1. Lemma 3.1.3 took care of the case  $\lambda_1 \neq -k, -k-1, \dots$ , etc. Suppose now that  $\tau$  is homogeneous of degree  $\lambda = -k - n^*$  ( $n^* = 0, 1, 2, \dots$ ). Then  $\widehat{\tau}$  will be homogeneous of degree  $n^*$  (cfr. Neri [1]). Consequently, by Lemma 3.1.3, we have  $\widehat{\tau} = r^{n^*} h$ , where  $h \in D'(\Sigma)$ . Let us expand  $h$  in spherical harmonic (Proposition 1.2.1).

$$(3.1.6) \quad h = \sum_{\substack{n^*-n=2l \\ l=0, 1, \dots}} a_{nm} Y_{nm} \sum_{\substack{n^*-n \neq 2l \\ l=0, 1, \dots}} a_{nm} Y_{nm}.$$

By remark 1.3.1, we obtain

$$(r^{n^*} (\sum_{n^*-n=2l} a_{nm} Y_{nm}(x')))^\wedge = P_{n^*}(D) \delta$$

where  $P_{n^*}$  is a homogeneous polynomial of degree  $n^*$ .

The Fourier inversion formula, the continuity and linearity of  $h \rightarrow r^{n^*} h$  (Lemma 3.1.2), and of the Fourier transform, together with (1.3.3) and (3.1.6) imply that

$$(3.1.6') \quad \tau(-x) = \frac{1}{(2\pi)^k} \left[ P_{n^*}(D) \delta + \left( \sum_{\substack{n^*-n \neq 2l \\ l=0, 1, 2}} (-i)^n \pi^{k/2} 2^{n^*+k} a_{nm} \frac{\Gamma\left(\frac{n^*+n+k}{2}\right)}{\Gamma\left(\frac{n-n^*}{2}\right)} Y_{nm}(x') \right) r^{-k-n^*} \right]$$

where the expression in parenthesis is a distribution  $f$  on  $\Sigma_{k-1}$ . The proof of Theorem 3.1.1 is thus complete.

DEFINITION 3.1.5. The distribution  $f$  on  $\Sigma_{k-1}$  associated to  $\tau$  by formula (3.1.1) or (3.1.1') will be called the *characteristic* of  $\tau$ . (This agrees with the usual terminology employed in the theory of singular integral equations).

DEFINITION 3.1.6. If  $f \in D'(\Sigma)$ , a homogeneous distribution of the form  $r^\lambda f$  will be called *simple*.

### 3.2. Fourier Transform of Homogeneous Distributions Locally in $L_2^s$ .

We introduce first some notation. If  $n^*$  is a real number  $D'(n^*)$  will denote the set of all distributions  $f \in D'(\Sigma)$  whose expansion in spherical

harmonics is of the form (cfr. (1.3.1'))

$$(3.2.1) \quad \sum_{\substack{n^*-n \neq 2l \\ l=0, 1, \dots, \text{etc.}}} a_{nm} Y_{nm} .$$

In other words,  $f \in D'(n^*)$  if the homogeneous distribution  $r^\lambda f$  is regular at  $\lambda = -k - n^*$ . We will write also  $L_p^s(n^*) = L_p^s(\Sigma) \cap D'(n^*)$  and  $D(n^*) = D(\Sigma) \cap D'(n^*)$ . It is clear that if  $n^* \neq 0, 1, 2, \dots, \text{etc.}$   $D(n^*) = D(\Sigma)$ .

$D'(n^*)$  is a closed subspace of  $D'(\Sigma)$  with finite codimension. If  $s$  and  $t$  are arbitrary reals it follows by (1.2.14) and by Proposition 2.3.2 that

$$J^t [L_2^s(n^*)] = L_2^{s+t}(n^*) .$$

Moreover, if  $h$  is a distribution on  $\Sigma$  the associated linear mapping  $T_h$ , (cfr. (1.2.6)), from  $D'(\Sigma)$  to  $D'(\Sigma)$  maps  $D'(n^*)$  into  $D'(n^*)$  for every  $n^*$ .

We give now an improvement of Lemma 3.1.2.

LEMMA 3.2.1. Let  $H_{n^*}$  be the space of functions analytic in the complex  $\lambda$  plane with exception of the points  $-k - n^* - 1, -k - n^* - 2, \text{etc.}$  Let us provide  $H_{n^*}$  with the topology of uniform convergence on compact sets and  $D'(n^*)$  with the topology of a closed subspace of the strong dual of  $D(\Sigma)$ . Then if  $\Phi \in \mathcal{D}(\mathbb{E}^k)$  the linear mapping

$$f \mapsto \langle r^\lambda f, \Phi \rangle$$

of  $D'(n^*)$  into  $H_{n^*}$  is continuous.

PROOF. By Lemma 3.1.2 if  $f_\nu \rightarrow f_0$  in  $D'(\Sigma)$  and  $\text{Re } \lambda > -k$ , then  $\langle r^\lambda f_\nu, \Phi \rangle \rightarrow \langle r^\lambda f_0, \Phi \rangle$ . Hence, if  $f_\nu \rightarrow 0$  and  $\langle r^\lambda f_\nu, \Phi \rangle$  is convergent, we deduce that  $r^\lambda f_\nu \rightarrow 0$ , i. e., the linear mapping has a closed graph. The result follows then by the Closed Graph theorem.

COROLLARY 3.2.2. The linear mapping  $f \rightarrow r^{-k-n^*} f$  is continuous from  $D(n^*)$  into  $\mathcal{D}'$  for the weak and strong topologies.

PROOF. In this case the weak and strong continuities are equivalent, hence it suffices to consider the strong topology. If  $f_\nu \rightarrow f_0$  strongly in  $D(n^*)$ , then if  $\Phi \in \mathcal{D}$ , the analytic function  $\lambda \rightarrow \langle r^\lambda f_\nu, \Phi \rangle$  converges to  $\lambda \rightarrow \langle r^\lambda f_0, \Phi \rangle$  uniformly on compact set of the  $\lambda$ -plane slit along the reals  $\leq -k - n^* - 1$  (Lemma 3.2.1). In particular, for  $\lambda = -k - n^*$ ,  $\langle r^{-k-n^*} f_\nu, \Phi \rangle \rightarrow \langle r^{-k-n^*} f_0, \Phi \rangle$  and the corollary again follows from the Closed Graph theorem. ■



LEMMA 3.2.2. Let  $\beta_{nm}$  be the components of a distribution  $h \in D'(\Sigma)$  and let  $T_h$  be the linear map from  $D'(n^*)$  into  $D'(n^*)$  associated to it (cfr. § 1.2). Then  $T_h$ , restricted to  $L_2^s(n^*)$ , is continuous from  $L_2^s(n^*)$  into  $L_2^{s+t}(n^*)$  if and only if:  $(\beta_{nm}) = 0(n^{-t})$ .

PROOF. If  $(\beta_{nm}) = 0(n^{-t})$  then  $(n(n+k-2) + L)^{t/2} \beta_{nm} = 0(1)$ . Hence, the map  $T_h \circ J^{-t}$ , given by

$$T_h \circ J^{-t}(\Sigma a_{nm} Y_{nm}) = \Sigma a_{nm} (\beta_{nm}) (n(n+k-2) + L)^{t/2} Y_{nm}$$

is bounded from  $L_2(n^*)$  to  $L_2(n^*)$ . This implies that

$$T_h = J^{s+t}(T_h J^{-t}) J^{-s}$$

is bounded from  $L_2^s(n^*)$  to  $L_2^{s+t}(n^*)$ .

Conversely, if  $T_h$  is continuous from  $L_2^s(\Sigma)$  to  $L_2^{s+t}(\Sigma)$  for every  $s$  real, then  $T_h J^{-t}$  is continuous from  $L_2(\Sigma)$  to  $L_2(\Sigma)$ . Consequently,

$$(n(n+k-2) + L)^{t/2} \beta_{nm} = 0(1)$$

and so  $(\beta_{nm}) = 0(n^{-t})$ . ▣

THEOREM 3.2.4. Let  $F$  be a homogeneous distribution on  $E^k$  of degree  $\lambda$  and let  $G = \widehat{F}$ . Then, if the characteristic  $f$  of  $F$  belongs to  $L_2^s$ , the characteristic  $g$  of  $G$  belongs to  $L_2^{s-k/2-\text{Re } \lambda}$ .

PROOF. Suppose that  $\lambda$  is not an integer, hence  $F = r^\lambda f$  and  $G = r^{-\lambda-k} g$  (Theorem 3.1.1). Let us expand  $f$  in spherical harmonics (Proposition 1.2.1),

$$f = \Sigma a_{nm} Y_{nm} .$$

The continuity of  $f \rightarrow r^\lambda f$  (Corollary 3.2.2) and of the Fourier transform, together with formula (1.3.3), imply that

$$g = \Sigma (-i)^n 2^{\lambda+k} \frac{\Gamma\left(\frac{n+\lambda+k}{2}\right)}{\Gamma\left(\frac{n-\lambda}{2}\right)} a_{km} Y_{nm} .$$

The asymptotic expansion of the  $\Gamma$  function (i.e., Stirling's formula) (cfr. Bourbaki [2], p. 181) implies that, as  $n \rightarrow \infty$ ,

$$(3.2.2) \quad \beta_{nm} = (-i)^n 2^{\lambda+k} \frac{\Gamma\left(\frac{n+\lambda+k}{2}\right)}{\Gamma\left(\frac{n-\lambda}{2}\right)} = 0(n^{\text{Re } \lambda + k/2}) .$$

Hence, by Lemma (3.2.3), the map  $f \rightarrow g$  transforms  $L_2^s(n^*)$  continuously into  $L_2^{s-\text{Re } \lambda - k/2}(n^*)$  which is what we wanted to prove.

If  $\lambda$  is an integer  $n^*$ , we note that the linear mapping

$$T_\lambda : f = \sum a_{nm} Y_{nm} \rightarrow g = \sum \beta_{nm} a_{nm} Y_{nm}$$

is well defined from  $D'(|n^*|)$ , into  $D'(|n^*|)$ , and, by formula (3.2.2), maps  $L_2^s(|n^*|)$  continuously into  $L_2^{s-\text{Re } \lambda - k/2}(|n^*|)$ .

A moment of reflection shows that there are two functions  $f_1$  and  $g_1$  in  $D(\Sigma)$  such that  $f - f_1 \in D'(|n^*|)$ ,  $g - g_1 \in D'(|n^*|)$ , and

$$T_\lambda(f - f_1) = g - g_1.$$

Consequently, if  $f \in L_2^s$  then  $g \in L_2^{s-\text{Re } \lambda - k/2}$ , and the proof is complete.  $\blacksquare$

**DEFINITION 3.2.5.** A homogeneous distributions is said to be *locally* in  $L_p^s$  if for arbitrary  $\Psi \in C_0^\infty(E^k - \{0\})$ ,  $\Psi F \in L_p^s$ .

**COROLLARY 3.2.6.** The Fourier transform of a homogeneous distribution  $F$  of degree  $\lambda$  that is locally in  $L_2^s$  is a homogeneous distribution  $G$  of degree  $-\lambda - k$  locally in  $L_2^{s-\text{Re } \lambda - k/2}$ .

**PROOF.** By Proposition 2.3.2  $F$  is locally in  $L_2^s$  if and only if its characteristic  $f \in L_2^s(\Sigma)$ . By Theorem 3.2.4, the characteristic  $g$  of  $G$  belongs to  $L_2^{s-\text{Re } \lambda - k/2}$  if  $f \in L_2^s(\Sigma)$ . Hence,  $G$  is locally in  $L_2^{s-\text{Re } \lambda - k/2}$  if  $F$  is locally in  $L_2^s$ .

In particular, the Fourier transform of a homogeneous distribution that is  $C^\infty$  in the complement of the origin is a homogeneous distribution  $C^\infty$  in the complement of the origin (i. e., with singular support at the origin). For  $\lambda = -k$  this is a well-known result of Calderón and Zygmund (cf. [1], p. 314).

**COROLLARY 3.2.7.** With the notations above, if  $\lambda = -k/2$  and  $f$  is given by

$$f = \sum a_{nm} Y_{nm}.$$

Thus,  $g$  is given by

$$(3.2.3) \quad g = \sum_{n,m} (-i)^n \pi^{k/2} 2^{k/2} a_{nm} Y_{nm}$$

and, (due to Hörmander and Calderón for  $s = 0$  (cfr. Preface)),

$$(3.2.4) \quad \|f\|_{2s} = \frac{1}{(2\pi)^{k/2}} \|g\|_{2s}.$$

Proof. The expression for  $g$  is a consequence of (1.3.3) and the continuity of  $f \rightarrow r^{-k|2} f$ . Formula (3.2.4) is a consequence of the definition of the norms in  $L_2^s(\Sigma)$  and of the expression (1.2.14) for the  $J^s$ . ■

### 3.3. Fourier Transform of Homogeneous Distributions Locally in $L_p^s$ .

In this section we study the action of the operator  $A$  on homogeneous distributions. As a consequence we derive a necessary and sufficient condition for a homogeneous distribution  $F$  to have its Fourier transform  $G$  locally in  $L_p^s$ .

If  $f \in D'(\Sigma)$  then the analytic family  $r^\beta f$  will be regular at  $\beta = t - \lambda - k$  if and only if:

a)  $t - \lambda$  is not an integer  $\leq 0$ ,

or

b)  $\lambda - t = n^* \geq 0$  and  $f \in D'(n^*)$ .

If one of the condition a) or b) holds and  $r^\lambda f$  is a distribution we may define

$$(3.3.0) \quad [A^t(r^\lambda f)]^\wedge = r^t (r^\lambda f)^\wedge.$$

In other words,  $A^t(r^\lambda f)$  is defined when  $f \in D'(-\lambda - k) \cap D'(\lambda - t)$ .

If we want the distribution  $A^{-t}(r^{\lambda-t} f)$  to be simple we must assume in addition that  $f \in D'(-\lambda - k + t)$ .

We introduce the notation  $D'(\lambda, t)$  for the space of distributions  $f \in D'(\Sigma)$  such that  $A^t(r^\lambda f)$  and  $A^{-t}(r^{\lambda-t} f)$  are simple homogeneous distributions. We will also use the notations:  $L_p^s(\lambda, t) = L_p^s(\Sigma) \cap D'(\lambda, t)$ , and

$$(3.3.1) \quad A^t(r^\lambda f) = r^{\lambda-t} T_\lambda^t(f), \quad (f \in D'(\lambda, t)).$$

**THEOREM 3.3.1.** The operator  $T_\lambda^t$ , given by (3.3.1), has the following properties:

a)  $T_\lambda^t$  is an isomorphism from  $L_p^s(\lambda, t)$  onto  $L_p^{s-\text{Re } t}(\lambda, t)$  if  $1 < p < \infty$  and  $-\infty < s < \infty$ .

b) If  $t$  is real,  $T_\lambda^t$  is an isomorphism from  $L_p^s(\lambda, t)$  onto  $L_p^{s-t}$ ,  $1 \leq p \leq \infty$  and  $-\infty < s < \infty$ .

**NOTE:** The theorem, roughly speaking, says that if the characteristic  $f$  of a homogeneous distribution  $F$  belongs to  $L_p^s$ , then the characteristic of  $A^t F$  belongs to  $L_p^{s-\text{Re } t}$ .

PROOF. The definition of  $T_\lambda^t$  and of  $D'(\lambda, t)$  shows that  $T_\lambda^t$  is continuous, one-to-one and onto from  $D'(\lambda, t)$  to  $D'(\lambda, t)$ , and also that its inverse is  $T_{\lambda-t}^{-t}$ .

(i) Let us assume  $t = 2l$  ( $l = 0, 1, 2, \dots$ , etc.). In this case  $A^t = -A^l$  is a derivation so the values of  $A^t(r^\lambda f)$  in an arbitrary open set  $\Omega$  depend only on the values of  $r^\lambda f$  in  $\Omega$ . Hence we need only to show (by Proposition 1.3.2) that, if  $\Psi \in C_0^\infty\{|x| > 0\}$ ,  $A^t \Psi(r^\lambda f)$  belongs to  $L_p^{s-t}(\mathbb{E}^k)$  if and only if  $\Psi(r^\lambda f)$  belongs to  $L_p^s$ . But this was proved in Corollary 2.3.6. So we have proved that  $T_\lambda^{2l}$  is an isomorphism from  $L_p^s(\lambda, 2l)$  onto  $L_p^{s-2l}(\lambda, 2l)$ . This implies that the inverse  $T_{\lambda-2l}^{-2l}$  is also an isomorphism from  $L_p^s(\lambda, 2l)$  onto  $L_p^{s+2l}(\lambda, 2l)$ .

(ii) Let us assume that  $\text{Re } \lambda < 0$ ,  $\text{Re } t < 0$ ,  $s > k$ , and  $1 < p < \infty$ .  $\Psi \in C_0^\infty\left\{1 < |x| < 1 \frac{1}{4}\right\}$  and  $\Psi \equiv 1$  in a neighborhood of  $\Sigma_{k-1}$ . We consider a partition of unity by positive  $C^\infty$  radial functions  $\Phi_1, \Phi_2, \Phi_3$  such that

I)  $\Phi_2(x) \equiv 1$  on  $\frac{1}{2} < |x| < 1 \frac{1}{4}$  and vanishes in  $|x| < \frac{1}{4}$  and  $|x| > 1 \frac{1}{2}$ ,

II)  $\Phi_3(x) \equiv 0$  in a neighborhood of  $|x| < 1 \frac{1}{4}$  and  $\equiv 1$  in  $|x| > 1 \frac{1}{2}$ , and  $\Phi_3 = k^2$  where  $k$  satisfies the same conditions of  $\Phi_3$ ,

III)  $\Phi_1 + \Phi_2 + \Phi_3 = 1$ .

Let  $v_i = A^t(\Phi_i r^\lambda f)$ , where  $i = 1, 2, 3$ . We want to show that  $\Psi v_i$  ( $i = 1, 2, 3$ ) belongs to  $L_p^{s-\text{Re } t}$  if and only if  $\Psi r^\lambda f$  belongs to  $L_p^s$ ,

a)  $v_1$  has support contained in  $|x| < \frac{1}{2}$ , hence, by Corollary 2.2.5, it follows that  $\Psi v_1 = \Psi A^s \Phi_1 r^\lambda f$  belongs to  $C_0^\infty$ .

b)  $v_2 = A^t(\Phi_2 r^\lambda f)$  belongs to  $L_p^{s-\text{Re } t}$  if and only if  $\Phi_2 r^\lambda f \in L_p^s$ . In fact we may write  $I^{-t} = A^t + S_1^*$ , where  $S_1^*$  is the convolution with a bounded  $C^\infty$  function. Hence  $\Psi A^t(\Phi_2 r^\lambda f) = \Psi [(I^{-t} - S_1^*) \Phi_2 r^\lambda f] \in L_p^{s-\text{Re } t}$  if and only if  $\Psi I^{-t} \Phi_2(r^\lambda f) \in L_p^{s-\text{Re } t}$  and this, by Corollary 2.2.7, is equivalent to the condition  $r^\lambda f \in L_p^s$  in a neighborhood of  $\Sigma_{k+1}$ , i. e.,  $r^\lambda f \in L_p^s$  locally.

c) We will finally prove that, under our hypothesis,  $\Psi v_3 = \Psi A^t(\Phi_3 r^\lambda f)$  belongs to  $C_0^\infty$ . To do so, we consider an arbitrary distribution  $\tau$  with support in  $|x| < 1 \frac{1}{4}$ . Then,

$$(3.2.2) \quad \langle \tau, \Psi A^t(\Phi_3 r^\lambda f) \rangle = \langle k A^t \Psi \tau, k r^\lambda f \rangle,$$

(since  $\Phi_3 = k^2$ ). By Corollary 2.2.5  $k A^t \Psi \tau \in \mathcal{S}_1$  and, by Corollary 2.2.4 b),  $k r^\lambda f$  belongs to  $L_\infty$ . Hence (3.3.2) is bounded, so  $A^t(\Phi_3 r^\lambda f)$  belongs to the dual of  $\mathcal{E}'\left(|x| < 1 \frac{1}{4}\right)$ , i. e., it is a  $C^\infty$  function in  $|x| < 1 \frac{1}{4}$ .

It follows that  $T_\lambda^t$  is an isomorphism from  $L_p^s(\lambda, t)$  onto  $L_p^{s-\text{Re } t}(\lambda, t)$  if  $\text{Re } \lambda < 0$ ,  $s > k$  and  $t > 0$  ( $1 < p < \infty$ ). Hence its inverse  $T_{\lambda-t}^{-t}$  will also be an isomorphism.

By the same reasoning, if  $t$  is real, we obtain the same conclusion also for  $p = 1$  and  $p = \infty$ . Summing up our results, we have proved that the theorem holds if  $\text{Re } \lambda < 0$  and  $t$  is arbitrary ( $t$  real if  $p = 1$  or  $\infty$ ), or if  $t$  is an even integer and  $\lambda$  is arbitrary.

Taking  $n$  such that  $\lambda = 2n + \beta$  ( $\text{Re } \beta < 0$ ) and subtracting, if necessary, a finite number of terms from the expansion of  $f$  in spherical harmonics to make sure that the expressions are defined, we may write

$$A^t(r^\lambda f) = A^{-2n} A^t(A^{2n}(r^\lambda f))$$

and the theorem follows.  $\square$

**THEOREM 3.3.2.** Let  $f = \sum a_{nm} Y_{nm} \in D'(\Sigma)$  and  $\lambda$  a complex number  $\neq 0, 1, 2, \dots$ , etc. If  $1 < p < \infty$  and

$$(3.3.3) \quad (r^\lambda f)^\wedge = g r^{-\lambda-k}$$

then

- a)  $g \in L_p^{s-\text{Re } \lambda - k/2}(\Sigma)$  if and only if  $f^* = \sum (-i)^n a_{nm} Y_{nm}$  belongs to  $L_p^s$ .
- b) If  $\lambda = 0, 1, \dots, n, \dots$  conclusion a) is still true but with  $g$  given by

$$(3.3.3') \quad (r^\lambda f)^\wedge = g r^{-\lambda-k} + P_{-\lambda-k}(D) \delta$$

where  $P_{-\lambda-k}(D)$  is a homogeneous polynomial of degree  $-\lambda - k$  (cfr. Theorem (3.1.1)).

- c) If  $p = 1$  or  $p = \infty$  a) and b) hold provided that  $\lambda$  is real.

**PROOF.** Clearly it is enough to prove the relations for the inverse Fourier transform  $\sim$  instead of the Fourier transform  $^\wedge$ . By (3.2.3),

$$(2\pi)^{-k} (r^{-k/2} f) = (r^{-k/2} f^*)^\wedge.$$

Dropping, if necessary, a finite number of terms in the expansion of  $f$ , we may assume that  $f \in D'(-k/2, \lambda + k/2)$ . Then,

$$(r^\lambda f)^\sim = (r^{\lambda+k/2} (r^{-k/2} f))^\sim = C (r^{\lambda+k/2} (r^{-k/2} f^*)^\wedge)^\sim = C A^{\lambda+k/2} (r^{-k/2} f^*).$$

From this and Theorem 3.3.1 the conclusions follow.

In the case  $k = 2$ , Theorem 3.3.2 can be improved in the following way.

**THEOREM 3.3.3.** If  $k = 2$  and  $f$  and  $g$  are related by (3.3.3) or (3.3.3'), then

a)  $g \in L_p^{s-\text{Re}\lambda-k/2}$  ( $1 < p < \infty$ ) if and only if  $f \in L_p^s$ .

b) If  $\lambda$  is real and  $f$  is even, conclusion a) holds also for  $p = 1$  or  $p = \infty$ .

We begin with the following lemmas.

**LEMMA 3.3.4.** If  $k = 2$  and  $f$  is an even function in  $L_p(\Sigma)$ ,  $1 \leq p \leq \infty$ , the  $f^*$  also belongs to  $L_p$ .

**PROOF.** Let  $f = \Sigma (a_{2n} \cos 2n\theta + b_{2n} \sin 2n\theta)$ . It is clear that  $\bar{f}$  defined by

$$\bar{f}(\theta) = f(\theta/2)$$

belongs to  $L_p$ , i. e.,  $\bar{f} = \Sigma (a_{2n} \cos n\theta + b_{2n} \sin n\theta) \in L_p$ .

The odd and the even part of  $\bar{f}$  belongs to  $L_p$ . So,

$$f_1 \Sigma (a_{2n} \cos 4n\theta + b_{4n} \sin 4n\theta)$$

and

$$f_2 = \Sigma (a_{4n+2} \cos (4n+2)\theta + b_{4n+2} \sin (4n+2)\theta)$$

belong to  $L_p$ , and hence  $f^* = -(f_1 - f_2) \in L_p$ . ■

**LEMMA 3.3.5.** If  $k = 2$  and  $f$  is odd, let  $g$  be given by

$$(r^{-2} f)^{\wedge} = g r^0.$$

Then,  $f \in L_p^s$  if and only if  $\frac{d}{d\theta} g \in L_p^s$ .

**PROOF.** It is known (cfr. Neri [2], p. 110) that if  $f$  is continuous  $g$  is given by

$$g(\theta) = -\frac{\pi i}{2} \int_{-\frac{\pi}{2} + \theta}^{+\frac{\pi}{2} + \theta} f(t) dt.$$

Consequently,  $\frac{d}{d\theta} g = f\left(\theta + \frac{\pi}{2}\right) - f\left(-\frac{\pi}{2} + \theta\right) = 2f\left(\theta + \frac{\pi}{2}\right)$ . From the continuity of the Fourier transform and the derivative the lemma follows.

In particular,  $f \in L_p^s (1 < p < \infty)$  if and only if  $g \in L_p^{s+1}$ . ■

PROOF OF THEOREM 3.3.3. We first remark that the following relations are equivalent.

$$(3.3.4) \quad \begin{aligned} a) \quad A^s (r^\lambda f) &= r^{\lambda-s} h \\ b) \quad A^s (r^\lambda f^*) &= r r^{\lambda-s} h^* . \end{aligned}$$

Assume first that  $f \in L_p^s$  is even. Dropping, if necessary, a finite number of terms of the expansion of  $f$ , we may assume that  $f \in D'(-1, s)$ . By Theorem 3.3.1, there is an  $h \in L_p$  such that :

$$(3.3.5) \quad A^s (r^{-1} f) = r^{-s} (r^{-1} h).$$

By Lemma 3.3.4,  $h^* \in L_p$  and then, by (3.3.4) and (3.3.5),

$$A^s (r^\lambda f^*) = r^{\lambda-s} h^* .$$

Using Theorem 3.3.1 we see that  $f^* \in L_p^s$ . By Theorem 3.3.2 if  $g$  is given by (3.3.3) on (3.3.3'), then  $g \in L_p^{s-\text{Re}\lambda-1}$ .

If  $f$  is odd, we will prove the relation for the inverse Fourier transform  $\sim$  instead of the Fourier transform  $\wedge$ . We have,

$$(r^\lambda f)^\sim = (r^{\lambda+2} ((r^{-2} f)^\sim)^\wedge)^\sim = A^{\lambda+2} (r^{-2} f)^\sim .$$

By Lemma 3.3.5 and Theorem 3.3.1, the characteristic of  $A^{\lambda+2} (r^{-2} f)^\wedge$  will belong to  $L_p^{s-\text{Re}\lambda-1}$ ,  $1 < p < \infty$ . ■

REMARK 3.3.5. We may now give a different proof Theorem 3.24. In fact in fact if  $f \in L_2^s(\Sigma)$  dropping a finite number of terms in the expansion of  $f$  in spherical harmonics (if necessary) we may assume the existence of  $h \in L_2(\Sigma)$  such that

$$(3.3.7) \quad A^{-s} (r^{-k/2} h) = r^{-k/2-s} f .$$

If  $h \in L(\Sigma)$  it is obvious that  $h^* \in L^2(\Sigma)$ . From (3.3.7) we obtain

$$(3.3.8) \quad A^{-s} (r^{-k/2} h^*) = r^{-k/2-s} f^*$$

but then Theorem 3.3.1 implies that  $f^* \in L_2^s$  and the conclusion follows by Theorem 3.3.2. ■

3.4. A Counterexample.

We prove in this section that we cannot give necessary and sufficient conditions in terms of the space  $L_p^s$  for the singular convolution

$$(3.4.1) \quad \Psi \rightarrow (r^{-k} f)_* \Psi, f \in D'(\Sigma), \Psi \in D(\mathbb{E}^k), \langle f, 1 \rangle = 0$$

to be continuous on  $L_2(\mathbb{E}^k)$ .

In fact (3.4.1) is continuous in  $L_2$  if and only if  $(r^{-k} f)^\wedge = g r^0 \in L_\infty$ , i. e.,  $g \in L_\infty(\Sigma)$ .

Let  $L$  be the space of  $f \in D'(0)$  such that (3.4.1) is  $L_2$  continuous, and suppose that  $H = L_p^s$  for some  $p$  and some  $s$ . By Theorem 3.3.2 the linear mapping  $T^*: f \rightarrow f^*$  defines an isomorphism from  $H$  onto  $L_\infty^{-k/2}$ . This implies that  $H$  cannot be reflexive and cannot be separable, i. e.,  $H = L_\infty^{-k/2}$ . The proof of Theorem 3.3.3 shows that  $T^*$  should define an isomorphism from  $L_\infty$  onto  $L_\infty$ . However, the following example shows that  $T^*$  is not such an isomorphism.

Let us consider on  $\Sigma_2$  the function defined by:

$$f(x, y, z) = \int_0^{2\pi} Lg(1 + (iz - y \sin \theta - x \cos \theta)) d\theta, x^2 + y^2 + z^2 = 1,$$

where we take the principal branch of  $Lg$ . It is clear that in  $z \neq 0$ ,  $f$  is continuous. Furthermore  $f(x, y, z)$  depends only on  $z$ , if  $x^2 + y^2 + z^2 = 1$ . In fact, if  $\alpha x = y$ ,  $y \sin \theta + x \cos \theta = \sqrt{x^2 + y^2} \cos(\theta - \arctan \alpha)$ .

Our goal now is to show that  $\lim f(x, y, z)$  exists. Since we have observed that  $f$  is independent of  $y$  we assume  $y = 0$ , i. e.,

$$f(x, 0, z) = \int_0^{2\pi} Lg(1 + (iz - x \cos \theta)) d\theta.$$

When  $z \rightarrow 0$ , the integrand tends uniformly on compact subsets of  $]0, \pi[$ ,  $]\pi, 2\pi[$  to  $Lg(1 - \cos \theta)$  whose integral is  $2\pi \lg(1/2)$  (cfr. Handbook of Mathematical Tables and Formulas, Buriogtan integral # 403). So,  $f \in L_\infty$ .

We recall that if  $P_{nj}$  and  $Q_{nj}$  are given by:

$$(3.4.3) \quad (z + iy \sin \theta + ix \cos \theta)^n = \sum_{j=0}^n P_{nj}(x, y, z) \cos j\theta + \sum_{j=1}^n Q_{nj}(x, y, z) \sin j\theta,$$



then  $P_{nj}(x, y, z)$   $Q_{nj}(x, y, z)$  form a base of the space  $\{Q_n\}$  of the spherical harmonics of degree  $n$  (cfr. Horváth [1]).

From this and the expansion

$$\lg(1+x) = \sum (-1)^n \frac{x^n}{n},$$

we obtain

$$\begin{aligned} f(x, y, z) &= \int_0^{2\pi} \sum_{n=1}^{\infty} (-1)^n (iz - y \sin \theta - x \cos \theta)^n d\theta \\ &= \int_0^{2\pi} \sum_n (-i)^n \left( \sum_{j=0}^n P_{nj}(x, y, z) \cos j\theta + \sum_{j=1}^n Q_{nj}(x, y, z) \sin j\theta \right) d\theta \end{aligned}$$

and

$$\begin{aligned} (3.4.4) \quad f^*(x, y, z) &= \int_0^{2\pi} \left( \sum_{n=1}^{\infty} (-1)^n \sum_{j=0}^n P_{nj}(x, y, z) \cos j\theta + \sum_{j=1}^n Q_{nj}(x, y, z) \sin j\theta \right) d\theta \\ &= \int_0^{2\pi} \sum_{n=1}^{\infty} (-1)^n (z + iy \sin \theta + ix \cos \theta)^n d\theta \\ &= \int_0^{2\pi} \text{Log}(1 + (z + iy \sin \theta + ix \cos \theta)) d\theta. \end{aligned}$$

It is clear from (3.4.4) that when  $z \rightarrow -1$ ,  $f^*$  is not bounded.

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