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THE CURVATURE GROUPS OF A SPACE FORM

by Izu Vaisman

Following an idea, developed in another manner in [7], we shall define the curvature groups of a connection on a principal bundle and we shall study these groups for the case of a Riemannian connection of constant curvature.

1. First, we shall remember some formulas, which can be found in Chern [2]. Let $B \xrightarrow{p} X$ be a differentiable, principal bundle (X is a C^{∞}-manifold of dimension n; differentiable will always mean C^{∞}), with group G and let g_{UV} be his transition functions, corresponding to a covering $\{U\}$ of X by coordinate neighbourhoods. Then a connection on B is defined by a collection $\{\theta_U\}$ of 1-forms on U with values in the Lie algebra $\mathfrak{g}$ of G and such that, in $U \cap V$, we have

$$\theta_V = \omega(g_{UV}) + (ad\, g_{UV}^{-1})\, \theta_U . \tag{1}$$

Here ω is the $\mathfrak{g}$-valued form on G, attaching to each tangent vector of G the corresponding left invariant field.

Let R be a linear representation of G on a finit-dimensional linear space E. A tensor p-form on X, of type R is a collection $T = \{T_U\}$ of p-forms on U with values in E and such that

$$T_U = R(g_{UV})\, T_V . \tag{2}$$

For instance, the 2-forms

$$\Theta_U = d\theta_U + \frac{1}{2}[\theta_U, \theta_U] \tag{3}$$

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(the notations are those of CHERN [2]), define a tensor form Θ of type ad G, which is the curvature form of the connection θ.

For the tensor forms, there is an operation of covariant differentiation given by

$$DT_U = dT_U + \widetilde{R}(\theta_U) \wedge T_U , \tag{4}$$

where $\widetilde{R}$ is the representation of $\mathfrak{g}$ associated to R and the following formulas hold

$$D\,\Theta_U = 0, \tag{5}$$

$$D^2 T_U = \widetilde{R}(\Theta_U) \wedge T_U . \tag{6}$$

The formula (2) shows that the tensor p-forms of type R on X define a module $\mathcal{T}^p$ over the ring $\mathcal{F}$ of differentiable functions on X and (6) shows that the p-forms $\{D^2T\}$ define an $\mathcal{F}$-submodule $\mathcal{D}^p$ of $\mathcal{T}^p$.

Following the classical case of KODAIRA-SPENCER [5], we will define a *tensorial p-jet-form* of type R on X as a pair (T, S) of tensor forms of type R and of degrees p and $p+1$ respectively and will denote by J^p the $\mathcal{F}$ module of these forms.

Consider now the formula [7]

$$D(T, S) = (DT - S,\ D^2T - DS). \tag{7}$$

It is easy to see, that it defines an operator (homomorphism) $D: J^p \to J^{p+1}$, whose square is zero and such that the cochain complex $\left(\bigoplus_{p=0}^{n} J^p ,\ D\right)$ is acyclic.

Let K^p be the submodule of J^p defined by jet-forms (T, S) such that $S \in \mathcal{D}^{p+1}$. Then $K = \left(\bigoplus_{p=1}^{n} K^p,\ D\right)$ is a subcomplex of the preceeding one, which is no more acyclic.

Let

$$H^i(X, \theta, R) = H^i(K) = Ker\ D^i/Im\ D^{i-1} \quad (i = 1, 2, \dots , n) \tag{8}$$

be the cohomology groups of the complex K. We shall say that these groups are the *curvature groups* of type R of the connection θ.

Suppose now θ is a linear connection on X defined by the matrices of 1-forms $\theta_U = (-\omega_i^j)$ and with curvature forms $\Theta_U = (\Omega_i^j)$ and let the tensor forms of type R employed be the usual vector forms on X [3, 7] denoted for instance by $\lambda(\lambda^i)$ where λ^i are scalar p-forms on U. The other notations remain as above. Then we have the formulas

$$(9) \qquad \Omega_i^j = -d\omega_i^j + \omega_i^k \wedge \omega_k^j,$$

$$(10) \qquad D\lambda^i = d\lambda^i + \omega_j^i \wedge \lambda^j,$$

$$(11) \qquad D^2\lambda^i = -\Omega_j^i \wedge \lambda^j.$$

The corresponding curvature groups will be noted by $H^i(X, \theta)$ and called the *curvature groups of the linear connection* θ. Part of them were called in [7] the cohomology groups of θ and some of their properties were established.

We shall remark that on X there is a canonical 1-form $\omega\,(dx^i)$ where x^i are coordinates in U and, if θ is without torsion, $D\omega = 0$. If we note by A^p the module of scalar p-forms on X, one may consider the monomorphism

$$(12) \qquad h:\ A^{p-1} \longrightarrow J^p$$

given by

$$h(a) = (a \wedge dx^i,\ 0) \qquad (a \in A^{p-1})$$

and it is easy to find

$$hD = Dh.$$

It follows that there is a homomorphism

$$(13) \qquad h^*;\ H^{p-1}(X, R) \longrightarrow H^p(X, \theta),$$

$H^{p-1}(X, R)$ being the real cohomology groups of X.

A vector form of the type $a \wedge dx^i$ will still be noted $a \wedge \omega$. More general, the sign $\wedge$ will always note componentwise exterior product.

If θ is a Riemannian connection, his curvature groups will be called the curvature groups of the respective Riemannian space; it is in particular the case of a Riemannian space of constant curvature, which is our object of study in the next.

2. Let $X = V_n$ be a pseudo-Riemannian space with metric $g\,(g_{ij})$ and let θ be the corresponding Levi-Civita connection. By the formulas

$$(14) \qquad \Omega_i^j = \frac{1}{2} R_{i\ kh}^{\ j}\, dx^k \wedge dx^h, \quad R_{ijkh} = g_{js} R_{i\ kh}^{\ s},$$

we get the curvature tensor of the space. It is known [8] that the space V_n has constant (sectional) curvature k if and only if

$$R_{ijkh} = k\,(g_{ik}\,g_{jh} - g_{jh}\,g_{jk}). \tag{15}$$

Hence

$$\Omega_i^j = k\,\varepsilon_i \wedge dx^j, \tag{16}$$

where $\varepsilon_i = g_{ik}\,dx^k$ is a covector ε with Pfaff forms as components; the calculus with such tensorial forms will be employed in the next without other references.

The formula (11) becomes now

$$D^2\lambda^i = k\,dx^i \wedge a, \quad a = \varepsilon_i \wedge \lambda^i \tag{17}$$

and the module $\mathcal{D}^p$ is defined by vector p-forms ξ of the type

$$\xi = k\,\omega \wedge a, \tag{18}$$

a being a scalar $(p-1)$-form. In fact, for $p \geq 1$, it is easy to see that $a = \varepsilon_i \wedge \nu^i$, hence $\xi = D^2\nu$.

For $k \neq 0$, the forms (18) may be written as

$$\xi = \omega \wedge b \tag{19}$$

and conversely. But, for $k = 0$, the forms (18) vanish. In that case, we shall also consider the modules $\mathcal{D}^p$ given by (19) and shall get cohomology groups by the scheme (8); they will be called the *special groups* of the corresponding flat space.

It is clear that $\omega \wedge a = 0$ if and only if $a = 0$, because $deg\,a \leq n-1$, hence if ξ is of the form (19) the corresponding b is uniquely determined. We find then, that the modules $K^p\,(p = 1, \ldots, n-1)$ are isomorphic with the modules L^p defined by *pairs*

$$(\lambda, a), \tag{20}$$

where λ is a vector p-form and a a scalar p-form.

Applying the formula (7), we find that, by the above isomorphism, D induces on L^p an operator, noted again by D, whose square is zero and which is given by the formula

$$D\,(\lambda, a) = (D\lambda - \omega \wedge a,\; da + k\,\varepsilon_i \wedge \lambda^i). \tag{21}$$

(Remember that θ is, at us, without torsion).

It is natural now to extend the definitions of L^p and D for $p = 0$ and $p = n$ and to obtain the cochain complex $\left(L = \bigoplus_{p=0}^{n} L^p , D\right)$. We remark that, for a direct verification of $D^2 = 0$ it is necessary to employ the relations

$$D\,\varepsilon_i = 0, \quad \varepsilon_i \wedge dx^i = 0, \tag{22}$$

which follow from the properties of g_{ij} to be covariantly constant and symmetrical.

As a conclusion of the above discussion we will consider the groups

$$H^i(V_n , L) = H^i(L) \qquad (i = 0, 1, \dots , n), \tag{23}$$

i.e. the cohomology groups of the complex L and, in the next, they will be the curvature groups of the space V_n, or the special groups in the case of a flat space.

We shall now show that the special groups of a flat space can be interpreted as cohomology groups of V_n with coefficients in a sheaf.

Let us begin with the remark that, in an euclidian n-space and in cartesian coordinates x^i, the equation

$$\frac{\partial^2 F}{\partial x^i\, \partial x^j} = C\,\delta_{ij} \qquad (C = const.) \tag{24}$$

caracterises the functions $F(x)$, such that $F(x) = const.$ is a hypersphere.

We can try then to define hyperspheres in a general pseudo Riemannian space V_n as hypersurfaces which may be given by an equation $F(x) = const.$ such that

$$F_{/ij} = f(x)\, g_{ij}. \tag{25}$$

But, by a derivation of (25) we get

$$R_{i\ kj}^{\ s}\, F_{/s} = f_{/k}\, g_{ij} - f_{/j}\, g_{ik}$$

and contracting with g^{ih}, then contracting $h = j$ and noting

$$r_k^s = g^{ij}\, R_{i\ ki}^{\ s}$$

we get

$$r_k^s\, F_{/s} = (n-1)\, f_{/k}, \tag{26}$$

whence, by a new derivation and by employing Bianchi's identities

$$g^{ij}\, R_{i\ km/j}^{\ s}\, F_{/s} = 0 \tag{27}$$

Hence, in the general case, F is constant on the connected components of V_n and, from (25), $f(x) = 0$. It follows that such hyperspheres as we intended to define do not generally exist. However, for a space of constant curvature, (27) is verified identically and we may hope to find such hyperspheres. (More generally, (27) is verified for any symetric space V_n).

In this case, by (15) and (26) we get $f_{/h} = -k F_{/h}$, whence $f = -kF + C$ and (25) becomes

$$F_{/ij} = (-kF + C)\, g_{ij}\,; \tag{28}$$

if $k = 0$, we get an equation equivalent to (24). The functions F on V_n, verifying (28), or equivalently (25), will be called S-functions. For a flat space and in cartesian coordinates these functions are locally

$$F = C\,(\delta_{ij}\, x^i x^j + a_i\, x^i + b) \qquad (a_i,\ b,\ C = const.). \tag{29}$$

Each S-function F determines a field of contravariant vectors, i.e. a vector 0-form λ — the normal field of the hyperspheres $F = const.$, which, by (25), is verifying the condition

$$\lambda^i_{/j} = f\,\delta^i_j. \tag{30}$$

Conversely, if a vector 0-form λ satisfies (30), we get

$$\lambda_{i/j} = f g_{ij}$$

which shows that λ_i is the gradient of an S-function F. The vector 0-forms λ, characterized by (30), will be called *S-fields*.

The S-fields of V_n define an additive abelian group but not an $\mathcal{F}$-module. It is clear that, for a given S-field λ, the function f in (30) is uniquely determined and is of the form $-kF + C$.

Let us note by S the group of S-fields on V_n. Then, by the above remarks we get a monomorphism

$$i:\ S \to L^0 \tag{31}$$

given by $i(\lambda) = (\lambda, f)$ and, if we note by $\mathcal{S}$, $\mathcal{L}^p$ the sheaves of germs associated to S, L^p respectively, we get a sequence of sheaves and homomorphisms

$$0 \to \mathcal{S} \xrightarrow{i} \mathcal{L}^0 \xrightarrow{D} \mathcal{L}^1 \xrightarrow{D} \dots \xrightarrow{D} \mathcal{L}^n \xrightarrow{D} 0, \tag{32}$$

which is exact at $\mathcal{S}$ and where the sheaves $\mathcal{L}^p$ are fine. Moreover we have $D^2 = Di = 0$ (because of (30) and $f = -kF + C$).

Suppose now we are in the case of a flat space, hence $k = 0$, and suppose we have an element (λ, a) of L^p which belongs to the kernel of D, i.e.

$$D\lambda = \omega \wedge a \qquad da = 0. \tag{33}$$

Then, by the Poincaré lemma we have locally $a = db$. Moreover, it is known that we can choose, locally, cartesian coordinates and get $D\lambda^i = d\lambda^i$, such that the first relation (33) will become

$$d\lambda^i = dx^i \wedge db,$$

whence

$$\lambda^i = d\sigma^i - dx^i \wedge b = D\sigma^i - dx^i \wedge b.$$

Hence (33) implies locally $(\lambda, a) = D(\sigma, b)$ and we conclude that the sequence (32) is exact (the exactity at $\mathcal{L}^0$ is trivial).

So, we got the interpretation looked for: *the special groups of a paracompact, flat, pseudo-Riemannian space are the cohomology groups of the space with coefficients in the sheaf of germs of the S-fields of the space.*

We still remark here that for the curvature groups of a space V_n the homomorphisms (13) remain valid.

3. In this section we shall suppose that the space V_n is Riemannian (g-positive definite), compact and oriented, i.e. V_n is a compact oriented space form and shall obtain a theory of « harmonic forms » for the curvature groups.

First we remember [2, 6] that, in this case, we have a global scalar product of scalar p-forms given by

$$\langle a, b \rangle = \int_{V_n} a \wedge * b \tag{34}$$

and the pairs of adjoint operators d, δ; $e(a)$, $i(a)$, the last two being respectively the exterior and interior products; they are linked by

$$\delta = (-1)^{np+n+1} * d *, \quad i(a) = (-1)^{n(p+1)} * e(a) * \tag{35}$$

when acting on p forms and deg $a = 1$. Then, we have the Laplace operator

$$\Delta = d\delta + \delta d, \tag{36}$$

which is self-adjoint and commutes with $d, \delta, *$.

Let $\lambda(\lambda^i)$ be a vector p-form. Then, it is easy to see that the forms $*\lambda^i$ define a vector $(n-p)$-form $*\lambda$, $e(a)\lambda^i$ define a vector form $e(a)\lambda$ and $i(a)\lambda^i$ a vector form $i(a)\lambda$.

For the vector p-forms $\lambda(\lambda^i)$, $\mu(\mu^i)$ we have again a global scalar product given by [5]

$$\langle \lambda, \mu \rangle = \int_{V_n} g_{ij}\, \lambda^i \wedge * \mu^j, \tag{37}$$

which is clearly positive definite and commutative and, with respect to it, the operators $e(a)$ and $i(a)$ are again adjoints.

Further we have

$$0 = \int_{V_n} d(g_{ij}\, \lambda^i \wedge * \mu^j) = \int_{V_n} D(g_{ij}\, \lambda^i \wedge * \mu^j) = \int_{V_n} g_{ij}\, D\lambda^i \wedge * \mu^j +$$

$$+ (-1)^{p-1} \int_{V_n} g_{ij}\, \lambda^i \wedge **^{-1} D * \mu^j,$$

whence the operator

$$\bar{D} = (-1)^p *^{-1} D * = (-1)^{np+n+1} * D * \tag{38}$$

is adjoint to D and satisfies the relation

$$\langle D\lambda, \mu \rangle = \langle \lambda, \bar{D}\mu \rangle. \tag{39}$$

We get then the following expressions for D and $\bar{D}$

$$D\lambda^i = d\lambda^i + e(\omega_j^i)\lambda^j, \quad \bar{D}\lambda^i = \delta\lambda^i - i(\omega_j^i)\lambda^j. \tag{40}$$

Let now (λ, a) and (μ, b) be elements of L^p. Then, the formula

$$\langle (\lambda, a), (\mu, b) \rangle = \langle \lambda, \mu \rangle + \langle a, b \rangle \tag{41}$$

defines a positive definite and commutative scalar product on L^p and we wish to find the adjoint of the operator D on pairs.

We remark, first, that $e(dx^i) = e(\omega)$ and $e(\varepsilon_i) = e(\varepsilon)$ may be applied componentwise to tensor forms, this operation leading again to tensor forms with a contravariant, respectively covariant, index more. This implies that $i(\omega)$

and $i(\varepsilon)$ may also be applied to tensor forms. A straightforward calculation gives

$$\langle e(\omega)\, a, \mu \rangle = \langle a, i(\varepsilon_j)\, \mu^j \rangle, \tag{42}$$
$$\langle e(\varepsilon_i)\, \lambda^i, b \rangle = \langle \lambda, i(\omega)\, b \rangle.$$

(Of course the degrees of the forms are taken in the necessary manner).

The following calculation holds now

$$\langle D(\lambda, a), (\mu, b) \rangle = \langle D\lambda, \mu \rangle - \langle e(\omega)\, a, \mu \rangle + \langle da, b \rangle +$$
$$+ k \langle e(\varepsilon_i)\, \lambda^i, b \rangle = \langle \lambda, \bar{D}\mu \rangle - \langle a, i(\varepsilon_j)\, \mu^j \rangle = \langle a, \delta b \rangle +$$
$$+ k \langle \lambda, i(\omega)\, b \rangle = \langle (\lambda, a), \bar{D}(\mu, b) \rangle,$$

where

$$\bar{D}(\mu, b) = (\bar{D}\mu + ki(\omega)\, b,\ \delta b - i(\varepsilon_j)\, \mu^j) \tag{43}$$

is the adjoint operator of D.

Further we shall consider according to the known scheme the self adjoint Laplace operator on L^p

$$\square = D\bar{D} + \bar{D}D, \tag{44}$$

and the pairs in $Ker\ \square$ will be called *harmonic pairs.*

We get

$$\square(\lambda, a) = (\square\lambda + kDi(\omega)\, a - \bar{D}e(\omega)\, a - e(\omega)\, \delta\, a + ki(\omega)\, da +$$
$$+ e(\omega)\, i(\varepsilon_j)\, \lambda^j + k^2\, i(\omega)\, e(\varepsilon_j)\, \lambda^j,\ \Delta a - di(\varepsilon_j)\, \lambda^j + \tag{45}$$
$$+ k\delta\, e(\varepsilon_j)\, \lambda^j + ke(\varepsilon_i)\, \bar{D}\, \lambda^i - i(\varepsilon_i)\, D\, \lambda^i + k^2\, e(\varepsilon_j)\, i(dx^j)\, a + i(\varepsilon_j)\, e(dx^j)\, a),$$

where $\square\lambda$ is given by (44) applied to vector-forms.

The formulas (40) and (45) show that $\square$ is a strongly elliptic operator as considered for instance in [1]. Hence, by classical theorems [1] we have the decomposition in direct sum

$$L^p = Ker\ \square \oplus Im\ \square, \tag{46}$$

which gives the decomposition

$$L^p = Ker\ \square \oplus Im\ D \oplus Im\ \bar{D} \tag{47}$$

the three terms being mutually orthogonal. Moreover, we have

$$Ker \square = Ker\, D \cap Ker\, \bar{D} \tag{48}$$

and it is a real linear space of a finite dimension.

If we note by $\square^p$ the action of $\square$ on L^p we see from (47) and (8) that there are isomorphisms

$$H^p(V_n, L) \simeq Ker \square^p \qquad (p = 0, 1, \dots, n), \tag{49}$$

hence

$$dim\, H^p(V_n, L) = l_p(V_n)$$

are finite, non negative, integers. They will be called the *curvature numbers* of V_n. The sum

$$\chi(V_n, L) = \sum_{p=0}^{n} (-1)^p\, l_p \tag{50}$$

will be the *curvature characteristic of* V_n.

The problem of effective calculation of the numbers l_p seems to be difficult. We shall remark that this is the problem of finding the number of independent solutions of the equation

$$\square(\lambda, a) = 0, \tag{51}$$

or of the equivalent system

$$D\lambda - \omega \wedge a = 0,\ da + k\varepsilon_i \wedge \lambda^i = 0,\ \bar{D}\lambda + ki(\omega)\, a = 0, \tag{52}$$

$$\delta a - i(\varepsilon_j)\, \lambda^j = 0.$$

Another equation equivalent to (51), is

$$\langle \square(\lambda, a),\ (\lambda, a) \rangle = 0, \tag{53}$$

which, after some calculations are done, gives

$$\begin{aligned} &\langle \square \lambda, \lambda \rangle + \langle i(\varepsilon_j)\, \lambda^j,\ i(\varepsilon_j)\, \lambda^j \rangle + k^2 \langle e(\varepsilon_j)\, \lambda^j, \\ &e(\varepsilon_j)\, \lambda^j \rangle + \langle \Delta a, a \rangle + (pk^2 + n - p) \langle a, a \rangle + 2 \langle a, k\, \delta\, e(\varepsilon_j)\, \lambda^j + \\ &+ ke(\varepsilon_j)\, \bar{D}\, \lambda^j - di(\varepsilon_j)\, \lambda^j - i(\varepsilon_j)\, D\, \lambda^j \rangle = 0. \end{aligned} \tag{54}$$

From (52), we get that if $k = 0$ and $p = n$, a pair $(0, a)$ is harmonic if and only if a is so; hence $l_n \geq 1$. In the same case $p = n$, even if $k \neq 0$, there are no non vanishing harmonic pairs of the form $(\lambda, 0)$. In the general case, $\lambda = 0$ for a pair implies $a = 0$. We still remark that the last term in (54) does not allow us to apply the methods used for instance in [4].

Finally, we make another remark which holds good for the general case of a pseudo-Riemannian manifold of constant curvature.

In [3] it is given a relation of *F-Relatedness* between the vector forms of two manifolds linked by an application $F: X \rightarrow Y$. From the definition given there it follows that, if F is a convering mapping, to every vector p-form on Y there corresponds a form on X which is F-related to the given one. Now, if X, Y are pseudo-Riemannian spaces of constant curvature and F a covering, locally isomorphic mapping, it is obvious that we get, in the indicated manner, monorphisms

$$H^i(Y, L) \rightarrow H^i(X, L). \tag{56}$$

If X, Y are Riemannian, compact, orientable spaces, this means

$$l_n(Y) \leq l_n(X). \tag{57}$$

The same relations for Betti numbers are classical [4].

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