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SOME NON-HOMOGENEOUS SYMBOLS AND ASSOCIATED PSEUDO-DIFFERENTIAL OPERATORS

S. Zaidman

1. Introduction.

In connection with a recent paper by Kohn and Nirenberg [1] we consider here a class of symbols $a(x, \xi)$ not necessarily homogeneous with respect to the ξ-variable and its associated pseudo-differential operators. Estimates of the norm of a pseudo-differential operator modulo lower order operators through the numbers $\max_{x \in R^n} \overline{\lim}_{|\xi| \to \infty} |a(x, \xi)|$ and $\overline{\lim}_{|\xi| \to \infty} \max_{x \in R^n} |a(x, \xi)|$ are obtained, following ideas of [1] and of a preliminary version of it.

A main technical tool is a partition of the unity constructed in [2], for similar (but there different) purposes.

2. Notation and Definitions.

We start by defining the space $\mathcal{S}$ consisting of C^∞ complex valued functions $u(x)$, $x = (x_1 \dots x_n)$, defined in R^n, which, together with all their derivatives die down faster than any power of $|x|$ at infinity. The Fourier transform $\tilde{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix \cdot \xi} u(x)\, dx$ of $u(x) \in \mathcal{S}$ is, as a function of $\xi = (\xi_1, \dots \xi_n)$ again in $\mathcal{S}$. Here we denote $x \cdot \xi = \sum_{j=1}^{n} x_j \xi_j$, $dx = dx_1 \dots dx_n$, $d\xi = d\xi_1 \dots d\xi_n$.

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Denote by $\mathcal{S}'$ the dual of $\mathcal{S}$, the space of temperate distributions in R^n. The Fourier transform is defined on $\mathcal{S}'$ too and maps $\mathcal{S}'$ onto itself. For any real numbr s we define the norm $\| \quad \|_s$ by

$$(2.1) \qquad \| u \|_s^2 = \int (1 + |\xi|^2)^s \, |\tilde{u}(\xi)|^2 \, d\xi \,, \qquad u \in \mathcal{S}$$

and denote by H_s the Hilbert space obtained by the completion of $\mathcal{S}$ in this norm.

We make use of the familiar notation

$$D_j = -\sqrt{-1}\,\frac{\partial}{\partial x_j}\,,\; D = (D_1 \dots D_n)\,,\; D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}\,,\; \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}\,,$$

for $\alpha = (\alpha_1 \dots \alpha_n)$ a multiindex with the α_j integers ≥ 0. Then $\widetilde{D^\alpha u} = \xi^\alpha \tilde{u}$ and, by Parseval's theorem, if s is a non-negative integer, then

$$C \| u \|_s^2 \leq \sum_{|\alpha| \leq s} \| D^\alpha u \|^2 \leq C' \| u \|_s^2 \,,$$

where C, c' are positive constants. Here $\| \quad \|$ denotes the L^2 norm and $\| u \| = \| u \|_0$ (by Parseval). A linear operator L from $\mathcal{S}$ into $\mathcal{S}'$ is said to have order r, or to be of order r, if for each real s there exists a constant C_s, such that $\| Lu \|_s \leq C_s \| u \|_{s+r}$, for $u \in \mathcal{S}$.

Following a notation in [1], if $\Phi(\xi)$ is a given function of ξ we shall denote by $\Phi(D)$ the operation of multiplying the Fourier transform of a function by $\Phi(\xi)$ and then applying the inverse Fourier transformation.

3. Pseudo-Differential Operators of Order Zero.

Let $a(x, \xi)$ be a complex-valued function defined for all x and $\xi \neq 0$. Assume that $a(x, \xi)$ has a limit $a(\infty, \xi)$ as $x \to \infty$ for each $\xi \neq 0$, and that $a'(x, \xi) = a(x, \xi) - a(\infty, \xi)$, as a function of x, defines a temperate distribution for any $\xi \neq 0$. About the Fourier transform $\tilde{a}'(\eta, \xi)$, $\xi \neq 0$ of $a(x, \xi) - a(\infty, \xi)$, we suppose it to be a function of η such that

$$\text{(i)} \qquad |\tilde{a}'(\eta, \xi)| \leq k(\eta), \; \xi \neq 0, \; \eta \in R^n$$

$$\text{(ii)} \qquad |\tilde{a}'(\eta, \xi) - \tilde{a}'(\eta, \tau)| \leq k(\eta)\, |\xi - \tau|\, (|\xi| + |\tau|)^{-1}, \; \xi, \tau \neq 0,$$

where $k(\eta)$ is a measurable function such that

$$(1 + |\eta|^2)^{\frac{p}{2}} k(\eta) \in L^1 \text{ for } p = 1, 2, \dots .$$

About the function $a(\infty, \xi)$ we assume that it is bounded in $R^n - \{0\}$ and that

(iii) $$|a(\infty, \xi) - a(\infty, \eta)| \leq C |\xi - \eta| (|\xi| + |\eta|)^{-1}.$$

Finally we suppose that, for $x \in R^n$ and $\xi \neq 0$, the formula

(iv) $$a(x, \xi) = a(\infty, \xi) + (2\pi)^{-n/2} \int e^{ix \cdot \eta} \tilde{a}'(\eta, \xi) \, d\eta$$ holds.

It follows from the representation (iv) that every symbol $a(x, \xi)$ is indefinitely differentiable with respect to x and we have with constants C_α that

(3.2) $$|D^\alpha a(x, \xi)| \leq C_\alpha, \; \xi \neq 0, \; x \in R^n.$$

REMARK 1. Let $a(x, \xi)$ be a complex-valued function, defined for $x \in R^n$ and $\xi \neq 0$, such that $D_x^\alpha \partial_\beta^\xi a(x, \xi)$ exists for each multi-index α and for each multi-index β with $|\beta| \leq 1$.

Suppose $a(x, t\xi) = a(x, \xi)$ for $t > 0$, $a(\infty, \xi) = \lim_{|x| \to \infty} a(x, \xi)$ exist for $\xi \neq 0$ and $a(\infty, \xi) \in C^1(R_\xi^n - \{o\})$.

Let us suppose also

$$|(1 + |x|^2)^p D_\xi^\alpha \partial_\xi^\beta (a(x, \xi) - a(\infty, \xi))| \leq C_{p,\alpha,\beta}, \quad \text{for } x \in R^n, \; |\xi| = 1, \; |\beta| \leq 1.$$

(that is $a(x, \xi)$ is a homogeneous symbol in Kohn-Nirenberg's sense [1]).

It can be proved that such a symbol is also a symbol in our sense; (see for the proof our forthcoming paper [4].

REMARK 2. Let us give a non-trivial example of a non-homogeneous (in our sense) symbol.

Put $a(x, \xi) = a(x) f(\xi)$, when $a(x) \in \mathcal{S}$ and $f(\xi) = \begin{cases} |\xi|, & |\xi| \leq 1 \\ 1, & |\xi| > 1. \end{cases}$

It can be proved (see [5]) that $|f(\xi) - f(\eta)| \leq C |\xi - \eta| (|\xi| + |\eta|)^{-1}$; then the other conditions are easily verified.

Define an operator $a(x, D) = A$ associated to $a(x, \xi)$ and mapping $\mathcal{S}$ in

$\mathcal{S}'$, by the formula

$$(3.3)\qquad \widetilde{a(x, D)\, u}(\xi) = a(\infty, \xi)\, \tilde{u}(\xi) + (2\pi^{-n/2} \int \tilde{a}'(\xi - \eta, \xi)\, \tilde{u}(\eta)\, d\eta,\ u \in \mathcal{S};$$

the Fourier transform of $a(x, D)\, u$ is taken in $\mathcal{S}'$. But the right hand side is, for $u \in \mathcal{S}$, bounded by

$$\sup_{\xi \neq 0} a(\infty, \xi)\, |\tilde{u}(\xi)| + c \int k(\xi - \eta)\, |\tilde{u}(\eta)|\, d\eta,$$

an integrable function of ξ. So we can take the classical inverse Fourier transform and get $a(x, D)\, u$, a bounded continuous function of $x \in R^n$, for each $u \in \mathcal{S}$.

We see also easily that for $u \in \mathcal{S}$, the useful formula

$$(3.4)\qquad (2\pi)^{-n/2} \int e^{-ix \cdot \xi}\, a(x, \xi)\, u(x)\, dx = \widetilde{a(x, D)\, u}(\xi) \qquad \text{holds.}$$

Using the definition (3.3) we shall give first a proof for

THEOREM 1. A *has order zero.*

Proof. Since obviously the operator $a(\infty, D)$ maps H_s boundedly into H_s we shall only consider the remaining term in (3.3). We have to estimate the L^2-norm of

$$(2\pi)^{-n/2} (1 + |\xi|^2)^{s/2} \int \tilde{a}'(\xi - \eta, \xi)\, \tilde{u}(\eta)\, d\eta =$$

$$(2\pi)^{-n/2} \int \left(\frac{1 + |\xi|^2}{1 + |\eta|^2}\right)^{s/2} \tilde{a}'(\xi - \eta, \xi)\, (1 + |\eta|^2)^{s/2}\, \tilde{u}(\eta)\, d\eta$$

in terms of the L^2-norm of $(1 + |\eta|^2)^{s/2}\, \tilde{u}(\eta)$.

This is done easily, as in [1], applying the elementary inequality (Min kowski)

$$(3.5)\qquad \left(\int \left(\int |f(x, y)|\, dx\right)^2 dy\right)^{\frac{1}{2}} \leq \int \left(\int |f(x, y)|^2\, dy\right)^{\frac{1}{2}} dx,$$

together with the simple one (Peetre)

$$(3.6)\qquad (1 + |\xi|^2)^{s/2} \leq 2^{\frac{|s|}{2}} (1 + |\eta|^2)^{s/2} (1 + |\xi - \eta|^2)^{\frac{|s|}{2}}$$

which is found in [1] and [6] pag. 39 and the property (i) from the Definition of a symbol.

4. The main estimate (I).

From (3.2) with $\alpha = 0$, it follows that a symbol $a(x, \xi)$ is bounded for $x \in R^n$, $\xi \neq 0$. Denote

$$\overline{\lim_{|\xi| \to \infty}} \max_{R^n} |a(x, \xi)| = K. \tag{4.1}$$

Then

THEOREM 2. *For any $\varepsilon > 0$ there is a constant C_ε such that*

$$\| Au \|_0 \leq (K + \varepsilon) \| u \|_0 + C_\varepsilon \| u \|_{-\frac{1}{2}}, \; u \in \mathcal{S} \tag{4.2}$$

holds.

Proof. Some Lemmas will be involved in the proof.

Consider the function $\varphi(\xi) = 0$ for $|\xi| \leq 1$, $= 1$ for $|\xi| \geq 2 = |\xi| - 1$ for $1 \leq |\xi| \leq 2$; for real $t > 0$ consider the operator $\varphi\left(\frac{1}{t} D\right)$ defined on $\mathcal{S}$ by;

$$\widetilde{\varphi\left(\frac{1}{t} D\right) u}(\xi) = \varphi\left(\frac{\xi}{t}\right) \tilde{u}(\xi). \tag{4.3}$$

We have

LEMMA 4.1. *Suppose that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ and $C_\varepsilon > 0$ such that*

$$\| \varphi(D/t_\varepsilon) a(x, D) u \|_0 \leq (K + \varepsilon) \| u \|_0 + C_\varepsilon \| u \|_{-\frac{1}{2}}, \; u \in \mathcal{S}. \tag{4.4}$$

Then Theorem 2 follows.

We have in fact for $\varepsilon > 0$

$$a(x, D) u = \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D) u + \left(1 - \varphi\left(\frac{D}{t_\varepsilon}\right)\right) a(x, D) u, \; u \in \mathcal{S} \tag{4.5}$$

This gives, by use of (4.4)

$$\| a(x. D) u \|_0 \leq (K + \varepsilon) \| u \|_0 + C_\varepsilon \| u \|_{-\frac{1}{2}} + \left\| \left(1 - \varphi\left(\frac{D}{t_\varepsilon}\right)\right) a(x, D) u \right\|_0. \tag{4.6}$$

Remark now that

$$\left\| \left(1 - \varphi\left(\frac{D}{t_\varepsilon}\right)\right) a(x, D) u \right\|_0^2 = \tag{4.7}$$

$$\int (1+|\xi|^2)^{-\frac{1}{2}}(1+|\xi|^2)^{\frac{1}{2}}\left(1-\varphi\left(\frac{\xi}{t_\varepsilon}\right)\right)^2 |\widetilde{a(x,D)u}(\xi)|^2 \, d\xi =$$

$$\int_{|\xi|<2t_\varepsilon} (1+|\xi|^2)^{\frac{1}{2}}(1+|\xi|^2)^{-\frac{1}{2}} |\widetilde{a(x,D)u}(\xi)|^2 \, d\xi \leq (1+4t_\varepsilon^2)^{\frac{1}{2}} \| a(x,D)u \|^2_{-\frac{1}{2}}$$

$$\leq c\,(1+4t_\varepsilon^2)^{\frac{1}{2}} \| u \|^2_{-\frac{1}{2}},$$

(by Th. 1); then, from (4.6), (4.7), we deduce

$$\| a(x,D)u \|_0 \leq (K+\varepsilon)\| u \|_0 + (C_\varepsilon + c^{\frac{1}{2}}(1+4\,t_\varepsilon^2))^{\frac{1}{4}} \| u \|_{-\frac{1}{2}}, \tag{4.8}$$

hence the lemma.

A main tool in the following proof is a certain partition of the unity in the ξ-space which was constructed in [2] for similar purposes.

Precisely, it is a sequence $\{\psi_\alpha(\xi)\}_{\alpha=0}^{\infty}$ of indefinitely differentiable functions with compact support in R^n_ξ, such that

(a) $$\psi_\alpha(\xi) \geq 0, \sum_{\alpha=0}^{\infty} \psi_\alpha^2(\xi) = 1, \xi \in R^n.$$

For $\alpha > 0$ all ψ_α are null in a fixed neighborhood of the origin.

(b) If $C = \frac{3}{2}\sqrt{n}$ and $\xi, \eta \in \operatorname{supp} \psi_\alpha$, then $\left| \sqrt{|\xi|} - \sqrt{|\eta|} \right| \leq C$

and $|\xi - \eta| \leq 2C\sqrt{|\xi|} + C^2$

(c) $$\left(\sum_0^{\infty} |\psi_\alpha(\xi) - \psi_\alpha(\eta)|^2\right)^{\frac{1}{2}} \leq \text{const.} \frac{|\xi-\eta|}{\sqrt{|\eta|}}, \text{ if } |\xi-\eta| < \frac{1}{2}|\eta|.$$

We have, for real s and $u \in \mathcal{S}$ (and also for $u \in H_s$)

$$\| u \|_s^2 = \int (1+|\xi|^2)^s \left(\sum_{\alpha=0}^{\infty} \psi_\alpha^2(\xi)\right) |\tilde{u}(\xi)|^2 \, d\xi = \tag{4.9}$$

$$\sum_{\alpha=0}^{\infty} \int (1+|\xi|^2)^s \psi_\alpha^2(\xi) |\tilde{u}(\xi)|^2 \, d\xi =$$

$$\sum_{\alpha=0}^{\infty} \| \psi_\alpha(D)u \|_s^2, \text{ where } \widetilde{\psi_\alpha(D)u}(\xi) = \psi_\alpha(\xi)\tilde{u}(\xi).$$

Hence, for any $t > 0$ and $u \in \mathcal{S}$

$$(4.10) \qquad \left\| \varphi\left(\frac{D}{t}\right) a(x, D)\, u \right\|_0^2 = \sum_{\alpha=0}^{\infty} \left\| \psi_\alpha(D)\, \varphi\left(\frac{1}{t} D\right) a(x, D)\, u \right\|_0^2 .$$

Take t_0 such that supp $\psi_0 \subseteq \{\xi ; |\xi| \leq t_0\}$.

We may have: supp $\psi_\alpha \subseteq \{\xi ; |\xi| \leq t_0\}$ for other values of α; in that case

$$(4.11) \qquad \varphi\left(\frac{D}{t}\right) \psi_\alpha(D) = \theta \text{ if } t \geq t_0 \ \left(\text{because } \varphi\left(\frac{\xi}{t}\right) \psi_\alpha(\xi) = 0,\ \xi \in R^n\right).$$

We deduce, from (4.10) that

$$(4.12) \qquad \left\| \varphi\left(\frac{D}{t}\right) a(x, D)\, u \right\|_0^2 = \Sigma' \left\| \psi_\alpha(D)\, \varphi\left(\frac{D}{t}\right) a(x, D)\, u \right\|_0^2, \quad u \in \mathcal{S} \text{ and } t \geq t_0$$

the sum Σ' being taken over those α such that supp $\psi_\alpha \cap \{\xi ; |\xi| > t_0\} \neq \Phi$.

Moreover we remark that

$$(4.13) \qquad \psi_\alpha(D)\, \varphi\left(\frac{D}{t}\right) a(x, D) = \varphi\left(\frac{D}{t}\right) a(x, D)\, \psi_\alpha(D) + \varphi\left(\frac{D}{t}\right) [\psi_\alpha(D), a(x, D)]$$

where, as usual

$$[\psi_\alpha(D), a(x, D)] = \psi_\alpha(D)\, a(x, D) - a(x, D)\, \psi_\alpha(D),$$

the commutator of two operators $\psi_\alpha(D)$ and $a(x, D)$.

Hence from the relation $(a + b)^2 \leq a^2(1 + \delta^2) + b^2\left(1 + \frac{1}{\delta^2}\right)$, which is true for any $\delta > 0$, we obtain, from (4.13)

$$(4.14) \qquad \left\| \psi_\alpha(D)\, \varphi\left(\frac{D}{t}\right) a(x, D)\, u \right\|_0^2 \leq (1 + \delta^2) \left\| \varphi\left(\frac{D}{t}\right) a(x, D)\, \psi_\alpha(D)\, u \right\|_0^2 +$$
$$\left(1 + \frac{1}{\delta^2}\right) \left\| \varphi\left(\frac{D}{t}\right) [\psi_\alpha(D), a(x, D)]\, u \right\|_0^2, \ u \in \mathcal{S},\ \forall\, \delta > 0.$$

From (4.12) we get, for any $\delta > 0$

$$(4.15) \qquad \left\| \varphi\left(\frac{D}{t}\right) a(x, D)\, u \right\|_0^2 = \Sigma' \left\| \psi_\alpha(D)\, \varphi\left(\frac{D}{t}\right) a(x, D)\, u \right\|_0^2 \leq$$

$$(1+\delta^2)\,\Sigma' \left\| \varphi\left(\frac{D}{t}\right) a(x,D)\,\psi_\alpha(D)\,u \right\|_0^2 +$$

$$\left(1+\frac{1}{\delta^2}\right) \sum_0^\infty \left\| \varphi\left(\frac{D}{t}\right) [\psi_\alpha(D), a(x,D)]\,u \right\|_0^2, \; u \in \mathcal{S},\; t \geq t_0 .$$

An estimation of the second term in the right hand side, a result similar to one in [2], is given here in

LEMMA 4.2. *We have for* $t \geq t_0$ *a constant* C_{t_0}, *such that*

$$\sum_0^\infty \left\| \varphi\left(\frac{D}{t}\right) [\psi_\alpha(D), a(x,D)]\,u \right\|_0^2 \leq C_{t_0} \| u \|_{-\frac{1}{2}}^2, \; u \in \mathcal{S}. \tag{4.16}$$

Remark first that for $u \in \mathcal{S}$ the Fourier transform $I_\alpha(\xi)$ of the function $\varphi\left(\frac{D}{t}\right) [\psi_\alpha(D), a(x,D)]\,u$ is given by

$$I_\alpha(\xi) = (2\pi)^{-n/2} \int \varphi(\xi/t)\, \tilde{a}'(\xi-\eta,\xi)\,[\psi_\alpha(\xi)-\psi_\alpha(\eta)]\,\tilde{u}(\eta)\,d\eta . \tag{4.17}$$

Consider now an arbitrary sequence $(W_\alpha(\xi))_{\alpha=0}^\infty$, of functions $W_\alpha(\xi) \in L^2$, such that $\sum_{\alpha=0}^\infty \| W_\alpha \|_0^2 < \infty$. We shall estimate the expression (where $(\,,\,)_0$ is the L_2 scalar product)

$$\sum_{\alpha=0}^\infty (I_\alpha(\xi), W_\alpha(\xi))_0 = \sum_{\alpha=0}^\infty (2\pi)^{-n/2} \iint \varphi\left(\frac{\xi}{t}\right) \tilde{a}'(\xi-\eta,\xi)\cdot (\psi_\alpha(\xi)-\psi_\alpha(\eta))\, W_\alpha(\xi)\, \tilde{u}(\eta)\, d\eta\, d\xi . \tag{4.18}$$

We have:

$$\left| \sum_{\alpha=0}^\infty (I_\alpha(\xi), W_\alpha(\xi))_0 \right| \leq c \iint \varphi\left(\frac{\xi}{t}\right) k(\xi-\eta) \left(\sum_{\alpha=0}^\infty | \psi_\alpha(\xi)-\psi_\alpha(\eta) |^2 \right)^{\frac{1}{2}} \left(\sum_{\alpha=0}^\infty | W_\alpha(\xi) |^2 \right)^{\frac{1}{2}} | \tilde{u}(\eta) |\, d\eta\, d\xi . \tag{4.19}$$

Remember property (c) of the partition of the unity $1 \equiv \sum_{\alpha=0}^\infty \psi_\alpha^2(\xi)$, that is

$$\left(\sum_{\alpha=0}^\infty | \psi_\alpha(\xi)-\psi_\alpha(\eta) |^2 \right)^{\frac{1}{2}} \leq c\, \frac{|\xi-\eta|}{\sqrt{|\eta|}}, \text{ if } |\xi-\eta| \leq \frac{1}{2} |\eta|$$

In (4.19) we may consider $|\xi| \geq t \geq t_0$. When $|\xi - \eta| < \frac{1}{2}|\eta|$, it follows $\frac{|\eta|}{2} \geq t_0 - |\eta|, |\eta| \geq \frac{2}{3} t_0$; then, for $|\xi - \eta| \leq \frac{|\eta|}{2}$, we have

$$(4.20) \qquad \left(\sum_{\alpha=0}^{\infty} |\psi_\alpha(\xi) - \psi_\alpha(\eta)|^2\right)^{\frac{1}{2}} \leq C_{t_0}(1 + |\xi - \eta|^2)^{\frac{1}{2}}(1 + |\eta|^2)^{-\frac{1}{4}}.$$

On the other hand, if $|\xi - \eta| > \frac{1}{2}|\eta|$, we obtain $1 + |\xi - \eta| \geq \geq \frac{1}{2}(1 + |\eta|)$ hence $2(1 + |\xi - \eta|)(1 + |\eta|)^{-1} \geq 1$ and

$$(4.21) \qquad \sum_{\alpha=0}^{\infty} |\psi_\alpha(\xi) - \psi_\alpha(\eta)|^2 \leq 2 \sum_{\alpha=0}^{\infty} (\psi_\alpha^2(\xi) + \psi_\alpha^2(\eta)) = 4 \leq$$

$$\leq 8(1 + |\xi - \eta|)(1 + |\eta|)^{-1} \leq C(1 + |\xi - \eta|^2)^{\frac{1}{2}}(1 + |\eta|^2)^{-\frac{1}{4}}.$$

Hence for any $|\xi| \geq t \geq t_0$

$$(4.22) \qquad \sum_{\alpha=0}^{\infty} |\psi_\alpha(\xi) - \psi_\alpha(\eta)|^2 \leq C_{t_0}(1 + |\xi - \eta|^2)^{\frac{1}{2}}(1 + |\eta|^2)^{-\frac{1}{4}}.$$

We get now, from (4 19)

$$(4.23) \qquad \left|\sum_{\alpha=0}^{\infty} (I_\alpha(\xi), W_\alpha(\xi))_0\right| \leq C_{t_0} \iint \varphi(\xi/t)\, k(\xi - \eta)(1 + |\xi - \eta|^2)^{\frac{1}{2}} \cdot$$

$$(1 + |\eta^2)^{-\frac{1}{4}} |\tilde{u}(\eta)| \times \left(\sum_{\alpha=0}^{\infty} |W_\alpha(\xi)|^2\right)^{\frac{1}{2}} d\xi\, d\eta.$$

We apply the simple inequality

$$(4.24) \qquad \left|\iint K(x - y) f(x)\, g(y)\, dx\, dy\right| \leq \|K\|_{L^1} \|f\|_{L^2} \|g\|_{L^2},$$

taking

$$K(\lambda) = k(\lambda)(1 + |\lambda|^2)^{\frac{1}{2}}, \; f(\eta) = (1 + |\eta|^{2^{-\frac{1}{4}}} |\tilde{u}(\eta)|$$

$g(\xi) = \left(\sum_{\alpha=0}^{\infty} |W_\alpha(\xi)|^2 \right)^{\frac{1}{2}}$ and obtain, as $0 \leq \varphi \leq 1$

$$(4.25) \qquad \left| \sum_{\alpha=0}^{\infty} (I_\alpha(\xi), W_\alpha(\xi))_0 \right| \leq C_{t_0} \| u \|_{-\frac{1}{2}} \left(\sum_{\alpha=0}^{\infty} \| W_\alpha \|_0^2 \right)^{\frac{1}{2}}.$$

This implies the estimate

$$(4.26) \qquad \sum_{\alpha=0}^{\infty} \| I_\alpha \|_0^2 \leq C_{t_0} \| u \|^2_{-\frac{1}{2}}, \quad \forall\, u \in \mathcal{S}$$

that is (4.16).

From (4.15) we deduce that for any $\delta > 0$, the relation

$$(4.27) \qquad \left\| \varphi\left(\frac{D}{t}\right) a(x, D)\, u \right\|_0^2 \leq (1 + \delta^2)\, \Sigma' \left\| \varphi\left(\frac{D}{t}\right) a(x, D)\, \psi_\alpha(D)\, u \right\|_0^2 + \left(1 + \frac{1}{\delta^2}\right) C_{t_0} \| u \|^2_{-\frac{1}{2}}, \quad t \geq t_0$$

holds, for $u \in \mathcal{S}$, $t \geq t_0$, the sum Σ' being taken over that α for which supp. $\psi_\alpha \cap \{\xi ; |\xi| > t_0\} \neq \Phi$.

We prove now the main step in the Theorem 2, namely the

LEMMA 4.3. *For any $\varepsilon > 0$ there is a constaut $C > 0$ and $t_\varepsilon \geq t_0$, such that for $u \in \mathcal{S}$ the estimate*

$$(4.28) \qquad \left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, \psi_\alpha(D)\, u \right\|_0 \leq (K + \varepsilon) \| \psi_\alpha(D)\, u \|_0 + C \| \psi_\alpha(D)\, u \|_{-\frac{1}{2}}$$

is verified, for α such that supp $\psi_\alpha \cap \{\xi ; |\xi| \geq t_\varepsilon\} \neq \Phi$.

In fact, as $K = \overline{\lim}_{|\xi| \to \infty} \max_{R^n} |a(x, \xi)|$, it follows that, $\varepsilon' > 0$ being given, there is $t_{1,\varepsilon'}$ such that $|a(x, \xi)| \leq K + \varepsilon'$ for $|\xi| > t_{1\varepsilon'}$, $x \in R^n$. Moreover, for $C = \frac{3\sqrt{n},}{2}\ \frac{2C\sqrt{|\xi|} + C^2}{|\xi|} \leq \varepsilon'$ for $|\xi| \geq t_{2,\varepsilon'}$.

Let $t_{\varepsilon'} = \max(t_0, t_{1,\varepsilon'}, t_{2,\varepsilon'})$, and let us estimate the expression $\left\| \varphi\left(\frac{D}{t_{\varepsilon'}}\right) a(x, D)\, \psi_\alpha(D)\, u \right\|_0$, that is by (3.4), (remarking that for $u \in \mathcal{S}$, $\psi_\alpha(D) u \in \mathcal{S}$), the L^2 norm of

$$I_\alpha(\xi) = \varphi\left(\frac{\xi}{t_{\varepsilon'}}\right) (2\pi)^{-n/2} \int e^{-ix\cdot\xi}\, a(x, \xi)\, \psi_\alpha(D)\, u\, dx.$$

Take $\xi_\alpha \in \operatorname{supp} \psi_\alpha$, $|\xi_\alpha| \geq t_{\varepsilon'}$, otherwise arbitrary; *we examine hence forth the case when* $\operatorname{supp} \psi_\alpha \cap \{\xi ; |\xi| \geq t_{\varepsilon'}\} \neq \Phi$.

Write now

$$I_\alpha(\xi) = \varphi(\xi/t_{\varepsilon'})(2\pi)^{-n/2} \int e^{-ix\cdot\xi} (a(x,\xi) - a(x,\xi_\alpha)) \psi_\alpha(D) u \, dx +$$

$$\varphi\left(\frac{\xi}{t_{\varepsilon'}}\right)(2\pi)^{-n/2} \int e^{-ix\cdot\xi} \, a(x,\xi_\alpha) \psi_\alpha(D) u \, dx = I_{\alpha,1} + I_{\alpha,2} .$$

By Plancherel we get, as

$$|I_{\alpha,2}| \leq (2\pi)^{-\frac{n}{2}} \left| \int e^{-ix\cdot\xi} \, a(x,\xi_\alpha) \psi_\alpha(D) u \, dx \right|$$

$$\|I_{\alpha,2}\|_0^2 = \|a(x,\xi_\alpha)\psi_\alpha(D) u\|_0^2 = \tag{4.29}$$

$$\int |a(x,\xi_\alpha)|^2 |\psi_\alpha(D) u(x)|^2 dx \leq (K+\varepsilon')^2 \|\psi_\alpha(D) u\|_0^2$$

so

$$\|I_{\alpha,2}\|_0 \leq (K+\varepsilon') \|\psi_\alpha(D) u\|_0, \text{ as } |\xi_\alpha| \geq t_{\varepsilon'} > t_{1,\varepsilon'} . \tag{4.30}$$

To estimate the L_2-norm of $I_{\alpha,1}(\xi)$ we write it again in the form

$$I_{\alpha,1}(\xi) = \varphi(\xi/t_{\varepsilon'})(a(\infty,\xi) - a(\infty,\xi_\alpha))\psi_\alpha(\xi)\tilde{u}(\xi) + \tag{4.31}$$

$$(2\pi)^{-\frac{n}{2}} \int_{\operatorname{supp} \psi_\alpha} \varphi(\xi/t_{\varepsilon'})(\tilde{a}'(\xi-\eta,\xi) - a'(\xi-\eta,\xi_\alpha))\psi_\alpha(\eta)\tilde{u}(\eta) \, d\eta =$$

$$I_{\alpha,3}(\xi) + I_{\alpha,4}(\xi).$$

We have $I_{\alpha,3}(\xi) = 0$ for $|\xi| \leq t_{\varepsilon'}$ or $\xi \notin \operatorname{supp} \psi_\alpha$ and

$$|I_{\alpha,3}(\xi)| \leq \frac{|\xi - \xi_\alpha|}{|\xi| + |\xi_\alpha|} |\psi_\alpha(\xi)| \, |\tilde{u}(\xi)|$$

for $|\xi| \geq t_{\varepsilon'}$ and $\xi \in \operatorname{supp} \psi_\alpha$

In this last case, we have

$$|\xi - \xi_\alpha| \leq 2C\sqrt{|\xi_\alpha|} + C^2 ;$$

hence, for any ξ

$$|I_{\alpha,3}(\xi)| \leq \varepsilon' |\psi_\alpha(\xi)| \, |\tilde{u}(\xi)| \tag{4.32}$$

which gives

$$\| I_{\alpha,3} \|_0 \leq \varepsilon' \, \| \psi_\alpha(D)\, u \|_0 \,. \tag{4.33}$$

Consider now the term $I_{\alpha,4}(\xi)$; we have the obvious estimate

$$| I_{\alpha,4}(\xi) | \leq \tag{4.34}$$

$$\leq \varphi\left(\frac{\xi}{t_{\varepsilon'}}\right)(2\pi)^{-\frac{n}{2}} \int\limits_{\operatorname{supp}\psi_\alpha} k(\xi-\eta)\,|\xi-\xi_\alpha|\,(|\xi|+|\xi_\alpha|)^{-1}\,|\psi_\alpha(\eta)|\,|\tilde{u}(\eta)|\,d\eta$$

Denote by S_α the sphere $\{\xi\,;\,|\xi-\xi_\alpha| \leq 2\,(2C\sqrt{|\xi_\alpha|}+C^2)$.

Then, we have

$$I_{\alpha,4}(\xi) = 0 \ \text{ for } \ |\xi| \leq t_{\varepsilon'} \tag{4.35}$$

and

$$| I_{\alpha,4}(\xi) | \leq (2\pi)^{-n/2}\, 2 \int\limits_{\operatorname{supp}\psi_\alpha} k(\xi-\eta)\,|\psi_\alpha(\eta)|\,\frac{2C\sqrt{|\xi_\alpha|}+C^2}{|\xi_\alpha|}\,|\tilde{u}(\eta)|\,d\eta \tag{4.36}$$

$$\leq 2^{-\frac{n}{2}+1}(\pi)^{-\frac{n}{2}}\,\varepsilon' \int\limits_{\operatorname{supp}\psi_\alpha} k(\xi-\eta)\,|\psi_\alpha(\eta)|\,|\tilde{u}(\eta)|\,d\eta, \quad \text{for } \xi \in S_\alpha \,.$$

Consider finally the case $\xi \notin S_\alpha$. Then, first of all, using *i*) of n. 3 in (4.31), we get

$$| I_{\alpha,4}(\xi) | \leq (2\pi)^{-\frac{n}{2}}\, 2 \int\limits_{\operatorname{supp}\psi_\alpha} k(\xi-\eta)\,|\psi_\alpha(\eta)|\,|\tilde{u}(\eta)|\,d\eta \,. \tag{4.37}$$

Now, for $|\xi-\xi_\alpha| \geq (2C\sqrt{|\xi_\alpha|}+C^2).2$ and $\eta \in \operatorname{supp}\psi_\alpha$ we have as is easily seen;

$$\xi_\alpha \in \operatorname{supp}\psi_\alpha\,,\ \eta \in \operatorname{supp}\psi_\alpha\,,\ |\xi-\xi_\alpha| \geq 2(2C\sqrt{|\xi_\alpha|}+C^2).$$

Then

$$\sqrt{|\eta|} \leq \left|\sqrt{|\eta|}-\sqrt{|\xi_\alpha|}\right| + \sqrt{|\xi_\alpha|} \leq C + \sqrt{|\xi_\alpha|}$$

(by b) pag. 6). Then $\sqrt{|\xi_\alpha|} \leq \frac{1}{4C} |\xi - \xi_\alpha| - \frac{C}{2}$; consequently

$$\sqrt{|\eta|} \leq \frac{C}{2} + \frac{1}{4C} |\xi - \xi_\alpha| .$$

On the other part

$$|\xi - \xi_\alpha| \leq |\xi - \eta| + |\eta - \xi_\alpha| \leq |\xi - \eta| + 2C\sqrt{|\eta|} + C^2$$

and henceforth

$$\sqrt{|\eta|} \leq \frac{C}{2} + \frac{1}{4C} |\xi - \eta| + \frac{1}{2}\sqrt{|\eta|} + \frac{C}{4}, \quad \sqrt{|\eta|} \leq C + \frac{C}{2} + \frac{1}{2C} |\xi - \eta|$$

which gives

$$1 \leq \frac{2}{C} \frac{1}{\sqrt{|\eta|}} (1 + |\xi - \eta|^2)^{\frac{1}{2}} . \tag{4.38}$$

The estimate is done for $\alpha \neq 0$, as we supposed:

$$\operatorname{supp} \psi_0 \subset \{\xi ; |\xi| \leq t_0\}, \; t_{\varepsilon'} \geq t_0 \text{ and } \operatorname{supp} \psi_\alpha \cap \{\xi ; |\xi| \geq t_{\varepsilon'}\} \neq \Phi .$$

Then, for a fixed constant δ independent of α, $\psi_\alpha(\eta) = 0$ for $|\eta| \leq \delta$. Then, from (4.38), for $\xi \notin S_\alpha$ and $\eta \in \operatorname{supp} \psi_\alpha$, we obtain

$$1 \leq \text{const.} \; (1 + |\eta|^2)^{-\frac{1}{4}} (1 + |\xi - \eta|^2)^{\frac{1}{2}} . \tag{4.39}$$

Hence, for $\xi \notin S_\alpha$, we obtain, using (4.37) and (4.39)

$$|I_{\alpha,4}(\xi)| \leq C_0 \int_{\operatorname{supp} \psi_\alpha} (1 + |\xi - \eta|^2)^{\frac{1}{2}} k(\xi - \eta) (1 + |\eta|^2)^{-\frac{1}{4}} |\psi_\alpha(\eta)| \, |\tilde{u}(\eta)| \, d\eta. \tag{4.40}$$

From (4.35), (4.36), (4.40) we obtain « easily » (when $\operatorname{supp} \psi_\alpha \cap \{\xi ; |\xi| \geq t_{\varepsilon'}\} \neq \Phi$):

$$\| I_{\alpha,4} \|_0 \leq C\varepsilon' \| \psi \; (D) u \|_0 + C_1 \| \psi_\alpha (D) u \|_{-\frac{1}{2}} \tag{4.41}$$

C and C_1 being absolute constants.

From (4.30), (4.33), (4.41) we get, for these α

$$\| I_\alpha \|_0 \leq (K + 2\varepsilon' + C\cdot\varepsilon') \| \psi_\alpha(D) u \|_0 + C_1 \| \psi_\alpha(D) u \|_{-\frac{1}{2}}, \tag{4.42}$$

$u \in \mathcal{S}$, $C, \mathcal{C}_1$ being absolute constants.

Hence, $\varepsilon' > 0$ being given, we found $t_{\varepsilon'}$ such that

$$\left\| \varphi\left(\frac{D}{t_{\varepsilon'}}\right) a(x, D)\, \psi_\alpha(D) u \right\|_0 \leq (K + (2 + C)\varepsilon') \| \psi_\alpha(D) u \|_0 + C_1 \| \psi_\alpha(D) u \|_{-\frac{1}{2}},$$

for $u \in \mathcal{S}$ and α such that

$$\operatorname{supp} \psi_\alpha \cap \{\xi ;\ |\xi| \geq t_{\varepsilon'}\} \neq \Phi .$$

Take $\varepsilon > 0$, and choose $\varepsilon' = \dfrac{\varepsilon}{2 + C}$; for $\operatorname{supp} \psi_\alpha \cap \left\{\xi ;\ |\xi| \geq t_{\frac{\varepsilon}{2+C}}\right\} \neq \Phi$ we have (4.28) verified.

We continue now the proof of Theorem 2. From (4.27) and Lemma 4.3 we have, given $\varepsilon > 0$, for any $\delta > 0$, after an easy passage using (4.16), for $t = t_\varepsilon \geq t_0$

$$\left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D) u \right\|_0^2 \leq (1 + \delta^2)\, \Sigma' \left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, \psi_\alpha(D) u \right\|_0^2 + C_{\delta, t_0} \| u \|_{-\frac{1}{2}}^2$$

where Σ' is taken for α such that $\operatorname{supp} \psi_\alpha \cap \{\xi ;\ |\xi| \geq t_0\} \neq \Phi$. Let us separate Σ' in $\Sigma_1' + \Sigma_2'$, where Σ_1' is taken for α such that

$$\operatorname{supp} \psi_\alpha \cap \{\xi ;\ |\xi| \geq t_\varepsilon\} \neq \Phi$$

and Σ_2' for α such that

$$\operatorname{supp} \psi_\alpha \subset \{\xi ;\ |\xi| \leq t_\varepsilon\} \quad \text{and} \quad \operatorname{supp} \psi_\alpha \cap \{\xi ;\ |\xi| \geq t_0\} \neq \Phi.$$

So

$$\left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D) u \right\|_0^2 \leq (1 + \delta^2)\, \Sigma_1' \left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, \psi_\alpha(D) u \right\|_0^2 +$$

$$(1 + \delta^2)\, \Sigma_2' \left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, \psi_\alpha(D) u \right\|_0^2 + C_{\delta, t_0} \| u \|_{-\frac{1}{2}}^2 .$$

Applying Lemma 4.3 to the sum Σ_1' we get,

$$\left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, u \right\|_0^2 \leq (1+\delta^2)\, \Sigma_1' \left((K+\varepsilon)\, \| \psi_\alpha(D)\, u \|_0 + C\, \| \psi_\alpha(D)\, u \|_{-\frac{1}{2}}\right)^2$$

$$+ (1+\delta^2)\, \Sigma_2' \left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, \psi_\alpha(D)\, u \right\|_0^2 + C_{\delta, t_0} \| u \|_{-\frac{1}{2}}^2 .$$

Then, for the terms in Σ_2' we have

$$\varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, \psi_\alpha(D)\, u = \psi_\alpha(D)\, \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D) u +$$

$$+ \varphi\left(\frac{D}{t_\varepsilon}\right) [a(x, D), \psi_\alpha(D)]\, u = \varphi\left(\frac{D}{t_\varepsilon}\right) [a(x, D), \psi_\alpha(D)]\, u,$$

and (by 4.16)

$$\Sigma_2' \left\| \varphi\left(\frac{D}{t_\varepsilon}\right) [a(x, D), \psi_\alpha(D)]\, u \right\|_0^2 \leq c\, \| u \|_{-\frac{1}{2}}^2 ;$$

so we get (4.43)

$$\left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, u \right\|_0^2 \leq (1+\delta^2) \sum_1^\infty \left((K+\varepsilon)\, \| \psi_\alpha(D)\, u \|_0 + C \| \psi_\alpha(D)\, u \|_{-\frac{1}{2}}\right)^2 \tag{4.43}$$

$$+ C_{t_0, \delta} \| u \|_{-\frac{1}{2}}^2 , \qquad \text{for} \quad u \in \mathcal{S}.$$

As $2\, ab \leq \varepsilon_0^2\, a^2 + \frac{1}{\varepsilon_0^2}\, b^2$ for any $\varepsilon_0 > 0$, we obtain from (4.43)

$$\left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, u \right\|_0^2 \leq (1+\delta^2) \sum_0^\infty (1+\varepsilon_0^2)\, (K+\varepsilon)^2\, \| \psi_\alpha(D)\, u \|_0^2 + \tag{4.44}$$

$$C^2 (1+\delta^2) \left(1 + \frac{1}{\varepsilon_0^2}\right) \| \psi_\alpha(D)\, u \|_{-\frac{1}{2}}^2 + \left(1 + \frac{1}{\delta^2}\right) C_{t_0} \| u \|_{-\frac{1}{2}}^2 =$$

$$(1+\delta^2)\, (1+\varepsilon_0^2) (K+\varepsilon)^2 \| u \|_0^2 + \left(C^2 (1+\delta^2) \left(1 + \frac{1}{\varepsilon_0^2}\right) + \left(1 + \frac{1}{\delta^2}\right) C_{t_0}\right) \| u \|_{-\frac{1}{2}}^2 .$$

As $\sqrt{a^2 + b^2} \leq a + b$ for $a, b > 0$, we obtain

$$(4.45) \qquad \left\| \varphi\left(\frac{D}{t_\varepsilon}\right) a(x, D)\, u \right\|_0 \leq (K + \varepsilon)(1 + \delta)(1 + \varepsilon_0) \| u \|_0 + C_{\delta, \varepsilon_0, t_0} \| u \|_{-\frac{1}{2}} =$$

$$= [(K + \varepsilon) + (\delta + \varepsilon_0 + \varepsilon_0 \delta)(K + \varepsilon)] \| u \|_0 + C_{\delta, \varepsilon_0, t_0} \| u \|_{-\frac{1}{2}}, \quad u \in \mathcal{S}$$

For $(\delta + \varepsilon_0 + \varepsilon_0 \delta) K < \varepsilon$, $\delta + \varepsilon_0 + \varepsilon_0 \delta < 1$, hence for δ and ε_0 sufficiently small we have

$$(4.46) \qquad \| \varphi(D/t_\varepsilon)\, a(x, D)\, u \|_0 \leq (K + 3\varepsilon) \| u \|_0 + C_{\varepsilon, t_0} \| u \|_{-\frac{1}{2}}$$

or

$$(4.47) \qquad \| \varphi(D/t_{\varepsilon/3})\, a(x, D)\, u \|_0 \leq (K + \varepsilon) \| u \|_0 + C_{\varepsilon/3, t_0} \| u \|_{-\frac{1}{2}}, \quad u \in \mathcal{S}.$$

Hence, by Lemma 4.1, Theorem 2 is proved.

5. The main estimate (II).

We give here a certain extension of Theorem 2.

Let $\zeta(\xi)$ be a C^∞ non negative function which equals one for $|\xi| > 1$ and vanishes for $|\xi| \leq 1/2$. Set

$$(5.1) \qquad \zeta_\sigma(\xi) = \zeta(\xi) |\xi|^\sigma, \quad \sigma \text{ real}$$

and denote the corresponding operator by $\zeta_\sigma(D)$. Let $a(x, \xi)$ be a symbol and

$$(5.2) \qquad A_\sigma = \zeta_\sigma(D)\, a(x, D).$$

Then

THEOREM 3. *Denote* $K = \overline{\lim}_{|\xi| \to \infty} \max_{x \in R^n} |a(x, \xi)|$. *Then, for* s *real and any* $\varepsilon > 0$ *there is a constant* $C_{\varepsilon, s}$ *such that*

$$(5.3) \qquad \| A_\sigma u \|_{s-\sigma} \leq (K + \varepsilon) \| u \|_s + C \| u \|_{s-\frac{1}{2}}, \quad u \in \mathcal{S}.$$

We first note that A_σ differs from $(1 + |D|^2)^{\frac{\sigma}{2}} a(x, D)$ by an operator of order $\sigma - 1$. This follows from Theorem 1 and

LEMMA 5.1. *For any real* s *and* $u \in \bigcap_{s \geq 0} H_s$ *the estimate*

$$\| [\zeta_\sigma(D) - (1 + |D|^2)^{\sigma/2}] u \|_s \leq c \| u \|_{s+\sigma-1}, \tag{5.4}$$

holds.

It is enough to see that for $|\xi| \geq 1$, $||\xi|^\sigma - (1 + |\xi|^2)^{\frac{\sigma}{2}}| \leq c (1 + |\xi|^2)^{\frac{\sigma-1}{2}}$. To obtain this inequality we take $F(t) = (t + |\xi|^2)^{\frac{\sigma}{2}}$ and apply the mean value theorem between $t = 0$ and $t = 1$ together with some elementary remarks.

Hence it suffices to consider the operator $(1 + |D|^2)^{\frac{\sigma}{2}} a(x, D)$. Since

$$\| (1 + |D|^2)^{\frac{\sigma}{2}} a(x, D) u \|_{s-\sigma} = \| a(x, D) u \|_s, \quad \text{for} \quad u \in \mathcal{S},$$

it is enough to consider the case $\sigma = 0$.
We have now

LEMMA 5.2. *For any real* s *and* $u \in \mathcal{S}$, *the estimate*

$$\| [a(x, D), (1 + |D|^2)^{s/2}] u \|_0 \leq c \| u \|_{s-1}, \tag{5.5}$$

holds where

$$[a(x, D), (1 + |D|^2)^{s/2}] = a(x, D)(1 + |D|^2)^{s/2} - (1 + |D|^2)^{s/2} a(x, D).$$

We have the elementary inequality: for $0 < \theta < 1$

$$(1 + |\eta + \theta(\xi - \eta)|^2)^{\frac{s-1}{2}} \leq C (1 + |\eta|^2)^{\frac{s-1}{2}} (1 + |\xi - \eta|^2)^{\frac{|s-1|}{2}}.$$

It follows readily that

$$|(1 + |\xi|^2)^{s/2} - (1 + |\eta|^2)^{s/2}| \leq C (1 + |\xi - \eta|^2)^{\frac{|s-1|+1}{2}} (1 + |\eta|^2)^{\frac{s-1}{2}}.$$

Then we remark that

$$\widetilde{[a(x, D), (1 + |D|^2)^{s/2}] u}(\xi) =$$

$$= (2\pi)^{-s/2} \int \widetilde{a}'(\xi - \eta, \xi) [(1 + |\eta|^2)^{s/2} - (1 + |\xi|^2)^{s/2}] \widetilde{u}(\eta) \, d\eta$$

Estimating the L^2-norm of this expression as in Theorem 1 we get the Lemma.

PROOF OF THEOREM 3. We saw already that it is enough to consider the case $\sigma = 0$. Now

$$(5.6) \qquad \| a(x, D)\, u \|_s = \| (1 + | D |^2)^{s/2}\, a(x, D)\, u \|_0 =$$

$$\| a(x, D)(1 + | D |^2)^{s/2}\, u \|_0 + 0\,(\| u \|_{s-1}),$$

by Lemma 5.2 and $\| u \|_s = \| (1 + | D |^2)^{s/2}\, u \|_0$.
Then

$$(5.7) \qquad \| A_\sigma u \|_{s-\sigma} \leq \| [A_\sigma - (1 + | D |^2)^{\sigma/2}\, a(x, D)]\, u \|_{s-\sigma} + \| a(x, D)\, u \|_s$$

$$\leq c \| a(x, D)\, u \|_{s-1} + \| a(x, D)(1 + | D |^2)^{s/2}\, u \|_0 + c \| u \|_{s-1}.$$

Remark that $u \in \mathcal{S}$ gives $(1 + | D |^2)^{s/2}\, u \in \mathcal{S}$ too. Apply (4.2) and get for any $\varepsilon > 0$ a constant C_ε such that

$$(5.8) \qquad \| A_\sigma u \|_{s-\sigma} \leq C_s \| u \|_{s-1} + (K + \varepsilon) \| (1 + | D |^2)^{s/2}\, u \|_0 +$$

$$C_\varepsilon \| (1 + | D |^2)^{s/2}\, u \|_{-\frac{1}{2}} \leq (K + \varepsilon) \| u \|_s + C_{\varepsilon, s} \| u \|_{s-\frac{1}{2}},$$

as $\| \quad \|_t$ is increasing with t.
This proves our theorem 3.

6. The main estimate (III).

We give here another extension of Theorem 2 when instead of the whole R^n arbitrary open sets in R^n are taken into account. We have precisely the

THEOREM 4. *Let Ω be an open set in the «x-space» R^n and $K_\Omega = \overline{\lim}_{|\xi| \to \infty} \max_{\overline{\Omega}} | a(x, \xi) |$. Then, for any $\varepsilon > 0$ there is a constant C_ε such that*

$$(6.1) \qquad \| a(x, D)\, u \|_0 \leq (K_\Omega + \varepsilon) \| u \|_0 + C_\varepsilon \| u \|_{-\frac{1}{2}}, \qquad u \in C_0^\infty(\overline{\Omega})$$

holds.

To prove it we need the following

LEMMA 6.1. *Let $a(x, \xi)$ be a symbol, Ω an open set in R^n, K_Ω as before. Then, for any $\varepsilon > 0$ there is an open set $\Omega_\varepsilon \supset \bar{\Omega}$, such that $K_{\Omega_\varepsilon} \leq K_\Omega + \varepsilon$.*

As is seen from (3.2), we have, with an absolute constant

$$(6.2) \qquad |a(x, \xi) - a(x_0, \xi)| \leq c\,|x - x_0| < \varepsilon \quad \text{if} \quad |x - x_0| < \frac{\varepsilon}{c} = \delta_\varepsilon .$$

Consider then, for each $x_0 \in \partial\Omega$ (boundary of Ω) the sphere $S(x_0, \delta_\varepsilon)$. Take $\Omega_\varepsilon = \Omega\, U\,(\bigcup_{x_0 \in \partial\Omega} S(x_0, \delta_\varepsilon))$.

Then, if $y \in \Omega_\varepsilon$ we have $y \in \Omega$ or $y \in S(x^*, \delta_\varepsilon)$ for an $x^* \in \partial\Omega$. The first case will give $|a(y, \xi)| \leq \max_{\bar{\Omega}} |a(x, \xi)|$, the second one gives

$$|a(y,\xi)| \leq |a(y,\xi) - a(x^*,\xi)| + |a(x^*,\xi)| \leq \varepsilon + |a(x^*,\xi)| \leq \varepsilon + \max_{\bar{\Omega}} |a(x,\xi)| .$$

Hence anyway

$$\max_{\bar{\Omega}_\varepsilon} |a(y, \xi)| \leq \varepsilon + \max_{\bar{\Omega}} |a(x, \xi)|, \quad \xi \neq 0$$

and, as easily seen

$$K_{\Omega_\varepsilon} = \overline{\lim_{|\xi| \to \infty}} \; \max_{\bar{\Omega}_\varepsilon} |a(y, \xi)| \leq \varepsilon + \overline{\lim_{|\xi| \to \infty}} \; \max_{\bar{\Omega}} |a(x, \xi)| = K_\Omega + \varepsilon$$

that is, the Lemma.

PROOF OF THEOREM 4. Given $\varepsilon > 0$ constract Ω_ε given by the Lemma. There exists a C_0^∞ function $\zeta_\varepsilon(x)$ equal to one on $\bar{\Omega}$, to zero outside Ω_ε, such that $0 \leq \zeta_\varepsilon \leq 1$.

Remark that C_0^∞-functions like $\zeta_\varepsilon(x)$, are symbols in our sense (a very special kind indeed) and it is also easy to see that $a(x, \xi)$ being a symbol. $\zeta_\varepsilon(x)\, a(x, \xi)$ is again a symbol, null outside Ω_ε. Hence

$$\max_{R^n} |\zeta_\varepsilon(x)\, a(x, \xi)| \leq \max_{\Omega_\varepsilon} |a(x, \xi)| \quad \text{and} \quad \overline{\lim_{|\xi| \to \infty}} \; \max_{R^n} |\zeta_\varepsilon(x)\, a(x, \xi)| \leq K_{\Omega_\varepsilon},$$

Call $\gamma_\varepsilon(x, \xi) = \zeta_\varepsilon(x)\; a(x, \xi)$ and $\Gamma_\varepsilon(x, D)$ the pseudo-differential operator associated to it. We have

$$\Gamma_\varepsilon(x, D) = A(x, D)\,(\zeta_\varepsilon(x)) .$$

In fact, $\forall\, u \in \mathcal{S}$

$$\widehat{\Gamma_\varepsilon u}\,(\xi) = (2\pi)^{-n/2} \int e^{-ix\cdot\xi}\, \gamma_\varepsilon(x,\xi)\, u(x)\, dx =$$

$$= (2\pi)^{-n/2} \int e^{-ix\cdot\xi}\, a(x,\xi)\,(\zeta_\varepsilon u)(x)\, dx = \widehat{A(x,D)(\zeta_\varepsilon u)}\,(\xi).$$

Then $\forall\, u \in \mathcal{S}$, $u = \zeta_\varepsilon u + (1-\zeta_\varepsilon)\, u$ and we can hence forth write the decomposition

$$a(x,D)\,u = a(x,D)(\zeta_\varepsilon u) + a(x,D)(1-\zeta_\varepsilon)\, u = \Gamma_\varepsilon(x,D)\, u$$

$$+ (a(x,D))(1-\zeta_\varepsilon)\, u, \quad \text{for} \quad u \in \mathcal{S}.$$

In particular, when, as in our hypothesis, u belongs to $C_0^\infty(\overline{\Omega})$, we obtain, as $1-\zeta_\varepsilon = 0$ in $\overline{\Omega}$, that $(1-\zeta_\varepsilon)\, u \equiv 0$ in R^n. It follows

$$a(x,D)\,u = a(x,D)(\zeta_\varepsilon u) = \Gamma_\varepsilon u, \qquad \forall\, u \in C_0^\infty(\Omega).$$

We apply Theorem 2 and get

$$\| a(x,D)\, u \|_0 = \| \Gamma_\varepsilon(x,D)\, u \|_0 \leq (K_{\Omega_\varepsilon} + \varepsilon)\, \| u \|_0 + C_\varepsilon \| u \|_{-\frac{1}{2}} \leq$$

$$\leq (K_\Omega + 2\varepsilon)\, \| u \|_0 + C_\varepsilon \| u \|_{-\frac{1}{2}}, \qquad \text{for any } u \in C_0^\infty(\overline{\Omega})$$

that is inequality (6.1) is proved.

7. Norms modulo lower order operators.

Our next results concern the estimate of the norm of pseudo-differential operators modulo operators of order $-\infty$ (an upper estimate) and modulo operators of order $-\dfrac{1}{2}$ (a lower estimate). Remember (see [1]) that a linear operator T from $\mathcal{S}$ into $\mathcal{S}'$ is of order $-\infty$ when

$$\| Tu \|_s \leq C\, \| u \|_{s-t}, \qquad \text{for any } u \in \mathcal{S} \text{ and } t > 0.$$

Also remember that T is of order $-\frac{1}{2}$ if

$$\| Tu \|_s \leq C_s \| u \|_{s-\frac{1}{2}}, \quad \text{for any} \quad u \in \mathcal{S}.$$

The upper estimate is quite a simple corollary of Theorem 2. We express it is the form of

THEOREM 5 (I). *Let $a(x, \xi)$ be a symbol, $a(x, D)$ its associated pseudo-differential operator, $\mathcal{T}_{-\infty}$ the class of operators of order $-\infty$.*

Denote

$$K^1_{R^n} = \overline{\lim_{|\xi| \to \infty}} \max_{x \in R^n} | a(x, \xi) |.$$

Then, the relation

$$\inf_{T \in \mathcal{T}_{-\infty}} \| a(x, D) + T \| \leq K^1_{R^n} \tag{7.1}$$

holds, the norm $\| \quad \|$ being the operator norm from $L^2(R^n)$ into itself.

We have to show that for any $\varepsilon > 0$ there is an operator T of order $-\infty$ such that

$$\| (a(x, D) + T_\varepsilon) u \|_0 \leq (K^1_{R^n} + \varepsilon) \| u \|_0, \qquad \forall u \in L^2(R^n). \tag{7.2}$$

To construct such an operator T_ε, consider a C^∞ function $\Phi_R(\xi)$, depending on $R > 0$, such that $0 \leq \Phi_R \leq 1$, $\Phi_R(\xi) = 1$ on $|\xi| \leq R$ and $\Phi_R(\xi) = 0$ for $|\xi| > 2R$.

The operator $T_R = -a(x, D)\Phi_R(D)$ has order $-\infty$. Precisely (using Theorem 1) we have the following estimate (for any $u \in \mathcal{S}$)

$$\begin{aligned} \| T_R u \|_s &= \| a(x, D) \Phi_R(D) u \|_s \leq C_s \| \Phi_R(D) u \|_s = \\ & C_s \int (1 + |\xi|^2)^s \Phi_R^2(\xi) | \tilde{u}(\xi) |^2 d\xi \leq C_s \int_{|\xi| \leq 2R} (1 + |\xi|^2)^s | \tilde{u}(\xi) |^2 d\xi \\ &= C_s \int_{|\xi| \leq 2R} (1 + |\xi|^2)^{s-t} | \tilde{u}(\xi) |^2 (1 + |\xi|^2)^t d\xi \leq \\ & C_s (1 + 4R^2)^t \| u \|^2_{s-t} = C_{s,t,R} \| u \|^2_{s-t}, \quad \text{for any } t > 0. \end{aligned} \tag{7.3}$$

We can apply (4.2) (replacing K by $K^1_{R^n}$), and derive, for any $\varepsilon > 0$, that

$$\| (a(x, D) + T_R) u \|_0 = \| a(x, D)(I - \Phi_R(D)) u \|_0 \tag{7.4}$$

$$\leq (K^1_{R^n} + \varepsilon) \| (I - \Phi_R(D)) u \|_0 + C_\varepsilon \| (I - \Phi_R(D)) u \|_{-\frac{1}{2}}, \quad u \in \mathcal{S}.$$

Let us remark now that

$$\| (I - \Phi_R(D)) u \|_0 = \left(\int (1 - \Phi_R(\xi))^2 \, | \tilde{u}(\xi) |^2 \, d\xi \right)^{\frac{1}{2}} \leq \tag{7.5}$$

$$\| u \|_0, \text{ for any } u \in \mathcal{S}$$

and furthemore, that

$$\| (I - \Phi_R(D)) u \|_{-\frac{1}{2}} = \left(\int (1 - \Phi_R(\xi))^2 \, | \tilde{u}(\xi) |^2 (1 + | \xi |^2)^{-\frac{1}{2}} \, d\xi \right)^{\frac{1}{2}} \tag{7.6}$$

$$= \left(\int_{|\xi| \geq R} (1 + |\xi|^2)^{-\frac{1}{2}} \, | \tilde{u}(\xi) |^2 \, d\xi \right)^{\frac{1}{2}} \leq (1 + R^2)^{-\frac{1}{4}} \| u \|_0 .$$

Take R_ε such that $(1 + R^2_\varepsilon)^{-\frac{1}{4}} C_\varepsilon \in \varepsilon$. Then, from (7.4), (7.5), (7.6) we deduce

$$\| (a(x, D) + T_{R_\varepsilon}) u \|_0 \leq (K^1_{R^n} + \varepsilon) \| u \|_0 + \varepsilon \| u \|_0 = (K^1_{R^n} + 2\varepsilon) \| u \|_0 \tag{7.7}$$

for any $u \in \mathcal{S}$ and hence for any $u \in L^2$.

THEOREM 5 (II). *Let $a(x, \xi)$ be a symbol $a(x, D)$ its associated pseudodifferential operator, $\mathcal{T}_{-\frac{1}{2}}$ the class of operators of order $-1/2$.*

Denote

$$K^2_{R^n} = \max_{x \in R^n} \overline{\lim_{|\xi| \to \infty}} \, | a(x, \xi) |$$

Then, the relation

$$\inf_{T \in \mathcal{T}_{-\frac{1}{2}}} \| a(x, D) + T \| \geq K^2_{R^n} . \tag{7.8}$$

holds, the norm $\| \ \|$ being the operator norm from $L^2(R^n)$ into itself.

A main tool in the proof is the following

LEMMA 7.1. *Let* $a(x, \xi)$ *be a symbol and* $\lim_{p \to \infty} a(x_0, \xi_p) = c_0$ *for some* $x_0 \in R^n$ *and for a sequence* $\xi_p \longrightarrow \infty$. *Then for any* $\varepsilon > 0$ *there exists a* C_0^∞ *function* $u_\varepsilon(x)$ *such that* $\| u_\varepsilon \|_0 \neq 0$ *and*

$$\| (a(x, D) - c_0) u_\varepsilon \|_0 \leq \varepsilon \| u_\varepsilon \|_0 \tag{7.9}$$

$$\| u_\varepsilon \|_{-\frac{1}{2}} \leq \varepsilon \| u_\varepsilon \|_0 . \tag{7.10}$$

We postpone a bit the proof of this result and derive the final proof of Theorem 5 (II).

Suppose by absurd that

$$\inf_{T \in \mathcal{T}_{-\frac{1}{2}}} \| a(x, D) + T \| = k^* < K_{R^n}^2 .$$

Then for some operator T_k of order $-1/2$, corresponding to any k, $k^* < k < K_{R^n}^2$ we have

$$\| (a(x, D) + T_k) u \|_0 \leq k \| u \|_0 , \quad \forall u \in L^2. \tag{7.11}$$

There exists at least an $x_0 \in R^n$ such that

$$d_0 = \overline{\lim_{|\xi| \to \infty}} \, | a(x_0, \xi) | > k.$$

There is a sequence $\xi'_p \longrightarrow \infty$ such that

$$\lim_{p \to \infty} | a(x_0, \xi'_p) | = d_0 > k.$$

But the set $\{ a(x_0, \xi'_p) \}$ is bounded. We can extract a subsequence $(\xi_p)_1^\infty \subset (\xi'_p)_1^\infty$, such that $\lim_{p \to \infty} a(x_0, \xi_p) = c_0$, exists.

It follows, $\lim_{p \to \infty} | a(x_0, \xi_p) | = | c_0 | = d_0 > k$. As T_k is order $-1/2$ we have

$$\| T_k u \|_0 \leq C \| u \|_{-\frac{1}{2}}, \text{ for any } u \in L^2.$$

Then, applying the Lemma 7.1, we have

(7.12) $$(|c_0| - \varepsilon) \| u_\varepsilon \|_0 \leq \| a(x, D) u_\varepsilon \|_0 \leq \| (a(x, D) + T_k) u_\varepsilon \|_0 + \| T_k u_\varepsilon \|_0 \leq k \| u_\varepsilon \|_0 + C \| u_\varepsilon \|_{-\frac{1}{2}} \leq (k + C\varepsilon) \| u_\varepsilon \|_0 .$$

Hence $|c_0| - \varepsilon \leq k + C\varepsilon$, $|c_0| - k \leq (C+1)\varepsilon$, absurde for small ε as $|c_0| > k$. This ends the proof of the Theorem 5 (II), if Lemma 7.1 is assumed.

PROOF OF LEMMA 7.1. We start by considering $\varepsilon' > 0$. For $|x - x_0| < \delta_{\varepsilon'}$, we have (see 6.2) that $|a(x, \xi) - a(x_0, \xi)| \leq \varepsilon'$, $\xi \neq 0$; consider a fixed function $\Phi_{\varepsilon'}(x)$ indefinitely differentiable, with support in the sphere $\{x ; |x - x_0| \leq \delta_{\varepsilon'}\}$ and the sequence of functions

(7.14) $$u_{p,\varepsilon'}(x) = e^{ix \cdot \xi_p} \Phi_{\varepsilon'}(x),$$

ξ_p being the sequence indicated in the lemma. We start by considering an estimate for the expression

(7.15) $$\| (a(x, D) - c_0) u_{p,\varepsilon'} \|_0 .$$

Let $f(x)$ be a C^∞ function, $= 1$ for $|x| \leq 1$, $= 0$ for $|x| > 2$; write then $\psi_p(\xi) = f\left(\frac{\xi - \xi_p}{\sqrt{|\xi_p|}}\right)$; we see that, with an absolute constant c, the estimate

(7.16) $$|\operatorname{grad} \psi_p| \leq \frac{c}{\sqrt{|\xi_p|}}, \text{ holds, } p = 1, 2, \ldots .$$

Consider the operator $\psi_p(D)$ defined as usual by $\widetilde{\psi_p(D) u}(\xi) = \psi_p(\xi) \tilde{u}(\xi)$ and remark the obvious decomposition

(7.17) $$a(x, D) v = a(x_0, \xi_p) v + \psi_p(D)(a(x, D) - a(x_0, \xi_p) I) v + (I - \psi_p(D))(a(x, D) - a(x_0, \xi_p) I) v .$$

We deduce from it, when $v = u_{p,\varepsilon'}$ that

(7.18) $$\| (a(x, D) - c_0) u_{p,\varepsilon'} \|_0 \leq | a(x_0, \xi_p) - c_0 | \, \| u_{p,\varepsilon'} \|_0 + \| \psi_p(D)(a(x, D) - a(x_0, \xi_p)) u_{p,\varepsilon'} \|_0 + \| (I - \psi_p(D))(a(x, D) - a(x_0, \xi_p)) u_{p,\varepsilon'} \|_0$$

and, as $\lim_{p \to \infty} a(x_0, \xi_p) = c_0$,

$$\| a(x, D) - c_0) u_{p, \varepsilon'} \|_0 \leq \varepsilon' \| u_{p, \varepsilon'} \|_0 + \| u^2_{p, \varepsilon'} \|_0 + \| u^3_{p, \varepsilon'} \|_0 \quad \text{for } p \geq p_1(\varepsilon'). \tag{7.19}$$

Consider now the term $u^2_{p, \varepsilon'} = \psi_p(D)(a(x, D) - a(x_0, \xi_p) I) u_{p, \varepsilon'}$ which we write in the form

$$u^2_{p, \varepsilon'} = \psi_p(D)(a(x, D) - a(x_0, D)) u_{p, \varepsilon'} + \psi_p(D)(a(x_0, D) - a(x_0, \xi_p)) u_{p, \varepsilon'} = u^4_{p, \varepsilon'} + u^5_{p, \varepsilon'}. \tag{7.20}$$

We have now

$$\| u^5_{p, \varepsilon'} \|_0 = \left(\int | \psi_p(\xi) |^2 \, | a(x_0, \xi) - a(x_0, \xi_p) |^2 \, | \widetilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 \, d\xi \right)^{\frac{1}{2}}. \tag{7.21}$$

Remark that from the definition of a symbol (N. 3) we may deduce

$$| a(x_0, \xi) - a(x_0, \xi_p) | \leq c \frac{| \xi - \xi_p |}{| \xi_p |}. \tag{7.22}$$

Introducing in (7.21) we obtain

$$\| u^5_{p, \varepsilon'} \|_0 \leq c \left(\int_{| \xi - \xi_p | \leq 2 \sqrt{| \xi_p |}} | \xi - \xi_p |^2 \frac{1}{| \xi_p |^2} | \widetilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 \, d\xi \right)^{\frac{1}{2}} \leq \frac{4c}{\sqrt{| \xi_p |}} \| u_{p, \varepsilon'} \|_0 \leq \varepsilon' \| u_{p, \varepsilon'} \|_0 \quad \text{for } p \geq p_2(\varepsilon'). \tag{7.23}$$

Let us consider now the term $u^4_{p, \varepsilon'}$. We have first of all, as $\| \psi_p(D) \|_{L^2 \to L^2} \leq 1$

$$\| u^4_{p, \varepsilon'} \|_0 \leq \| (a(x, D) - a(x_0, D)) u_{p, \varepsilon'} \|_0. \tag{7.24}$$

Call $b(x, \xi) = a(x, \xi) - a(x_0, \xi)$ the symbol associated to $a(x, D) - a(x_0, D)$. If $S(x_0, \delta_{\varepsilon'})$ is the sphere of center x_0 and radius $\delta_{\varepsilon'}$, we have

$$\max_{\substack{x \in S(x_0, \delta_{\varepsilon'}) \\ \xi \in R^n - \{0\}}} | b(x, \xi) | \leq \varepsilon' \quad \text{and} \quad \overline{\lim_{| \xi | \to \infty}} \max_{S(x_0, \delta_{\varepsilon'})} | b(x, \xi) | \leq \varepsilon' \quad \text{too}. \tag{7.25}$$

On the other hand the functions $u_{p, \varepsilon'} = e^{ix \cdot \xi_p} \Phi_{\varepsilon'}(x)$ belong to $C_0^\infty(S(x_0, \delta_{\varepsilon'}))$. Apply Theorem 4 for $\varepsilon' > 0$ and get that there is a constant $C_{\varepsilon'}$ such that

$$\| (a(x, D) - a(x_0, D)) u_{p, \varepsilon'} \|_0 \leq 2\varepsilon' \| u_{p, \varepsilon'} \|_0 + C_{\varepsilon'} \| u_{p, \varepsilon'} \|_{-\frac{1}{2}}, \; p = 1, 2 \ldots \tag{7.26}$$

Let us now estimate the term $u^3_{p,\varepsilon'} = (I - \psi_p(D))(a(x, D) - a(x_0, \xi_p))\, u_{p,\varepsilon'} = (a(x, D) - a(x_0, \xi_p))(I - \psi_p(D))\, u_{p,\varepsilon'} + [a(x, D), \psi_p(D)]\, u_{p,\varepsilon'}$; the second term in the commutator of $a(x, D)$ and $\psi_p(D)$. We obtain at once (Theorem 1) that

$$\| u^3_{p,\varepsilon'} \|_0 \leq c\, \| (I - \psi_p(D))\, u_{p,\varepsilon'} \|_0 + \| [a(x, D), \psi_p(D)]\, u_{p,\varepsilon'} \|_0 . \tag{7.27}$$

Then, first of all

$$\| (I - \psi_p(D))\, u_{p,\varepsilon'} \|_0 = \left(\int | (1 - \psi_p(\xi) |^2 \, | \widetilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 \, d\xi \right)^{\frac{1}{2}} = \tag{7.28}$$

$$\left(\int\limits_{|\xi - \xi_p| > \sqrt{|\xi_p|}} | \widetilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 \, d\xi \right)^{\frac{1}{2}} \leq \varepsilon' \, \| u_{p,\varepsilon'} \|_0 \quad \text{if} \quad p \geq p_3(\varepsilon', \widetilde{\Phi}_{\varepsilon'}) = p_3(\varepsilon').$$

The estimate for the commutator is given by using the formula

$$\widetilde{[a(x, D), \psi_p(D)]\, u_{p,\varepsilon'}}(\xi) = \tag{7.29}$$

$$(2\pi)^{-n/2} \int \widetilde{a}'(\xi - \eta, \xi)\, (\psi_p(\xi) - \psi_p(\eta))\, \widetilde{u}_{p,\varepsilon'}(\eta)\, d\eta .$$

Hence, by using the definition of a symbol, the mean value theorem and (7.16) we get

$$| \widetilde{[a(x, D)\, \psi_p(D)]\, u_{p,\varepsilon'}}(\xi) | \leq \frac{c}{\sqrt{|\xi_p|}} \int k(\xi - \eta)(1 + |\xi - \eta|^2)^{1/2} \, | \widetilde{u}_{p,\varepsilon'}(\eta) | \, d\eta. \tag{7.30}$$

This last inequlity implies, by means of some elementary calculations, that

$$\| [a(x, D), \psi_p(D)]\, u_{p,\varepsilon'} \|_0 \leq \frac{c_1}{\sqrt{|\xi_p|}} \| u_{p,\varepsilon'} \|_0 \leq \varepsilon' \, \| u_{p,\varepsilon'} \|_0 \ \text{ for } p \geq p_4(\varepsilon'). \tag{7.31}$$

The last inequalities that we got, give us the desired bound for $u^3_{p,\varepsilon'}$, namely from (7.27), (7.28) and (7.31)

$$\| u^3_{p,\varepsilon'} \|_0 \leq (c + 1)\, \varepsilon' \, \| u_{p,\varepsilon'} \|_0, \quad \text{when } p \geq \max\,(p_3(\varepsilon'), p_4(\varepsilon')) = p_5(\varepsilon'). \tag{7.32}$$

If we sum up the preceding estimates (7.19), (7.23), (7.26), (7.32) we get

$$\| (a(x, D) - c_0) u_{p,\varepsilon'} \|_0 \leq \varepsilon' \| u_{p,\varepsilon'} \|_0 + \varepsilon' \| u_{p,\varepsilon'} \|_0 + \tag{7.33}$$

$$2\varepsilon' \| u_{p,\varepsilon'} \|_0 + C_{\varepsilon'} \| u_{p,\varepsilon'} \|_{-\frac{1}{2}} + (c+1)\varepsilon' \| u_{p,\varepsilon'} \|_0, \quad \text{for } p \geq p_6(\varepsilon').$$

Let us show now that: *for any* $\varepsilon'' > 0$ *there is* $\tilde{p}(\varepsilon'', \varepsilon')$ *such that*

$$\| u_{p,\varepsilon'} \|_{-\frac{1}{2}} \leq \sqrt{2}\, \varepsilon'' \| u_{p,\varepsilon'} \|_0, \quad p \geq \tilde{p}(\varepsilon', \varepsilon''). \tag{7.34}$$

We have

$$\| u_{p,\varepsilon'} \|^2_{-\frac{1}{2}} = \int (1 + |\xi|^2)^{-\frac{1}{2}} | \tilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 d\xi = \tag{7.35}$$

$$\int_{|\xi - \xi_p| \leq r} (1 + |\xi|^2)^{-\frac{1}{2}} | \tilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 d\xi + \int_{|\xi - \xi_p| \geq r} (1 + |\xi|^2)^{-\frac{1}{2}} | \tilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 d\xi \leq$$

$$\int_{|\xi - \xi_p| < r} (1 + |\xi|^2)^{-\frac{1}{2}} | \tilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 d\xi + \int_{|\xi - \xi_p| > r} | \tilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 d\xi, \text{ for any } r > 0.$$

Now, given $\varepsilon'' > 0$, there is $r^*(\varepsilon', \varepsilon'')$ such that

$$\int_{|\xi - \xi_p| > r} | \tilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 d\xi \leq \varepsilon''^2 \| u_{p,\varepsilon'} \|^2_0 . \tag{7.36}$$

Remark moreover that if $|\xi - \xi_p| < r^*$, then $|\xi| \geq |\xi_p| - r^*$; consequently for p large,

$$\int_{|\xi - \xi_p| < r^*} (1 + |\xi|^2)^{-\frac{1}{2}} | \tilde{\Phi}_{\varepsilon'}(\xi - \xi_p) |^2 d\xi \leq \frac{1}{(1 + (|\xi_p| - r^*)^2)^{\frac{1}{2}}} \| u_{p,\varepsilon'} \|^2_0 .$$

For big enough $\tilde{p}$ depending on $r^*(\varepsilon', \varepsilon'')$, hence on ε' and ε'' we get $(1 + (|\xi_p| - r^*)^2)^{-\frac{1}{2}} \leq \varepsilon''^2$. Then, from (7.35) and (7.36) we obtain

$$\| u_{p,\varepsilon'} \|_{-\frac{1}{2}} \leq \sqrt{2}\, \varepsilon'' \| u_{p,\varepsilon'} \|_0, \quad \text{for } \tilde{p} \geq p(\varepsilon', \varepsilon''), \quad \text{that is (7.34).} \tag{7.37}$$

Then, from (7.33) and (7.34) we get

$$(7.38) \qquad \| (a(x, D) - c_0) u_{p,\varepsilon'} \|_0 \leq (c+5)\varepsilon' \| u_{p,\varepsilon'} \|_0 + \sqrt{2}\,\varepsilon'' C_{\varepsilon'} \| u_{p,\varepsilon'} \|_0$$

for $p \geq \max(p_6(\varepsilon'), \tilde{p}(\varepsilon', \varepsilon''))$.

Take then $\varepsilon'' = \min\left(\frac{\varepsilon'}{\sqrt{2}}, \frac{\varepsilon'}{\sqrt{2}\, C_{\varepsilon'}}\right)$.

We deduce:

$$(7.39) \qquad \| (a(x, D) - c_0) u_{p,\varepsilon'} \|_0 \leq (c+6)\varepsilon' \| u_{p,\varepsilon'} \|_0, \quad \text{for} \quad p \geq P(\varepsilon')$$

and

$$(7.40) \qquad \| u_{p,\varepsilon'} \|_{-\frac{1}{2}} \leqq \varepsilon' \| u_{p,\varepsilon'} \|_0, \; p \geqq P(\varepsilon'),$$

where P depends on ε' only. Take now $\varepsilon > 0$ and $\varepsilon' = \frac{\varepsilon}{c+6}$. Then the whole sequence of functions $u_{p,\varepsilon}(x) = e^{ix\cdot\xi_p}\, \Phi_{\frac{\varepsilon}{c+6}}(x)$, $p \geq P\left(\frac{\varepsilon}{c+6}\right)$ will satisfy conditions of the Lemma 7.

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