ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

J. DUGUNDJI

Comparison of homologies

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 20, nº 4 (1966), p. 745-751

<http://www.numdam.org/item?id=ASNSP_1966_3_20_4_745_0>

COMPARISON OF HOMOLOGIES

by J. DUGUNDJI [1]

To Reinhold Baer for his 65^{th} birthday

Our purpose in this paper is to present a simple principle (3.1) for the comparison of homology theories. As applications, we will obtain elementary and particularly short (in fact, without computation) proofs of (1) the various invariance properties of the simplicial homology groups of a polytope, of (2) Serre's « Vietoris mod $\mathcal{C}$ » theorem [3 ; 270] when the base is a finite polytope, and of (3) a special case of Leray's theorem [3 ; 213] on nerves of coverings.

Even though the principle itself (and its proof) is entirely elementary, the principle has apparently not been explicitly formulated or used before ; it is vaguely related to the acyclic model theorem [3 ;29] but it is generally easier to use and, as the above-mentioned applications will show, its use can give some deep results rather trivially. Since there is no requirement that the complexes be bounded below, or that the dimension axiom (or even the homotopy axiom) be fulfilled, it can be also applied to extraordinary homology theories.

1. Structures.

1.1 DEFINITION. Let X be a topological space. A structure for X is a lattice [2] $\mathcal{K} \subset \mathcal{P}(X)$ containing X and $\varnothing$, which satisfies the descending chain condition [3].

Pervenuto alla Redazione il 18 Marzo 1966 ed in forma definitiva il 13 Luglio 1966.

(1) This research was partially supported by the Universität Frankfurt and by the Centro Ricerche Fisica e Matematica.

(2) For all lattices of sets in this paper, the lattice operations are understood to be union and intersection.

(3) Explicitly, we require of $\mathcal{K} = \{A\}$ that, if $A, B \in \mathcal{K}$, then $A \cup B$, $A \cap B$ also belong to $\mathcal{K}$, and that every descending sequence $A_1 \supset A_2 \supset \dots \supset A_n \supset \dots$ is finite.

An element $S \in \mathcal{K}$ is called indecomposable if it is not expressible in the form $S = A \cup B$, where $A, B \in \mathcal{K}$ and $A \neq S$, $B \neq S$.

It is clear that

1.2 If P is a finite polytope, then the set of all its subpolytopes constitutes a structure $\mathcal{K}$ for P, and the indecomposables are precisely the closed simplexes $\bar{\sigma}$.

Further examples of structures can be found by using the trivial

1.3 Let $\mathcal{K} = \{X_\alpha \mid \alpha \in \mathcal{R}\}$ be a structure for X and let $p : E \longrightarrow X$ be any surjective map. Then $p^{-1}(\mathcal{K}) = \{p^{-1}(X_\alpha) \mid \alpha \in \mathcal{R}\}$ is a structure for E (called the structure in E induced by p) and the indecomposables of $p^{-1}(\mathcal{K})$ are precisely the sets $p^{-1}(S)$ where S is an indecomposable of $\mathcal{K}$.

2. Homology theories on structure categories.

2.1 DEFINITION. If $\mathcal{K}$ is a structure for X, the structure category $C(\mathcal{K})$ is that in which (*a*) the objects are all pairs (A, B) such that $A, B \in \mathcal{K}$, $A \supset B$, and (*b*) the set of morphisms $(A, B) \longrightarrow (C, D)$ consists of the inclusion map if $A \subset C$, $B \subset D$, and is empty otherwise.

As is customary, we abbreviate $(A, \emptyset)$ by A.

2.2 DEFINITION. By a homology theory on a structure category $C(\mathcal{K})$ is meant a sequence $h = \{h_q \mid q \in Z\}$ of covariant functors $C(\mathcal{K}) \longrightarrow \mathcal{G}$ ($\mathcal{G}$ an) abelian category) together with natural trasformations $\partial_q : h_q(A, B) \longrightarrow h_{q-1}(B)$ one for each q and (A, B), such that

(*a*). For each $(A, B) \in C(\mathcal{K})$, the sequence

$$\ldots \longrightarrow h_q(B) \xrightarrow{i_q} h_q(A) \xrightarrow{j_q} h_q(A, B) \xrightarrow{\partial_q} h_{q-1}(B) \longrightarrow \ldots$$

is exact (i_q, j_q are the morphisms corresponding to the inclusion maps i, j),

(*b*). For each $A, B \in \mathcal{K}$, the inclusion map $i : (A, A \cap B) \longrightarrow (A \cup B, B)$ corresponds to an isomorphism (« excision isomorphism »)

$$i_q : h_q(A, A \cap B) \approx h_q(A \cup B, B)$$

for every $q \in Z$.

Observe that the morphisms in the category $C(\mathcal{K})$ are simply the inclusion maps, so that no homotopy invariance is required of h.

By the standard dualizing process [1 ; 15] the notion of a « cohomology theory on $C(\mathcal{K})$ » is clear.

3. The comparison principle.

A subcategory $\mathcal{S}$ of a category $\mathcal{X}$ is called *full* whenever (1) $Y \in \mathcal{S}$ and $Z \approx Y$ implies $Z \in \mathcal{S}$ (2) $Y, Z \in \mathcal{S}$ implies $\mathrm{Mor}_{\mathcal{S}}(Y, Z) = \mathrm{Mor}_{\mathcal{X}}(Y, Z)$ and (3) the composition law in $\mathcal{S}$ is that induced by the one in $\mathcal{X}$. A *Serre subcategory* $\mathcal{R}$ of an abelian category $\mathcal{G}$ is a full subcategory having the property: whenever $0 \to G_0 \to G_1 \to G_2 \to 0$ is exact in $\mathcal{G}$, then $G_1 \in \mathcal{R}$ if and only if both G_0 and G_2 belong to $\mathcal{R}$. If $\mathcal{R}$ is a Serre subcategory of $\mathcal{G}$, a morphism u in $\mathcal{G}$ is called an $\mathcal{R}$-isomorphism whenever both ker u and coker u belong [4] to $\mathcal{R}$. Using this terminology, we now state the comparison principle:

3.1 THEOREM. Let $C(\mathcal{K})$ be a structure category in a space X, let $h: A \to h(A), \widehat{h}: A \to \widehat{h}(A)$ be two homology theories on $C(\mathcal{K})$ with values in the abelian category $\mathcal{G}$, and let $\mathcal{R}$ be any Serre subcategory of $\mathcal{G}$. Assume that there is a natural transformation $t: h \to \widehat{h}$ and that $t(S): h(S) \to \widehat{h}(S)$ is an $\mathcal{R}$-isomorphism for each indecomposable $S \in \mathcal{K}$. Then $t(A,B): h(A,B) \to \widehat{h}(A,B)$ is an $\mathcal{R}$-isomorphism for every $(A,B) \in C(\mathcal{K})$ and, in particular, $t(X): h(X) \to \widehat{h}(X)$ is an $\mathcal{R}$-isomorphism.

PROOF. We first show that $t(A)$ is an $\mathcal{R}$-isomorphism for every $A \in \mathcal{K}$. The proof is by contradiction, so we assume that there is some $B \in \mathcal{K}$ such that $t(B)$ is not an $\mathcal{R}$-isomorphism.

Then there exists a $C \in \mathcal{K}$ such that $t(C)$ is not an $\mathcal{R}$-isomorphism, but $t(D)$ is an $\mathcal{R}$-isomorphism for every proper subset $D \in \mathcal{K}$ of C. Indeed, starting with B, if there is some proper subset $B_1 \subset B$ such that $t(B_1)$ is not an $\mathcal{R}$-isomorphism, replace B by B_1 and repeat the search; this gives a descending chain $B \supset B_1 \supset B_2 \supset \dots \supset B_n$ which, by 1.1, is finite and so $C = B_n$ is the required set.

By the hypothesis, C is not an indecomposable, so $C = D \cup E$, where $D, E \in \mathcal{K}$ are proper subsets of C and, consequently, $D \cap E \in \mathcal{K}$ is also a proper subset of C. Because $t(D)$ and $t(D \cap E)$ are $\mathcal{R}$-isomorphisms, the commutative diagram of exact sequences

$$\begin{array}{ccccccccccc}
\dots \to & h_q(D \cap E) & \to & h_q(D) & \to & h_q(D, D \cap E) & \to & h_{q-1}(D \cap E) & \to & h_{q-1}(D) & \to \dots \\
& \downarrow t(D \cap E) & & \downarrow t(D) & & \downarrow t(D, D \cap E) & & \downarrow t(D \cap E) & & \downarrow t(D) & \\
\dots \to & \widehat{h}_q(D \cap E) & \to & \widehat{h}_q(D) & \to & \widehat{h}_q(D, D \cap E) & \to & \widehat{h}_{q-1}(D \cap E) & \to & \widehat{h}_{q-1}(D) & \to \dots
\end{array}$$

---

[4] Alternatively stated: If $T: \mathcal{G} \to \mathcal{G}/\mathcal{R}$ is the canonical functor from $\mathcal{G}$ to the quotient category $\mathcal{G}/\mathcal{R}$, a morphism u in $\mathcal{G}$ is an $\mathcal{R}$-isomorphism if, and only if, Tu is an isomorphism in $\mathcal{G}/\mathcal{R}$.

and the 5-Lemma of $\mathcal{C}$-theory (a proof of which is given in [4 ; 263]) shows that $t(D, D \cap E)$ is an $\mathcal{R}$-isomorphism. Because of excision, we find that $t(D \cup E, E)$ is also an $\mathcal{R}$-isomorphism. Writing down the analogous diagram as above for the $(D \cup E, E)$ exact sequences, recalling that $t(E)$ is an $\mathcal{R}$-isomorphism, and using the 5-Lemma of $\mathcal{C}$-theory once again, we find that $t(D \cup E)$ is an $\mathcal{R}$-isomorphism. Since $C = D \cup E$, this is the required contradiction.

Thus, $t(A)$ is an $\mathcal{R}$-isomorphism for every $A \in \mathcal{K}$ and, applying the 5-Lemma of $\mathcal{C}$-theory once again, this time to the diagram of (A, B) exact sequences, it follows that $t(A, B)$ is an $\mathcal{R}$-isomorphism for every $(A, B) \in C(\mathcal{K})$. This completes the proof.

In view of 1.2, we obtain the immediate

3.2 COROLLARY. Let P be a finite polytope and let $C(\mathcal{K})$ be the structure category of its subpolytopes. Let $h, \widehat{h}$ be two homology theories on $C(\mathcal{K})$ and assume that there is a natural trasnformation $t: h \longrightarrow \widehat{h}$. Then, if $t(\bar{\sigma})$ is an $\mathcal{R}$-isomorphism for every closed simplex $\bar{\sigma}$, $t(A, B)$ is an $\mathcal{R}$-isomorphism for every $(A, B) \in C(\mathcal{K})$.

3.3 REMARK. Because the proof of 3.1 is functorial, it is clear that 3.1 and 3.2 are also valid whenever $h, \widehat{h}$ are cohomology theories on $C(\mathcal{K})$.

4. Applications.

(*a*). Topological invariance of simplicial homology on a finite polytope.

Let $C(\mathcal{K})$ be the structure category of the subpolytopes of P, let $h(K, L)$ represent the simplicial homology of $(K, L) \in C(\mathcal{K})$ and let $\widehat{h}(K, L)$ represent its singular homology. It is well known that h has the strong excision property; and so also does $\widehat{h}$, since subpolytopes are strong neighborhood deformation retracts [1; 72,31]. There is [1; 200] a natural trasformation $\beta : h \longrightarrow \widehat{h}$ induced by associating with each oriented $\bar{\sigma}$ the singular simplex $T: \Delta_n \longrightarrow P$ that maps Δ_n affinely on $\bar{\sigma}$.

4.1 THEOREM. $\beta : h(K, L) \approx \widehat{h}(K, L)$ for every $(K, L) \in C(\mathcal{K})$.

PROOF. We first note that $\beta : h(\bar{\sigma}) \approx \widehat{h}(\bar{\sigma})$ for every closed simplex $\bar{\sigma}$ since (using augmented groups to cut down calculations) both sides are zero. By 3.2, the proof is complete.

In the same way, one can show that the homology of ordered chains on P is isomorphic to the homology of oriented chains on P.

(*b*). Invariance of simplicial homology under barycentric subdivision.

Structure the complex P by its subcomplexes, and for each $(K, L) \in C(\mathcal{K})$ let $h(K, L)$ denote the simplicial homology of (K, L). Let P' be the barycentric subdivision of P and for each $(K, L) \in C(\mathcal{K})$, let $\widehat{h}(K, L)$ denote the simplicial homology of the pair (K', L'). There is a well-known [1; 177] natural transformation $Sd : h \longrightarrow \widehat{h}$.

4.2 **Theorem.** $Sd : h \approx \widehat{h}$.

Proof. We note that $Sd : h(\bar{\sigma}) \approx \widehat{h}(\bar{\sigma})$ for each simplex $\bar{\sigma}$ since (using augmented groups) we have $h(\bar{\sigma}) = 0$ because $\bar{\sigma}$ is a cone over a face and $\widehat{h}(\bar{\sigma}) = 0$ since $(\bar{\sigma})'$ is a cone over its boundary. By 3.2, the proof is complete.

(*c*). Block decomposition.

Let P be an oriented complex. A block is an integral chain $b = \sigma_1 + + \ldots + \sigma_n$, where each σ_i is positively oriented and the σ_i need not have same dimension; the support of b is the set $s(b) = |\bar{\sigma}_1| \cup |\bar{\sigma}_2| \cup \ldots \cup |\bar{\sigma}_n|$. Call a collection $\{b\}$ of blocks a block-decomposition of P if (1) $P = \bigcup s(b)$, (2) each σ belongs to a unique b, and (3) ∂b is a block-chain for each b.

Now let $\{b\}$ be a block-decomposition of P, and structure P by block-subcomplexes (i. e., unions of sets $s(b)$). On $C(\mathcal{K})$ we place two homology theories: h is the block-chain homology, and $\widehat{h}$ is the simplicial homology. Since there is a natural trasformation $t : h \longrightarrow \widehat{h}$ induced by setting $t(b) = = t(\sigma_1 + \ldots + \sigma_n) = \sigma_1 + \ldots + \sigma_n$, and since the sets $s(b)$ are the indecomposables, we find from 3.1 that

4.3 **Theorem.** If $t : h[s(b)] \approx \widehat{h}[s(b)]$ for each block b, then $t : h(P) \approx \widehat{h}(P)$.

(*d*). Serre's « Vietoris mod $\mathcal{C}$ » theorem.

4.4 **Theorem.** Let (E, p, B) be a Serre fibration, with B a finite connected polytope and each fiber $F = p^{-1}(b)$ path-connected.

Let H denote augmented singular homology, and let $\mathcal{R}$ be any Serre class. Then if $H_i(F) \in \mathcal{R}$ for all $i > 0$, the projection $p_+ : H_i(E) \longrightarrow H_i(B)$ is an $\mathcal{R}$-isomorphism for all $i \geq 0$.

Proof. Structure E by $p^{-1}(\mathcal{K})$, where $\mathcal{K}$ is the structure of all subpolytopes of B (see 1.3). On the structure category $C(p^{-1}(\mathcal{K}))$, define two homology theories by setting

$$h(p^{-1}(K), p^{-1}(L)) = H(p^{-1}(K), p^{-1}(L))$$

$$\widehat{h}(p^{-1}(K), p^{-1}(L)) = H(K, L)$$

and observe that the projection p induces a natural transformation $\widetilde{p} : h \longrightarrow \widehat{h}$. According to 1.3, the indecomposables of $C(p^{-1}(\mathcal{K}))$ are the sets $p^{-1}(\overline{\sigma})$. For $i = 0$, we have $h_0(p^{-1}(\overline{\sigma})) = \widehat{h}_0(p^{-1}(\overline{\sigma})) = 0$ because augmented groups are used; and for $i > 0$, we have that $\widetilde{p} \mid p^{-1}(\overline{\sigma}) : H_i(p^{-1}(\overline{\sigma})) \longrightarrow H_i(\overline{\sigma}) = 0$ is an $\mathcal{R}$-isomorphism because, in Serre fibrations, $H_i(p^{-1}(\overline{\sigma})) \approx H_i(p^{-1}(b))$ for any vertex $b \in \overline{\sigma}$. Thus, $\widetilde{p} \mid p^{-1}(\overline{\sigma})$ is an $\mathcal{R}$-isomorphism for each $p^{-1}(\overline{\sigma})$ so, by 3.1, the proof is complete.

Observe that 4.4 is actually an extension of Serre's proposition 6B, in that we do not require that the base B be simply-connected; however, we do require that B be a finite polytope. Note also that the proof of 4.4 applies whenever p is merely an h-fibration, i. e. for each $\overline{\sigma} \in B$ and each vertex $b \in \overline{\sigma}$, the inclusion $h[p^{-1}(b)] \longrightarrow h[p^{-1}(\overline{\sigma})]$ is an isomorphism.

(*e*). Leray's theorem.

Let X be a compact metric space, and $\{F_i \mid i \in M\}$ a finite closed covering. Let $N(F)$ be the nerve of $\{F_i \mid i \in M\}$ and, for each $\sigma \in N(F)$, let $K(\sigma) \subset X$ be the intersection of the sets corresponding to the vertices of σ. Let $Tr(\sigma)$ be the closed traverse (5) of $\sigma \in N(F)$; in another paper we have proved (under much more general conditions) that there exists a continuous map $\lambda : X \longrightarrow N(F)$ such that $\lambda^{-1}(Tr\,\sigma) = K(\sigma)$ for each $\sigma \in N(F)$.

4.5 THEOREM. Let X be a connected compact metric space and let $\{F_i \mid i \in M\}$ be a finite closed covering. Let H denote augmented Čech cohomology. Then, if each finite intersection of the F_i is $\mathcal{R}$-acyclic (6), $\lambda^{+} : H(N) \longrightarrow H(X)$ is an $\mathcal{R}$-isomorphism.

PROOF. Structure $N(F)$ by traverse-complexes, and let $\lambda^{-1}(\mathcal{K})$ be the induced structure on X. Define two cohomology theories on $C(\lambda^{-1}(\mathcal{K}))$ by setting

$$h(\lambda^{-1}(K), \lambda^{-1}(L)) = H(K, L)$$

$$\widehat{h}(\lambda^{-1}(K), \lambda^{-1}(L)) = H(\lambda^{-1}K), \lambda^{-1}(L)).$$

(5) If $[\sigma]$ represents the barycenter of σ, and $\sigma < \tau$ that σ is a proper face of τ, then the simplexes of the barycentric subdivision N' of N are all sequences $([\sigma_1], [\sigma_2], \dots, [\sigma_r])$ such that $\sigma_1 < \sigma_2 < \dots < \sigma_r$. The closed traverse $Tr(\sigma)$ of $\sigma \in N$ is the closed subcomplex of N' consisting of all simplexes $([\sigma], [\sigma_1], [\sigma_2], \dots, [\sigma_r])$ such that $\sigma < \sigma_1 < \sigma_2 < \dots < \sigma_r$, together with all faces of such simplexes.

(6) Precisely: $H_n(F_{i_1} \cap \dots \cap F_{i_s}) \in \mathcal{R}$ for every combination $(i_1, \dots, i_s)$ and every $n \geq 0$.

We note that $\widehat{h}$ has the strong excision property because Čech cohomology is invariant under relative homeomorphisms of compact pairs [1; 266]. To apply 3.1, we need observe only that (1) $\lambda^{+} : h \longrightarrow \widehat{h}$ provides a natural transformation, (2) the indecomposables are the sets $\lambda^{-1}(Tr\,\sigma) = K(\sigma)$ and (3) $H(Tr\,\sigma) = 0$ since $Tr\,\sigma$ is contractible, whereas $H(K(\sigma)) \in \mathcal{R}$ so that

$$(\lambda \mid K(\sigma))^{+} : H(Tr\,\sigma) \longrightarrow H(K(\sigma))$$

is an $\mathcal{R}$-isomorphism for each $K(\sigma)$. This completes the proof.

(*f*). We remark that, if K, L are two complexes, and if the cartesian product $K \times L$ is structured by $\{K_\lambda \times L\}$, where $\{K_\lambda\}$ is the of subcomplexes of K, then the use of 3.1 permits a simple proof of the Künneth formula in the form $H_n(K \times L) \approx \sum_{i=0}^{n} H_i(K, H_{n-i}(L))$; the obvious details are omitted.

Università di Pisa

BIBLIOGRAPHY

1. S. Eilenberg and N. E. Steenrod: *Foundations of Algebraic Topology.* Princeton University Press, 1952.
2. R. Godement: *Topologie Algébrique et Théorie des faisceaux.* Hermann, Paris, 1958.
3. J. P. Serre: *Groupes d'homotopie et classes de groupes abéliens.* Ann. Math. 58 (1953) 258-294.
4. E. H. Spanier: *Duality and the suspension category.* Sympos. Internac. Topologia Algebraica (1958) 259-272.