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D. D. J. Hacon

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MANIFOLDS OF THE HOMOTOPY TYPE OF A BOUQUET OF SPHERES

By D. D. J. HACON

1. Introduction.

This note is concerned with manifolds homotopy equivalent to $\vee S_i$ (a bouquet of spheres of varying dimensions). In this connection a useful concept is that of *thickening* which is a homotopy generalization of the idea of regular neighbourhood. The reader is referred to [7] for definitions in the general case. Here we shall be concerned with the specific problem of describing the set of thickenings of $\vee S_i$ and it will be convenient to adopt a definition of thickening that differs slightly from that to be found in [6] (see § 2).

In [3] Haefliger classified thickenings of (simply-connected) bouquets of spheres subject to certain dimensional restrictions. Our purpose here is to improve on Haefliger's result in the piecewise-linear case and deal with the non-simply-connected case, reducing it to the problem of classifying concordance classes of embeddings of a number of solid tori in a certain manifold, as follows.

Denote by P^n the solid n-pretzel i.e. an n-ball with a finite number of 1-handles attached orientably. Then the classification of thickenings is reduced to the classification of concordance classes of embeddings of the disjoint union of solid tori in ∂P, which is a simpler question. For istance, if the bouquet in question is simply-connected then ∂P is a sphere and the problem is now to classify concordance classes of embeddings of solid tori in a sphere. See [3]. If, on the other hand, the bouquet consists of a circle and a sphere we need to look at knots of a solid torus in $S^1 \times S^q$ (q being dim $P - 2$).

Suppose $f : \vee S_i \rightarrow W$ is a homotopy equivalence (and hence a simple homotopy equivalence since the Whitehead group of a free group is trivial). If f is homotopic to a piecewise linear embedding $g : \vee S_i \rightarrow W$ proceed as follows. Take a regular neighbourhood N of $g \vee S_i$ in Int W, the interior of

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W (if $g \vee S_i$ meets ∂W isotop it into Int W). N is homeomorphic to W, for by the s-cobordism theorem [1], W-Int N is homeomorphic to $\partial N \times [0, 1]$. We thus obtain a handlebody decomposition of W suffixed by the cell structure of $\vee S_i$.

In general, however, there exist homotopy equivalences $f : \vee S_i \to W$ which are *not* homotopic to an embedding and consequently the above procedure cannot be followed. But a theorem of Stallings [5] allows us to factor $f : \vee S_i \to W$ up to homotopy through a simple homotopy equivalence $f' : \vee S_i \to N$ where N is a $p_N + 1$-dimensional polyhedron in W. We seek a simple description of N in terms of $\vee S_i$ which will (as in the case when f is an embedding) provide a handlebody decomposition of W suffixed by the cell structure of $\vee S_i$. In fact it will be shown (§ 3) that, if $f : \vee S_i \to W$ is a homotopy equivalence, W may be expressed as P plus handles of index two or more and that handles of sufficiently large index are attached disjointly i. e. after a certain point in the construction of W the order in which handles are subsequently attached is immaterial.

2. The main theorem.

Throughout we restrict ourselves to the piecewise linear (PL) category [7].

Write $\cup S_i$ for the disjoint union $S_1 \cup, \dots, \cup S_N$ of spheres $S_1, \dots, S_N$ of dimensions $p_1, \dots, p_N$ subject to the condition $1 \leq p_1 \leq, \dots, \leq p_N$. Let $* = (*, \dots, *)$ be a point of $S_1 \times, \dots, \times S_N$. Then $\vee S_i = S_1 \vee, \dots, \vee S_N$ is the subpolyhedron

$$(S_1 \times \{*\} \times, \dots, \times \{*\}) \cup, \dots, \cup (\{*\} \times, \dots, \times \{*\} \times S_N) \text{ of } S_1 \times, \dots, \times S_N.$$

Let $\pi : \cup S_i \to \vee S_i$ be the obvious identification map. If we write $\vee S_i$ in the form $S^1 \vee, \dots, \vee S^1 \vee S_1 \vee, \dots, \vee S_N$ it is understood that $p_1 \geq 2$.

Now let M be a compact, connected, oriented manifold with nonempty boundary ∂M and such that

$$\text{(1)} \qquad \dim M \geq \text{Max}\,(6, \dim \cup S_i + 3)$$

$$\text{(2)} \qquad \partial M \subset M \text{ induces an isomorphism of fundamental groups.}$$

We will be considering pairs (M, f), M as above and $f : \cup S_i \to M$ homotopic to $g \circ \pi$ where $g : \vee S_i \to M$ is a homotopy equivalence. We call such a pair (M, f) a *thickening*.

REMARKS (1). Since M is assumed connected any map $f: \cup S_i \to M$ factors up to homotopy through $\pi: \cup S_i \to \vee S_i$.

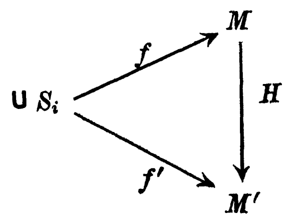
(2) Suppose $f: \cup S_i \to M$ factors up to homotopy through a homotopy equivalence $g: S_i \to M$. Let $g': \vee S_i \to M$ be any other homotopy factorization. Then g' is also a homotopy equivalence.

(3) If $S_1, \ldots, S_N$ are all circles then all thickenings (M, f) are equivalent in the sense below.

(M, f) and (M', f') are said to be *equivalent* if M M' are homeomorphic and the homeomorphism H can be chosen to preserve all the data i. e..

(*a*) H is orientation-preserving

(*b*) the diagram homotopy commutes.



REMARK. In the simply-connected case ($p_1 \geq 2$) these definitions coincide with those of Haefliger [3].

Let P be the manifold defined in the introduction. That is a ball plus 1-handles. Let $\dim P = q$ where $q \geq p_N + 3$ and $p_1 \geq 2$. Any inessential orientation-preserving embedding $h: \bigcup_1^N \partial \Delta^{p_i} \times \Delta^{q-p_i} \to \partial P$ determines an oriented manifold of the homotopy type of a bouquet of spheres. The manifold is P plus handles $\Delta^{p_i} \times \Delta^{q-p_i}$ attached by means of the given embedding. Since the embedding h was assumed inessential we obtain a well-defined homotopy class of maps $f: \cup S_i \to P +$ handles. It is easily seen that this f is in fact a thickening.

REMARK. The restriction that h be inessential is only a restriction if $p_1 = 2$, since $\pi_i(P) = 0$ for $i > 1$.

Suppose h, k are concordant embeddings. It is an immediate consequence of the concordance extension theorem [4] that the two thickenings defined by h and k are equivalent.

Thus there is defined a function Φ from the set of concordance classes of embeddings of solid tori in ∂P to the set of equivalence classes of thickenings of $\vee S_i$.

Here is the main result of this note.

THEOREM. If, for the bouquet $S^1 \vee \ldots \vee S^1 \vee S_1 \vee \ldots \vee S_N$, $\dim P = q \geq$ $\geq \mathrm{Max}(6, p_N + 3)$ then

(1) Φ is surjective if $2p_N - q + 1 < p_1$

(2) Φ is injective if $2p_N - q + 2 < p_1$.

The proof is deferred to section 3.

3. The factorization lemma.

In this section it will be shown that any thickening $f: \cup S_i \to W$ factors up to homotopy through a homotopy equivalence $g: \vee S_i \to N$ where N is a subpolyhedron of Int W with special properties. But first some notation. Write X for $\vee S_i$ and filter X by $* = X_0 \subset, \dots, \subset X_N = X$ where $X_i = S_1 \vee, \dots, \vee S_i$ and $*$ is the base point of X. Let $f_k = f \mid S_k$. Finally, denote by Σf the singular set of f i. e. the closure of the set $\{x \in X \mid f^{-1} f x \neq x\}$.

LEMMA 1. (factorization lemma) Let $f: X \to \text{Int } W$ be a thickening (or more accurately let $f \circ \pi$ be one). Suppose that f is nondegenerate and that for $k = 1, \dots, N$, $\dim \Sigma f \cap S_k \leq p_k + p_N - q$. Then there exist polyhedra $Y_0 \subset Y_1 \subset, \dots, \subset Y_N$ in Int W such that

(1) $f \mid X_k \to Y_k \cup f S_k$ is a homotopy equivalence

(2) Y_{k+1} collapses to $Y_k \cup f S_k$

(3) $\dim Y_k \cap f S_{k+i} \leq p_k + p_N - q \ (i > 0)$

(4) $\Sigma f \cap S_k \subset f_k^{-1} Y_k$ and the latter collapses to $*$.

REMARK 1. If the first few S_i of X have small enough dimension then f embeds them (by the general position hypotheses on f) and the first few Y_i are defined by $Y_i = f X_{i-1}$ satisfying conditions (1), ..., (4).

REMARK 2. Lemma 1 yields a minimal handlebody decomposition for W as follows. Define inductively handlebodies H_k in Int W $(k = 1, \dots, N)$ by first triangulating $f: X \to W$ so that $Y_0, \dots, Y_N$ appear as subcomplexes (to be denoted by the same symbols). Take the barycentric second-derived suddivisions of X, W. f remains simplicial being non-degenerate. Define $H_k = N^2(Y_k \cup f S_k; W)$, the simplicial neighbourhood of $Y_k \cup f S_k$ in the second-derived subdivision of W. Then H_{k+1} is H_k plus a handle. For $(Y_{k+1} \cup f S_{k+1}) - \text{Int } N^2(Y_{k+1}; W) = f S_{k+1} - \text{Int } N^2(Y_{k+1}; W)$ and the latter is a p_{k+1} — disk in Int W that meets $N^2(Y_{k+1}; W)$ in its boundary. This follows from condition (4).

And $H_{k+1} - \text{Int } N^2(Y_{k+1}; W)$ is a ball meeting $N^2(Y_{k+1}; W)$ in $N^2(f S_{k+1}; W) \cap \partial N^2(Y_{k+1}; W)$ which is a solid torus. By regular neighbouhoods theory, H_{k+1} is homeomorphic to

$$N^2(Y_{k+1};\, W) \underset{\psi}{\cup} \Delta^{p_{k+1}} \times \Delta^{q-p_{k+1}}$$

where $\psi : \partial\Delta \times \Delta \rightarrow \partial N^2(Y_{k+1};\, W)$ is an embedding. But by condition (2) Y_{k+1} collapses to $Y_k \cup f S_k$ and so $N^2(Y_{k+1};\, W)$ and $H_k = N^2(Y_k \cup f S_k;\, W)$ are homeomorphic. Therefore H_{k+1} is H_k plus a handle and we obtain a handlebody decomposition of H_N (and hence of W) suffixed by the cell structure of X. Furthermore, (1) implies that the thickening $f : X \rightarrow W$ is filtered by a series of thickenings $f \mid X_k \rightarrow N^2(Y_k \cup f S_k;\, W)$.

As observed in the introduction, it is possible (under certain dimensional restrictions) to find a handlebody decomposition of W in which the handles are attached independently of one another after a certain stage. To show this we need a modification of the factorization lemma.

LEMMA 2. Let f satisfy the hypothesis of lemma 1. Suppose, in addition, that $p_{k+1} \geq 2p_N - q + 2$ for some $k\,(1 \leq k < N)$, and that $Y_0, \ldots$ $\ldots, Y_k$ have been found satisfying conditions (1) through (4) of Lemma 1. Then there exists a polyhedron Y in Int W such that

(*a*) Y collapses to $Y_k \cup f S_k$

(*b*) $f : X \rightarrow Y \cup f X$ is a homotopy equivalence

(*c*) $\Sigma f \subset f^{-1} Y$

(*d*) $f_{k+i}^{-1} Y$ collapses to $*$ for all $i > 0$.

REMARK. As before we have a handlebody decomposition of W. Triangulate $f : X \rightarrow W$ so that $Y_0, \ldots, Y_k$, Y are subcomplexes of W. Define

$$H_j = N^2(Y_j \cup f S_j;\, W \qquad (0 \leq j \leq k)$$

and

$$H = N^2(Y \cup f X;\, W).$$

Then the handles $N^2(f S_j;\, W) - \text{Int}\, N^2(Y;\, W)$ are attached independently to $N^2(Y;\, W)$ i.e. $N^2(f S_j;\, W) \cap \partial N^2(Y;\, W)$ are disjoint solid tori $(k+1 \leq$ $\leq j \leq N)$. For, by (*c*) of Lemma 2, f embeds

$$f^{-1}\left\{\bigcup_{j=k+1}^{N} f S_j - \text{Int}\; N^2(Y;\, W)\right\} \text{ and } S_j - f^{-1}\, \text{Int}\; N^2(Y;\, W)$$

$$= S_j - \text{Int}\, N^2(f_j^{-1} Y;\, W)$$

$$= S_j \text{ minus the interior of a ball, by } (d)$$

$$= a \text{ ball}, \ (j = k+1, \ldots, N);$$

and the ball $fS_j - \text{Int } N^2(Y;W)$ meets $N^2(Y;W)$ in its boundary only, by (*c*). Thus $N^2(Y;W) \cup fX$ is $N^2(Y;W)$ plus balls $fS_j - \text{Int } N^2(Y;W)$ attached disjointly to $\partial N^2(Y;W)$. This completes the proof that H is H_k plus disjointly attached handles.

To prove Lemma 1 and 2 we will need some general position and engulfing lemmas.

DEFINITION. If Y_0, Y, Z are polyhedra in the manifold M and $Y_0 \subset Y$, then $Y - Y_0$ is said to be in *general position with respect to* Z if $\dim (Y - Y_0) \cap Z \leq \dim Y - Y_0 + \dim Z - \dim M$.

DEFINITION. If Y is a polyhedron and M a manifold, a map $f\colon Y \to M$ is in *general position* if

(1) f is non-degenerate

(2) $\dim \Sigma f < 2 \dim Y - \dim M$.

COROLLARY TO THEOREM 15 [7]. If $Y_0, Y, A_1, \ldots, A_n$ are polyhedra in a manifold M with $Y_0 \subset Y$ and $Y - Y_0 \subset \text{int } M$, then there exists a homeomorphism $h: M \to M$ such that

(1) $h \mid Y_0 \cup \partial M = \text{Identity}$

(2) $h(Y - Y_0)$ is in general position with respect to $A_1, \ldots, A_n$.

PROOF. By induction on $\dim A_1 \cup, \ldots, \cup A_n$.

COROLLARY TO THEOREM 18 [7]. Let $f: Y \to \text{Int } M$ be a map and Y_0 a subpolyhedron of Y. Suppose $f \mid Y_0$ is in general position. Then f is homotopic to g, a map in general position, by an arbitrarily small homotopy that keeps Y_0 fixed.

LEMMA 3. If $f: X \to M$, X a sphere-bouquet, M a manifold, then f is homotopic to $g: X \to \text{Int } M$ where g is in general position and $\dim \Sigma g \cap S_k \leq \dim S_k + \dim X - \dim M$ $(k = 1, \ldots, N)$.

PROOF. First homotop fX into Int M and then use induction on N, the number of spheres in the bouquet. If $N = 1$ apply the second corollary above. If not, the inductive step is proved by homotoping $f \mid S_N$ into general position keeping $f*$ fixed and then applying the first corollary to minimize the dimension of $fS_N \cap fX_{N-1}$ by putting $f(S_N - *)$ into general position with respect to fX_{N-1} keeping $f*$ fixed.

To state the engulfing lemmas we need

DEFINITION. A subpolyhedron C of a manifold M is called a *k-spine* of M if the pair M, C is k-connected.

DEFINITION. A polyhedron is called *t-collapsible* if it can be collopsed to a polyhedron of dimension not greater than t. The following lemma is a special case of Theorem 21 [7].

LEMMA 4 (Zeeman). Let C be an m-3-collapsible k-spine of the manifold M (dim M being m), Y a polyhedron in M and

$$\dim\ Y \cap \partial M < \dim Y \leq k \leq m-3.$$

Then Y may be engulfed from C relative to ∂M i. e. there exists C^+ in M such that $C \cup Y \subset C^+$, $(C \cup Y) \cap \partial M = C^+ \cap \partial M$, C^+ collapses to C, and $\dim\ C^+ - C \leq \dim\ Y + 1$.

ADDENDUM TO LEMMA 4. Suppose that $A_1, \dots, A_n$ are polyhedra in M. By the corollary to Theorem 15 we may insist that $C^+ - (C \cap Y)$ be in general position with respect to $A_1, \dots, A_n$.

LEMMA 5. Let C be an m-3-collapsible k-spine of M and D a q-3-collapsible $k+1$-spine of Q and let $f: M, C \to Q, D$ be non-degenerate and proper (i.e. $f^{-1}\partial Q = \partial f^{-1} Q$). Suppose that $\dim (f^{-1}D - C) = x \leq k \leq m - - 3 \leq q - 6$ and that $\partial M \cap (f^{-1}D - C)$ is empty.

Then there exist polyhedra $C^+ \subset M, D^+ \subset Q$ such that

(A) $C^+ = f^{-1}D^+$ (i.e. $\dim f^{-1}D^+ - C^+ < 0$)

(B) $C^+ \cap \partial M = C \cap \partial M;\ D^+ \cap \partial Q = D \cap \partial Q$

(C) C^+ collapses to C; D^+ collapses to D

(D) $\dim\ C^+ - C \leq x + 1$; $\dim\ D^+ - D \leq x + 2$.

If, further, $A_1, \dots, A_n \subset Q$ are polyhedra in general position with respect to fM, then C^+, D^+ may be chosen to satisfy (A), ..., (D) and the extra condition

(E) $D^+ - D$ is in general position with respect to $A_1, \dots, A_n$.

PROOF. The proof resembles that of Lemma 63 [7]. We will define inductively polyhedra $C_i \subset M, D_i \subset Q$ such that

(a) $fC_i \subset D_i$ and $\dim f^{-1}D_i - C_i \leq x - i$.

(b) C_i collapses to C; D_i collapses to D.

(c) $C_i \cap \partial M = C \cap \partial M$; $D_i \cap \partial Q = D \cap \partial Q$.

(d) $\dim C_i - C_{i-1} \leq x + 2 - i$; $\dim D_i - D_{i-1} \leq x + 3 - i$.

(e) $D_i - D_{i-1}$ is in general position with respect to $A_1, \dots, A_n$.

The induction starts with $C_i = C$, $D_i = D$ $(i \leq 0)$ and finishes with $i = x + 1$ because then $\dim f^{-1} D_i - C_i < 0$. Condition (E) will be satisfied because $D^+ - D = \bigcup_{i \geq 0} (D_{i+1} - D_i)$ and each $D_{i+1} - D_i$ is in general position with respect to $A_1, \dots, A_n$.

The inductive step $(i \geq 0)$.

Assume that C_j, D_j have been chosen satisfying (a), ..., (e) for $j \leq i$.

By (a) $\dim f^{-1} D_i - C_i \leq x - i$.

By (b) C_i is an $m - 3$-collapsible k-spine of M (since C is).

So by Lemma 4 there exists $C_{i+1} \subset M$ such that C_{i+1} collapses to C_i, $f^{-1} D_i \subset C_{i+1}$, $\partial M \cap C_{i+1} = \partial M \cap f^{-1} D_i$, $\dim C_{i+1} - C_i \leq x + 1 - i$ and $C_{i+1} - f^{-1} D_i$ is in general position with respect to $f^{-1} A_1, \dots, f^{-1} A_n$. This implies that $\dim f C_{i+1} - D_i \leq \dim f(C_{i+1} - C_i) \leq x + 1 - i$; also that $\partial M \cap C_{i+1} = \partial M \cap C_i \cup \partial M \cap f^{-1} D_i$. But

$$\begin{aligned}
&\partial M \cap f^{-1} D_i = \\
&= f^{-1}(\partial Q \cap D_i) && (f \text{ is proper}) \\
&= f^{-1}(\partial Q \cap D) && (\text{by } (c)) \\
&= \partial M \cap f^{-1} D \\
&= \partial M \cap C && (\text{by initial hypothesis}).
\end{aligned}$$

By (b) D_i is a $q - 3$-collapsible $k + 1$-spine of Q. So by Lemma 4, there exists $D_{i+1} \subset Q$ such that D_{i+1} collapses to D_i, $f C_{i+1} \subset D_{i+1}$, $\dim D_{i+1} - D_i \leq x + 2 - i$, $\partial Q \cap D_{i+1} = \partial Q \cap (D_i \cup f C_{i+1})$ and $D_{i+1} - (D_i \cup f C_{i+1})$ is in general position with respect to $fM, A_1, \dots, A_n$. This implies that $\dim f^{-1} D_{i+1} - C_{i+1} \leq \dim f f^{-1} D_{i+1} - f C_{i+1} = \dim fM \cap (D_{i+1} - f C_{i+1}) = \dim fM \cap (D_{i+1} - (f C_{i+1} D_i)) \leq x + 2 - i - 3$. Also we have that $\partial Q \cap D_{i+1} = \partial Q \cap D_i \cup \partial Q \cap f C_{i+1}$. But $\partial Q \cap f C_{i+1} = f(\partial M \cap C_{i+1}) = f(\partial M \cap C) \subset \partial Q \cap D$. So $\partial Q \cap D_{i+1} = \partial Q \cap D$. We have thus defined C_{i+1}, D_{i+1} satisfying (a), ..., (d). The⁄ also satisf⁄ (e); for, $D_{i+1} - (f C_{i+1} \cup D_i)$ is in general position with respect to $A_1, \dots, A_n$; and $f C_{i+1} - D_i = f(C_{i+1} - f^{-1} D_i)$ is in general position with respect to $A_1, \dots, A_n$ since $A_1, \dots, A_n$ are (by hypothesis) in general position with respect to fM and $C_{i+1} - f^{-1} D_i$ was chosen to be in general position with respect to $f^{-1}A_1, \dots, f^{-1}A_n$. This completes the proof of the inductive step and hence of lemma 5.

PROOF OF LEMMA 1. Let us write $Z_k = Y_k \cup f S_k$. Construct Y_k (and hence Z_k) inductively starting with $Y_0 = Z_0 = fX_0 = f*$. Suppose that we have found $Y_0, \dots, Y_k$ satisfying conditions (1), ..., (4). By lemma 4 and the fact that dim $\Sigma f \cap S_{k+1} \leq p_{k+1} - 3$ there exists C_{k+1} in S_{k+1} such that $\Sigma f \cap S_{k+1} \subset C_{k+1}$, C_{k+1} collapses to $*$, and dim $C_{k+1} \leq 1 + p_N + p_{k+1} - q$. Now Z_k is a $p_{k+1} - 1$-spine of Int W and $1 + p_N + p_{k+1} - q \leq p_{k+1} - 2$. Therefore by lemma 4 there exists D_{k+1} in Int W such that $fC_{k+1} \subset D_{k+1}$, D_{k+1} collapses to Z_k, dim $D_{k+1} - Z_k \leq 2 + p_N + p_{k+1} - q$ and $D_{k+1} - (Z_k \cup fC_{k+1})$ is in general position with respect to $fS_{k+1}, \dots, fS_N$. This and condition (3) imply that for $i > 1$

$$p_N + p_{k+1} - q \geq$$

$$\geq \dim fS_{k+i} \cap [D_{k+1} - (Z_k \cup fC_{k+1}) \cup Z_k \cup fC_{k+1}]$$

$$\geq \dim fS_{k+i} \cap D_{k+1}.$$

Now $f_{k+1} : S_{k+1}, C_{k+1} \to W, D_{k+1}$, C_{k+1} is a $p_{k+1} - 2$-spine of S_{k+1}, D_{k+1} is a $p_{k+1} - 1$-spine of Int W and dim $f_{k+1}^{-1} D_{k+1} - C_{k+1} \leq p_N + p_{k+1} - q$. So, by lemma 5, there exists Y_{k+1} in Int W such that Y_{k+1} collapses to D_{k+1}, $f_{k+1}^{-1} Y_{k-1}$ collapses to $*$ and dim $fS_{k+i} \cap (Y_{k+1} - D_{k+1}) \leq p_N + p_{k+1} - q$ $(i > 1)$. It follows that dim $fS_{k+i} \cap Y_{k+1} \leq p_N + p_{k+1} - q \, (i > 1)$. Thus Y_{k+1} is defined and satisfies (2) (3) and (4).

The proof of the induction step will be complete once it has been shown that $f \mid X_{k+1} \to Z_{k+1}$ is a homotopy equivalence. First triangulate $f : X \to W$ and pass to the barycentric second derived triangulations of X, W. f remains simplicial.

We showed that $f_{k+1}^{-1} Y_{k+1}$ collapsed to $*$. Thus $N^2(f_{k+1}^{-1} Y_{k+1}; S_{k+1}) = f_{k+1}^{-1} N^2(Y_{k+1}; W)$ is a ball.

Further, $\Sigma f \cap S_{k+1} \subset f_{k+1}^{-1} Y_{k+1}$ and so f_{k+1} maps

$$S_{k+1} - \text{Int } N^2(f_{k+1}^{-1} Y_{k+1}; S_{k+1})$$

homeomorphically onto $Z_{k+1} - \text{Int } N^2(Y_{k+1}; W)$.

To prove that $f \mid X_{k+1} \to Z_{k+1}$ is a homotopy equivalence, we show that

(*) $f \mid X_{k+1} \to N^2(Y_{k+1}; W) \cup fS_{k+1}$ is a homotopy equivalence.

(**) $N^2(Y_{k+1}; W) \cup fS_{k+1}$ collapses to $Y_{k+1} \cup fS_{k+1}$.

Composing (**) with (*), we obtain a homotopy equivalence:

$$X_{k+1} \xrightarrow{f|} N^2(Y_{k+1}; W) \cup fS_{k+1} \supset Y_{k+1} \cup fS_{k+1}.$$

PROOF OF (*). f maps the pair X_{k+1}, $X_k \cup N^2(f_{k+1}^{-1} Y_{k+1}; S_{k+1})$ into the pair $fS_{k+1} \cup N^2(Y_{k+1}; W)$, $N^2(Y_{k+1}; W)$.

$f | X_k \to N^2(Y_{k+1}; W)$ is a homotopy equivalence because $f | X_k \to Z_k$ is one and $N^2(Y_{k+1}; W)$ collapses to Z_k via Y_{k+1}.

Let us write $U(\ )$ for «universal cover of». All spaces to which $U(\ )$ is applied will have isomorphic fundamental groups for [by Remark (1) following Lemma 1] p_{k+1} may be assumed to be greater than one. Therefore the map $f | X_{k+1}$ induces homology excision isomorphisms between $H_*(U(X_{k+1}), U(X_k \cup N^2(f_{k+1}^{-1} Y_{k+1}; S_{k+1})))$ and

$$H_*(U(fS_{k+1} \cup N^2(Y_{k+1}; W)),\ U(N^2(Y_{k+1}; W))).$$

So, by the 5-Lemma and Whitehead's theorem, the map $fX_{k+1} \to N^2(Y_{k+1}; W) \cup \cup fS_{k+1}$ induces isomorphisms of homotopy groups in all dimensions and is thus a homotopy equivalence.

PROOF OF (**). $N^2(Y_{k+1}; W) \cup fS_{k+1}$ collapses to $Y_{k+1} \cup fS_{k+1}$ because we may factor the collapse from $N^2(Y_{k+1}; W)$ to Y_{k+1} through $Y_{k+1} \cup \cup N^2(Y_{k+1} \cap fS_{k+1}; fS_{k+1})$. This proves (**) and completes the proof of Lemma 1.

PROOF OF LEMMA 2. Suppose polyhedra $Y_0, \ldots, Y_k$ have been found satisfying conditions (1), ..., (4) of Lemma 1. Recall that for $i > 0$ dim $\Sigma f \cap S_{k+1} \leq p_N + p_{k+i} - q$ and dim $f_{k+i}^{-1} Z_k \leq p_N + p_k - q$. So by Lemma 4 there exists C in $S_{k+1} \cup, \ldots, \cup S_N$ such that $(\Sigma f \cup f^{-1} Z_k) \cap (S_{k+1} \cup, \ldots, \cup S_N) \subset C$, C collapses to $*$ and dim $C \cap S_{k+i} \leq p_N + p_{k+i} - q + 1$. Now Z_k is a $p_{k+1} - 1$-spine of Int W and by hypothesis $1 + 2p_N - q \leq p_{k+1} - 1$. Therefore by Lemma 4 there exists D such that $fC \subset D$, D collapses to Z_k dim $D - Z_k \leq 2 + 2p_N - q$ and $D - (fC \cup Z_k)$ is in general position with respect to $fS_{k+1}, \ldots, fS_N$.

Then

$$\begin{aligned} \dim f_{k+i}^{-1} D - (C \cap S_{k+i}) &= \dim f S_{k+i} \cap (d - (f C \cup Z_k)) \\ &\leq p_{k+i} + 2 + 2p_N - q - q \\ &\leq p_{k+1} - 3. \end{aligned}$$

Now $C \cap S_{k+i}$ is a (collapsible) $p_{k+1} - 2$-spine of S_{k+i} and D is a $p_{k+1} -$

— 1-spine of Int W and so there exists Y in Int W such that Y collapses to Z_k, $\Sigma f \subset f^{-1} Y$, and $f_{k+i}^{-1} Y$ collapses to $*$ $(i > 0)$. The proof of lemma 2 is completed by showing that (as in lemma 1) $f: X \to Y \cup f X$ is a homotopy equivalence.

It remains to prove the theorem of § 2.

PROOF OF THEOREM. (1) Surjectivity of Φ. If $f: X \to W$ is a thickening, homotop f into general position in the sense of lemma 3 and use lemma 2 to obtain a manifold W_0 in Int W such that $f X \subset \operatorname{Int} W$ and $f: X \to W_0$ is a thickening representing an element in the image of Φ. See Remark after lemma 2. The S cobordism theorem provides us with an equivalence between the thickenings $f: X \to W_0$ and $f: X \to W$ and so Φ is surjective.

(2) Injectivity of Φ. Consider the special case $X = S^1 \vee, \dots, \vee S^1 \vee S^p$; the proof for more spheres is similar.

Let $*$ be the barycenter of the simplex Δ. Let h_0, h_1 be two embeddings of the solid torus $\partial \Delta \times \Delta$ in ∂P ($\dim \partial \Delta = p - 1$ and $\dim \partial \Delta \times \Delta = \dim \partial P$). Let the handlebody corresponding to h_i be $H(h_i) = P \bigcup_{h_i} \Delta \times \Delta$ $(i = 0, 1)$. Let $\delta_i: \Delta \times \Delta \to H(h_i)$ and $p_i: P \to H(h_i)$ be the associated embeddings (thus $p_i^{-1} \delta_i = h_i$ i. e. $\forall\, x \in \partial \Delta \times \Delta$, $\delta_i x = p_i h_i x$). Suppose that $h_0 h_1$ determine equivalent thickenings (the equivalence being a homeomorphism $G: H(h_1) \to$ $\to H(h_0)$. Then a relative version of the proof of surjectivity shows that there exist embeddings

$$\alpha: \Delta \times \Delta \times [0\,1] \to H(h_0) \times [0\,1]$$

$$\beta: P \times [0\,1] \to H(h_0) \times [0\,1] \text{ such that}$$

1) $$\alpha(x, 0) = (\delta_0 x, 0)$$

$$\alpha(x, 1) = (G\, \delta_1 x, 1)$$

$$\beta(x, 0) = (p_0 x, 0)$$

and

$$\beta P \times \{1\} = G p_1 P \times \{1\}$$

2) $$\alpha^{-1} \operatorname{Im} \beta = \partial \Delta \times \Delta \times [0, 1].$$

Thus we have a concordance $\alpha^{-1} \circ \beta \mid \partial \Delta \times \Delta \times [0\,1] \to \partial P \times [0, 1]$ between h_0 and $\lambda \circ h_1$ where $\lambda: P \to P$ is a self equivalence (i. e. an orientation — preserving homeomorphism homotopic to the identity). We need to show that $\lambda \circ h_1$ and h_1 are concordant.

First we choose λ of a special type. Let $I^1 = [-1, +1]$ and $I^k = I^1 \times \times \dots \times I^1 \subset \mathbb{R}^k$. Then if $q \geq 3$ we take $P^q = P^3 \times I^{q-3}$. $P^3 = B \cup H$ is the union of a 3-ball B and disjointly-attached 1-handles.

Let C be the union of the set of cores of these handles. Then the reader may verify the following.

PROPOSITION. Any self-equivalence $\lambda : P^3 \times I^k \to P^3 \times I^k$ is concordant to one of the form $\mu \times \text{Id}$, where $\mu \mid B \cup C = \text{Id}$. As for h_1, we may clearly assume that Im h_1 lies in Int $P^q \times \{-1\} \subset \partial (P^q \times I^1)$. It will suffice then to prove the following.

LEMMA. If $\lambda : P^q \to P^q$ is a self-equivalence and $\Sigma^p \subset \text{Int}\, P$ a sphere ($p \leq q-3$) then λ is concordant to λ where λ' fixes (pointwise) a neighbourhood of Σ in P.

PROOF. By the proposition above choose $\lambda = \mu \times \text{Id}$, with $\mu \mid B \cup C = \text{Id}$. Thus $\lambda \mid B \times I^{q-3} \cup C \times I^{q-3} = \text{Id}$. The result of [2] is easily generalized to show that Σ can be compressed (by an ambient isotopy) into $B \times I^{q-3} \cup \cup C \times I^{q-3}$ [the intersection of Σ with $C \times I^{q-3}$ being a set of disjoint cylinders (= homeomorphs of $S^{p-1} \times [0\ 1]$). Thus λ fixes Σ. It remains to show that after an isotopy λ fixes not only Σ but some neighbourhood of Σ in P.

Let $\widetilde{P}$ be the universal cover of P with covering projection $\pi : \widetilde{P} \to P$. Since Σ is inessential in P choose a connected component $\widetilde{\Sigma}$ of $\pi^{-1} \Sigma$; thus $\Sigma, \widetilde{\Sigma}$ are homeomorphic via π. Furthermore, in a neighbourhood of $\widetilde{\Sigma}$, π is (1-1). Let $\widetilde{\lambda} : \widetilde{P} \to \widetilde{P}$ be the lift of λ that fixes $\widetilde{\Sigma}$ pointwise i. e. $\pi \circ \widetilde{\lambda} = \lambda \circ \pi$ and $\widetilde{\lambda} \mid \widetilde{\Sigma} = \text{Id}$.

Since $\lambda \mid B \times I^{q-3} = \text{Id}$ there is a q-ball R in $\widetilde{P}$ with $\widetilde{\lambda} R = R$ and $\widetilde{\Sigma} \subset \text{Int}\, R$.

It follows from Lemma 59 of [7] that $\widetilde{\lambda}$ is isotopic (fixing $\widetilde{\Sigma}$ to $\Lambda : \widetilde{P} \to \to \widetilde{P}$ that fixes R pointwise. Projecting down by π we see that there is an ambient isotopy of P that takes λ to λ' where λ' is the inclusion in a neighbourhood of Σ. This completes the proof of the lemma and hence of the theorem.

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