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# THE GEGENBAUER FUNCTION

by W. R. SCHUCANY (*)

In the notes of the American Mathematical Monthly, [1] G. G. Bilodeau gives a new expression for the Gegenbauer polynomials by employing the concept of a fractional derivative. It is interesting to note that by employing the results obtained in [3], one can obtain a single formula for both the classical polynomials and the Gegenbauer functions ($C_\beta^\mu$) as defined by Bateman [2].

The only definition necessary for clarity is the following:

If i) $\int_a^x (x-t)^{n-\alpha-1} f(t)\, dt \in C^{(n)}$ on $(a, b)$,

ii) $\alpha > 0$ and $n - \alpha > 0$,

where $n$ is a positive integer, then the Holmgren-Riesz transform of $f$ is given by

$$\overset{x}{\underset{a}{I}}{}^{-\alpha} f(x) = {}_aD_x^\alpha f(x) = \frac{D_x^n}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt.$$

Now one solution of the equation

$$(1) \qquad (1-x^2)\, y'' - (2\mu+1)\, xy' + \beta(\beta+2\mu)\, y = 0,$$

may be written in its $H - R$ transform form as

$$(2) \qquad y = k(1-x^2)^{-\mu+1/2}\, {}_1D_x^\beta (1-x^2)^{\beta+\mu-1/2}, \quad \beta > 0 \quad \text{and} \quad x > 1,$$

(See [3]).

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If the arbitrary constant $k$ equals

$$\frac{(-1)^{\beta}\,\Gamma(\mu+1/2)\,\Gamma(\beta+2\mu)}{2^{\beta}\Gamma(\beta+1)\,\Gamma(2\mu)\,\Gamma(\mu+\beta+1/2)},$$

then when $\beta = n$, equation (2) becomes the standard Rodrigues formula for the Gengenbauer polynomials. Also for positive non-integer values of $\beta$, $\mu > -1/2$ and $\mu \neq 0$, equation (2) can be shown to be equivalent to

$$y = C_{\beta}^{\mu}(x) = \frac{\Gamma(\beta+2\mu)}{\Gamma(\beta+1)\,\Gamma(2\mu)}\,F\left[\beta+2\mu,\,-\beta\,;\,\mu+1/2\,;\,\frac{1-x}{2}\right]$$

for $x \in (1, 3)$, where $F[a, b\,; c\,; z]$ is the Hypergeometric function. (See [2]).

Consider (2) when $\beta \neq m$, an integer, and note that when $n = |[-\beta]|$,

$${}_1D_x^{\beta}\,(1-x^2)^{\beta+\mu-1/2} = \frac{D_x^n}{\Gamma(n-\beta)}\int_1^x (x-t)^{n-\beta-1}\,(1-t^2)^{\beta+\mu-1/2}\,dt.$$

Letting $z = \dfrac{t-1}{x-1}$, the right hand side of the above equation becomes,

$$\frac{D_x^n}{\Gamma(n-\beta)}\int_0^1 [x-1+z(1-x)]^{n-\beta-1}\,[z(1-x)]^{\beta+\mu-1/2}\,[2-z(1-x)]^{\beta+\mu-1/2}\,(x-1)\,dz$$

$$= \frac{(-1)^{n-\beta}\,2^{\beta+\mu-1/2}}{\Gamma(n-\beta)}\,D_x^n\,(1-x)^{n+\mu-1/2}\int_0^1 (1-z)^{n+\beta-1}\,z^{\beta+\mu-1/2}\,\cdot$$

$$\cdot\left[1-z\left(\frac{1-x}{2}\right)\right]^{\beta+\mu-1/2}dz.$$

Then setting $w = \dfrac{1-x}{2}$ we have,

$${}_1D_x^{\beta}(1-x^2)^{\beta+\mu-1/2} = \frac{(-1)^{\beta}\,2^{\beta+2\mu-1}\,\Gamma(\beta+\mu+1/2)}{\Gamma(n+\mu+1/2)}\,\cdot$$

$$\cdot\,D_w^n\,\{w^{n+\mu-1/2}\,F[-\beta-\mu+1/2,\ \beta+\mu+1/2\,;\ n+\mu+1/2\,;\ w]\},$$

$$= \frac{(-1)^{\beta}\,2^{\beta+2\mu-1}\,\Gamma(\beta+\mu+1/2)}{\Gamma(\mu+1/2)}\left(\frac{1-x}{2}\right)^{\mu-1/2} F\left[-\beta-\mu+1/2,\ \beta+\mu+1/2\,;\right.$$

$$\left.\mu+1/2\,;\ \frac{1-x}{2}\right].$$

Now applying a linear transformation formula we get

$$
{}_1D_x^{\beta}\,(1-x^2)^{\beta+\mu-1/2}
$$

$$
=\frac{(-1)^{\beta}\,2^{\beta}\,\Gamma(\beta+\mu+1/2)}{\Gamma(\mu+1/2)}\,(1-x^2)^{\mu-1/2}\,F\left[\beta+2\mu,\ -\beta;\ \mu+1/2;\ \frac{1-x}{2}\right].
$$

Therefore

$$
\frac{(-1)^{\beta}\,\Gamma(\mu+1/2)\,\Gamma(\beta+2\mu)}{2^{\beta}\,\Gamma(\beta+1)\,\Gamma(2\mu)\,\Gamma(\mu+\beta+1/2)}\,(1-x^2)^{-\mu+1/2}\,{}_1D_x^{\beta}\,(1-x^2)^{\beta+\mu-1/2}
$$

$$
=\frac{\Gamma(\beta+2\mu)}{\Gamma(\beta+1)\,\Gamma(2\mu)}\,F\left[\beta+2\mu,\ -\beta;\ \mu+1/2;\ \frac{1-x}{2}\right]=C_{\beta}^{\mu}\,(x).
$$

Hence the single expression (2) with the given value of $k$ is a valid formula for the Gegenbauer polynomials, and with the generality afforded by the $H-R$ transform the expression may be interpreted as including the associated function as well.

## REFERENCES

[1] Bilodeau, G. G., « *Gegenbauer Polynomials and Fractional Derivatives* », American Mathematical Monthly, Vol. 71, No. 9, 1964.

[2] Erdelyi, A., et al. *Higher Transcendental Functions,* Vols. 1 and 2, McGraw-Hill, 1953.

[3] Gray, H. L., « *Application of the Holmgren-Riesz Transform* », Annali della Scuola Normale Superiore di Pisa Scienze Fisiche e Matematiche, Serie III, Vol. XVIII, Fasc. I, 1964.