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REMARKS ON INTEGRAL INEQUALITIES ON COMPLEX MANIFOLDS (*)

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Let M be a connected orientable and oriented differentiable manifold of class C^∞, endowed with a complete riemannian metric. The action of the Laplace operator Δ on any q-form u of class C^2 on M can be expressed locally by

$$\Delta u = - \nabla_i \nabla^i u + \varkappa u.$$

In this formula ∇_i and ∇^i stand for covariant derivatives with respect to the riemannian connection, and $\varkappa$ is a mapping of the space of real valued q-forms into itself, which is linear over the ring of real continuous functions on M. The operator $\varkappa$ is symmetric with respect to the scalar product, $\langle\, , \rangle_x$, defined by the riemannian metric at each point $x \in M$. Setting

$$| u |_x^2 = \langle u, u \rangle_x$$

we call $| u |_x$ the *lenght* of the form u at the point x. We introduce also the L_2 norm

$$\| u \|^2 = \int_M | u |^2 \, dX,$$

dX being the volume element of the riemannian metric of M.

The following theorem has been proved in [5]:

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THEOREM. — *Let the symmetric form* $\langle \varkappa\,,\ \rangle_x$, *acting on the space of q-forms on M, be positive semidefinite at each point* $x \in M$ *outside a compact* $K \subset M$. *Any q-form* φ *of class* C^2 *on M, such that* $\| \varphi \| < \infty$, $\| \Delta\varphi \| < \infty$, *satisfies the inequality*

$$\operatorname*{Sup}_{M} | \varphi | \leq \operatorname*{Sup}_{K \cup \mathrm{Supp}(\Delta\varphi)} | \varphi | .$$

The proof depends on an integral inequality estimating the L_2 norm $\| \nabla u \|$ of the covariant derivatives of a q-form u in terms of $\| du \|$, $\| \delta u \|$ and of the integral $\int_M \langle \varkappa u, u \rangle \, dX$.

In this paper we extend the above theorem to vector bundle-valued (p, q)-forms u on a complex manifold.

In the proof we establish an integral inequality estimating the L_2 norm of *all* the covariant derivatives of u in terms of $\| \bar\partial u \|$, $\| \vartheta u \|$ and of an integral of type $\int_X \langle \varkappa u, u \rangle \, dX$. A few direct applications of that inequality are listed in n. 8. The first section (nn. 1-3) contains some preliminary properties whose proofs can be found in [1] or in [5].

§ 1. — **Preliminaries.**

1. Let X be a complex manifold of complex dimension n, and let $E \xrightarrow{\pi} X$ be a holomorphic vector bundle of rank m on X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open coordinate covering of X such that, on each U_i, $E_{|U_i}$ is isomorphic to the trivial bundle. The bundle E is defined, with respect to this covering, by a system $\{e_{ij}\}$ of holomorphic transition functions

$$e_{ij} : U_i \cap U_i \longrightarrow GL(m, \mathbb{C}),$$

satisfying the compatibility condition

$$e_{ij} \cdot e_{jk} \cdot e_{ki} = Id \qquad \text{on} \quad U_i \cap U_j \cap U_k .$$

The dual bundle E^* of E is defined on the covering $\mathcal{U}$ by the system of holomorphic transition functions $\{e^*_{ij}\}$ expressed by

$$e^*_{ij} = {}^t e_{ij}^{-1} .$$

Let $C^{pq}(X, E)$ be the vector space of continuous (p, q)-forms with values in E. Any element φ of $C^{pq}(X, E)$ is defined on U_i by a continuous vector valued (p, q)-form $\varphi_i = {}^t(\varphi_i^1, \dots, \varphi_i^m)$ such that

$$\varphi_i = e_{ij}\, \varphi_j \qquad \text{on} \qquad U_i \cap U_j .$$

A metric along the fibers of E is defined by a positive definite hermitian scalar product $h(u, v)$ $(u, v \in \pi^{-1}(x),\ x \in X)$ on the fibers of E depending differentiably of class C^∞ on the point $x \in X$. If on the coordinate neighbourhood U_i, $u = \xi_i = {}^t(\xi_i^1, \dots, \xi_i^m)$ $v = \eta_i = {}^t(\eta_i^1, \dots, \eta_i^m)$, then the local expression of $h(u, v)$ on U_i is given by

$$h(u, v) = {}^t\overline{\eta_i}\, h_i\, \xi_i ,$$

where h_i is a positive definite hermitian matrix of class C^∞ on U_i.

The metric h along the fibers of E enables us to define an antiisomorphism

$$\# : C^{pq}(X, E) \longrightarrow C^{qp}(X, E^*),$$

which is local, i.e. preserves the supports. For any form $\varphi = \{\varphi_i\}$ of $C^{pq}(X, E$ we have

$$(\# \varphi)_i = \overline{h_i\, \varphi_i} \qquad \text{on} \quad U_i .$$

2. The local forms

$$l_i = h_i^{-1}\, \partial h_i$$

define a ∂-connection on E, and hence an absolute differentiation of any C^1 section of E in the following way.

The connection form l is expressed, in terms of a local complex coordinates system $(z^1, \dots, z^m)$ by an $m \times m$ matrix of C^∞ (1, 0)-forms

$$l = (l_b^a)_{a,\, b=1,\dots,m}, \quad l_b^a = l_{b\alpha}^a\, dz^\alpha .$$

A C^1 section t of E is locally represented by an m-vector of class C^1

$$t = {}^t(t_1, \dots, t_m).$$

We define the covariant derivatives of t, setting locally

$$\begin{aligned} \nabla_\alpha t^a &= \partial_\alpha t^a + l_{b\alpha}^a t^b \\ \nabla_{\bar\alpha} t^a &= \partial_{\bar\alpha} t^a . \end{aligned} \qquad \left(\partial_\alpha = \frac{\partial}{\partial z^\alpha}, \quad \partial_{\bar\alpha} = \frac{\partial}{\partial \overline{z^\alpha}} \right).$$

Let Θ be the holomorphic tangent bundle on X. The vector ${}^t(\nabla_\alpha t^a)_{\alpha=1,\dots,n;\ a=1,\dots,m}$, represents locally a global continuous section, $\nabla' t$, of the holomorphic vector bundle $E \otimes \Theta^*$.

Similarly ${}^t(\nabla_{\bar\alpha} t^a)_{\alpha=1,\dots,n;\ a=1,\dots,m}$ represents locally a global continuous section $\nabla'' t$ of the vector bundle $E \otimes \overline{\Theta}^*$.

The conjugate $\bar l = \{\bar l_i\}$ of the ∂-connection form l on E defines a $\bar\partial$-connection in the antiholomorphic vector bundle $\bar E$. If u is a C^1 section of $\bar E$, we define the covariant derivatives $\nabla' u$ and $\nabla'' u$ in terms of the covariant derivatives of the section $\bar u$ of E, setting

$$\overline{\nabla' u} = \nabla'' \bar u, \qquad \overline{\nabla'' u} = \nabla' \bar u.$$

Using the ∂— and $\bar\partial$—connection forms we can define covariant derivatives of C^1 sections of tensor products of holomorphic and antiholomorphic vector bundles.

The metric h on E, considered as a C^∞ section of $E \otimes \bar E$, has all its covariant derivatives zero.

The curvature form of the ∂-connection form l is given locally by a $m \times m$ matrix

$$s = \bar\partial\, l = (s^a_b)_{a,\, b=1,\dots,m}$$

of scalar C^∞ $(1, 1)$-forms

$$s^a_b = s^a_{b\bar\beta\alpha}\, \overline{dz^\beta} \wedge dz^\alpha .$$

Letting t be a C^2 section of E, we have

$$(\nabla_{\bar\beta} \nabla_\alpha - \nabla_\alpha \nabla_{\bar\beta})\, t^a = s^a_{b\bar\beta\alpha}\, t^b \qquad \text{(Ricci identity).}$$

3. We assume now a C^∞ metric along the fibers of Θ. This is equivalent to saying that a positive definite hermitian differential form of class C^∞, $g_{\alpha\bar\beta}\, dz^\alpha\, \overline{dz^\beta}$, is assigned on X. This form induces a C^∞ positive definite riemannian metric on the underlying C^∞ manifold of X. The $*$ operator defined by the riemannian metric of X maps scalar (p, q)-forms into scalar $(n - q, n - p)$-forms, and extends trivially to an isomorphism

$$* : C^{pq}(X, E) \longrightarrow C^{n-q,\, n-p}(X, E).$$

The ∂-connection determined by the hermitian metric on X is a symmetric connection if, and only if, the hermitian metric is a Kähler metric. In that case, the curvature form of the ∂-connection form coincides with the Riemannian curvature form of the underlying riemannian metric.

Let $\varphi, \psi \in C^{pq}(X, E)$. Then $\varphi \wedge * \,\#\, \psi$ is a scalar (n, n)-form. If dX denotes the volume element of the hermitian metric of X, $\varphi \wedge * \,\#\, \psi$ can be written as

$$\varphi \wedge * \,\#\, \psi = A(\varphi, \psi)\, dX.$$

$A(\varphi, \psi)$ acts, at each point of X, as a sesquilinear positive definite hermitian scalar product on the space $C^{pq}(X, E)$.

We set

$$|\varphi| = \sqrt{A(\varphi, \varphi)},$$

and we call $|\varphi|$ the *lenght* of the form φ.

Let $\mathcal{D}^{pq}(X, E)$ be the space of compactly supported C^∞ (p, q)-forms with values in E.

The scalar product

$$(\varphi, \psi) = \int_X \varphi \wedge * \,\#\, \psi$$

gives $\mathcal{D}^{pq}(X, E)$ the structure of a complex pre-Hilbert space over $\mathbb{C}$. Let $\mathcal{L}^{pq}(X, E)$ denote the completion of $\mathcal{D}^{pq}(X, E)$ with respect to the norm $\|\varphi\| = (\varphi, \varphi)^{\frac{1}{2}}$.

We denote by ϑ the formal adjoint of the $\bar{\partial}$ operator, i.e. the linear operator

$$\vartheta : \mathcal{D}^{pq+1}(X, E) \longrightarrow \mathcal{D}^{pq}(X, E),$$

such that

$$(\bar{\partial}\varphi, \psi) = (\varphi, \vartheta\psi) \text{ for all } \varphi \in \mathcal{D}^{pq}(X, E),\ \psi \in \mathcal{D}^{pq+1}(X, E).$$

Let us consider the scalar product on $\mathcal{D}^{pq}(X, E)$

$$a(\varphi, \psi) = (\varphi, \psi) + (\bar{\partial}\varphi, \bar{\partial}\psi) + (\vartheta\varphi, \vartheta\psi) \qquad (\varphi, \psi \in \mathcal{D}^{pq}(X, E)),$$

and let N be the norm defined by $N(\varphi)^2 = a(\varphi, \varphi)$.

We denote by $W^{pq}(X, E)$ the Hilbert space completion of $\mathcal{D}^{pq}(X, E)$ with respect to the norm N.

PROPOSITION 1 [1, 5]. — *If the hermitian metric of X is complete, $W^{pq}(X, E)$ can be identified with the space of forms $\varphi \in \mathcal{L}^{pq}(X, E)$ which admit a $\bar{\partial}\varphi \in \mathcal{L}^{pq+1}(X, E)$ and a $\vartheta\varphi \in \mathcal{L}^{pq-1}(X, E)$ (in the sense of distributions).*

Let us introduce the Laplace-Beltrami operator $\square = \bar{\partial}\vartheta + \vartheta\bar{\partial}$.

PROPOSITION 2 [1, 5]. — *If the hermitian metric of X is complete, then for any (p, q)-form φ with values in E and of class C^2 on X, and for any positive constant σ, we have*

$$\| \bar{\partial}\varphi \|^2 + \| \vartheta\varphi \|^2 \leq \sigma \| \Box \varphi \|^2 + \frac{1}{\sigma} \| \varphi \|^2.$$

COROLLARY 3. — *Under the same hypotheses of proposition 2, if* $\| \varphi \| < \infty$, $\| \Box \varphi \| < \infty$, *then* $\varphi \in W^{pq}(X, E)$.

§ 2. — Integral inequalities.

4. We suppose that the complex manifold X is equipped with a (positive definite, C^∞) hermitian metric. We choose also a metric along the fibers of the holomorphic vector bundle E.

We denote by ∇' and ∇'' the covariant derivatives with respect to the given metrics. We shall use the same symbols ∇' and ∇'' to denote covariant derivatives of sections of different bundles.

Let φ be a (p, q)-form with values in E, of class C^2 on X; φ is locally represented by a vector form of class C^2

$$\varphi = \frac{1}{p\,!\,q\,!} \varphi^a_{A\bar{B}}\, dz^A \wedge \overline{dz^B} \qquad (a = 1, \dots, m),$$

where A and B denote blocks of p and q indices $A = (\alpha_1, \dots, \alpha_p)$, $B = (\beta_1, \dots, \beta_q)$ and $dz^A = dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p}$, $dz^B = dz^{\beta_1} \wedge \dots \wedge dz^{\beta_q}$. In terms of the covariant derivatives ∇', ∇'', the operator $\bar{\partial}$ has the expression

$$\bar{\partial} = \hat{\bar{\partial}} + S,$$

where

$$(\hat{\bar{\partial}} \varphi)^a{}_{A\bar{\beta}_1 \dots \bar{\beta}_{q+1}} = (-1)^p \sum_{r=1}^{q+1} (-1)^{r-1} \nabla_{\bar{\beta}_r} \varphi^a{}_{A\bar{\beta}_1 \dots \hat{\bar{\beta}}_r \dots \bar{\beta}_{q+1}},$$

and

$$(S\varphi)^a{}_{A\bar{\beta}_1 \dots \bar{\beta}_{q+1}} = (-1)^p \sum_{r=1}^{q+1} (-1)^{r-1} \overline{S^{\alpha}_{\beta_i \beta_r}}\, \varphi^a{}_{A\bar{\beta}_1 \dots (\bar{\alpha})_i \dots \hat{\bar{\beta}}_r \dots \bar{\beta}_{q+1}},$$

$S^\alpha_{\beta\gamma}$ being the torsion tensor of the connection.

Analogously we have

$$\vartheta = \hat{\vartheta} + T,$$

where

$$(\widehat{\vartheta}\,\varphi)^a{}_{A\bar\beta_1\cdots\bar\beta_{q-1}} = (-1)^{p-1}\nabla_\alpha\,\varphi^a{}_A{}^{\alpha}{}_{\bar\beta_1\cdots\bar\beta_{q-1}}\,,$$

$$T = -*\,\#^{-1}S\,\#\,*\,.$$

Setting $\widehat{\Box} = \widehat{\partial}\,\widehat{\vartheta} + \widehat{\vartheta}\,\widehat{\partial}$ we have

$$(1)\qquad (\widehat{\Box}\,\varphi)^a{}_{A\bar B} = -\nabla_\alpha\nabla^\alpha\varphi^a{}_{A\bar B} + \sum_{r=1}^{q}(-1)^{r-1}(\nabla_\alpha\nabla_{\bar\beta_r} - \nabla_{\bar\beta_r}\nabla_\alpha)\,\varphi^a{}_A{}^{\alpha}{}_{\bar B'_r}\,,$$

where

$$\nabla^\alpha = g^{\alpha\bar\beta}\,\nabla_{\bar\beta}\,,$$

and

$$B'_r = (\beta_1\,,\ldots,\widehat{\beta_r}\,,\ldots,\beta_q)\qquad (r = 1,\ldots,q).$$

If the hermitian metric on X is a Kähler metric then $S \equiv 0$, hence $T \equiv 0$, and therefore

$$\widehat{\partial} = \bar\partial\,,\quad \widehat{\vartheta} = \vartheta\,,\quad \widehat{\Box} = \Box\,.$$

In general

$$(2)\qquad \Box = \widehat{\Box} + \widehat{\partial}\,T + T\,\widehat{\partial} + \widehat{\vartheta}\,S + S\,\widehat{\vartheta} + ST + TS\,.$$

By the Ricci identity, the last summand of (1) can be espressed by

$$(3)\qquad \sum_{r=1}^{q}(-1)^{r-1}(\nabla_\alpha\nabla_{\bar\beta_r} - \nabla_{\bar\beta_r}\nabla_\alpha)\,\varphi^a{}_A{}^{\alpha}{}_{\bar B'_r} = (\varkappa\varphi)^a{}_{A\bar B}\,.$$

where $\varkappa$ is a hermitian mapping

$$\varkappa:\ C^{pq}(X,E)\longrightarrow C^{pq}(X,E),$$

which is linear over the ring $\mathcal{F}$ of complex valued continuous functions on X and hermitian with respect to the scalar product $A(\,,\,)$. Its local expression involves linearly (with integral coefficients) only the coefficients of the curvature forms of the metrics on E and on X. If $q = 0$, then $\varkappa = 0$.

A direct computation yields

$$(\varkappa\varphi)^a{}_{A\bar B} = \sum_{r=1}^{q}(-1)^{r-1}\,s^a_{b\bar\beta_r\alpha}\,\varphi^b{}_A{}^{\alpha}{}_{\bar B'_r} + (\varkappa^0\,\varphi)^a{}_{A\bar B}$$

where $\varkappa^0$ involves only the curvature tensor of the hermitian metric on X.

It has been shown in [1] (see also [5]) that there exist universal positive constants c_1, c_2 such that, if $\varphi \in \mathcal{D}^{pq}(X, E)$, then

$$\| \nabla'' \varphi \|^2 + c_1 (\varkappa \varphi, \varphi) \leq c_2 (\| \bar{\partial} \varphi \|^2 + \vartheta \varphi \|^2). \tag{4}$$

If the hermitian metric on X is a Kähler metric, then we can choose $c_1 = c_2 = 1$; fur thermore with this choice the above inequality becomes an equality

$$\| \nabla'' \varphi \|^2 + (\varkappa \varphi, \varphi) = \| \bar{\partial} \varphi \|^2 + \| \vartheta \varphi \|^2 \qquad (\varphi \in \mathcal{D}^{pq}(X, E)). \tag{4'}$$

5. We shall now establish an integral inequality on X involving both the ∇' and ∇'' derivatives.

We have

$$| \varphi |^2 = \frac{1}{p!\, q!} \varphi^a{}_{A\bar{B}} (\# \varphi)_a{}^{A\bar{B}}.$$

Consider the tangent vector field on X

$$\xi = (\xi^\alpha, \xi^{\bar{\alpha}}),$$

where

$$\xi^\alpha = \nabla^\alpha | \varphi |^2 = g^{\alpha\bar{\beta}} \nabla_{\bar{\beta}} | \varphi |^2, \qquad \xi^{\bar{\alpha}} = 0.$$

An easy computation shows that

$$\operatorname{div} \xi = \nabla_\alpha \xi^\alpha - 2 S^\beta_{\alpha\beta} \xi^\alpha = \nabla_\alpha \nabla^\alpha | \varphi |^2 - 2 S^\beta_{\alpha\beta} \nabla^\alpha | \varphi |^2 .$$

We have

$$\nabla_\alpha \nabla^\alpha | \varphi |^2 = | \nabla' \varphi |^2 + | \nabla'' \varphi |^2 + \\ + \frac{1}{p!\, q!} [(\nabla_\alpha \nabla^\alpha \varphi)^a{}_{A\bar{B}} (\# \varphi)_a{}^{A\bar{B}} + \varphi^{\,a}_{A\bar{B}} (\# \nabla_{\bar{\alpha}} \nabla^{\bar{\alpha}} \varphi)_a{}^{A\bar{B}}],$$

with $\nabla^{\bar{\alpha}} = g^{\beta\bar{\alpha}} \nabla_\beta$.

By the Ricci identity

$$(\nabla_{\bar{\alpha}} \nabla^{\bar{\alpha}} \varphi)^a{}_{A\bar{B}} = (\nabla_\alpha \nabla^\alpha \varphi)^a{}_{A\bar{B}} + (\varkappa_1 \varphi)^a{}_{A\bar{B}},$$

where

$$(\varkappa_1 \varphi)^a{}_{A\bar{B}} = s^{a\ \bar{\gamma}}_{b\gamma} \varphi^b{}_{A\bar{B}} + (\varkappa_1^0 \varphi)^a{}_{A\bar{B}};$$

here $\varkappa_1^0$ involves only the curvature tensor of the hermitian metric of X.

Let us introduce the $\mathcal{F}$-linear hermitian operator

$$\varkappa_2 = 2\varkappa + \varkappa_1 : \ C^{pq}(X, E) \longrightarrow C^{pq}(X, E).$$

We have by (1) and (3)

$$(5) \quad \nabla_\alpha \nabla^\alpha \,|\, \varphi \,|^2 = |\, \nabla' \varphi \,|^2 + |\, \nabla'' \varphi \,|^2 - A\,(\widehat{\Box}\, \varphi, \varphi) - A\,(\varphi, \widehat{\Box}\, \varphi) + A\,(\varkappa_2\, \varphi, \varphi).$$

A direct computation shows that

$$A\,(\varkappa_2\, \varphi, \varphi) = \frac{1}{p\,!\,q\,!} \{ s^{a\ \ \bar\gamma}_{b\bar\gamma}\, \varphi^b{}_{A\bar B}\, (\#\, \varphi)_a^{A\bar B} - 2q s^{a\ \ \bar\beta}_{b\bar\gamma}\, \varphi^b{}_{A\bar\beta\bar B'} (\#\, \varphi)_a^{A\bar\gamma\bar B'} \} + A\,(\varkappa_2^0\, \varphi, \varphi)$$

$$(B' = \beta_1 \dots \beta_{q-1}),$$

where $\varkappa_2^0$ involves only the curvature tensor of the hermitian metric on X.

Let the hermitian metric be a Kähler metric. For $\varphi \in \mathcal{D}^{pq}(X, E)$ we have

$$\int_X \operatorname{div} \xi \; dX = \int_X \nabla_\alpha \nabla^\alpha \,|\, \varphi \,|^2 \, dX = 0,$$

i.e. by (5)

$$(6) \quad \|\, \nabla' \varphi \,\|^2 + \|\, \nabla'' \varphi \,\|^2 + (\varkappa_2\, \varphi, \varphi) = 2\,(\Box\, \varphi, \varphi) = 2\,(\,\|\, \bar\partial\, \varphi \,\|^2 + \|\, \vartheta\, \varphi \,\|^2).$$

In the general case (i.e. if the hermitian metric on X is not necessarily Kähler), we have for any $\varphi \in \mathcal{D}^{pq}(X, E)$,

$$\int_X \operatorname{div} \xi \; dX = \int_X (\nabla_\alpha \nabla^\alpha \,|\, \varphi \,|^2 - 2 S^\beta_{\alpha\beta}\, \nabla^\alpha \,|\, \varphi \,|^2)\, dX = 0$$

i.e.

$$\|\, \nabla' \varphi \,\|^2 + \|\, \nabla'' \varphi \|^2 + (\varkappa_2\, \varphi, \varphi) = 2\,(\,\|\, \bar\partial\, \varphi \,\|^2 + \|\, \vartheta\, \varphi \,\|^2) +$$

$$+ ((\widehat{\Box} - \Box)\, \varphi, \varphi) + (\varphi, (\widehat{\Box} - \Box)\, \varphi) + 2 \int_X S^\beta_{\alpha\beta}\, \nabla^\alpha \,|\, \varphi \,|^2 \, dX.$$

We shall now estimate the last three summands on the right hand side. There exists a C^∞ function $g\,(x) \geq 0$ on X such that

$$A\,(\bar\partial\; T\varphi, \varphi) \leq g\,(x)\,(\,|\, \varphi \,|^2 + |\, \nabla'' \varphi \,|\,|\, \varphi \,|\,);$$

the function $g(x)$ can be so chosen to involve only the torsion tensor and its first covariant derivatives. Repeating the same argument for all terms of the expression of $\widehat{\square} - \square$ appearing in (2) and for $S^{\beta}_{\alpha\beta} \nabla^{\alpha} |\varphi|^2$, we see that there exist C^{∞} functions $f_i(x) \geq 0$ $(i = 1, 2, 3)$ on X such that

$$|A((\widehat{\square} - \square)\varphi, \varphi)| + |S^{\beta}_{\alpha\beta} \nabla^{\alpha} |\varphi|^2| \leq f_1(x)|\varphi|^2 + \\ + f_2(x)|\varphi||\nabla'\varphi| + f_3(x)|\varphi||\nabla''\varphi|;$$

hence, for any $\sigma > 0$

$$\begin{aligned} |A((\widehat{\square} - \square)\varphi, \varphi)| + |S^{\beta}_{\alpha\beta} \nabla^{\alpha} |\varphi|^2|| &\leq f_1(x)|\varphi|^2 + \\ + \sigma(|\nabla'\varphi|^2 + |\nabla''\varphi|^2) &+ \frac{1}{\sigma}(f_2(x)^2 + f_3(x)^2)|\varphi|^2 \qquad (7) \\ \leq \left[f_1(x) + \frac{1}{\sigma} f_2(x)^2 + \frac{1}{\sigma} f_3(x)^2 \right] &|\varphi|^2 + \sigma(|\nabla'\varphi|^2 + |\nabla''\varphi|^2). \end{aligned}$$

The functions f_i can be so chosen to involve only the torsion tensor and its first covariant derivatives. When the hermitian metric on X is a Kähler metric, we may assume $f_1 \equiv f_2 \equiv f_3 \equiv 0$ on X.

Setting $\sigma = \frac{1}{4}$, and

$$\varkappa_3 \varphi = \varkappa_2 \varphi - 2[f_1(x) + 4f_2(x)^2 + 4f_3(x)^2]\varphi, \qquad (8)$$

we can state the following

PROPOSITION 4. — *Every $\varphi \in \mathcal{D}^{pq}(X, E)$ satisfies the inequality*

$$\|\nabla'\varphi\|^2 + \|\nabla''\varphi\|^2 + 2(\varkappa_3\varphi, \varphi) \leq 4(\|\bar{\partial}\varphi\|^2 + \|\vartheta\varphi\|^2).$$

If the metric on X is a Kähler metric, then φ satisfies equality (6).

6. We assume now that X satisfies the following condition:

a) There is a complete hermitian metric on X and a compact set $K \subset X$ such that the hermitian form $A(\varkappa_3 \varphi, \varphi)$ acting on the space $C^{pq}(X, E)$ is positive semidefinite at each point of $X - K$.

If follows from proposition 4 that there exists a constant $c \geq 0$ such that, for every $\varphi \in \mathcal{D}^{pq}(X, E)$,

$$\| \nabla' \varphi \|^2 + \| \nabla'' \varphi \|^2 + 2 (\varkappa_3 \varphi, \varphi)_{X-K} \leq 2c \| \varphi \|^2_K + 4 (\| \bar{\partial} \varphi \|^2 + \| \vartheta \varphi \|^2),$$

whence, by Corollary 3,

LEMMA 5. — *If X satisfies condition* a), *every (p, q)-form φ of class C^2 on X, with values in E, for which $\| \varphi \| < \infty$, $\| \square \varphi \| < \infty$, is such that $\| \nabla' \varphi \| < \infty$, $\| \nabla'' \varphi \| < \infty$, $(\varkappa_3 \varphi, \varphi) < \infty$.*

Let $\lambda = \lambda(t)$ be a real C^∞ function on $\mathbb{R}$. Setting $\dot{\lambda}(t) = \dfrac{d\lambda}{dt}$, $\ddot{\lambda}(t) = \dfrac{d^2 \lambda}{dt^2}$, we assume that $\dot{\lambda}(t) \geq 0$, $\ddot{\lambda}(t) \geq 0$ on $\mathbb{R}$, and that $\ddot{\lambda}(t) \equiv 0$ outside a bounded interval of $\mathbb{R}$.

LEMMA 6. — *Let φ be a (p, q)-form of class C^2 on X, with values in E, such that $\| \varphi \| < \infty$, $\| \square \varphi \| < \infty$, $(\varkappa_2 \varphi, \varphi) < \infty$. If condition* a) *is satisfied, the following inequality holds*

$$\begin{aligned} &2 \int_X \ddot{\lambda}(|\varphi|^2) | \nabla' | \varphi |^2 |^2 \, dX + (\dot{\lambda}(|\varphi|^2) \nabla' \varphi, \nabla' \varphi) + \\ (9) \qquad &(\dot{\lambda}(|\varphi|^2) \nabla'' \varphi, \nabla'' \varphi) + 2 (\dot{\lambda}(|\varphi|^2) \varkappa_3 \varphi, \varphi) \leq \\ &\leq 2 (\dot{\lambda}(|\varphi|^2) \square \varphi, \varphi) + 2 (\dot{\lambda}(|\varphi|^2) \varphi, \square \varphi). \end{aligned}$$

PROOF. Consider the tangent vector field ξ on X locally defined by

$$\xi^\alpha = \nabla^\alpha \lambda(|\varphi|^2), \quad \xi^{\bar{\alpha}} = 0.$$

We have

$$\operatorname{div} \xi = \nabla_\alpha \xi^\alpha - 2 S^\beta_{\alpha\beta} \xi^\alpha = \nabla_\alpha \nabla^\alpha \lambda(|\varphi|^2) - 2 S^\beta_{\alpha\beta} \nabla^\alpha \lambda(|\varphi|^2)$$

$$= \ddot{\lambda}(|\varphi|^2) \nabla_\alpha |\varphi|^2 \cdot \nabla^\alpha |\varphi|^2 + \dot{\lambda}(|\varphi|^2) (\nabla_\alpha \nabla^\alpha |\varphi|^2 - 2 S^\beta_{\alpha\beta} \nabla^\alpha |\varphi|^2).$$

Since $\ddot{\lambda}(t)$ vanishes outside a bounded interval, then $\dot{\lambda}$ is bounded on $\mathbb{R}$ by a constant $c_1 > 0$. On the other hand there exists a positive constant c_2 such that

$$| \nabla' |\varphi|^2 | = | \nabla'' |\varphi|^2 | \leq c_2 |\varphi| \cdot | \nabla'' \varphi | \leq c_2 (|\varphi|^2 + | \nabla'' \varphi |^2);$$

hence

$$|\xi| \leq c_1 c_2 (|\varphi|^2 + |\nabla'' \varphi|^2).$$

Furthermore by (5)

$$|\nabla_\alpha \nabla^\alpha |\varphi|^2| \leq |\nabla'\varphi|^2 + |\nabla''\varphi|^2 + A(\varkappa_2 \varphi, \varphi) + 2|A(\widehat{\Box}\varphi, \varphi)|$$

$$\leq |\nabla'\varphi|^2 + |\nabla''\varphi|^2 + A(\varkappa_2 \varphi, \varphi) + 2|A(\Box\varphi, \varphi)| + 2|A((\widehat{\Box} - \Box)\varphi, \varphi)|$$

$$\leq |\nabla'\varphi|^2 + |\nabla''\varphi|^2 + A(\varkappa_2 \varphi, \varphi) + |\varphi|^2 + |\Box\varphi|^2 + 2|A((\widehat{\Box} - \Box)\varphi, \varphi)|.$$

Hence by (7) $\left(\text{with } \sigma = \frac{1}{4}\right)$

$$|\nabla_\alpha \nabla^\alpha |\varphi|^2 - 2S^\beta_{\alpha\beta}\nabla^\alpha|\varphi|^2| \leq \frac{3}{2}(|\nabla'\varphi|^2 + |\nabla''\varphi|^2) +$$

$$+ A(\varkappa_2\varphi, \varphi) + |\Box\varphi|^2 + (1 + F(x))|\varphi|^2,$$

with

$$F(x) = 2(f_1(x) + 4f_2(x)^2 + 4f_3(x)^2).$$

Let c_3 be a positive constant such that $\ddot{\lambda}(t) = 0$ when $t > c_3$. We have

$$|\operatorname{div}\xi| \leq c_2^2 c_3^2 \ddot{\lambda}(|\varphi|^2)|\nabla''\varphi|^2 + \dot{\lambda}(|\varphi|^2)\left\{\frac{3}{2}(|\nabla'\varphi|^2 + |\nabla''\varphi|^2) + \right.$$

$$\left. + A(\varkappa_2\varphi, \varphi) + |\Box\varphi|^2 + (1 + F(x))|\varphi|^2\right\}.$$

By (8) we have

$$F(x)|\varphi|^2 = A(\varkappa_2\varphi, \varphi) - A(\varkappa_3\varphi, \varphi).$$

Thus, by lemma 5,

$$\int_X F(x)|\varphi|^2\, dX < \infty.$$

We conclude that

$$\int_X |\xi|\, dX < \infty, \qquad \int_X |\operatorname{div}\xi|\, dX < \infty.$$

It follows from a theorem of M. P. Gaffney [2] that

$$\int_X \operatorname{div} \xi \, dX = 0,$$

i.e.

$$\int_X \ddot{\lambda}(|\varphi|^2) |\nabla' |\varphi|^2|^2 \, dX + (\dot{\lambda}(|\varphi|^2) \nabla'\varphi, \nabla'\varphi) +$$

$$+ (\dot{\lambda}(|\varphi|^2 \nabla''\varphi, \nabla''\varphi) + (\dot{\lambda}(|\varphi|^2 \varkappa_2 \varphi, \varphi) = (\dot{\lambda}(|\varphi|^2) \widehat{\square} \varphi, \varphi) +$$

$$+ (\dot{\lambda}(|\varphi|^2 \varphi, \widehat{\square} \varphi) + 2 \int_X \dot{\lambda}(|\varphi|^2) S^{\beta}_{\alpha\beta} \nabla^{\alpha} |\varphi|^2 \, dX.$$

Applying again (7) $\left(\text{with } \sigma = \frac{1}{4}\right)$ we obtain inequality (9).

Q.E.D.

Remark 1. — If the complete hermitian metric on X is a Kähler metric outside the compact K, then $\varkappa_2 = \varkappa_3$ on $X - K$. Hence, by lemma 5, inequality (9) holds whenever $\|\varphi\| < \infty$, $\|\square \varphi\| < \infty$.

2. If φ has compact support, then $\int_X \operatorname{div} \xi \, dX = 0$ for any choice of λ. Hence inequality (9) holds for all $\varphi \in \mathcal{D}^{pq}(X, E)$ and for all real C^∞ functions $\lambda = \lambda(t)$, with $\dot{\lambda}(t) \geq 0$.

§ 3. — Applications.

7. A maximum principle. Theorem I. — *Let X be a connected complex manifold satisfying the following condition.*

a) *There exists a complete hermitian metric on X and a compact set $K \subset X$ such that the hermitian form $A(\varkappa_3 \varphi, \varphi)$, acting on $C^{pq}(X, E)$, is positive semidefinite at each point of $X - K$.*

Let φ be a (p, q) form, with values in E, of class C^2 on X, such that

$$\|\varphi\| < \infty, \ \|\square \varphi\| < \infty, \ (\varkappa_2 \varphi, \varphi) < \infty. \tag{10}$$

Then at each point of X

$$|\varphi| \leq \sup_{K \cup \operatorname{Supp}(\square\varphi)} |\varphi|. \tag{11}$$

PROOF. Let $c_0 = \operatorname{Sup} |\varphi|$ on $K \cup \operatorname{Supp}(\square \varphi)$, and suppose that c_0 is finite. Let $\lambda = \lambda(t)$ be a real C^∞ function on $\mathbb{R}$ such that

$$\lambda(t) = 0 \qquad \text{for} \quad t \leq c_0^2,$$

$$\dot{\lambda}(t) > 0 \qquad \text{for} \quad t > c_0^2$$

$$\ddot{\lambda}(t) \geq 0 \quad \text{on } \mathbb{R}, \quad \text{and } \ddot{\lambda}(t) \equiv 0 \quad \text{outside a bounded interval.}$$

The right hand side of (9) vanishes, while the left hand side yields

$$(\dot{\lambda}(|\varphi|^2) \nabla' \varphi, \nabla' \varphi) + (\dot{\lambda}(|\varphi|^2) \nabla'' \varphi, \nabla'' \varphi) \leq 0.$$

Let $|\varphi| > c_0$ at some point of X. Since $\dot{\lambda}(t) > 0$ for $t > c_0$, it follows from the above inequality that $\nabla' \varphi = 0$, $\nabla'' \varphi = 0$, in a neighbourhood of that point. Hence $\nabla' |\varphi|^2 = 0$, $\nabla'' |\varphi|^2 = 0$ and therefore $|\varphi|^2$ is constant in that neighbourhood. But this is absurd, since X is connected and $|\varphi|^2$ is continuous on X. Q.E.D.

In view of remark 1 of n. 6, if the hermitian metric of X is a Kähler metric on $X - K$ then condition $(\varkappa_2 \varphi, \varphi) < \infty$ may be dropped. Hence

THEOREM I′. — *Under the same hypotheses of theorem* I *and if furthermore the complete hermitian metric on X is a Kähler metric on $X - K$, inequality* (11) *holds, provided that* $\| \varphi \| < \infty$, $\| \square \varphi \| < \infty$.

8. If $K = \emptyset$ and if the hermitian metric on X is a complete Kähler metric, the results of n. 7 can be sharpened. The most interesting result in this direction concerns the case of a square summable holomorphic section of E.

Let X be a complete connected Kähler manifold. Assume a metric along the fibers of E and consider the corresponding curvature form

$$s = (s^a_{b \bar{\beta} \alpha} \, \overline{dz^\beta} \wedge dz^\alpha) \qquad (a, b = 1, \dots, m = \operatorname{rank} E; \quad \alpha, \beta = 1, \dots, n = \dim_{\mathbb{C}} X).$$

PROPOSITION 8. — *If the hermitian form*

$$s^{b \bar{\gamma}}_{a \gamma} \, u^a (\# u)_b \tag{12}$$

is positive semidefinite (*possibly* $\equiv 0$) *at each point of X then every holomorphic section ψ of E such that $\| \psi \| < \infty$ has constant lenght on X. If the form* (12) *is positive definite at some point of X, or if X has infinite volume* (*with respect to the Kähler metric*), *then* $\psi \equiv 0$.

PROOF. The metric on X being a Kähler metric, a direct computation shows that, for every $\varphi \in C^{00}(X, E)$,

$$A(\varkappa_3 \varphi, \varphi) = s_{a\bar{\gamma}}^{b\bar{\gamma}} \varphi^a (\# \varphi)_b .$$

Since this hermitian form is positive semidefinite, proposition 4 yields:

$$(13) \quad \| \nabla' \varphi \|^2 + \| \nabla'' \varphi \|^2 \leq \| \nabla' \varphi \|^2 + \| \nabla'' \varphi \|^2 + 2(\varkappa_3 \varphi, \varphi) \leq 4 \| \bar{\partial} \varphi \|^2$$

for every $\varphi \in \mathcal{D}^{00}(X, E)$. Hence every $\varphi \in W^{00}(X, E)$ of class C^1 admits covariant derivatives $\nabla' \varphi$, $\nabla'' \varphi$, such that $\| \nabla' \varphi \| < \infty$, $\| \nabla'' \varphi \| < \infty$. Furthermore such a φ satisfies (13).

The form ψ is of type $(0, 0)$ and holomorphic. Thus

$$\bar{\partial} \psi = 0, \qquad \vartheta \psi = 0,$$

whence (Proposition 1): $\psi \in W^{00}(X, E)$. It follows from (13), that $\nabla' \psi = 0$, $\nabla'' \psi = 0$, and therefore $|\psi|$ is constant on X, $\psi \equiv 0$ if vol $X = \infty$. If (12) is positive definite at some point of X, then $\psi \equiv 0$ (in a neighbourhood of that point and therefore) on the whole manifold X. Q.E.D.

An immediate consequence of proposition 8 is the following

COROLLARY 9. — *Under the hypotheses of Proposition* 8 *the space of square integrable holomorphic sections of E has finite dimension d, with* $d \leq m = \operatorname{rank} E$ *if* $\operatorname{Vol}(X) < \infty$, $d = 0$ *otherwise.*

If E is the trivial bundle, and if the trivial metric is chosen on it, (12) vanishes identically on X. Proposition 8 yields:

If a holomorphic function on the connected manifold X is square summable with respect to a complete Kähler metric, then the function is constant on X, equal zero if the volume of X is infinite.

Let E be the holomorphic vector bundle of C^∞ $(p, 0)$-forms (with scalar values), and assume on E the metric induced by the Kähler metric of X.

The hermitian form (12) becomes, apart from an inessential positive constant factor,

$$(14) \qquad R_\beta^\alpha \, u_{\alpha A'} \, \overline{u^{\beta \bar{A}'}}$$

$R_{\beta\bar{\alpha}}$ being the Ricci tensor of X. If (14) is positive semidefinite at each point of X all square summable holomorphic p-forms on X have constant lenght on X.

The space spanned by these forms has finite dimension, which is zero if $\mathrm{Vol}(X)=\infty$, or $\leq \binom{n}{p}$ if $\mathrm{Vol}(X)<\infty$. The extreme value $\binom{n}{p}$ is attained, for istance, when X is a complex torus.

If $p=n$, E can be identified with the canonical bundle on X. The metric induced on E by the Kähler metric on X is defined locally by the function $(\det(g_{\alpha\bar{\beta}}))^{-1}$. In view of this choice, we have that

$$A(\varphi, \varphi)\, dX = \varphi \wedge \overline{\varphi}\,.$$

Thus the fact that a form is square integrable is independent of the choice of the metric on X [4]. The hermitian form (14) becomes, apart from an inessential positive constant factor,

$$R\,|\,u\,|^2\,,$$

R being the riemannian curvature of X.

Hence:

If the connected complete Kähler manifold X has riemannian curvature $R \geq 0$ everywhere on X, then every square summable holomorphic $(n, 0)$ *form φ on X has constant lenght on X. If $R > 0$ at some point of X, or if* $\mathrm{Vol}(X)=\infty$, *then $\varphi \equiv 0$.*

REFERENCES

[1] A. ANDREOTTI - E. VESENTINI, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds,* Publications Mathématiques de l'Institut des Hautes Études Scientifiques, n⁰ 25 (1965), 81-130; *Erratum,* ibd. n⁰ 27, 153-155.

[2] M. P. GAFFNEY, *A special Stokes's theorem for complete riemannian manifolds,* Ann. of Math., **60** (1954), 140-145.

[3] L. HÖRMANDER, *L^2 estimates and existence theorems for the $\bar{\partial}$-operator,* Acta Mathematica 113 (1965), 89-152.

[4] S. KOBAYASHI, *Geometry of bounded domains,* Trans. Amer. Math. Soc., 92 (1959), 267-290.

[5] E. VESENTINI, *Levi convexity of complex manifolds and cohomology vanishing theorems* Lecture notes, Tata Institute of fundamental Research (to appear).

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