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ON THE LOCAL ZETA FUNCTION OF A CUBIC THREEFOLD

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I. The object of this paper is to determine the Zeta function of a non-singular cubic threefold in projective four space, defined over a finite field, to give a canonical interpretation of it, and to show that it satisfies the Riemann hypothesis.

Let $\mathbf{F}_q$ denote the finite field of q elements, let $V/\mathbf{F}_q$ be a complete non-singular threefold in projective four-space $\mathbf{P}^4$, defined over $\mathbf{F}_q$, and let $\nu_n = \nu_n(V)$ denote the number of points of V defined over $\mathbf{F}_{q^n}$. It follows from the theorem of Dwork [2] that

$$\nu_n = (q^{4n} - 1)/(q^n - 1) - \sum_{h=1}^{10} \omega_h^n . \tag{1}$$

where the ω_h are algebraic integers, depending on V and q but not on n. Among other things, we shall show that

$$| \omega_h | = q^{\frac{3}{2}} \qquad \text{(hypothesis of Riemann-Weil)} \tag{2}$$

and indicate how to determine the ω_h by solving a simpler problem.

We remark that in order to prove (2) we may freely allow ourselves finite field extensions, for it is obvious from (1) that replacing the field $\mathbf{F}_q$ by $\mathbf{F}_{q^m}$ replaces the ω_h by ω_h^m, and the new version of (2) is equivalent to the old. This very fruitful device, which enables one to assume that subsidiary objects obtained in the course of the proof are defined over the ground field, is due to Davenport and Lewis [1]. In this way it is possible

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to obtain a simple proof of (2) by reducing our problem to a problem about the Zeta function of a curve, to which we can apply Weil's well-known results. However, in order to give a canonical form to the Zeta function of V we need some information about the geometry of V. Our result is expressed by means of the Albanese variety $A(X)$ of the variety $X/\mathbf{F}_q$ parametrizing the lines of V. We shall prove the following

THEOREM 1. *We have, if char* $(\mathbf{F}_q) \neq 2$,

$$\nu_n = (q^{4n} - 1)/(q^n - 1) - q^n \, Tr(\pi^n), \tag{3}$$

where $Tr(\pi_n)$ *denotes the trace of the n-th power of the Frobenius endomorphism* π *of* $A(X)/\mathbf{F}_q$*, in the endomorphism algebra of* $A(X)$ *over the rationals* Q.

By standard arguments and by Weil's results on the endomorphism algebra of an abelian variety, we know that (3) implies the validity of the Weil conjectures for the Zeta function $Z(t; V/\mathbf{F}_q)$; see for instance [4], Ch. V_3, Th. 2.

It might be useful to give some indications about the general plan of this paper and the methods of proof of our Theorem 1. Our techniques are of three kinds. Firstly, a projective study of the variety X and some associated geometric objects, i.e. properties depending on the embedding. Secondly, a study from a birational point of view, and mostly up to isogenies, of the abelian varieties associated to the geometric objects previously introduced. Thirdly, there is the proof of Theorem 1, where we make use of arithmetical methods related to finite field techniques. Finally, in the last section of this paper we shall discuss some conjectures and problems arising in a natural way from our Theorem 1. In particular, we shall point out the relationships of our result to the theory of higher jacobians, as introduced by Weil [6].

The language of this paper will be that of Weil's Foundations, and for concepts related to abelian varieties we refer to Lang [4].

We end this introduction by expressing our indebtedness to Professor Davenport, who is responsible for our originally attacking the problem and who has given us valuable advice and encouragement.

II. In this section we shall study the geometry of the variety X of lines of V. It is convenient to look at the more general case where V is defined over a field k, of characteristic not 2. This variety X has been studied in some detail by Fano [3] when $k = \mathbf{C}$, the field of complex numbers. His methods can be easily extended to the case $char(k) \neq 2$, while if $char(k) = 2$ there are some difficulties because of presence of inseparable field extensions. We believe that our method can be modified so as to give a proof of Theorem 1, even when $char(k) = 2$.

We begin by proving a result (Lemma 3 below) which was explicitly used by Fano [3] but for which we have found no satisfactory reference.

We consider X as the algebraic set which parametrizes the lines of V, embedded in the appropriate Grassmannian. For this model X we have

LEMMA 1. *X is a disjoint union of absolutely irreducible algebraic surfaces, complete and non-singular. The algebraic set X is defined over k.*

PROOF. Let u be a point of X, and consider the local ring $\mathcal{O}_u$ of all functions on X regular at u. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_u$, and let $\mathbf{k}_u = \mathcal{O}_u/\mathfrak{m}$ denote the residue class field of $\mathcal{O}_u$. We shall prove that for every u the Zariski tangent space $\mathfrak{m}/\mathfrak{m}^2$ at u is a vector space of dimension 2 over $\mathbf{k}_u$. In other words, the local ring $\mathcal{O}_u$ has exactly two generators for every point u of X.

We show first how Lemma 1 follows from this result.

The algebraic set X is defined over k (obvious); also every component of X is complete, because every specialization over k of a line of V is a linear space of dimension at least one, hence a line because V cannot contain planes by the hypothesis of non-singularity. Let $X_1, X_2, \dots, X_m$ be the absolutely irreducible components of X and choose a simple point $u \in X_i$ such that $u \notin X_j$ for $j \neq i$. The local ring $\mathcal{O}_u$ is regular because u is simple and does not belong to any X_j with $j \neq i$; also we know that $\mathcal{O}_u$ has two generators. Hence $\mathcal{O}_u$ has dimension two and X_i is a surface. Hence every $\mathcal{O}_u$ has dimension two, so it is regular because it has exactly two generators. Let $u \in X_i \cap X_j$ where $i \neq j$, supposing the intersection is not empty; then the local ring $\mathcal{O}_u$ has zero divisors and it is not regular, a contradiction. Hence the surfaces X_i are disjoint, complete and non-singular, the latter because every $\mathcal{O}_u$ is regular. This will prove Lemma 1 and it remains to show thas the vector space $\mathfrak{m}/\mathfrak{m}^2$ has dimension two over $\mathbf{k}_u$.

Without loss of generality we may assume that k is algebraically closed and that the line L_u corresponding to the point u of X is given by the system of equations

$$x_1 = x_2 = x_3 = 0,$$

in the ambient projective space $(x_1, x_2, \dots, x_5)$ of V. A generic point (v) of G over k, where G is the Grassmannian of lines of the ambient space of V, corresponds to a line L_v meeting the hyperplanes $x_4 = 0$ and $x_5 = 0$ in two points $P = (y_1, y_2, y_3, 0, 1)$ and $Q = (z_1, z_2, z_3, 1, 0)$ respectively, and we take $(y_1, y_2, y_3, z_1, z_2, z_3)$ as local parameters at (v). The birational map

$$(v) \longrightarrow (y_1, y_2, y_3, z_1, z_2, z_3)$$

is biregular at (0, 0, 0, 0, 0, 0) which corresponds to the point $u \in X$ previously defined, so that these parameters are indeed local uniformizing parameters on G at the point u. The condition that the line PQ lies on V is just that its generic point $\lambda P + \mu Q$ lies on V; substituting into the defining equation of V and equating coefficients of λ^3, $\lambda^2 \mu$, $\lambda\mu^2$, μ^3 separately to 0 we obtain four conditions

$$f_i(y_1, y_2, y_3, z_1, z_2, z_3) = 0,$$

$i = 1, 2, 3, 4$, where the f_i are polynomials defined over k. Also, in this case the residue class field $\mathbf{k}_u$ is the field k, and it follows that $\mathfrak{m}/\mathfrak{m}^2$ is the vector space over k generated by $y_1, y_2, \ldots, z_3$ together with the relations $g_i = 0$, where g_i denotes the linear part of f_i. The linear part of f_i clearly comes from the terms in the defining equation for V which are quadratic in x_4, x_5 together and linear in x_1, x_2, x_3 together. We now split cases.

Case 1. The coefficients of x_4^2, $x_4 x_5$, x_5^2 in the equation for V are linear forms in x_1, x_2, x_3 linearly independent over k.

Then by a linear change of variables we may take the terms we are interested in to be

$$x_1 x_4^2 + x_2 x_4 x_5 + x_3 x_5^2 .$$

Then $g_1 = z_1$, $g_2 = y_1 + z_2$, $g_3 = y_2 + z_3$, $g_4 = y_3$ and clearly $\mathfrak{m}/\mathfrak{m}^2$ has dimension two over k.

Case 2. The coefficients of x_4^2, $x_4 x_5$, x_5^2 in the equation for V are linear forms in x_1, x_2, x_3 linearly dependent over k.

They cannot be all multiples of the same form; for if they were then by a linear transformation over k on x_4, x_5 only we could reduce the coefficient of x_5^2 in the equation for V to 0, and then (0, 0, 0, 0, 1) would be a singular point of V, which is contrary to hypothesis. Hence we may make a linear transformation on x_4, x_5 only so that the coefficients of x_4^2 and x_5^2 are linearly independent, and then by a linear transformation on x_1, x_2, x_3 only we may write the terms we are interested in the form

$$x_1 x_4^2 + (ax_1 + bx_2) x_4 x_5 + x_2 x_5^2 .$$

Now $g_1 = z_1$, $g_2 = y_1 + az_1 + bz_2$, $g_3 = ay_1 + by_2 + z_2$, $g_4 = y_2$ and again we obtain that $\mathfrak{m}/\mathfrak{m}^2$ has dimension two over k, provided $ab \neq 1$. Finally if $ab = 1$, then $(0, 0, 0, a, -1)$ is a singular point of V, contrary to hypothesis.

This completes the proof of Lemma 1.

We shall show later that X is connected, so that by Lemma 1 X is a complete, non-singular, absolutely irreducible surface defined over k. In order to prove this it is convenient to study the geometry of some curves C_u and Γ_u, which we are going to define.

Let C_u denote the algebraic set, embedded in X, which parametrizes all lines on V meeting a line L_u corresponding to a point u of X, with the convention that the point u itself is a point of C_u if and only if there is a plane tangent at V along the line L_u. Note that if such a plane exists, it is unique.

Let v be a point of C_u: the two lines L_u, L_v are coplanar and determine uniquely a plane which will meet V residually in a third line, say L_w. Il is clear that $w \in C_u$, so that writing $j_u(v) = w$ we have defined an application $j_u : C_u \longrightarrow C_u$, such that $j_u^2 = identity$. Plainly j_u is an equivalence relation on the algebraic set C_u and the quotient C_u/j_u parametrizes all planes through L_u which meet V in three lines. The planes through L_u are parametrized in a natural way (Plücker coordinates) by a projective plane $\mathbf{P}_u^2$, whence the algebraic set C_u/j_u has a natural model Γ_u in this plane $\mathbf{P}_u^2$.

It is easily seen, by the same argument used in the proof of Lemma 1, that every component of the algebraic sets C_u and Γ_u is complete.

LEMMA 2. *We have*

(a) *Γ_u is a (possibly reducible) plane curve, of degree* 5 ;

(b) *the singular points of Γ_u, if there are any, are ordinary double points but not cusps. The double points of Γ_u correspond to the fixed points of j_u on C_u.*

REMARK. In the proof of this lemma we need the hypothesis $char(k) \neq 2$.

COROLLARY. *Every component of the algebraic set C_u is a curve.*

PROOF OF COROLLARY TO LEMMA 2. Apply (a) of Lemma 2 to $C_u/j_u = \Gamma_u$ and note that the application j_u is everywhere defined.

PROOF OF LEMMA 2. The statements of Lemma 2 do not involve the fields of definition of the various geometric objects we are studying, so that we may suppose that K is an algebraically closed field of definition for V and L_u, and that L_u is given by the system of equations

$$x_1 = x_2 = x_3 = 0,$$

in the ambient projective space $(x_1, \dots, x_5)$ of V. For this reason we shall write L, C, Γ for L_u, C_u, Γ_u.

Any plane through L is defined by the system of equations

$$x_1/\lambda_1 = x_2/\lambda_2 = x_3/\lambda_3, \tag{4}$$

and $(\lambda_1, \lambda_2, \lambda_3)$ are homogeneous Plücker coordinates of this plane; the algebraic set Γ is embedded in the projective plane of the λ_i. The residual section of the plane (4) with V is a conic Q, and to obtain the equation for Q we write $x_1 = \lambda_1 t$, $x_2 = \lambda_2 t$, $x_3 = \lambda_3 t$ and use (t, x_4, x_5) as homogeneous coordinates on the plane (4) containing Q. A simple calculation then shows that the equation for Q can be written in the form

$$At^2 + Btx_4 + Ctx_5 + Dx_4^2 + Ex_4 x_5 + Fx_5^2 = 0, \tag{5}$$

where $A, \dots, F$ are homogeneous polynomials in the λ_i, A being cubic, B and C quadratic, and D, E, F linear. The conic Q splits in two lines if and only if

$$4ADF + BCE - AE^2 - FB^2 - DC^2 = 0, \tag{6}$$

and equation (6) defines the model Γ of C/j we are studying. An immediate consequence of (6) is assertion (a) of Lemma 2, unless Γ is the entire plane of the λ_i.

The fact that equation (6) is not identically satisfied will be clear from the following analysis, which at the same time will prove assertion (b) of Lemma 2.

We consider the five ways Q can degenerate; these are

(i) two distinct lines whose intersection is not on L;
(ii) two distinct lines which meet on L;
(iii) L and another line distinct from it;
(iv) L as a double line;
(v) a double line other than L.

Our curve Γ is in one-to-one correspondence with the degenerate Q, and our aim is to show that the left hand side of (6) is not identically 0 and that in cases (i), (ii), (iii) we have a simple point of Γ, while in cases (iv) and (v) we have an ordinary double point at which the branches have distinct tangents. In cases (iv) and (v) we need the fact that $char(K) \neq 2$.

By a linear change of variables we may assume that the plane containing the degenerate Q is $x_2 = x_3 = 0$, and we use the linear change of variables to make such further simplifications as we can. Also we make repeated use of the fact that V is non-singular, which implies easily that the left hand side of (5) is at least quadratic in λ_1 and neither D nor F can vanish identically.

Case (i). We may take Q to be $x_4 x_5 = 0$; thus the coefficient of λ_1 in E is non-zero. Moreover A, B, C, D, F all vanish at $(1, 0, 0)$ so that D and F are independent of λ_1, B and C are at most linear in λ_1 and A is at most quadratic. Hence A must genuinely be quadratic in λ_1. Now the left hand side of (6) contains a non-zero multiple of λ_1^4, arising from the term AE^2; since (6) represents a quintic equation, it follows that $(1, 0, 0)$ is a non-singular point of Γ.

Cases (ii) and (iii). Here we may take Q to be $x_4(x_4 + t) = 0$ or $x_4 t = 0$ respectively. The proof that $(1, 0, 0)$ is a simple point of Γ is similar to that for case (i), the only difference being that the non-zero multiple of λ_1^4 arises from the term FB^2.

Case (iv). Now Q is $t^2 = 0$; thus the coefficient of λ_1^3 in A is non-zero, D, E, F are independent of λ_1 and B and C are at most linear in λ_1. There is no term in λ_1^4 in the left hand side of (6), and the term in λ_1^3 arises from $A(4DF - E^2)$. To prove that $(1, 0, 0)$ is an ordinary double point of Γ, therefore, we have to show that $4DF - E^2$, viewed as a quadratic form in λ_2, λ_3, is neither zero nor a perfect square; note that here we make use of the hypothesis $char(K) \neq 2$.

Suppose that $4DF - E^2$ is zero or a square in the field $K(\lambda_2, \lambda_3)$. This implies that the quadratic form in x_4, x_5: $Dx_4^2 + Ex_4 x_5 + Fx_5^2$ factorizes over $K(\lambda_2, \lambda_3)$ and since D, E, F are linear in λ_2, λ_3 one of the factors must be defined over K. Let $c_5 x_4 - c_4 x_5$ be this factor; then $(0, 0, 0, c_4, c_5)$ is a singular point of V, which is impossible.

Case (v). We may take Q to be $x_4^2 = 0$, so that the coefficient of λ_1 in D is non-zero. Moreover E and F are independent of λ_1, B and C are at most linear in λ_1 and A is quadratic in λ_1. Suppose that the coefficient of λ_1^2 in A is a, which does not vanish identically, and that the coefficient of λ_1 in C is c, which may vanish identically. There is no term in λ_1^4 in the left hand side of (6) and the term in λ_1^3 arises from $D(4AF - C^2)$; to prove that $(1, 0, 0)$ is an ordinary double point of Γ, therefore, we must show that $4aF - c^2$ is neither zero nor a perfect square in the field $K(\lambda_2, \lambda_3)$, again because $char(K) \neq 2$.

As before, we obtain that $at^2 + ctx_5 + Fx_5^2$ has a linear factor defined over K; then if $c_5 t - c_1 x_5$ is this factor, the point $(c_1, 0, 0, 0, c_5)$ is a singular point of V, which is impossible.

This contradiction proves our assertion and completes the proof of Lemma 2.

LEMMA 3. *X is an absolutely irreducible surface, complete and non-singular, defined over k.*

PROOF. In view of Lemma 1, it is enough to prove that X is connected and we may assume that the field k is algebraically closed. Let $X_1, X_2, \ldots, X_m$ be the absolutely irreducible components of X and let u be a point of X. We prove first that $C_u \cap X_i$ is non-empty for every i, so that if X is not connected then no C_u is connected.

To prove this assertion, let D_i be a complete curve on X_i defined over k and such that $u \notin D_i$, let (y) be a generic point of D_i over k, and consider the surface F_i on V given by $F_i = \text{locus}_k\, L_{(y)}$. This surface is complete and ruled and there is at least one line on F_i passing through a given point of F_i, which is parametrized by a point of D_i. It is well-known that the cubic threefold V is regular, hence F_i is linearly equivalent to a positive multiple of a hyperplane section of V, hence the intersection $L_u \cap F_i$ is never empty. Hence there is a line of F_i, distinct from L_u, parametrized by a point y of D_i and meeting L_u, because by the large choice of the curve D_i we may suppose that L_u is not a line of F_i. This implies $y \in C_u \cap X_i$, as asserted.

We write C_u as the union of absolutely irreducible components C_u^i, $i = 1, \ldots n$, and noting that $j_u^2 = \textit{identity}$ we may describe the action of i_u on C_u as follows

(A) j_u maps C_u^i into itself for $i = 1, \ldots, r$;

(B) j_u interchanges C_u^i and C_u^{i+s} for $i = r+1, \ldots, r+s$ where $r + 2s = n$.

Now the quotient Γ_u may be written as the union of absolutely irreducible curves Γ_u^i, $i = 1, \ldots, r+s$, where $\Gamma_u^i = (C_u^i + j_u(C_u^i))/j_u$. By the theorem of Bézout, the intersection $\Gamma_u^i \cap \Gamma_u^j$ is non-empty for $i \neq j$, also every point of this intersection is a singular point of Γ_u because the curves Γ_u^i are all distinct. By assertion (b) of Lemma 2, these points come from fixed points of j_u.

Suppose first $r \geq 1$. Taking inverse images of $\Gamma_u^1 \cap \Gamma_u^i$ we find that, for every i, $C_u^1 \cap (C_u^i + j_u(C_u^i))$ contains a fixed point of j_u. This result proves that C_u is connected in case $r \geq 1$. A similar argument applies if $r = 0$ and $s = 2$, therefore if X is not connected we have $r = 0$ and $s = 1$, that is C_u always consists of two absolutely irreducible components which are interchanged by j_u. Hence X would consist of two absolutely irreducible components, each of them containing a component of C_u. Finally, we get a contradiction by destroying the symmetry of the previous result, which implies that two meeting lines of V belong to distinct algebraic systems; in fact, it is enough to consider three coplanar lines of V.

This contradiction proves that X is connected, and completes the proof of Lemma 3.

By Lemma 3 we have that C_u is a divisor on X, defined over $k(u)$. We are now interested in the behaviour of C_u when u is a generic point of X over k.

LEMMA 4. *If* (u) *is a generic point of* X *over* k *then* C_u *is absolutely irreducible and defined over* $k(u)$.

PROOF. The algebraic family of divisors $\{C_u\}$ is absolutely irreducible and in fact parametrized by X, becase it is easily seen that if $u \neq v$ then $C_u \neq C_v$. A simple consequence of this result is that if D is a divisor on X then the intersection number $(C_u \cdot D)$ is independent of u.

Now we recall that, given a non-singular cubic surface Σ, there are exactly five lines of Σ meeting two given skew lines of Σ. It follows easily from this remark and the well-known fact that the generic hyperplane section of V is a non-singular cubic surface, that if (u) and (v) are two independent generic points of X over k then C_u and C_v intersect properly in five points. In particular we have

$$(C_u \cdot C_u) = 5. \tag{7}$$

Suppose that if (u) is a generic point of X over k then C_u splits into absolutely irreducible components C_u^i, using the same notation as in the proof of Lemma 3. We have just shown that if (u) and (v) are two independent generic points of X over k then $C_u \cdot C_v$ is a positive 0-cycle on X of degree 5; it follows that for each i the algebraic system of divisors $\{C_u^i\}$ is absolutely irreducible, also

$$(C_u^i \cdot C_u^i) \geq 0 \tag{8}$$

for all i, j, and

$$\Sigma_{i,j}\,(C_u^i \cdot C_u^j) = 5. \tag{9}$$

On considering the action of j_u on C_u we find by the same argument used in the proof of Lemma 3 that

$$(C_u^i \cdot C_u^j) \geq (deg\, \Gamma_u^i)\,(deg\, \Gamma_u^j) \tag{10}$$

if $$i, j \leq r \quad \text{and} \quad i \neq j,$$

also

$$(C_u^i \cdot C_u^i) + (C_u^i \cdot C_u^{j+s}) \geq (deg\, \Gamma_u^i)\,(deg\, \Gamma_u^j) \tag{11}$$

if $$i \leq r, \quad r+1 \leq j \leq r+s,$$

and

$$(C_u^i \cdot C_u^j) + (C_u^i \cdot C_u^{j+s}) + (C_u^{i+s} \cdot C_u^j) + (C_u^{i+s} \cdot C_u^{j+s}) \geq (deg\, \Gamma_u^i)\,(deg\, \Gamma_u^j) \tag{12}$$

if $$r+1 \leq i, j \leq r+s \quad \text{and} \quad i \neq j.$$

These inequalities (8), (9), ..., (12) are compatible with the obvious relation (assertion (a) of Lemma 2)

$$\Sigma\,(deg\, \Gamma_u^i) = 5,$$

only if $r = 1,\ s = 0$ or $r = 0,\ s = 1$.

The first alternative says that C_u is absolutely irreducible. The second one implies that $C_u = C_u^1 + C_u^2$ where j_u interchanges the components C_u^i. We assert that this last alternative is impossible.

Let Σ be a non-singular hyperplane section of V containing L_u, so that Σ is a non-singular cubic surface, let L_v be another line of Σ not meeting L_u, and let $M_1, \ldots M_5$ be the five lines of Σ meeting L_u and L_v. Let M_i' be the residual section of Σ with the plane determined by M_i and L_u; the five lines M_i' are all distinct from the lines M_i, and there is a line of Σ, say $L_{v'}$, other than L_u, meeting the five lines M_i'. This implies that the C_u^i intersect properly with C_v and $C_{v'}$; also

$$(C_u^1 \cdot C_v) = (C_u^2 \cdot C_{v'}),$$

because j_u interchanges the components C_u^i of C_u.

By the remark made at the beginning of the proof of Lemma 4 we obtain

$$(C_u \cdot C_u) = (C_u^1 \cdot C_u) + (C_u^2 \cdot C_u) \equiv 0 \pmod 2,$$

a contradiction because $(C_u \cdot C_u) = 5$ is odd.

This contradiction completes the proof of Lemma 4.

LEMMA 5. *Let $A(X)$ be the k-Albanese variety of X. Then*

$$dim_k\, A\,(X) = 5.$$

PROOF. The dimension of $A(X)$ is the irregularity q of the surface X, so that we have to prove that $q = 5$. We shall consider here first the case where k is a field of zero characteristic and then apply a specialization argument when $char(k) \neq 0$.

By Lemma 3, we may apply surface theory to X, and in case $char(k) = 0$ we have

$$q = p_g - p_a,$$

where p_g and p_a are the geometric and arithmetic genus of X. These invariants of the surface X have been computed by Fano [3], who found $p_g = 10$ and $p_a = 5$. As Fano's paper is not easily available we shall sketch briefly his arguments.

It is proved first that the canonical system K of X is the class of a hyperplane section ([3], pag. 784) so that p_g is the dimension of the linear system of hyperplane sections of X, that is $p_g > 1$ is the dimension of the projective space where X is embedded. Obviously X is embedded in projective space P^9 because so is the Grassmannian of lines of P^4. The fact that X is not embedded in projective space P^8 is more difficult to prove ([3], pp. 782-783).

The computation of p_a is based on the classical formula

$$p_a + 1 = 1/12\,((K\cdot K) + \chi(X)), \tag{13}$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X. The value of the self-intersection $(K\cdot K)$ is easily found. Let L_u, L_v, L_w be three coplanar lines on V; then $C_u + C_v + C_w$ is a hyperplane section of X, hence a canonical divisor, and by the remark in the first paragraph of the proof of Lemma 4 and by equation (7) we obtain

$$(K\cdot K) = 9\,(C_u\cdot C_u) = 45. \tag{14}$$

Finally, the Euler-Poincaré characteristic $\chi(X)$ of X is computed by using the classical definition of the Zeuthen-Segre invariant $I = \chi(X) - 4$. One obtains ([3], pp. 786-788) $\chi(X) = 27$ whence $p_a = 5$ by equations (13) and (14).

Now suppose that $char(k) \neq 0$. The statement of Lemma 5 is independent of ground field extensions ([4], Ch. II_3, Th. 12) so that we may assume that k is algebraically closed. Let R denote the ring of Witt vectors over k. Then R is a valuation ring of characteristic zero and k is the residue class field of R. The cubic threefold V is a projective hypersurface $\Sigma c_{ijk}\, x_i x_j x_k = 0$ and if c'_{ijk} denotes the Teichmüller representative of c_{ijk} in R the cubic hypersurface $\Sigma c'_{ijk}\, x_i x_j x_k = 0$ is again a non-singular cubic threefold whose reduction modulo the maximal ideal I in R gives V. Let

$\bar{X}$ be the variety of lines of the lifted threefold $\bar{V}$. We know that every specialization mod I of a line of $\bar{V}$ is a linear space on V, hence a line because V is non-singular, and it follows from this that the reduction mod I of $\bar{X}$ is a component of X of dimension two. By Lemma 3, we deduce that X is the reduction of $\bar{X}$ mod I (O. Zariski: Theory of holomorphic functions, pp. 80-82). By Lemma 3 again, we know that both X and $\bar{X}$ are non-singular surfaces, where by the recent results of Grothendieck about Picard varieties we have that the Albanese varieties of X and $\bar{X}$ have the same dimension. In fact, $dim_k\ A(X)$ and $dim\ A(\bar{X})$ are respectively half of the first Betti number (in the l-adic sense, where $l \neq char(k)$) of X and $\bar{X}$, and these numbers are the same because X is non-singular, hence X is a « good » reduction of $\bar{X}$ mod I (see M. Artin - A. Grothendieck - Cohomologie étale des schémas, Sém. Géom. Alg., 1963/64 exp. XII, cor. 5.4). Otherwise, and more simply, one may use the fact that the part prime to $char(k)$ of the fundamental group is the same for X and $\bar{X}$ (see Sém. Géom. Alg., 1961, exp. X or Sém. Bourbaki, mai 1959, exp. 182 and exp. 236 p. 14).

This proves Lemma 5, because it has already been proved in case $char(k) = 0$, and the pair $(\bar{V}, \bar{X})$ is defined over a ring R of characteristic zero, with no zero-divisors.

III. We shall prove here some results on the Albanese varieties of C_u and Γ_u, which will be needed in the proof of Theorem 1.

Let $A(C_u)$ and $A(\Gamma_u)$ denote the $k(u)$-Albanese varieties of C_u and Γ_u, where (u) is a generic point of X over k; these exist because by Lemma 4 both C_u and Γ_u are absolutely irreducible. Let $f_u : C_u \times C_u \longrightarrow A(C_u)$ be a canonical admissible map defined over $k(u)$, and in the same way let $f : X \times X \longrightarrow A(X)$ be a canonical admissible map defined over k.

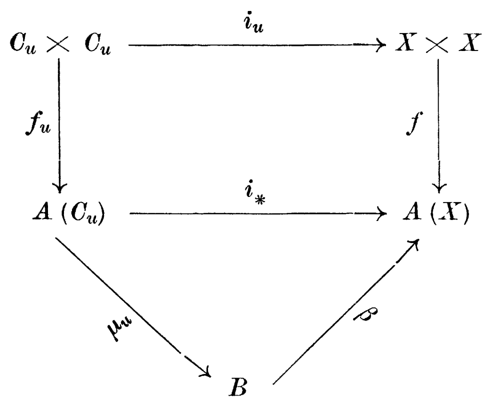
We shall denote by (A, λ_u) and (B, μ_u) the $k(u)/k$-Images of $A(\Gamma_u)$ and $A(C_u)$ respectively.

THEOREM 2. *There exist two homomorphisms $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow A(X)$, defined over k, such that the sequence of abelian varieties*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} A(X) \longrightarrow 0$$

is exact up to isogenies. The homomorphism β is the homomorphism induced by the inclusion map $i_u : C_u \longrightarrow X$ and the universal mapping properties of the Albanese variety and the $k(u)/k$-Image.

Let $i_u: C_u \times C_u \longrightarrow X \times X$ be the inclusion map, and consider the commutative diagram

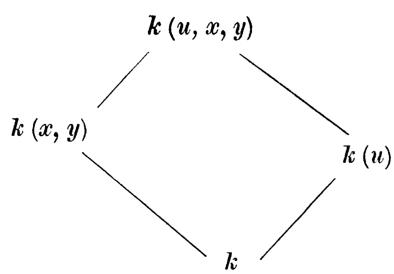


where i_* is the homomorphism induced by i_u (universal mapping property of the Albanese variety) and β is the homomorphism induced by i_* (universal mapping property of the $k(u)/k$-Image). Clearly i_* is defined over $k(u$ and β is defined over k.

LEMMA 6. *The homomorphism* $\beta: B \longrightarrow A(X)$ *is surjective.*

PROOF. From the commutativity of the triangle in our diagram it is enough to show that the homomorphism i_* is surjective.

Let (x), (y) be two independent generic points of C_u over $k(u)$; we assert that (x), (y) are two independent generic points of X over k. In fact, consider the tower of fields



We have $dim_k(u) = 2$ and $dim_{k(u)}(x) = dim_{k(u)}(y) = 1$ by our definitions, and as (x) and (y) are independent generic over $k(u)$ we have $dim_{k(u)}(x, y) = 2$, whence $dim_k(u, x, y) = 4$. On the other hand, the field $k(u, x, y)$ is algebraic over $k(x, y)$ because $(C_x \cdot C_y) = 5 > 0$ by equation (7). It follows that $dim_k(x, y) = 4$ and our assertion follows from this.

The image of the restriction of f to $C_u \times C_u$ is a subvariety of $A(X)$ going through the origin and defined over a regular extension $k(u)$ of k. The abelian subvariety of $A(X)$ generated by this subvariety is defined over $k(u)$, hence over k by Chow's theorem ([4], Ch. II_1, Th. 5). Also, it contains a point $f(x, y)$ where (x, y) is a generic point of $X \times X$ over k. Hence it contains the image of f in $A(X)$ and it follows that it coincides with $A(X)$. Going the other way round the square in our diagram we obtain that the homomorphism i_* is surjective, and Lemma 6 is proved.

Let Z be the three-dimensional variety defined by $Z = locus_k(u, v)$ where (u) is a generic point of X over k and (v) is a generic point of C_u over $k(u)$. By Lemma 4, Z is absolutely irreducible and is defined over k. Clearly Z is the graph on $X \times X$ of the divisorial correspondence $X \longrightarrow X$ defined by $u \longrightarrow C_u$. Our next step is to define an admissible rational map (i. e. vanishing on the diagonal) $h : Z \times Z \longrightarrow A(X) \times A(X) \times B'$, defined over k and such that the pair $(Z \times Z, h)$ generates $A(X) \times A(X) \times B'$. Here B' denotes the connected component of the kernel of β; it is an abelian variety, a priori defined over a purely inseparable extension of k, hence defined over k by Chow's theorem ([4], Ch. II_1, Th. 5). By Poincaré's complete reducibility theorem ([4], Ch. II_1, Th. 6) there exists a homomorphism $\eta : B \longrightarrow B'$ with the following properties:

(i) η is defined over k and is surjective ;

(ii) the homomorphism $\beta \times \eta : B \longrightarrow A(X) \times B'$ is an isogeny.

Let (u), (u') be two independent generic points of X over k and (v, v') a generic point of $C_u \times C_{u'}$ over $k(u, u')$. Let $P_1(u, u') + \dots + P_5(u, u')$ be the zero cycle of degree 5 on C_u and $C_{u'}$ given by the intersection $C_u \cdot C_{u'}$.

The point (u, v, u', v') is a generic point of $Z \times Z$ over k, and we define a rational map $h : Z \times Z \longrightarrow A(X) \times A(X) \times B'$ by

$$(h(u, v, u', v')) =$$

$$= \left(5f(u, u'), 5f(v, v'), \sum_{i=1}^{5} (\eta \mu_u f_u(v, P_i(u, u')) - \eta \mu_{u'} f_{u'}(v', P_i(u, u')))\right).$$

Lemma 7. *The rational map h is admissible and defined over k. The pair $(Z \times Z, h)$ generates $A(X) \times A(X) \times B'$.*

PROOF. It is clear that h is admissible because $h(u, v, u', v') + h(u', v', u, v) = (0, 0, 0)$, the origin of the abelian variety $A(X) \times A(X) \times B'$. Also it is defined over k because the zero cycle $C_u \cdot C_{u'}$ is defined over $k(u, u')$ whence the point $h(u, v, u', v')$ is defined over $k(u, v, u', v')$. It remains to prove that the pair $(Z \times Z, h)$ generates $A(X) \times A(X) \times B'$.

Let $h' : Z \times Z \to A(X) \times B$ be the rational admissible map defined by

$$h'(u, v, u', v') = (5f(u, u'), \sum_{i=1}^{5} (\mu_u f_u(v, P_i(u, u')) - \mu_{u'} f_{u'}(v', P_i(u, u')))).$$

We have the identities

$$(\beta \times \eta)\, \mu_u f_u = (f\, i_u) \times (\eta \mu_u f_u)$$

and

$$\sum_{i=1}^{5} (f(v, P_i(u, u')) - f(v', P_i(u, u'))) = 5f(v, v'),$$

hence

$$h = (e \times \beta \times \eta)\, h'$$

where e is the identity on the first factor of $A(X) \times B$. It follows from the fact that the homomorphism η is an isogeny that $(Z \times Z, h)$ generates $A(X) \times A(X) \times B'$ if and only if the pair $(Z \times Z, h')$ generates $A(X) \times B$.

For every point P of C_u the curve $locus_{k(u)} f_u(v, P)$ is a generating curve of the Albanese variety $A(C_u)$, whence the same is true for the curve $locus_{k(u, u')} (f_u(v, P_1(u, u')) + \ldots f_u(v, P_5(u, u')))$, for this last curve is a translation of the previous generating curve, multiplied by 5. It follows that the curve

$$locus_{k(u, u', v')}\, h'(u, v, u', v') =$$

$$= (5f(u, u')) \times (locus_{k(u, u', v')} \sum_{i=1}^{5} (\mu_u f_u(v, P_i(u, u')) - \mu_{u'} f_{u'}(v', P_i(u, u'))))$$

is a product of the point $5f(u, u')$ of $A(X)$ with a generating curve of B. Taking the sum of this curve with itself a sufficiently high number of times, say m times, we find that the pair $(Z \times Z, h')$ generates the subvariety $(5mf(u, u')) \times B$ of $A(X) \times B$. Taking the locus of this subvariety over k we see that the pair $(Z \times Z, h')$ generates $A(X) \times B$, and Lemma 7 is proved.

The following lemma is the key to our proof of Theorem 2.

LEMMA 8. *There exists a homomorphism $\alpha' : A \to B'$ defined over k and with finite kernel.*

PROOF. We begin with the very simple remark that if L_u and L_v are two meeting lines of V, then v is a point of C_u and conversely u is a point of C_v. It follows that the rational map $j : Z \longrightarrow Z$ defined by $j(u, v) = (v, u)$ is an involution on Z of order two and defined over k. We shall denote by Y the quotient variety of Z by the cyclic group of order two generated by j. Clearly Y is an algebraic variety (say Z is projective) defined over k, and we shall denote by $A(Y)$ the k-Albanese variety of Y. Let $A(Z)$ be the k-Albanese variety of Z, let $g : Z \times Z \longrightarrow A(Z)$ and $g' : Y \times Y \longrightarrow A(Y)$ be canonical admissible maps, and let $\varrho : Z \longrightarrow Y$ denote the covering previously defined.

Consider the commutative diagram

$$\begin{array}{ccccc} Y \times Y & \xrightarrow{\varrho^{-1} \times \varrho^{-1}} & Z \times Z & \xrightarrow{\varrho \times \varrho} & Y \times Y \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ A(Y) & \xrightarrow{\varrho^*} & A(Z) & \xrightarrow{\varrho_*} & A(Y) \end{array}$$

where $g(\varrho^{-1} \times \varrho^{-1}) : Y \times Y \longrightarrow A(Z)$ is the admissible rational map of $Y \times Y$ into $A(Z)$, given by the sum $g(Q_1) + g(Q_2) + g(Q_3) + g(Q_4)$ where $Q_1 + Q_2 + Q_3 + Q_4$ is the inverse image by $\varrho^{-1} \times \varrho^{-1}$ of a generic point P of $Y \times Y$ over k.

On applying the result in [4], App.$_1$, Th. 5 we see that $\varrho_* \varrho^* = 4\delta_{A(Y)}$, where $\delta_{A(Y)}$ is the identity map on $A(Y)$, because $\varrho \times \varrho : Z \times Z \longrightarrow Y \times Y$ is generically a covering of degree four. It is obvious that ϱ_* is surjective and defined over k, hence ϱ^* has finite kernel and is separable, the latter because *char* $(k) \neq 2$. Hence ϱ^* is defined over k.

We have a commutative diagram

$$\begin{array}{ccccc} Y \times Y & \xrightarrow{\varrho^{-1} \times \varrho^{-1}} & Z \times Z & \searrow{\scriptstyle h} & \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \\ A(Y) & \xrightarrow{\varrho^*} & A(Z) & \xrightarrow{h_*} & A(X) \times A(X) \times B', \end{array}$$

whence we obtain a homorphism defined over k

$$h_* \varrho^* : A(Y) \longrightarrow A(X) \times A(X) \times B' ;$$

we assert that this homomorphism has finite kernel.

We already know that ϱ^* has finite kernel, so that in order to prove our assertion we have to prove that the kernel of h_* is finite too. On the other hand, by Lemma 7 the pair $(Z \times Z, h)$ generates $A(X) \times A(X) \times B'$, whence h_* is surjective. Hence if h_* has finite kernel it must be an isogeny, and to prove both these facts it is enough to show that $A(Z)$ and $A(X) \times \times A(X) \times B'$ have the same dimension.

The variety Z is a fibre space over X, with generic fibre C_u, as we can see from the generically exact sequence of varieties

$$C_u \xrightarrow{i_u} Z \xrightarrow{p} X$$

where i_u is the inclusion map $i_u(v) = (u, v)$ and where p is the projection $p(u, v) = u$.

On applying the result in [4], Ch. VIII_6, Th. 13 we obtain that $A(Z)$ is isogenous to $A(X) \times B$ over the algebraic closure $\bar{k}$ of the field k; also B is isogenous over k to $A(X) \times B'$ (compare the proof of Lemma 7). Hence $A(Z)$ and $A(X) \times A(X) \times B'$ have the same dimension and our assertion is proved.

The image of $h_* \varrho^*$ in $A(X) \times A(X) \times B'$ is the abelian subvariety of $A(X) \times A(X) \times B'$ generated by

$$locus_k(h(u, v, u', v') + h(v, u, v', u') + h(u, v, v', u') + h(v, u, u', v'))$$

where (u, v, u', v') is a generic point of $Z \times Z$ over k, as we can see from the previous commutative diagram. On the other hand, by the definition of our rational map h we see that this locus is contained in $\Delta \times B'$, where Δ denotes the diagonal of $A(X) \times A(X)$. It is clear that Δ is an abelian subvariety of $A(X) \times A(X)$, isomorphic over k to $A(X)$, thus if $d : A(X) \longrightarrow A(X) \times A(X)$ is the diagonal map there exists a homomorphism φ over k such that the following diagram commutes

$$\begin{array}{ccc} A(Y) & \xrightarrow{\quad h_* \varrho^* \quad} & A(X) \times A(X) \times B' \\ \varphi \big\downarrow & \nearrow_{5d \times id} & \\ A(X) \times B' & & \end{array}$$

In particular φ has finite kernel because so has $h_* \varrho^*$.

On the other hand, Y is the variety which parametrizes the pairs of points (u, v) such that $v \in C_u$, without regard to the order of them. Hence Y parametrizes the pairs of incident lines L_u, L_v on V. Given any such pair, the plane determined by them cuts V residually in a third line, say L_w, defined over $k(u, v)$. The locus over $k(w)$ of the point $(u, v) + (v, u)$ of Y is simply the curve Γ_w. As before, we obtain a generically exact sequence of varieties

$$\Gamma_w \xrightarrow{i_w} Y \xrightarrow{p} X$$

where i_w is an inclusion map and p a projection, therefore from [4], Ch. VIII_6, Th. 13 we obtain that $A(Y)$ is isogenous to $A(X) \times A$ over the algebraic closure $\bar{k}$ of k.

To find an isogeny over k we proceed as in the proof of Lemma 7. Let $\varrho : Z \to Y$ be the Galois covering previously defined, and let $\varrho_u : C_u \to \Gamma_u$ be the Galois covering of degree two induced by the equivalence relation j_u on C_u. Let $(u, v; u', v')$ be a generic point of $Z \times Z$ over k, so that $(\varrho(u, v), \varrho(u', v'))$ is a generic point of $Y \times Y$ over k, and define a rational admissible map

$$h'' : Y \times Y \to A(X) \times A$$

by

$$h''(\varrho(u, v), \varrho(u', v')) = (f(w, w'); \sum_{i=1}^{5} \lambda_w g_w(\varrho(u, v), \varrho_w(P_i(w, w'))) -$$

$$- \lambda_{w'} g_{w'}(\varrho(u', v'), \varrho_{w'}(P_i(w, w')))$$

where w and w' are the two points of X uniquely determined by the pairs (u, v) and (u', v'), where $g_w : \Gamma_w \times \Gamma_w \to A(\Gamma_w)$ is a canonical admissible map and where $P_1(w, w') + \dots + P_5(w, w')$ is the 0-cycle on C_w and $C_{w'}$ given by the intersection.

By the same argument as in the proof of Lemma 7 we have that the rational map h'' is defined over k and that the pair $(Y \times Y, h'')$ generates $A(X) \times A$. Hence the induced homomorphism $h''_* : A(Y) \to A(X) \times A$ is defined over k and is surjective. Hence h''_* is an isogeny because we have already proved that $A(Y)$ and $A(X) \times A$ have the same dimension.

On combining this isogeny h''_* with the homomorphism φ previously found we get a homomorphism $\psi : A(X) \times A \to A(X) \times B'$ defined over k and with finite kernel.

Let A_1 and A_2 be the images of $A(X) \times 0$ and of $0 \times A$ by ψ. Then A_1 and A_2 are isogenous to $A(X)$ and A over k; also $A_1 \cap A_2$ is a finite group because ψ has finite kernel. Now by Poincaré's complete reducibility theorem ([4], Ch. III_1, Th. 6) there is an abelian subvariety B_1 of $A(X) \times B'$, defined over k and such that

(i) $A(X) \times B' = A_1 + A_2 + B_1$,

(ii) $(A_1 + A_2) \cap B_1$ is a finite group.

Hence $A(X) \times B'$ is isogenous over k to $(A_1 + A_2) \times B_1$, and so also to $A_1 \times A_2 \times B_1$ since $A_1 \cap A_2$ is finite. Since A_1 is isogenous to $A(X)$ over k, $A_2 \times B_1$ is isogenous to B' over k; hence there is a map $\tau : A_2 \rightarrow B'$ defined over k and with finite kernel. Let $\psi_2 : A \rightarrow A_2$ be the isogeny over k given by the restriction of ψ to the second factor; then Lemma 8 follows from the series of homomorphisms

$$A \xrightarrow{\psi_2} A_2 \xrightarrow{\tau} B'$$

defined over k and with finite kernel.

Proof of Theorem 2. Let $\alpha' : A \rightarrow B'$ be the homomorphism considered in Lemma 8, and let $i : B' \rightarrow B$ be the inclusion map. Let $\alpha = i\alpha'$.

We assert that the sequence of abelian varieties

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} A(X) \longrightarrow 0$$

is exact, up to isogenies.

In fact, α has finite kernel by Lemma 8 and β is surjective by Lemma 6. Again by Lemma 8 and the definition of α, we have $im(\alpha) \subseteq B'$, and B' was the connected component of $ker(\beta)$. The required exactness will follow if we show that $im(\alpha) = B'$, that is if we prove

$$dim_k B - dim_k A = dim_k A(X).$$

From the inclusion $im(\alpha) \subseteq B'$ we find

$$dim_k B - dim_k A \geq dim_k A(X). \tag{15}$$

The covering $\varrho_u : C_u \rightarrow \Gamma_u$ gives rise to a surjective homomorphism

$$(\varrho_u)_* : A(C_u) \rightarrow A(\Gamma_u),$$

and looking at the commutative diagram

$$\begin{array}{ccc} A(C_u) & \xrightarrow{(\varrho_u)_*} & A(\Gamma_u) \\ {\scriptstyle \mu_u}\downarrow & & \downarrow{\scriptstyle \lambda} \\ B & \xrightarrow{\gamma} & A \end{array}$$

where γ is the homomorphism induced by the universal mapping property of the $k(u)/k$-Image, we see that

$$\text{(16)} \qquad dim_k\, B - dim_k\, A \leq dim_{k(u)}\, A\,(C_u) - dim_{k(u)}\, A\,(\Gamma_u)$$

because the connected component of $ker\,(\gamma)$, together with the appropriate restriction of μ_u as canonical map, is the $k(u)/k$-Image of the connected component of $ker\,(\varrho_u)_*$.

By Lemma 2, (*b*) the covering ϱ_u is unramified at the simple points of Γ_u, and ramified with ramification index two at the singular points of Γ_u, which are ordinary double points but not cusps. Let C_u^* and Γ_u^* be $k(u)$-normalizations of C_u and let $\sigma_u : C_u^* \longrightarrow \Gamma_u^*$ be the induced covering. To a simple point of Γ_u there correspond one point of Γ_u^*, two distinct points of C_u and hence two points of C_u^*; for C_u^* is a double covering of Γ_u^*. If t is a singular point of Γ_u, there correspond to it two points t_1^*, t_2^* of Γ_u^* and only one point v of C_u. We wish to show that to v there correspond just two points v_1^*, v_2^* of C_u^*, that is the points t_1^*, t_2^* are branch points of the covering σ_u, with ramification index two.

In the notation of the proof of Lemma 2, we deal here with case (iv), the argument for case (v) being similar but less symmetric. By a change of variables on λ_2, λ_3 we may ensure that $4\,DF - E^2 = 4\,\lambda_2\lambda_3$ and then by a change of variables on x_4, x_5 we can also have $Dx_4^2 + E\,x_4\,x_5 + F\,x_5^2 = \lambda_2\,x_4^2 + \lambda_3\,x_5^2$. Now λ_2/λ_1 is a uniformizing parameter for one of the branches of Γ_u through t, and λ_3/λ_1 for the other. The function field of C_u is that quadratic extension of the function field of Γ_u which is needed to factorize the quadratic equation for the degenerate Q. Looking at the terms of (5) independent of x_5, we see that λ_2/λ_1 cannot be a uniformizing parameter at a point of C_u^* above v. Hence these points are branch points of the covering, and so there are just two points v_1^*, v_2^* of C_u^* above v.

Let d denote the number of double points of Γ_u; we have proved that the double covering $\sigma_u : C_u^* \longrightarrow \Gamma_u^*$ is ramified at exactly $2d$ points, where the ramification index is two. If g_1, g_2 are the genus of Γ_u and C_u respectively, on applying the Hurwitz-Zeuthen formula to the covering σ_u (note that $char(k(u)) \neq 2)$ we get

$$2g_2 - 2 = 2(2g_1 - 2) + 2d.$$

On the other hand, by Lemma 2, (a), Γ_u is a plane curve of degree 5 with d ordinary double points. Hence

$$g_1 = 6 - d$$

and combining this result with the previous equation we obtain $g_2 - g_1 = 5$, i.e.

$$dim_{k(u)} A(C_u) - dim_{k(u)} A(\Gamma_u) = 5. \tag{17}$$

On combining equation (17) with inequalities (15) and (16) we have

$$dim_k A(X) \leq dim_k B - dim_k A \leq 5.$$

By Lemma 5 and this last inequality, Theorem 2 is proved.

The following result will be needed in the proof of Theorem 1.

LEMMA 9. *Let u be a point of X such that C_u is absolutely irreducible. Then there are homomorphisms*

$$\alpha_u : A(\Gamma_u) \longrightarrow A(C_u), \quad i_* : A(C_u) \longrightarrow A(X)$$

defined over $k(u)$, such that the sequence of abelian varieties

$$0 \longrightarrow A(\Gamma_u) \xrightarrow{\alpha_u} A(C_u) \xrightarrow{i_*} A(X) \longrightarrow 0$$

is exact, up to isogenies.

The homomorphism i_* is the homomorphism induced by the inclusion map $i_u : C_u \longrightarrow X$ and the universal mapping property of the Albanese variety.

PROOF. Suppose first that u is a generic point of X over k, and let B_u and A_u denote the kernels of the homomorphisms $\mu_u : A(C_u) \longrightarrow B$ and $\lambda_u : A(\Gamma_u) \longrightarrow A$. Then B_u and A_u are abelian varieties defined over $k(u)$ by [4], Ch. VIII_1, prop. 3 and Chow's theorem. By Poincaré's complete reducibility theorem there are isogenies over $k(u)$

$$\varphi_u : A(C_u) \longrightarrow B \times B_u,$$

$$\psi_u : A(\Gamma_u) \longrightarrow A \times A_u.$$

Also, from the covering $\varrho_u : C_u \longrightarrow \Gamma_u$ we deduce by taking inverse images a homomorphism

$$\varrho_u^* : A(\Gamma_u) \longrightarrow A(C_u)$$

defined over $k(u)$ and with finite kernel (see the analogous construction of ϱ^*, at the beginning of the proof of Lemma 8).

We have also isogenies over $k(u)$

$$\varphi'_u : B \times B_u \longrightarrow A(C_u)$$

$$\psi'_u : A \times A_u \longrightarrow A(\Gamma_u),$$

thus we obtain a homomorphism with finite kernel

$$\varphi_u\, \varrho_u^*\, \psi'_u : A \times A_u \longrightarrow B \times B_u .$$

By [4], Ch. II_1, Th. 3 this homomorphism factorizes as a sum $\tau_u + \tau'_u$, where $\tau_u : A \longrightarrow B \times B_u$ and $\tau'_u : A_u \longrightarrow B \times B_u$ are the restrictions of $\varphi_u \varrho_u^* \psi'_u$ to the first and the second factors of $A \times A_u$. Let $p_1 : B \times B_u \longrightarrow B$ and $p_2 : B \times B_u \longrightarrow B_u$ be the projections. We obtain homomorphisms

$$p_1\, \tau'_u : A_u \longrightarrow B,$$

$$p_2\, \tau'_u : A_u \longrightarrow B_u$$

defined over $k(u)$, and $\tau'_u = (p_1\, \tau'_u , p_2\, \tau'_u)$ has finite kernel.

On the other hand, the $k(u)/k$-Image of A_u is a point, whence $p_1\, \tau'_u$ is the constant map of A_u into the origin of B. It follows at once that the homomorphism $p_2\, \tau'_u$ has finite kernel.

Let α be the homomorphism considered in Theorem 2 and let

$$\theta_u : A \times A_u \longrightarrow B \times B_u$$

be the product mapping $\theta_u = \alpha \times p_2\, \tau'_u$. Then θ_u is defined over $k(u)$ and has finite kernel. Finally define α_u by

$$\alpha_u = \varphi'_u\, \theta_u\, \psi_u .$$

We assert that the sequence

$$0 \longrightarrow A(\Gamma_u) \xrightarrow{\ \alpha_u\ } A(C_u) \xrightarrow{\ i_*\ } A(X) \longrightarrow 0$$

is exact, up to isogenies.

We have proved that α_u has finite kernel, that i_* is surjective, also we know that

$$dim_k A(X) = dim_{k(u)} A(C_u) - dim_{k(u)} A(\Gamma_u),$$

by equation (17) and Lemma 5.

Consider the commutative diagram

$$\begin{array}{ccccc} B \times B_u & \xrightarrow{\varphi'_u} & A(C_u) & \xrightarrow{i_*} & A(X) \\ & & \Big\downarrow{\scriptstyle \mu_u} & & \Big\| \\ & & B & \xrightarrow{\beta} & A(X) \end{array}$$

The homomorphism φ'_u factorizes as a sum $\gamma_u + \gamma'_u$, where $\gamma_u : B \longrightarrow A(C_u)$ and $\gamma'_u : B_u \longrightarrow A(C_u)$ are two homomorphisms defined over $k(u)$ and with finite kernel. We have $ker(i_*) = \mu_u^{-1}(ker(\beta))$, also $ker(\mu_u) = B_u$ and it follows that the connected component of $ker(i_* \varphi'_u)$ is $B' \times B_u$ because the homomorphism $\mu_u \gamma_u$ is a multiple of the identity on B and B' is the connected component of $ker(\beta)$.

On the other hand, the image of $\theta_u \psi_u : A(\Gamma_u) \longrightarrow B \times B_u$ is contained in $Image(\alpha) \times B_u$, hence in $B' \times B_u$, hence in the kernel of $i_* \varphi'_u$. This proves that $i_* \alpha_u$ is the constant map of $A(\Gamma_u)$ into the origin of $A(X)$, thus in view of the previous checking of dimensions in our sequence of abelian varieties we obtain the required exactness up to isogenies.

The proof of Lemma 9 finally is completed by an obvious specialization argument, because equation (17) has been proved not only when (u) is a generic point of X over k, but also when u is any point of X such that C_u is absolutely irreducible. This remark completes the proof of Lemma 9.

In the next section we shall show how a combination of arithmetical methods with the result of Lemma 9 will give the proof of our Theorem 1.

IV. Proof of Theorem 1. Let V/k be our cubic threefold, defined over a finite field $k = \mathbf{F}_q$ of $char(k) \neq 2$.

We may give V the structure of a fibre space over a projective plane in the following way. Let L_u be a line of V, defined over $k(u)$; the planes through L_u are parametrized by a projective plane $\mathbf{P}^2_u$ defined over $k(u)$, and a point w of $\mathbf{P}^2_u$ determines uniquely a conic Q_w on V which is the residual section of V with the plane through L_u determined by w. Also, by the obvious fact that a line and a point not on this line determine uniquely a plane, we have a projection $p_u : V \longrightarrow \mathbf{P}^2_u$, defined over $k(u)$ and regular on V except at L_u, by taking for a point x of V not on L_u, $p_u(x)$ to be the point of $\mathbf{P}^2_u$ corresponding to the plane determined by L_u and x.

Thus we have obtained a generically exact sequence of varieties

$$Q_w \xrightarrow{i_w} V \xrightarrow{p_u} \mathbf{P}^2_u$$

where i_w is the inclusion map and p_u the previously defined projection.

We already know that the generic curve C_u is absolutely irreducible, also X is absolutely irreducible. The points u on X such that C_u is absolutely irreducible then form a Zariski open set on X, whence from the well-known fact that the points u of X algebraic over k are dense in X we deduce that as soon as n is sufficiently large there is a point u of X such that $k(u) \subseteq k_n$ and C_u is absolutely irreducible. Here we have written k_n for $\mathbf{F}_{q^n}$.

We want to count the number $\nu_n(V)$ of points of V defined over $k_n = \mathbf{F}_{q^n}$, and in view of the exact sequence previously written our problem is equivalent to counting the number of points defined over k_n of the base space $\mathbf{P}^2_u$ and of the fibres Q_w, together with a more accurate analysis of the behaviour of the projection p_u along the line L_u. Here of course we take u to be such that $k(u) \subseteq k_n$, which is possible if n is sufficiently large.

The total number of points of $\mathbf{P}^2_n$ defined over k_n is exactly $q^{2n} + q^n + 1$ and it is the total number of fibres Q_w defined over k_n.

Let Q_w be one of the fibres defined over k_n. The number of points of Q_w defined over k_n is

(i) $q^n + 1$, if the conic Q_w is non-singular;

(ii) $2q^n + 1$, if the conic Q_w degenerates into two distinct lines, each defined over k_n;

(iii) 1, if the conic Q_w degenerates into two distinct lines, each defined over a quadratic extension of k_n;

(iv) $q^n + 1$, if the conic Q_w degenerates into a double line.

The behaviour of the degenerate fibres is most conveniently described by means of the two curves C_u and Γ_u defined earlier. In fact the curve C_u parametrizes the components of the degenerate fibres, while the curve Γ_u parametrizes the degenerate fibres themselves. It follows that if $\nu_n(C_u)$ and $\nu_n(\Gamma_u)$ denote the number of points defined over k_n and lying on C_u and Γ_u respectively, then the total number of points defined over k_n and lying on the degenerate conics Q_w is given by

$$q^n \nu_n(C_u) + \nu_n(\Gamma_u),$$

for every point u such that $k(u) \subseteq k_n$. Hence the total number of points defined over k_n and lying on the fibres of our fibre system is

$$(q^{2n} + q^n + 1 - \nu_n(\Gamma_n))(q^n + 1) + q^n \nu_n(C_n) + \nu_n(\Gamma_u),$$

for every point u such that $k(u) \subseteq k_n$, the first term being the contribution of the non-degenerate fibres and the remaing ones being the contribution of the degenerate conics.

This is not yet the number of points of V defined over k_n, because the projection p_u is not a regular map at L_u. Take a point x of L_u; a conic Q_w contains x if and only if the plane corresponding to w is tangent at V in the point x, and these planes are parametrized by a line of $\mathbf{P}^2_u$ defined over $k(u)(x) \subseteq k_n(x)$. It follows that in the previous counting of the points of the fibres of our fibre system the $q^n + 1$ points of L_u are counted $q^n + 1$ times each. Hence the total number of points of V defined over k_n is the total number of points defined over k^n of the fibres of our fibre system, minus $q^n(q + 1)$. We obtain

$$(18) \qquad \nu_n(V) = (q^{4n} - 1)/(q^n - 1) + q^n(\nu_n(C_n) - \nu_n(\Gamma_u))$$

for every point u such that $k(u) \subseteq k_n$.

Now suppose that C_u is absolutely irreducible, and let C_u^* and Γ_u^* be $k(u)$-normalizations of C_u and Γ_u. We assert that

$$(19) \qquad \nu_n(C_u) - \nu_n(\Gamma_u) = \nu_n(C_u^*) - \nu_n(\Gamma_u^*).$$

Let t be a simple point of Γ_u; we have proved in section III that to this point there correspond one point t^* of Γ_u^*, two distinct points v_1, v_2 of C_u, two distinct points v_1^*, v_2^*, of C_u^*, therefore t and t^* have the same field of rationality and the same is true for v_1, v_1^* and for v_2, v_2^*. Now let t be a singular point of Γ_u; we have proved in section II that t is an ordinary double point of Γ_u, not a cusp. To this point there correspond two points t_1^*, t_2^* of Γ_u^*, one point v of C_u, two points v_1^*, v_2^* of C_u^*, therefore t and v have the same field of rationality, and the same is true for t_1^*, v_1^* and t_2^*, v_2^*. It is easily seen that equation (19) follows from this.

By equations (18) and (19) and the fact C_u^* and Γ_u^* are non-singular we obtain using Weil's well-known results

$$(20) \qquad \nu_n(V) = (q^{4n} - 1)/(q^n - 1) - q^n(Tr(\pi^n; A(C_u)) - Tr(\pi^n; A(\Gamma_u)))$$

where $Tr(\pi^n; A)$ means the trace of the n-th power of the Frobenius endomorphism $\pi(x) = x^q$, in the endomorphism algebra over the rationals Q of the abelian variety A defined over k_n. Note we have proved equation (20) for every point u such that C_u is absolutely irreducible and $k(u) \subseteq k_n$.

Now from the work of Lang [5] we know that two abelian varieties defined over a finite field k and isogenous over k, have the same number of points defined over k, whence the traces of the Frobenius endomorphism in the corresponding endomorphism algebras are equal. By this remark and the exact sequence (up to isogenies) of Lemma 9 we find

$$Tr\,(\pi^n\,;\,A\,(X)) = Tr\,(\pi^n\,;\,A\,(C_u)) - Tr\,(\pi^n\,;\,A\,(\Gamma_u))$$

for every point u such that C_u is absolutely irreducible and $k\,(u) \subseteq k_n$. On combining this result with equation (20) we obtain Theorem 1 for all sufficiently large n, i. e. for every n such that there exists u such that C_u is absolutely irreducible and $k\,(u) \subseteq k_n$.

To prove Theorem 1 for all n, we use a simple argument like the one used by Davenport and Lewis [1]. Let η_i, $i = 1, 2, \ldots, 10$ be the 10 characteristic roots of the Frobenius endomorphism π in the endomorphism algebra of $A\,(X)$. We have for all n the equation

$$Tr\,(\pi^n\,;\,A\,(X)) = \eta_1^n + \ldots + \eta_{10}^n,$$

also by equation (1) (the theorem of Dwork) we find

$$\nu_n\,(V) = (q^{4n} - 1)/(q^n - 1) - \omega_1^n - \ldots - \omega_{10}^n =$$

$$= (q^{4n} - 1)/(q^n - 1) - (q\eta_1)^n - \ldots - (q\eta_{10})^n$$

for all sufficiently large n. Hence the equation between the middle and the last term is an identity true for all n, and by equation (1) we know that the equation between the first and the middle term is true for all n; this completes the proof of our Theorem 1.

If we had not proved that $dim_k\,A\,(X) = dim_{k(u)}\,A\,(C_u) - dim_{k(u)}\,A\,(\Gamma_u)$, but only the trivial $\leq$, then at this stage of the proof we would find that all the $q\eta_i$ were among the ω_i, but we would not have shown that every ω_i was a $q\eta_i$. We expect that in more general circumstances this will actually happen.

V. We end this paper with a few comments about our result. There is no doubt that the case of a cubic threefold, as was pointed out by Weil at the very end of the second edition of his Foundations, is a good special case to investigate for making conjectures about the equivalence problem for cycles of codimension ≥ 2, as well as its relation to the abelian varieties attached to the cohomology.

Our Theorem 1 is a very special case of a Lefschetz fixed point formula over the rationals, and such a formula is still lacking except for curves, abelian varieties, rational surfaces, varieties with « only the algebraic cohomology », and varieties obtained by products of varieties of the previous types. We believe that our result for the cubic threefold is of a new type and in fact closely related with Weil's theory [6] of higher jacobians.

To explain this, let $Z^1_{alg}(V)$ be the group of 1-cycles of V with integral coefficients algebraically equivalent to 0, and let $Z^1_{rat}(V)$ be the subgroup of $Z^1_{alg}(V)$ consisting of cycles rationally equivalent to 0. It seems that (though at this moment we have no complete proof) if x is any element of $Z^1_{alg}(V)$, then $x + Z^1_{rat}(V)$ contains an element which is a sum of lines of V taken with appropriate multiplicity. If this result is true, then there is a homomorphism

$$Z^1_{alg}(V) \longrightarrow A(X)$$

which is surjective and whose kernel is $Z^1_{rat}(V)$. Thus $A(X)$ appears to be the higher jacobian of V in dimension 1.

Let $f\colon V \longrightarrow V$ be a morphism defined over k. Then, accordingly to the previous discussion f would induce a (contravariant) homomorphism

$$f^0 : A(X) \longrightarrow A(X)$$

and we may expect that for every $l \neq p$ the trace

$Tr(f^* \mid H^3(V; Q_l))$ would be related in some way to the homomorphism f^0. The factor q^n in formula (3) may suggest that

$$Tr(f^* \mid H^3(V; Q_l)) = Tr(f^0 f^{0\prime} f^0) \tag{21}$$

where $f^{0\prime}$ is the endomorphism of $A(X)$ given by the involution in the endomorphism algebra of $A(X)$ determined by a polarization of $A(X)$. Of course the trace in the right hand side of (21) is taken in the endomorphism algebra of $A(X)$ tensorized with Q_l.

More generally, we are led to the following conjectures.

Let V be a non-singular projective variety of dimension n, defined over a field k, and let $Z^i_{alg}(V)$ be the group of i-cycles of V with integral coefficients and algebraically equivalent to 0. Accordingly to Weil's conjectures, we may expect that there are abelian varieties $J_i(V), J^i(V)$ defined over k, for $2i \leq n-1$, «functorially» attached to V and with the following properties.

A) $J_i(V)$ parametrizes the group $Z^i_{alg}(V)$ modulo an appropriate equivalence relation; these would be the Albanese varieties of V.

B) $J^i(V)$ parametrizes the group $Z^{n-1-i}_{alg}(V)$ modulo an appropriate equivalence relation; these would be the Picard varieties of V.

C) the varieties $J_i(V)$ and $J^i(V)$ are dual of each other.

D) if U and V are non-singular projective varieties of dimension m n respectively and defined over k, and if $f: U \to V$ is a morphism defined over k, then for $2i + 1 \leq min(m, n)$ we have an induced homomorphism

$$f_* : J_i(U) \to J_i(V)$$

which makes the following diagram commutative

$$\begin{array}{ccc} Z^i_{alg}(U) & \xrightarrow{f_i} & Z^i_{alg}(V) \\ \downarrow & & \downarrow \\ J_i(U) & \xrightarrow{f_*} & J_i(V) \end{array}$$

where f_i is the homomorphism obtained by direct images.

E) if U, V and f are as in D), then for $2i + 1 \leq min(m, n)$ we have, an induced homomorphism, dual to f_* in D),

$$f^* : J^i(V) \to J^i(U)$$

which makes the following diagram commutative

$$\begin{array}{ccc} Z^{n-1-i}_{alg}(V) & \xrightarrow{f^i} & Z^{m-1-i}_{alg}(U) \\ \downarrow & & \downarrow \\ J^i(V) & \xrightarrow{f^*} & J^i(U) \end{array}$$

where f^i is the homomorphism obtained by inverse images.

F) if V_u is a generic hyperplane section of V, defined over $k(u)$, and if $j: V_u \to V$ is the inclusion map then $(J^i(V), j_*)$ and $(J^i(V), j^*)$ are respectively the $k(u)/k$-Images and the $k(u)/k$-Traces of $J_i(V_u)$ and $J^i(V_u)$, for $2i + 1 \leq n - 1$.

Let V be as before, let $k = \mathbf{F}_q$ be the finite field of q elements and let

$$Z(t, V) = \prod_{i=0}^{2n} P_i(t, V)^{(-1)^{i-1}}$$

be the Zeta function of V, where $P_i(t, V)$ is the characteristic polynomial of the Frobenius endomorphism on $H^i(V, Q_l)$.

Then our conjecture can be expressed by saying that

« $P_1(q^i t, J_i(V)) = P_1(q^i t, J^i(V))$ *is a factor of* $P_{2i+1}(t, V)$ ».

Our Theorem 1 can be expressed in the following form.

« *If* V *is a non-singular cubic threefold defined over* $\mathbf{F}_q$, *and char* $(\mathbf{F}_q) \neq 2$ *then*

$$P_3(t, V) = P_1(qt, A(X)). \text{»}$$

We conclude with the remark that our result does not seem to be an isolated one, and that one of the authors has worked out other cases, for example the complete intersection of two quadrics, with almost the same result. In this latter case something more can be said and the results obtained confirm our conjectures about higher jacobians.

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