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FATOU'S THEOREM FOR GENERALIZED HALFPLANES

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The Poisson integral for generalized halfplanes was defined and studied in [2]. For holomorphic functions in the corresponding Hardy classes the existence of boundary values almost everywhere was shown in [4]. In the present paper we show that the Poisson integral of any L^∞ - function on a generalized halfplane has boundary values a. e. on the distinguished boundary. For certain special cases of generalized halfplanes which are symmetric domains in the sense of Cartan such results hold also for the Poisson integrals of L^p-functions ($p \geqq 1$), in some cases even for the Poisson integrals of measures ([7, Ch. XVII], [3], [6], [5]).

Following the notations of [2], let

$$D = \{(z_1, z_2) \in V_1 \times V_2 \mid \operatorname{Im} z_1 - \Phi(z_2, z_2) \in \Omega\}$$

be a generalized halfplane, where V_1, V_2 are complex vector spaces, Ω a regular cone in $\operatorname{Re} V_1$, and Φ an Ω-positive hermitian form. The distinguished boundary is

$$B = \{(z_1, z_2) \in V_1 \times V_2 \mid \operatorname{Im} z_1 - \Phi(z_2, z_2) = 0\}$$

and $\mathfrak{N}$ is the group of affine holomorphic automorphisms $g = (a, c)$ ($a \in \operatorname{Re} V_1$, $c \in V_2$) of D acting by

$$g : \begin{cases} z_1 \to z_1 + a + 2i\,\Phi(z_2, c) + i\Phi(c, c) \\ z_2 \to z_2 + c. \end{cases}$$

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It is immediate to check that the composition rule in $\mathfrak{N}$ is

$$(a, c)(a', c') = (a + a' + 2 \operatorname{Im} \Phi(c, c'), c + c'). \tag{1}$$

$\mathfrak{N}$ is clearly nilpotent of step 2. For $g = (a, c) \in \mathfrak{N}$ we define

$$\| g \| = \operatorname{Max} \{ | a |, | c |^2 \}$$

where $| a |, | c |$ denote some norm on V_1, V_2 which we consider fixed once and for all. It is easy to see from (1) that $\| g^{-1} \| = \| g \|$ and that there exists a constant C (depending only on Φ) such that

$$\| g g' \| \leqq C (\| g \| + \| g' \|) \tag{2}$$

for all $g, g' \in \mathfrak{N}$.

Let $G(\Omega, \Phi)$ be the group of all pairs $(A_1, A_2) \in GL(V_1) \times GL(V_2)$ such that $A_1 \cdot \Omega = \Omega$ and

$$\Phi(A_2 \zeta, A_2 \eta) = A_1 \Phi(\zeta, \eta)$$

for all $\zeta, \eta \in V_2$. This group acts on D in the obvious way, and together with $\mathfrak{N}$ it generates what is called the group of affine automorphisms of D.

For $t > 0$, $G(\Omega, \Phi)$ always contains the element $(tI, t^{\frac{1}{2}} I)$. For brevity we shall denote the result of applying this transformation to $z = (z_1, z_2)$ by z^t, i. e.,

$$z^t = (t z_1, t^{\frac{1}{2}} z_2).$$

Using the definition of P in [2] it is easy to check that, for all $t > 0$,

$$P(u, z) = P(u^t, z^t)\, t^n \tag{3}$$

where $n = \dim V_1 + \dim V_2$. In fact, with the same effort one can see that, for all $A = (A_1, A_2) \in G(\Omega, \Phi)$,

$$P(u, z) = P(Au, Az)(\det A_1)(\det A_2)^2$$

but this will not be used here. It is also easy to check that, for all $g \in \mathfrak{N}$,

$$P(u, z) = P(gu, gz). \tag{4}$$

Now let $\alpha > 0$ and let ω be a proper subcone of Ω, i. e. an open cone such that $\bar{\omega} \subset \Omega - \{0\}$. The set

$$\Gamma_{\alpha, \omega}(0) = \{g \cdot (iy, 0) \mid y \in \omega, g \in N, \| g \| < \alpha \mid y \|\}$$

will be called a *restricted admissible domain* at 0. If $u_0 = g_0 \cdot 0 \in B \; (g_0 \in \mathfrak{N})$, the set

$$\Gamma_{\alpha, \omega}(u_0) = g_0 \cdot \Gamma_{\alpha, \omega}(0)$$

will be called a *restricted admissible domain* at u_0.

Let f be a function on B and F a function on D. We say that *F converges to f restrictedly and admissibly a. e.* if for all α, ω and almost all $u_0 \in B$,

$$\lim_{\substack{z \to u_0 \\ z \in \Gamma_{\alpha, \omega}(u_0)}} F(z) = f(u_0).$$

This is the same notion as in [3], [6]; in the case of a product of one-dimensional halfplanes it reduces to restricted nontangential convergence in the sense of [7, Ch. XVII].

It is worth while to mention that restricted admissible convergence is a notion invariant under affine automorphisms of D. In fact, this is obvious from the definitions for elements of $\mathfrak{N}$. For elements $A = (A_1, A_2)$ in $G(\Omega, \Phi)$ one checks easily that there exists a constant K (depending on A) such that

$$A \cdot \Gamma_{\alpha, \omega}(0) \subset \Gamma_{K\alpha, A_1 \omega}(0)$$

for all α and ω, whence the assertion follows at once.

For $r > 0$ we define

$$B_r = \{u = g \cdot 0 \mid g \in \mathfrak{N}, \| g \| < r\}.$$

The proof of our main result is based on the following extension of the classical Lebesgue theorem. (We use the measure β defined in [2]).

LEMMA. If f is a locally integrable function on B, then for almost all $u_0 = g_0 \cdot 0 \in B$.

$$\lim_{r \to 0} \frac{1}{\beta(B_r)} \int_{B_r} | f(g_0 \cdot u) - f(u_0) | \, d\beta(u) = 0.$$

Proof. The function f can be lifted to $\mathfrak{N}$, (in fact B can even be identified with $\mathfrak{N}$); it is known [2] that β lifts then to a Haar measure on $\mathfrak{N}$. The assertion of the Lemma is equivalent to

$$\lim_{r \to 0} \frac{1}{m(N_r)} \int_{N_r} | f(g_0 g) - f(g_0) | \, dg = 0.$$

with $N_r = \{g \mid \| g \| < r\}$, and this follows by classical methods from an extension of the Hardy-Littlewood Maximal Theorem [1]. (In fact, this extension takes a particularly simple and natural form in the case of the nilpotent group $\mathfrak{N}$, and was obtained by the second named author independently of [1]; see e. g. [5]).

Theorem. Let f be a bounded measurable function on B and let F be its Poisson integral. Then F converges to f restrictedly and admissibly a. e.

Proof. Let $u_0 = g_0 \cdot 0$ $(g_0 \in \mathfrak{N})$ be a point for which the statement of the Lemma holds and let α, ω be given. Let $z = g_0 \, g \cdot (iy, 0) \in \Gamma_{\alpha, \omega}(u_0)$ $(g \in \mathfrak{N})$, and let $\varepsilon > 0$ be given. We shall show that $| F(z) - f(u_0) | < \varepsilon$ if $| y |$ is small enough.

We use a trivial estimate, then (4) and (3) with appropriate changes of variable to get

$$| F(z) - f(u_0) | = | F(g_0 \, g \cdot (iy, 0)) - f(u_0) | \leqq$$

$$\leqq \int_B | f(u) - f(u_0) | \, P(u, g_0 \, g \cdot (iy, 0)) \, d\beta(u) =$$

$$= \int_B | f(g_0 \, g \cdot u) - f(u_0) | \, P(u, (iy, 0)) \, d\beta(u) =$$

$$= \int_B | f(g_0 \, g \cdot u^{|y|}) - f(u_0) | \, P\left(u, \left(i \frac{y}{|y|}, 0\right)\right) d\beta(u).$$

This we write as

$$(5) \quad | F(z) - f(u_0) | \leqq \left(\int_{B - B_r} + \int_{Br} \right) | f(g_0 \, g \cdot u^{|y|}) - f(u_0) | \, P\left(u, \left(i \frac{y}{|y|}, 0\right)\right) d\beta(u)$$

where we choose $r > 0$ so that

$$\int_{B-B_r} P\left(u, \left(i\frac{y}{|y|}, 0\right)\right) d\beta(u) < \frac{\varepsilon}{4\|f\|_\infty}$$

for all $y \in \omega$. (This is possible since an r of this kind can be found for every fixed $y \in \omega$; by continuity this r then works for a whole neighborhood of y, and it only remains to notice that the set of the $\frac{y}{|y|}$ $(y \in \bar{\omega})$ is compact).

By the choice of r it is clear that the first integral in (5) is $< \frac{\varepsilon}{2}$.

Next we note that $P\left(u, \left(i\frac{y}{|y|}, 0\right)\right)$ is bounded by some number M for $y \in \omega, u \in B_r$, since it is a continuous function on a compact set. So the second integral in (5) is majorized by

$$M \int_{B_r} |f(g_0 g \cdot u^{|y|}) - f(u_0)|\, d\beta(u) = \frac{M}{|y|^n} \int_{B_{r|y|}} |f(g_0 g \cdot u) - f(u_0)|\, d\beta(u) =$$

$$= \frac{M}{|y|^n} \int_{g \cdot B_{r|y|}} |f(g_0 \cdot u) - f(u_0)|\, d\beta(u)$$

where we have made some changes of variable. By the definition of admissible domain and by (2) the last expression is further majorized by

$$\frac{M}{|y|^n} \int_{B_{C(\alpha+r)|y|}} |f(g_0 \cdot u) - f(u_0)|\, d\beta(u) =$$

$$= \frac{MC^n(\alpha + r)^n}{\beta(B_{C(\alpha+r)|y|})} \int_{B_{C(\alpha+r)|y|}} |f(g_0 \cdot u) - f(u_0)|\, d\beta(u).$$

By the Lemma, this integral is $< \frac{\varepsilon}{2}$ for small enough $|y|$, and the proof is finished.

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