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SUMMABILITY FACTORS FOR GENERALIZED ABSOLUTE RIESZ SUMMABILITY I

By Z. U. Ahmad

1.1. Let $\Sigma\, a_n$ be a given infinite series, and let λ_n be a sequence of positive, monotonically increasing numbers, diverging to infinity. We write

$$A_\lambda(t) = A_\lambda^0(t) = \sum_{\lambda_n \leq t} a_n\,,$$

$$A_\lambda(t) = 0,\ \text{for}\ t \leq \lambda_1\,;$$

and for $r > 0$,

$$A_\lambda^r(t) = \sum_{\lambda_n < t} (t - \lambda_n)^r\, a_n = r\int_{\lambda_1}^{t} (t - \tau)^{r-1} A_\lambda(\tau)\, d\tau = \int_{\lambda_1}^{t} (t - \tau)^r\, dA_\lambda(\tau).$$

Then $R_\lambda^r(t) \equiv A_\lambda^r(t)/t^r$ is called the *Riesz mean of type* λ_n and *order* r, while $A_\lambda^r(t)$ is called the Riesz sum of type λ_n and order r. We say that $\Sigma\, a_n$ is absolutely summable by this Riesz mean, or summable $|R, \lambda, r|$, $r \geq 0$, if $R_\lambda^r(t)$ is a function of bounded variation in (h, ∞) for some positive number h; or if

$$\int_h^\infty \left| \frac{d}{dt}\{R_\lambda^r(t)\}\right| dt < \infty \ ([10], [11]).$$

We say that $\Sigma\, a_n$ is summable $|R, \lambda, r|_p$, $p \geq 1$, $r > 0$, $rp' > 1$, and $1/p + 1/p' = 1$, if

$$\int_h^\infty t^{p-1} \left| \frac{d}{dt}\{R_\lambda^r(t)\}\right|^p dt < \infty, [7],$$

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where h is some positive number as before. Evidently, for $p = 1$, $|R, \lambda, r|_p$ is the same as $|R, \lambda, r|$.

1.2. Suppose that h is some positive number, and unless or otherwise stated *k is a positive integer.* We suppose further that $\Phi(t)$ and $\Psi(t)$ are functions with absolutely continuous $(k-1)\,th$ derivatives in every interval $[h, W]$, and that $\Phi(t)$ is non-negative and non-decreasing function of t for $t \geq h$, tending to infinity with t.

Without any loss of generality we take $\Phi(\lambda_1) = h = \lambda_1$.

By $B_k(t)$ we mean the Rieszian sum of type λ_n and order k of the series $\Sigma\, a_n \lambda_n$, and by $E_k(t)$ we mean the Rieszian sum of type $\Phi(\lambda_n)$ and order k of the series $\Sigma\, a_n\, \Psi(\lambda_n)\, \Phi(\lambda_n)$.

1.3. **Introduction.** Concerning $|R, \lambda, k| \Longrightarrow |R, \Phi(\lambda), k|$-summability factors, when k is a positive integer, the following theorem is known.

THEOREM A [3]. *If there is a function, $\gamma(t)$, defined and positive for $t \geq h$, such that, for $t \geq h$,*

$$\text{(i)} \qquad \gamma(t) = O(t),$$

$$\text{(ii)} \qquad t^n\, \Psi^{(n)}(t) = O\left[\left\{\frac{\gamma(t)}{t}\right\}^{k-n}\right], \text{ for } n = 0, 1, 2, \ldots, k;$$

and

$$\text{(iii)} \qquad \{\gamma(t)\}^n\, \Phi^{(n)}(t) = O\{\Phi(t)\}, \text{ for } n = 1, 2, \ldots, k;$$

and if the series $\Sigma\, a_n$ is summable $|R, \lambda, k|$, then the series $\Sigma\, \Psi(\lambda_n)\, a_n$ is summable $|R, \Phi(\lambda), k|$.

This is a generalization of a number of previously known results (See [3], [1], [2]). In particular, in the special cases in which (i) $\Psi(t) = 1, \gamma(t) = t$, (ii) $\Phi(t) = e^t$, $\Psi(t) = t^{-k}$, $\gamma(t) = 1$, it reduces respectively to the following theorems.

THEOREM B [4]. *If the series $\Sigma\, a_n$ is summable $|R, \lambda, k|$ and*

$$t^k\, \Phi^{(k)}(t) = O\{\Phi(t)\},$$

for $t \geq \lambda_1$, then the series $\Sigma\, a_n$ is summable $|R, \Phi(\lambda), k|$.

THEOREM C [12]. *If $k \geq 0$, and the series $\Sigma\, a_n$ is summable $|R, \lambda, k|$, then the series $\Sigma\, a_n \lambda_n^{-k}$ is summable $|R, l, k|$, where $l_n = e^{\lambda_n}$.*

Recently Mazhar has extended these theorems (Theorems B and C) for generalized absolute Riesz summability (defined in 1.1) in the form of

THEOREM D [8]. *If, for* $p \geq 1$, *and* $t \geq \lambda_1$,

(i) $t^k \Phi^{(k)}(t) = O\{\Phi(t)\}$,

(ii) $\{\Phi(t)/t\, \Phi^{(1)}(t)\}^{p-1} = O(1)$,

then any infinite series Σa_n *which is summable* $|R, \lambda, k|_p$, *is also summable* $|R, \Phi(\lambda), k|_p$.

THEOREM E [9]. *If* $p \geq 1$, *and* Σa_n *is summable* $|R, \lambda, k|_p$, *then* $\Sigma\, a_n \lambda_n^{-k+\frac{1}{p'}}$ *is summable* $|R, l, k|_p$, *where* $l_n = e^{\lambda_n}$ *and* $1/p + 1/p' = 1$.

The object of the present paper is to generalize Theorem A for generalized absolute Riesz summability so as to include Theorems D and E.

2.1. We establish the following theorem.

THEOREM. *If there is a function,* $\gamma(t)$, *defined and positive for* $t \geq h$, *such that, for* $t \geq h$,

(i) $\gamma(t) = O(t)$;

(ii) $t^n \Psi^{(n)}(t) = O\left[\left\{\frac{\gamma(t)}{t}\right\}^{k-n}\right]$, *for* $n = 0, 1, 2, \ldots k$;

(iii) $\{\gamma(t)\}^n \Phi^{(n)}(t) = O\{\Phi(t)\}$, *for* $n = 1, 2, \ldots, k$;

(iv) $\{\Phi(t)/\gamma(t)\, \Phi^{(1)}(t)\}^{p-1} = O(1)$,

and if the series Σa_n *is summable* $|R, \lambda, k|_p$, *then the series* $\Sigma \Psi(\lambda_n) a_n$ *is summable* $|R, \Phi(\lambda), k|_p$.

2.2. The following lemmas will be required for the proof of our theorem.

LEMMA 1 [6]. *For* $k > 0$,

$$w^{k+1} \frac{d}{dw} \{R_\lambda^k(w)\} = k\, B_{k-1}(w) = \frac{d}{dw} \{B_k(w)\}.$$

LEMMA 2 [5]. *If* k *is a positive integer, then*

$$A_\lambda(t) = \frac{1}{k!} \left(\frac{d}{dt}\right)^k A_\lambda^k(t).$$

LEMMA 3 ([13], p. 89). *If n is a positive integer and $m \neq 0$, then the nth derivative of $\{f(x)\}^m$ is a sum of constant multiples of a finite number of terms of the form:*

$$\{f(x)\}^{m-r} \prod_{s=1}^{n} \{f^{(s)}(x)\}^{\alpha_s},$$

where $1 \leq r \leq n$ and α's are zeros or positive integers such that

$$\sum_{s=1}^{n} \alpha_s = r \text{ and } \sum_{s=1}^{n} s\,\alpha_s = n.$$

If m is a positive integer, $1 \leq r \leq \min(m, n)$.

2.3. PROOF OF THE THEOREM:

Under the hypothesis of the theorem we have by Lemma 1, for $p > 1$ (*),

$$\int_{\lambda_1}^{\infty} t^{-(kp+1)} \,|\, B_{k-1}(t) \,|^p \, dt < \infty, \tag{2.3.1}$$

and we have to establish that

$$\int_{\Phi(\lambda_1)}^{\infty} w^{-(kp+1)} \,|\, E_{k-1}(w) \,|^p \, dw < \infty. \tag{2.3.2}$$

By writing $w = \Phi(t)$ in the above integral we find that the required inequality can be written in the form of

$$\int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}} \,|\, E_{k-1}(\Phi(t)) \,|^p \, dt < \infty. \tag{2.3.3}$$

Now, we have

$$E_{k-1}(\Phi(t)) = \int_{\Phi(\lambda_1)}^{\Phi(t)} (\Phi(t) - u)^{k-1} \, dE(u)$$

(*) For the case $p = 1$, the theorem is known (Theorem A).

$$= \int_{\lambda_1}^{t} (\Phi(t) - \Phi(u))^{k-1}\, dE(\Phi(u))$$

$$= \int_{\lambda_1}^{t} (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\,\Phi(u)}{u}\, dB(u)$$

$$= \left[(\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\,\Phi(u)}{u} B(u) \right]_{\lambda_1}^{t} -$$

$$- \int_{\lambda_1}^{t} B(u) \frac{d}{du} \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\,\Phi(u)}{u} \right\} du$$

$$= - \int_{\lambda_1}^{t} B(u) \frac{d}{du} \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\,\Phi(u)}{u} \right\} du.$$

Applying Lemma 2 and integrating $(k-1)$-times we get

$$E_{k-1}(\Phi(t)) = \frac{(1)^{k-1}}{(k-1)!} \left[B_{k-1}(u) \left(\frac{d}{du}\right)^{k-1} \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\,\Phi(u)}{u} \right]_{\lambda_1}^{t} +$$

$$+ \frac{(-1)^k}{(k-1)!} \int_{\lambda_1}^{t} B_{k-1}(u) \left(\frac{d}{du}\right)^{k} \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\,\Phi(u)}{u} \right\} du$$

$$= \frac{(-1)^{k-1}}{(k-1)!} B_{k-1}(t) \frac{\Psi(t)\,\Phi(t)}{t} \{\Phi^{(1)}(t)\}^{k-1} +$$

$$+ \frac{(-1)^k}{(k-1)!} \int_{\lambda_1}^{t} B_{k-1}(u) \left(\frac{d}{du}\right)^{k} \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\,\Phi(u)}{u} \right\} du$$

$$= \frac{(-1)^{k-1}}{(k-1)!} (\varepsilon_1(t) - \varepsilon_2(t)).$$

Thus, by virtue of Minkowski's inequality, it is sufficient to prove that

$$\int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} |\varepsilon_1(t)|^p \, dt < \infty, \tag{2.3.4}$$

and

$$\int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} |\varepsilon_2(t)|^p \, dt < \infty. \tag{2.3.5}$$

PROOF OF (2.3.4). We have

$$\int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} |\varepsilon_1(t)|^p \, dt$$

$$= \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} \left| \frac{\Psi(t)\,\Phi(t)}{t} \right|^p (\Phi^{(1)}(t))^{(k-1)p} |B_{k-1}(t)|^p \, dt$$

$$\leq K \int_{\lambda_1}^{\infty} \left\{ \frac{\gamma(t)\,\Phi^{(1)}(t)}{\Phi(t)} \right\}^{pk} \left\{ \frac{\Phi(t)}{\gamma(t)\,\Phi^{(1)}(t)} \right\}^{p-1} \left(\frac{\gamma(t)}{t} \right)^{p-1} t^{-(kp+1)} |B_{k-1}(t)|^p \, dt$$

$$\leq K \int_{\lambda_1}^{\infty} t^{-(kp+1)} |B_{k-1}(t)|^p \, dt \leq K\,(^*),$$

by hypotheses.

PROOF OF (2.3.5).

Since, by Leibnitz's formula and Lemma 3,

$$\vartheta(t, u) \equiv \left(\frac{d}{du} \right)^k \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\,\Phi(u)}{u} \right\}$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \Gamma(k-j+1)\, u^{-(k-j+1)} \times$$

$$\times \left(\frac{d}{du} \right)^j \{ (\Phi(t) - \Phi(u))^{k-1} \Psi(u)\,\Phi(u) \}$$

(*) Throughout K's denote absolute constants, not necessarily the same at each occurrence.

$$= \sum_{j=0}^{k} \sum_{r=0}^{j} (-1)^{k-j} \frac{\Gamma(k+1)}{\Gamma(r+1)\Gamma(j-r+1)} u^{-(k-j+1)} \times$$

$$\times \Phi^{(j-r)}(u) \left(\frac{d}{du}\right)^{r} \{(\Phi(t) - \Phi(u))^{k-1} \Psi(u)\}$$

$$= \sum_{j=0}^{k} \sum_{r=0}^{j} \sum_{s=0}^{r} (-1)^{k-j} \frac{\Gamma(k+1)}{\Gamma(s+1)\Gamma(r-s+1)\Gamma(j-r+1)} \times$$

$$\times u^{-(k-j+1)} \Phi^{(j-r)}(u) \Psi^{(r-s)}(u) \left(\frac{d}{du}\right)^{s} \{(\Phi(t) - \Phi(u))^{k-1}\}$$

$$= \sum_{j=0}^{k} \sum_{r=0}^{j} (-1)^{k-j} \frac{\Gamma(k+1)}{\Gamma(r+1)\Gamma(j-r+1)} u^{-(k-j+1)} \times$$

$$\times \Phi^{(j-r)}(u) \Psi^{(r)}(u) (\Phi(t) - \Phi(u))^{k-1} +$$

$$+ \sum_{j=1}^{k} \sum_{r=1}^{j} \sum_{s=1}^{r} \sum_{m=1}^{\min(s,k-1)} K_{j,r,s,m} u^{-(k-j+1)} \times$$

$$\times \Phi^{(j-r)}(u) \Psi^{(r-s)}(u) (\Phi(t) - \Phi(u))^{k-1-m} \prod_{i=1}^{s} (\Phi^{(i)}(u))^{\alpha_i}$$

where α's are zeros or positive integers, such that

$$\sum_{i=1}^{s} \alpha_i = m; \quad \sum_{i=1}^{s} i\alpha_i = s,$$

we have

$$\vartheta(t,u) = \sum_{j=0}^{k} \sum_{r=0}^{j} K_{j,r} F_1(u) u^{-(k+1)} \Phi(u) (\Phi(t) - \Phi(u))^{k-1} +$$

$$+ \sum_{j=1}^{k} \sum_{r=1}^{j} \sum_{s=1}^{r} \sum_{m=1}^{\min(s,k-1)} K_{j,r,s,m} F_2(u) \times$$

$$\times u^{-(k+1)} (\Phi(u))^{m} (\Phi(t) - \Phi(u))^{k-m-1},$$

where

$$F_1(u) = \frac{\{\gamma(u)\}^{j-r} \Phi^{(j-r)}(u)}{\Phi(u)} \frac{u^{r} \Psi^{(r)}(u)}{\{\gamma(u)/u\}^{k-r}} \left\{\frac{\gamma(u)}{u}\right\}^{k-},$$

$$0 \leq r \leq j \leq k,$$

$$F_2(u) = \frac{\{\gamma(u)\}^{j-r}\,\Phi^{(j-r)}(u)}{\Phi(u)}\,\frac{u^{r-s}\,\Psi^{(r-s)}(u)}{\{\gamma(u)/u\}^{k-r+s}}\times$$

$$\times\left(\frac{\gamma(u)}{u}\right)^{k-j}\prod_{i=1}^{s}\left[\frac{\{\gamma(u)\}^{i}\,\Phi^{(i)}(u)}{\Phi(u)}\right]^{\alpha_i},$$

$$1\le s\le r\le j\le k \quad\text{and}\quad 1\le m\le \min(s,\,k-1),$$

are bounded functions in $(\lambda_1\,,\infty)$, by hypotheses.

Therefore, in order to establish (2.3.5), by virtue of Minkowski's inequality we only need to show that, for $0\le r\le j\le k$,

$$J_1\equiv\int_{\lambda_1}^{\infty}\frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}}\,dt\,\left|\int_{\lambda_1}^{t}B_{k-1}(u)\,F_1(u)\,u^{-(k+1)}\times\right.$$

$$\left.\times\,\Phi(u)\,(\Phi(t)-\Phi(u))^{k-1}\,du\right|^{p}<\infty,$$

and, for $1\le s\le r\le j\le k$, $1\le m\le\min(s,\,k-1)$,

$$J_2\equiv\int_{\lambda_1}^{\infty}\frac{\Phi^{(1)}(t)}{\Phi(t)^{kp+1}}\,dt\left|\int_{\lambda_1}^{t}B_{k-1}(u)\,F_2(u)\,u^{-(k+1)}\times\right.$$

$$\left.\times\,\{\Phi(u)\}^{m}\,(\Phi(t)-\Phi(u))^{k-m-1}\,du\right|^{p}<\infty.$$

Now, applying Hölder's inequality, we observe that, for $0\le r\le j\le k$,

$$J_1\le\int_{\lambda_1}^{\infty}\frac{\Phi^{(1)}(t)}{\Phi(t)^{kp+1}}\,dt\left(\int_{\lambda_1}^{t}|B_{k-1}(u)|\,|F_1(u)|\,u^{-(k+1)}\,\Phi(u)(\Phi(t)-\Phi(u))^{k-1}du\right)^{p}$$

$$<\int_{\lambda_1}^{\infty}\frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}}\,dt\left(\int_{\lambda_1}^{t}|\,B_{k-1}(u)\,|^{p}\,|\,F_1(u)\,|^{p}\times\right.$$

$$\left.\times\,u^{-(k+1)p}\,(\Phi(u))^{k}\,(\Phi^{(1)}(u))^{-(p-1)}\,(\Phi(t)-\Phi(u))^{k-1}\,du\right)\times$$

$$\times\left(\int_{1}^{t}(\Phi(t)-\Phi(u))^{k-1}\,\Phi^{(1)}(u)\,du\right)^{p-1}$$

$$< K \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^2}\, dt \int_{\lambda_1}^{t} | B_{k-1}(u) |^p \, u^{-(kp+1)} \times$$

$$\times \left(\frac{\Phi(u)}{\gamma(u)\, \Phi^{(1)}(u)}\right)^p \left(\frac{\gamma(u)}{u}\right)^{p-1} \Phi(u) \left(1 - \frac{\Phi(u)}{\Phi(t)}\right)^{k-1} du$$

$$\leq K \int_{\lambda_1}^{\infty} u^{-(kp+1)} \, | B_{k-1}(u) |^p \, du \; \Phi(u) \int_{u}^{\infty} \left(1 - \frac{\Phi(u)}{\Phi(t)}\right)^{k-1} \frac{\Phi^{(1)}(t)}{(\Phi(t))^2}\, dt$$

$$\leq K \int_{\lambda_1}^{\infty} u^{-(kp+1)} \, | B_{k-1}(u) |^p \, du$$

$$\leq K,$$

by hypotheses.

Again, applying Hölder's inequality, we find that for $1 \leq s \leq r \leq j \leq k$ and $1 \leq m \leq \min(s, k-1)$,

$$J_2 < \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} (\Phi(t))^{(k-1)p}\, dt \left(\int_{\lambda_1}^{t} | B_{k-1}(u) | \times \right.$$

$$\left. \times \, | F_2(u) | \; \Phi(u)\, u^{-(k+1)} \left\{\frac{\Phi(u)}{\Phi(t)}\right\}^m \left\{1 - \frac{\Phi(u)}{\Phi(t)}\right\}^{k-m-1} du \right)^p$$

$$< \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}}\, dt \left(\int_{\lambda_1}^{t} | B_{k-1}(u) |^p \, | F_2(u) |^p \times \right.$$

$$\left. \times \, u^{-(k+1)p} (\Phi(u))^p \, (\Phi^{(1)}(u))^{1-p} \left\{1 - \frac{\Phi(u)}{\Phi(t)}\right\}^{k-m-1} du \right) \times$$

$$\times \left(\int_{\lambda_1}^{t} \left\{1 - \frac{\Phi(u)}{\Phi(t)}\right\}^{k-m-1} \Phi^{(1)}(u) \left\{\frac{\Phi(u)}{\Phi(t)}\right\}^{\frac{mp}{1-p}} du \right)^{p-1}$$

$$\leq K \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}}\, dt \int_{\lambda_1}^{t} u^{-(kp+1)} \, | B_{k-1}(u) |^p \, | F_2(u) |^p \times$$

$$\times \left\{\frac{\Phi(u)}{\gamma(u)\ \Phi^{(1)}(u)}\right\}^{p-1} \left\{\frac{\gamma(u)}{u}\right\}^{p-1} \Phi(u) \left\{1 - \frac{\Phi(u)}{\Phi(t)}\right\}^{k-m-1} du$$

$$\leq K \int_{\lambda_1}^{\infty} u^{-(kp+1)} \,|\, B_{k-1}(u) \,|^p\, du \int_{u}^{\infty} \left\{1 - \frac{\Phi(u)}{\Phi(t)}\right\}^{k-m-1} \Phi(u) \frac{\Phi^{(1)}(t)}{(\Phi(t))^2}\, dt$$

$$\leq K \int_{\lambda_1}^{\infty} u^{-(kp+1)} \,|\, B_{k-1}(u) \,|^p\, du$$

$$\leq K,$$

by hypotheses. This completes the proof of (2.3.5.).

Thus the proof of our theorem is completed.

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