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OPERATIONAL REPRESENTATIONS FOR THE LAGUERRE POLYNOMIALS

By S. K. CHATTERJEA

1. Carlitz [1] gave the following operational representation for the Laguerre polynomials:

$$(1.1) \qquad L_n^{(\alpha)}(x) = \frac{1}{n!} \prod_{j=1}^{n} (x D - x + \alpha + j) \cdot 1.$$

Recently Al-Salam [2] has given the operational formula

$$(1.2) \qquad L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha-n} e^x \{x(1 + x D)\}^n x^\alpha e^{-x},$$

which is closely related to the formula of Chak [3]:

$$(1.3) \qquad L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha-n-1} e^x (x^2 D)^n x^{\alpha+1} e^{-x}.$$

In a recent paper [4], M. K. Das has proved the operational formula

$$(1.4) \qquad L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha-n-k} e^x \{x(x D - k + 1)\}^n x^{\alpha+k} e^{-x}.$$

It may be noted that (1.2) and (1.3) are special cases of (1.4).

Moreover, one may note from (1.4) the following special case

$$(1.5) \qquad L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha-n-2} e^x \{x(x D - 1)\}^n x^{\alpha+2} e^{-x}.$$

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In [2, pp. 129-136] Al-Salam has proved several results involving Laguerre polynomials by the operational formula (1.2). The object of this paper is to derive some formulas of Laguerre polynomials by employing formulas of Carlitz and Chak, and such type of derivation does not seem to appear in the earlier investigation.

2. For our purpose, we write (1.1) in the form :

$$L_n^{(\alpha)}(x) = \frac{1}{n\,!}\, e^x (\delta + \alpha + 1)_n\, e^{-x}\,; \quad \delta \equiv x\, D \tag{2.1}$$

For Chak's operator we notice that [5]

$$e^{tx\delta} f(x) = f\left(\frac{x}{1 - t\,x}\right) \tag{2.2}$$

and

$$e^{t\delta} f(x) = f(x\, e^t). \tag{2.3}$$

Again it is easy to prove that

$$(x\,\delta)^n\, x^{\alpha+k} = (\alpha + k)_n\, x^{\alpha+k+n}\,. \tag{2.4}$$

First we shall prove the following generating function:

$$\sum_{n=0}^{\infty} \frac{t^n}{(\alpha+1)_n} L_n^{(\alpha)}(x) = e^t\, {}_0F_1[-\,;\alpha+1\,; -x\,t]. \tag{2.5}$$

We have

$$\sum_{n=0}^{\infty} \frac{t^n}{(\alpha+1)_n} L_n^{(\alpha)}(x)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n\,!\,(\alpha+1)_n}\, e^x (\delta + \alpha + 1)_n\, e^{-x}$$

$$= e^x\, {}_1F_1[\delta + \alpha + 1\,;\ \alpha + 1\,;\ t]\, e^{-x}$$

$$= e^{x+t}\, {}_1F_1[-\,\delta;\ \alpha + 1\,;\ -t]\, e^{-x}$$

$$= e^{x+t} \sum_{n=0}^{\infty} \frac{(-\,\delta)_n}{(\alpha+1)_n\, n\,!} (-\,t)^n\, e^{-x}$$

$$= e^{x+t} \sum_{n=0}^{\infty} \frac{(x\,t)^n}{(\alpha+1)_n\, n\,!} D^n\ e^{-x}$$

$$= e^{x+t} \sum_{n=0}^{\infty} \frac{(-x\,t)^n}{(\alpha+1)_n\, n!}\, e^{-x}$$

$$= e^t\, {}_0F_1\,[-\,;\,\alpha+1\,;\,-x\,t].$$

Another way of obtaining (2.5) is the following:

From (1.3) we have

$$\sum_{n=0}^{\infty} \frac{(x\,t)^n}{(\alpha+1)_n} L_n^{(\alpha)}(x)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(\alpha+1)_n\, n!}\, x^{-\alpha-1}\, e^x\, (x\,\delta)^n\, x^{\alpha+1}\, e^{-x}$$

$$= x^{-\alpha-1}\, e^x \sum_{n=0}^{\infty} \frac{t^n}{(\alpha+1)_n\, n!}\, (x\,\delta)^n\, x^{\alpha+1}\, e^{-x}$$

$$= x^{-\alpha-1}\, e^x\, {}_0F_1\,[-\,;\,\alpha+1\,;\,t\,x\,\delta]\, x^{\alpha+1}\, e^{-x}\,. \tag{2.6}$$

But

$${}_0F_1\,[-\,;\,\alpha+1\,;\,t\,x\,\delta]\, x^{\alpha+1}\, e^{-x}$$

$$= {}_0F_1\,[-\,;\,\alpha+1\,;\,t\,x\,\delta] \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\, x^{\alpha+r+1}$$

$$= x^{\alpha+1} \sum_{r=0}^{\infty} \frac{(-x)^r}{r!}\, {}_1F_1\,[\alpha+r+1\,;\,\alpha+1\,;\,x\,t]$$

$$= x^{\alpha+1}\, e^{xt} \sum_{r=0}^{\infty} \frac{(-x)^r}{r!}\, {}_1F_1\,[-r;\,\alpha+1\,;\,-x\,t]$$

$$= x^{\alpha+1}\, e^{xt-x}\, {}_0F_1\,[-\,;\,\alpha+1\,;\,-x^2\,t]. \tag{2.7}$$

It follows therefore from (2.6) and (2.7) that

$$\sum_{n=0}^{\infty} \frac{(x\,t)^n}{(\alpha+1)_n}\, L_n^{(\alpha)}(x) = e^{xt}\, {}_0F_1\,[-;\,\alpha+1\,;\,-x^2\,t], \tag{2.8}$$

which is (2.5).

We remark therefore that the method adopted at first in proving (2.5) is quite simple and interesting.

Next we shall prove the following generating function:

$$\sum_{n=0}^{\infty} \frac{(c)_n}{(\alpha+1)_n}\, t^n\, L_n^{(\alpha)}(x) = (1-t)^{-c}\, {}_1F_1\left[c;\; a+1;\; \frac{-x\,t}{1-t}\right]. \tag{2.9}$$

Here we have

$$\sum_{n=0}^{\infty} \frac{(c)_n}{(\alpha+1)_n} t^n L_n^{(\alpha)}(x)$$

$$= \sum_{n=0}^{\infty} \frac{(c)_n t^n}{(\alpha+1)_n n!} e^x (\delta+\alpha+1)_n e^{-x}$$

$$= e^x {}_2F_1 [\delta+\alpha+1, c;\ \alpha+1; t] e^{-x}$$

$$= e^x (1-t)^{\alpha+1-c-\delta-\alpha-1} {}_2F_1 [-\delta, \alpha+1-c;\ \alpha+1; t] e^{-x}$$

$$= e^x (1-t)^{-c} \left(\frac{1}{1-t}\right)^{\delta} \sum_{n=0}^{\infty} \frac{(\alpha+1-c)_n}{(\alpha+1)_n} \frac{(-x t)^n}{n!} D^n e^{-x}$$

$$= e^x (1-t)^{-c} \left(\frac{1}{1-t}\right)^{\delta} {}_1F_1 [\alpha+1-c;\ \alpha+1;\ x t] e^{-x} .$$

From (2.3) we notice that

$$a^{\delta} f(x) = f(ax).$$

Thus we derive

$$\sum_{n=0}^{\infty} \frac{(c)_n}{(\alpha+1)_n} t^n L_n^{(\alpha)}(x)$$

$$= e^x (1-t)^{-c} {}_1F_1 \left[\alpha+1-c;\ \alpha+1;\ \frac{x t}{1-t}\right] e^{-\frac{x}{1-t}}$$

$$= e^x (1-t)^{-c} e^{\frac{xt}{1-t}} {}_1F_1 \left[c;\ \alpha+1;\ \frac{-x t}{1-t}\right] e^{-\frac{x}{1-t}}$$

$$= (1-t)^{-c} {}_1F_1 \left[c;\ \alpha+1;\ \frac{-x t}{1-t}\right] .$$

Alternatively we derive (2.9) in the following manner:

From (1.3) we have

$$\sum_{n=0}^{\infty} \frac{(c)_n}{(\alpha+1)_n} (x t)^n L_n^{(\alpha)}(x)$$

$$= \sum_{n=0}^{\infty} \frac{(c)^n}{(\alpha+1)_n} \frac{t^n}{n!} x^{-\alpha-1} e^x (x\delta)^n x^{\alpha+1} e^{-x}$$

(2.10) $$= x^{-\alpha-1} e^x {}_1F_1 [c;\ \alpha+1;\ t x \delta] x^{\alpha+1} e^{-x} .$$

Now

$${}_1F_1\,[c;\ \alpha+1;\ t\,x\,\delta]\,x^{\alpha+1}\,e^{-x}$$

$$= x^{\alpha+1}\sum_{r=0}^{\infty}\frac{(-x)^r}{r\,!}\,{}_2F_1\,[r+\alpha+1,\ c;\ \alpha+1;\ x\,t]$$

$$= x^{\alpha+1}\sum_{r=0}^{\infty}\frac{(-x)^r}{r\,!}\,(1-x\,t)^{-c-r}\,{}_2F_1\,[-r,\alpha+1-c;\ \alpha+1;\ x\,t]$$

$$= x^{\alpha+1}\,(1-x\,t)^{-c}\sum_{r=0}^{\infty}\frac{\left(\frac{-x}{1-xt}\right)^r}{r!}\,{}_2F_1\,[-r,\alpha+1-c;\ \alpha+1\,;\ x\,t]$$

$$= x^{\alpha+1}\,(1-x\,t)^{-c}\,e^{\frac{-x}{1-xt}}\,{}_1F_1\left[\alpha+1-c;\ \alpha+1;\ \frac{x^2\,t}{1-x\,t}\right]$$

(2.11) $$= x^{\alpha+1}\,(1-x\,t)^{-c}\,e^{\frac{-x}{1-xt}+\frac{x^2t}{1-xt}}\,{}_1F_1\left[c;\ \alpha+1\,;\ -\frac{x^2\,t}{1-x\,t}\right].$$

Thus it follows from (2.10) and (2.11) that

(2.12) $$\sum_{n=0}^{\infty}\frac{(c)_n}{(\alpha+1)_n}\,(x\,t)^n\,L_n^{(\alpha)}(x) = (1-xt)^{-c}\,{}_1F_1\left[c;\ \alpha+1\,;\frac{-x^2\,t}{1-xt}\right],$$

which is (2.9).

Using the operational formula of Carlitz, we shall finally prove the Hardy-Hille formula:

(2.13) $$\sum_{n=0}^{\infty}\frac{n\,!}{(\alpha+1)_n}\,L_n^{(\alpha)}(x)\,L_n^{(\alpha)}(y)\,t^n$$

$$= (1-t)^{-\alpha-1}\exp\left[-\frac{(x+y)\,t}{1-t}\right]{}_0F_1\left[-;\ \alpha+1;\ \frac{x\,y\,t}{(1-t)^2}\right].$$

We observe that

$$\sum_{n=0}^{\infty}\frac{n!}{(\alpha+1)_n}\,t^n\,L_n^{(\alpha)}(x)\,L_n^{(\alpha)}(y)$$

$$= \sum_{n=0}^{\infty}\frac{n\,!}{(\alpha+1)_n}\,t^n\,\frac{1}{n\,!}\,e^x\,(\delta+\alpha+1)_n\,e^{-x}\,\frac{1}{n\,!}\,e^y\,(\theta+\alpha+1)_n\,e^{-y}$$

$$\left(\text{where } \delta = x\,\frac{d}{dx},\ \text{and } \theta = y\,\frac{d}{dy}\right)$$

$$= e^{x+y} \sum_{n=0}^{\infty} \frac{(\delta + \alpha + 1)_n (\theta + \alpha + 1)_n}{(\alpha + 1)_n} \frac{t^n}{n!} e^{-(x+y)}$$

$$= e^{x+y} {}_2F_1 [\delta + \alpha + 1, \theta + \alpha + 1 ; \alpha + 1 ; t] e^{-(x+y)}$$

$$= e^{x+y} (1 - t)^{-\delta-\theta-\alpha-1} {}_2F_1 [- \delta, - \theta; \alpha + 1; t] e^{-(x+y)}$$

$$= e^{x+y} (1 - t)^{-\alpha-1} (1 - t)^{-\delta-\theta} {}_0F_1 [- ; \alpha + 1; x y t] e^{-(x+y)}$$

$$= e^{x+y} (1 - t)^{-\alpha-1} \left(\frac{1}{1-t}\right)^{\delta} \left(\frac{1}{1-t}\right)^{\theta} {}_0F_1 [- ; \alpha + 1; x y t] e^{-(x+y)}$$

$$= e^{x+y} (1 - t)^{-\alpha-1} {}_0F_1 \left[- ; \alpha + 1 ; \frac{x y t}{(1-t)^2}\right] e^{\frac{-(x+y)}{1-t}}$$

$$= (1 - t)^{-\alpha-1} e^{\frac{-(x+y)t}{1-t}} {}_0F_1 \left[- ; \alpha + 1 ; \frac{x y t}{(1 - t)^2}\right].$$

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