

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 19,
n° 4 (1965), p. 465-479

http://www.numdam.org/item?id=ASNSP_1965_3_19_4_465_0

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A BOUNDARY VALUE PROBLEM ASSOCIATED WITH THE TRICOMI EQUATION

by ALBERT E. HEINS

I. Introduction. We shall be concerned here with a boundary value problem associated with the generalized Tricomi equation, that is, with

$$(1.1) \quad \sigma^m \left(\frac{\partial^2 \varphi}{\partial x^2} - \lambda^2 \varphi \right) + \frac{\partial^2 \varphi}{\partial \sigma^2} = 0$$

where $\operatorname{Re} \lambda \geq 0$, although we shall only discuss the case in which $\operatorname{Re} \lambda > 0$. When the parameter m is a positive and even integer, the equation (1.1) is elliptic in the x, σ plane. On the other hand, when m is odd and positive, this equation is either elliptic or hyperbolic depending on whether σ is positive or negative. The boundary data we supply, will be given on the line $\sigma = 0$ and we shall be concerned here with the elliptic domain, that is $\sigma \geq 0$. In this case, the only restriction on m is that it be positive. The example we shall consider in detail supplies $\varphi(x, \sigma)$ for $\sigma = 0, x < 0$ and $\frac{\partial \varphi}{\partial \sigma}$ for $\sigma = 0, x \geq 0$. Here we recognize that we are dealing with a boundary value problem which is a generalization of a classical example in two dimensional diffraction theory. Indeed, for $m = 0$, the above data (save for conditions at infinity) provide information for formulating the celebrated half-plane problem of diffraction theory.

Such a problem as we have described above falls in a class recently considered by C. C. Chang and T. S. Lundgren [1] and W. E. Williams [2]. These writers observed that since equation (1.1) has a variable coefficient which depends only on σ , that Fourier analysis might be applied with respect to the variable x . Aspects of this analysis have already been discussed

Pervenuto alla Redazione il 15 Marzo 1965.

by P. Germain and R. Bader [3]. For the case described above, we expect that the method of Wiener and Hopf would be available, although it would be difficult to justify it for the case $\lambda = 0$. As we shall see, however, there is no need for such machinery. In fact, if we take into account A. Weinstein's [4] basic remarks on the Tricomi equation and its fundamental solution and R. J. Weinacht's [5] discussion for the case $\lambda \neq 0$, as well as the recent function-theoretic efforts of R. C. MacCamy and the present writer [6, 7], we shall see that the boundary-value problem we have described is susceptible to the methods of analytic function theory. Indeed, the final results appear in a form which is different from the results of Williams and has the added advantage of being justifiable as well as somewhat simplified.

The method we employ here was inspired by one of the earliest fundamental papers on singular integral equations of T. Carleman [8]. A distinction which occurs between Carleman's method and the one we employ is the following. While he converted an Abel type integral equation with constant limits into one which depends on variable limits which could be solved explicitly, we have arranged matter so that there is no need to solve an integral equation. The solution of the boundary value problem is reduced to the determination of an analytic function in terms of data to be found on the line $\sigma = 0$, $-\infty < x < \infty$. The integral equation we derive serves only to provide properties of the analytic function.

II. Preliminary Reduction of Equation (1.1) and on Integral Representation. In order to reduce equation (1.1) to a more tractable form, we take advantage of the fact [4] that it may be cast into an axially symmetric wave equation by the substitution $y = A\sigma^\alpha$ where $\alpha = (2 + m)/2$ and $A = 1/\alpha$. With this change of variables, we have for equation (1.1)

$$(2.1) \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{k}{y} \frac{\partial \varphi}{\partial y} - \lambda^2 \varphi = 0$$

where $k = m/(m + 2)$ and $m > 0$.

We now ask whether we can find a representation for a solution of this equation in the half-plane $y > 0$ in terms of the boundary data on the line $y = 0$. Let us recall that Weinstein [4] has given the fundamental solution of equation (2.1) in the case $\lambda = 0$, while Weinacht [5] has given it for the case in which λ is pure imaginary. The modification for the present case is direct. Indeed, the function

$$Z(x, y, x', y') = S_{k-1} \int_0^\pi r^{-k/2} K_{k/2}(\lambda r) \sin^{k-1} \alpha \, d\alpha$$

where $r = [(x - x')^2 + y^2 + y'^2 - 2yy' \cos \alpha]^{1/2}$ is such a function. Here

$$S_{k-1}^{-1} = \int_0^\pi \sin^{k-1} \alpha \, d\alpha.$$

The function $K_\nu(x)$ is the MacDonald function, that is it is a singular solution of the equation

$$(2.2) \quad \frac{d^2 w}{dx^2} + \frac{1}{x} \frac{dw}{dx} - \left(1 + \frac{\nu^2}{x^2}\right) w = 0.$$

Indeed, if for $\nu > 0$, equation (2.2) has a solution $I_\nu(x)$ which vanishes at the origin when $\nu \neq 0$, we may define

$$K_\nu(x) = \frac{\pi}{2 \sin \nu \pi} [I_{-\nu}(x) - I_\nu(x)], \quad x > 0.$$

This fundamental solution of equation (2.1) has the following three important properties:

- (i) The y derivative of Z vanishes for $y = 0$.
- (ii) Z has a logarithmic singularity at $x = x'$, $y = y'$ and it therefore has a nonvanishing «residue» in the sense of two-dimensional potential theory. This leads us to expect a representation theorem of the Helmholtz type, that is, the case $m = 0$.
- (iii) For x', y' finite and $(x^2 + y^2)^{1/2} \rightarrow \infty$, $Z(x, y, x', y')$ is asymptotic to

$$\sqrt{\frac{\pi}{2\lambda}} (x^2 + y^2)^{-(k+1)/4} \exp[-\lambda(x^2 + y^2)^{1/2}].$$

A similar form exists upon interchanging x, y with x', y' . This function $Z(x, y, x', y')$ which we have given, can be constructed along lines suggested by Weinacht [5] and avoids the machinery of the Fourier integral theorem. The simple form of $Z(x, y, x', y')$ is noteworthy and it assumes an even more simple one at $y = 0$.

In order to derive the representation theorem, we rewrite equation (2.1) as

$$y^k \varphi_{xx} + (y^k \varphi_y)_y - \lambda^2 y^k \varphi = 0$$

and observe that Z satisfies the same equation save at the point $x = x'$, $y = y'$. We form the surface integral

$$(2.3) \quad \iint [Z \{(y^k \varphi_y)_y + y^k \varphi_{xx}\} - \varphi \{(y^k Z_y)_y + y^k Z_{xx}\}] \, dx \, dy$$

over an area bounded by the exterior closed curve C which is a portion of the x axis and a circular arc whose center is at the point $P(x', y')$ ($y' > 0$) and of radius R . The interior closed curve C_1 is a circle of radius $\varepsilon < y'$ and center at the point $P(x', y')$ is deleted, of course, because of the singularity which Z possesses at this point and therefore prevents a direct application of Green's theorem to the entire area interior to C . We shall assume, henceforth, that φ is twice continuously differentiable in the entire area inside C , and it is clear from the structure of Z that such is the case, save in the neighborhood of the point $P(x', y')$. The integral (2.3) can now be written as the sum of two line integrals

$$\int_C + \int_{C_1} y^k [Z \varphi_n - \varphi Z_n] ds$$

and because of the regularity properties of Z and φ , this line integral vanishes. The path C is traversed in a counter-clockwise sense, while C_1 is traversed in a clockwise sense. The normal derivatives are exterior ones and s is the arc-length parameter along C and C_1 .

We first ask what is the effect of permitting $\varepsilon \rightarrow 0$. Since

$$Z(x, y, x', y') = -2^{k+1} \lambda^{-k} \Gamma\left(\frac{k}{2}\right) S_{k-1} (yy')^{-\frac{k}{2}} \ln \varepsilon + Q(x, y)$$

where $y \neq 0$, $y' \neq 0$, $\varepsilon = [(x - x')^2 + (y - y')^2]^{1/2}$ and $Q(x, y)$ is a function which may be neglected when calculating the « residue », we obtain the following along C_1 . We parameterize C_1 by writing $x = x' + \varepsilon \cos \theta$, $y = y' + \varepsilon \sin \theta$ with $0 \leq \theta \leq 2\pi$ and note that $ds = \varepsilon d\theta$. Then since $y^k = (y')^k$ and $\varepsilon \ln \varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, the first line integral over C_1 vanishes. The second line integral over C_1 has the limit

$$\pi 2^{k/2} \lambda^{-k/2} \Gamma\left(\frac{k}{2}\right) S_{k-1} \varphi(x', y').$$

On the circular arc of radius R we shall assume that the leading terms in $\varphi(x, y)$ satisfy a « radiation condition » when $R \rightarrow \infty$. That is,

$$\lim_{R \rightarrow \infty} R^{(1+k)/2} [\varphi_R + \lambda \varphi] \rightarrow 0, \quad \lambda > 0,$$

uniformly for all $\theta = \arctan y/x$, $y > 0$. This radiation condition is adequate to eliminate the contribution along the circular arc of radius R as $R \rightarrow \infty$. It may be noted that the above asymptotic condition is the same for the fundamental solution.

Since $y^k \varphi_x = -y^k \varphi_y = A^k \varphi_\sigma$ on $y = 0$ (or $\sigma = 0$) and since $Z_y = 0$ on $y = 0$, the integral along the x axis takes the form

$$A^k \int_{-\infty}^{\infty} Z(x, 0, x', y') \varphi_\sigma(x, 0) dx$$

in the limit $R \rightarrow \infty$. Hence we have the representation [9]

$$(2.4) \quad \frac{2^{\frac{k}{2}-1}}{\pi} \varphi(x, y) = A^k \lambda^{\frac{k}{2}} \int_{-\infty}^{\infty} \frac{K_{k/2} [\lambda \sqrt{(x-x')^2 + y^2}] \varphi_\sigma(x', 0) dx'}{[(x-x')^2 + y^2]^{k/4}}$$

upon interchanging x and x' and y' by y . If $\varphi_\sigma(x', 0)$ is known for $-\infty < x' < \infty$ we have a means of calculating $\varphi(x, y)$ for all x and y since this representation is even in y . However, for the problem we wish to discuss, $\varphi(x, 0)$ is known for $x \leq 0$ and $\varphi_\sigma(x, 0)$ is known for $x \geq 0$. Hence (2.4) provides an integral equation for the unknown function $\varphi_\sigma(x, 0)$, $x \leq 0$.

III. The Boundary Value Problem and its Solution. On the line $y = 0$, equation (2.4) becomes a representation for $\varphi(x, 0)$ in terms of $\varphi_\sigma(x, 0)$. Indeed we have

$$(3.1) \quad B \varphi(x, 0) = \int_{-\infty}^{\infty} \frac{K_{k/2} [\lambda |x-x'|] \varphi_\sigma(x', 0) dx'}{|x-x'|^{k/2}}$$

where

$$B = \frac{2^{\frac{k}{2}-1}}{\pi} A^{-k} \lambda^{-k/2},$$

and $k = m/(m+2) < 1$ since m is a real positive number. Since $m > 0$, the kernel of the integral equation (3.1) is integrable in the neighborhood of the point $x = x'$. If $\varphi_\sigma(x, 0)$ is known for $-\infty < x < \infty$ we have a representation for finding $\varphi(x, 0)$. If on the other hand we assign $\varphi(x, 0) = f(x)$ for $x < 0$ and $\varphi_\sigma(x, 0) = g(x)$ for $x > 0$, equation (3.1) becomes an integral equation for $\varphi_\sigma(x, 0)$. For $x < 0$, we have

$$(3.2) \quad \int_{-\infty}^0 \frac{K_{k/2} [\lambda |x-x'|] \varphi_\sigma(x', 0) dx'}{|x-x'|^{k/2}} = Bf(x) - \int_0^{\infty} \frac{K_{k/2} [\lambda |x-x'|] g(x') dx'}{|x-x'|^{k/2}}.$$

If we know the first integral in (3.2) for $x > 0$, we have a means of computing $\varphi(x, 0)$ for $x > 0$ by equation (3.1). Further, it is possible to find $\varphi(x, y)$, $y > 0$ in the large by the Poisson type representation [10]

$$(3.3) \quad \varphi(x, y) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{\lambda y}{2}\right)^{1-\nu} \int_{-1}^1 I_{\nu-1}(\lambda y \sqrt{1-t^2}) (1-t^2)^{\frac{\nu-1}{2}} \varphi(x+iyt, 0) dt$$

where $2\nu = k$ and which incidentally requires the continuation of $\varphi(x, 0)$, $-\infty < x < \infty$ into the complex domain. A further alternative is available in that we may solve an integral equation of the Abel type for $\varphi_\sigma(x, 0)$ and may therefore use equation (2.4) to find $\varphi(x, y)$ in the large.

We shall concentrate our efforts in this paper on finding the unknown portion of $\varphi(x, 0)$, that is for $x > 0$. Following Carleman, we introduce an analytic function $F(z)$ which is regular in the plane cut along the negative real axis and is given by the integral

$$F(z) = \int_{-\infty}^0 \frac{K_{k/2}[\lambda(z-t)] \varphi_\sigma(t, 0) dt}{(z-t)^{k/2}}, \quad z = x + iy.$$

Let us assume, for the time being, that there does exist an appropriate $\varphi_\sigma(x, 0)$, $x < 0$ which will produce such a function $F(z)$ and inquire why such a form was chosen. Clearly for $x > 0$, $y = 0$, a knowledge of $F(x)$ will give us $\varphi(x, y)$ $x > 0$. In fact, for $x > 0$

$$B\varphi(x, 0) = F(x) + \int_0^\infty \frac{K_{k/2}[\lambda|x-x'|] g(x') dx'}{|x-x'|^{k/2}}.$$

From this we see that there are immediate integrability requirements on $g(x)$, although they are not particularly severe in the light of the asymptotic form of $K_{k/2}(\lambda x)$, $x \rightarrow \infty$.

We now define the following limits, assuming for the present, that they exist. Let

$$F(x + i0) = \lim_{\varepsilon \rightarrow 0} F(x + i\varepsilon), \quad \varepsilon > 0$$

and

$$F(x - i0) = \lim_{\varepsilon \rightarrow 0} F(x - i\varepsilon), \quad \varepsilon > 0.$$

These limits are, of course, dependent on $\varphi_\sigma(x, 0)$ for $x < 0$, a function which we do not know yet. For $x < 0$

$$(3.4) \quad F(x + i0) = \int_{-\infty}^x \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda(x-t)] dt}{(x-t)^{k/2}} \\ + \exp\left(\frac{-ik\pi}{2}\right) \int_x^0 \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda e^{i\pi}(t-x)] dt}{(t-x)^{k/2}}$$

and

$$F(x - i0) = \int_{-\infty}^x \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda(x-t)] dt}{(x-t)^{k/2}} \\ + \exp\left(\frac{ik\pi}{2}\right) \int_x^0 \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda e^{-i\pi}(t-x)] dt}{(t-x)^{k/2}}.$$

Upon simplifying the expression for $F(x + i0)$ and $F(x - i0)$, we get

$$F(x + i0) = \int_{-\infty}^x \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda(x-t)] dt}{(x-t)^{k/2}} \\ + \int_x^0 \frac{\varphi_\sigma(t, 0)}{(t-x)^{k/2}} \{ \exp(-ik\pi) K_{k/2}[\lambda(t-x)] - i\pi \exp(-ik\pi/2) I_{k/2}[\lambda(t-x)] \} dt$$

and

$$F(x - i0) = \int_{-\infty}^x \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda(x-t)] dt}{(x-t)^{k/2}} \\ + \int_x^0 \frac{\varphi_\sigma(t, 0)}{(t-x)^{k/2}} \{ \exp(ik\pi) K_{k/2}[\lambda(t-x)] + i\pi \exp(ik\pi/2) I_{k/2}[\lambda(t-x)] \} dt.$$

The integral equation is equivalent to

$$\psi(x) = \int_{-\infty}^x \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda(x-t)] dt}{(x-t)^{k/2}} + \int_x^0 \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda(t-x)] dt}{(t-x)^{k/2}}$$

where $\psi(x)$ is a known function of x which depends on $f(x)$ and $g(x)$. It is possible to eliminate all integrals in these last three equations and obtain a linear relation between $F(x+i0)$, $F(x-i0)$ and $\psi(x)$ which will be valid for $x < 0$. We obtain thus

$$(3.5) \quad F(x+i0) + \exp(-ik\pi)F(x-i0) = 2\psi(x)(\cos k\pi/2) \exp(-ik\pi/2).$$

For $x > 0$, $F(x+i0) - F(x-i0) = 0$.

A simplification can be effected if we put $F(z) = z^\beta \Phi(z)$ where β will be ultimately chosen conveniently. The branch of z^β which we choose is cut along the negative real axis, so that

$$\Phi(x+i0) = \Phi(x-i0), \quad x > 0.$$

For $x < 0$

$$F(x+i0) = (-x)^\beta e^{\beta i\pi} \Phi(x+i0)$$

and

$$F(x-i0) = (-x)^\beta e^{-\beta i\pi} \Phi(x-i0).$$

Hence equation (3.5) becomes

$$(3.6) \quad e^{\beta i\pi} \Phi(x+i0) + e^{-k i\pi - \beta i\pi} \Phi(x-i0) = \frac{2\psi(x) e^{-\frac{ik\pi}{2}} \cos \frac{k\pi}{2}}{(-x)^\beta}$$

If we now choose $2\beta + k = 1$, equation (3.6) becomes

$$(3.7) \quad \Phi(x+i0) - \Phi(x-i0) = \frac{2\psi(x) \cos \frac{k\pi}{2}}{i(-x)^\beta}, \quad x < 0$$

and

$$(3.8) \quad \Phi(x+i0) - \Phi(x-i0) = 0, \quad x > 0.$$

Hence we have information regarding $\Phi(z)$ on the line $y=0$ which will enable us to determine $\Phi(z)$, once we know the behavior of $\Phi(z)$ at infinity. This, of course, can be determined by a careful study of $F(z)$ in the closed sector $-\pi \leq \arg z \leq \pi$, recalling the assumption we stipulated for $\varphi_\sigma(x, 0)$, $x < 0$. Indeed, we find [6,7] that $F(z)$ possesses an exponential factor $\exp(-\lambda z)$ in this closed sector of $|z| \rightarrow \infty$ and we eliminate it by writing

$$\Phi(z) = \Psi(z) \exp(-\lambda z)$$

so that we get

$$(3.9) \quad \Psi(x+i0) - \Psi(x-i0) = \frac{2\psi(x) \exp(\lambda x) \cos \frac{k\pi}{2}}{i(-x)^\beta}, \quad x < 0$$

and

$$(3.10) \quad \Psi(x + i0) - \Psi(x - i0) = 0, \quad x > 0.$$

Now since $\Psi(z) = F(z) z^{-\beta} \exp(\lambda z)$ vanishes for $|z| \rightarrow \infty$, $-\pi \leq \arg z \leq \pi$, we can determine it with the aid of a Cauchy type representation as

$$(3.11) \quad \begin{aligned} \Psi(z) &= -\frac{1}{\pi} \int_{-\infty}^0 \frac{\psi(t) \exp(\lambda t) \cos \frac{k\pi}{2}}{(-t)^\beta (t-z)} dt + E(z) \\ &= \frac{1}{\pi} \int_0^{\infty} \frac{\psi(-t) \exp(-\lambda t) \cos \frac{k\pi}{2}}{t^\beta (t+z)} dt + E(z) \end{aligned}$$

or

$$\begin{aligned} F(z) &= \frac{z^\beta \exp(-\lambda z)}{\pi} \int_0^{\infty} \frac{\psi(-t) \exp(-\lambda t) \cos \frac{k\pi}{2}}{t^\beta (t+z)} dt \\ &\quad + z^\beta \exp(-\lambda z) E(z) \end{aligned}$$

where $E(z)$ is an entire function.

We have not yet put down any conditions on $f(t)$, $t < 0$ and $g(t)$, $t > 0$. Without these, we cannot demonstrate the validity of our calculations, for example that the $F(z)$ we found possesses appropriate properties which will generate an acceptable $\varphi_\sigma(x, 0)$, $x < 0$. We shall devote the next section to a discussion of such questions.

IV. Verification. Before we can demonstrate that the $F(z)$ we have given in (3.12) is an acceptable one, it is necessary to supply some conditions for $f(t)$ and $g(t)$ which will insure the existence of certain integrals. We also require some conditions which will give us some information about $F(x)$ for $|x| \rightarrow 0$ and $|x| \rightarrow \infty$. Such information will enable us to determine the asymptotic behavior of $\varphi(x, 0)$ for $x \rightarrow 0^-$ and $x \rightarrow \infty$ as well as some of the properties of $\varphi_\sigma(x, 0)$ when $x \rightarrow 0^-$ and $x \rightarrow -\infty$. One such set of conditions will be given but it will be clear that others may be found. We shall only treat the case for which $\varphi_\sigma(t, 0) = g(t) \equiv 0$ ($t > 0$). To include this term will only lead us to some detailed analysis which will not add any interesting ideas.

We start then, by examining the behavior of the function

$$(4.1) \quad F_1(z) = \frac{z^\beta \exp(-\lambda z)}{\pi} \int_0^\infty \frac{f(-t) \exp(-\lambda t) dt}{t^\beta (t+z)}, \quad z = x + iy$$

for $|x| \rightarrow 0$ and $|x| \rightarrow \infty$, $y = 0$. Let us note that the integral is either a Stieltjes transform and is a function of the complex variable z or it may be viewed as a Hilbert transform when $y = 0$. Suppose that $f(-t)$ is real, quadratically integrable for $0 \leq t \leq \infty$ and $\in C'$ and further that $f(-0) = A_1$ (a constant). Then a direct application of the Schwarz inequality guarantees, in view of the restriction on β , the existence of the integral

$$\int_0^t f(-\tau) \tau^{-\beta} \exp(-\lambda \tau) d\tau$$

for all positive t . We denote the limit of this integral as $t \rightarrow \infty$ by A_2 . Since we may define $f(-t) \equiv 0$, for $t < 0$, we observe that $f(-t) t^{-\beta} \exp(-\lambda t)$ is quadratically integrable along the entire t axis and therefore the integral in (4.1) defines a quadratically integrable function almost everywhere for real z [11, p. 121]. This assures that the integral in (4.1) vanishes for $|x| \rightarrow \infty$, $y = 0$.

We can, however, obtain more precise information with our hypotheses on $f(-t)$. According to an Abelian theorem for the ordinary Stieltjes transform [12, p. 185] (that is, the integral in (4.1) with $y = 0$, $x \geq 0$) we have that the integral in (4.1) is asymptotic to $B_1 x^{-\beta}$, $x \rightarrow 0^+$ and to $B_2 x^{-1}$, $x \rightarrow \infty$ where B_1 and B_2 are known constants. Hence, we have that

$$F_1(x) \sim B_1 / \pi, \quad x \rightarrow 0^+$$

and

$$F_1(x) \sim B_2 x^{\beta-1} \pi^{-1} \exp(-\lambda x), \quad x \rightarrow +\infty.$$

It follows that $F_1(x) \equiv \varphi(x, 0)$ obeys the « radiation condition » when $x \rightarrow +\infty$. It is possible to carry through this discussion for λ complex and suitably limited, that is $\text{Re } \lambda \geq 0$ but we shall not pursue these details here.

Further information is available from the theory of the ordinary Stieltjes transform. Let us suppose that $-\pi < \arg z < \pi$. Then the integral in (4.1) converges and in fact represents a single-valued analytic function of z which has a cut along the negative real axis, provided that $f(-t)$ is subject to the conditions we gave. For x real and negative, we have an example

of the limiting value of a Cauchy type integral from which we may derive for $x < 0$

$$(4.2) \quad F_1(x + i0) = (-x)^\beta \exp(\beta i\pi) \exp(-\lambda x) \cdot \left[\oint_0^\infty \frac{\exp(-\lambda t) f(-t) dt}{\pi t^\beta (t+x)} - \frac{if(x) \exp(\lambda x)}{(-x)^\beta} \right]$$

and

$$(4.3) \quad F_1(x - i0) = (-x)^\beta \exp(-\beta\pi) \exp(-\lambda x) \cdot \left[\oint_0^\infty \frac{\exp(-\lambda t) f(-t) dt}{\pi t^\beta (t+x)} + \frac{if(x) \exp(\lambda x)}{(-x)^\beta} \right].$$

The integrals are taken in the sense of the Cauchy principal value.

We shall have need of the asymptotic behavior of (4.1) in the limit $x \rightarrow 0^-$, ($y = 0$), since as we shall see presently, it will be needed to provide information regarding the behavior of $q_\sigma(x, 0)$, $x < 0$. In order to determine this behavior, we write

$$\begin{aligned} \oint_0^\infty \frac{f(-t) \exp(-\lambda t) dt}{t^\beta (t+x)} &= \oint_0^2 \frac{f(-t) \exp(-\lambda t) dt}{t^\beta (t+x)} \\ &+ \int_2^\infty \frac{f(-t) \exp(-\lambda t) dt}{t^\beta (t+x)} \end{aligned}$$

where $0 < -x < 2$. The last integral is clearly $O(1)$ as $x \rightarrow 0^-$. The second integral may be examined with the aid of the theory of Abelian asymptotics of such principal value integrals. We put $t = 1 + \tau$ and $-x = 1 + \zeta$. Then

$$\oint_0^2 \frac{f(-t) \exp(-\lambda t) dt}{t^\beta (t+x)} = \oint_{-1}^1 \frac{\exp(-\lambda - \lambda\tau) f(-1 - \tau) d\tau}{(1 + \tau)^\beta (\tau - \zeta)}.$$

We may write $f(-1 - \tau) = A_1 + \theta(\tau)$ where $\theta(-1) = 0$. If we now impose the additional requirement on $\theta(\tau)$, that it obeys a uniform Lipschitz

condition of positive order $\epsilon > \beta$, we may apply Tricomi's discussion to get

$$\oint_{-1}^1 \frac{\exp(-\lambda - \lambda\tau) f(-1 - \tau) d\tau}{(1 + \tau)^{-\beta} (\tau - \zeta)} \sim A_1 \cot \beta\pi (1 + \zeta)^{-\beta} + O(1), \zeta \rightarrow -1^-$$

or

$$\oint_0^\infty \frac{\exp(-\lambda t) f(-t) dt}{t^\beta (t + x)} \sim A_1 \cot \beta\pi (-x)^{-\beta} + O(1), x \rightarrow 0^-.$$

Now we have information which will enable us to examine $\varphi_\sigma(x, 0)$ for $x < 0$ and $E(z)$. To begin with, we note that

$$E(z) = z^{-\beta} \exp(\lambda z) [F(z) - B(\cos k\pi/2) F_1(z)].$$

This relation implies that $E(z)$, an entire function of z , vanishes at infinity and therefore vanishes everywhere. Hence

$$F(z) = B(\cos k\pi/2) F_1(z).$$

For $x < 0$, we have

$$\begin{aligned} & \int_{-\infty}^x \varphi_\sigma(t, 0) \frac{K_{k/2}[\lambda(x-t)] dt}{(x-t)^{k/2}} + \\ & \exp(-ik/2) \int_x^0 \frac{\varphi_\sigma(t, 0)}{(t-x)^{k/2}} \{ \exp(-ik\pi/2) K_{k/2}[\lambda(t-x)] - i\pi I_{k/2}[\lambda(t-x)] \} dt \\ & = B(-x)^\beta \exp(\beta i\pi) \exp(-\lambda x) (\cos k/2) \cdot \\ & \quad \cdot \left[\oint_0^\infty \frac{\exp(-\lambda t) f(-t) dt}{\pi t^\beta (t+x)} - \frac{if(x) \exp(\lambda x)}{(-x)^\beta} \right]. \end{aligned}$$

Upon using the definition of $K_{k/2}$ in terms of $I_{k/2}$ and $I_{-k/2}$ we get an

Abel-type integral equation for $\varphi_\sigma(x, 0)$, that is

$$\pi \int_x^0 \frac{I_{-k/2} [\lambda (t - x)]}{(t - x)^{k/2}} \varphi_\sigma(t, 0) dt = B (\sin k\pi/2) f(x). \tag{4.4}$$

$$- \frac{B (-x)^\beta (\cos k\pi/2) \exp(-\lambda x)}{\pi} \int_0^\infty \frac{f(-t) \exp(-\lambda t) dt}{t^\beta (t + x)}, x < 0.$$

In the neighborhood of the left side of the origin, that is $x \rightarrow 0^-$, $B (\sin k\pi/2) f(x) = B (\sin k\pi/2) A_1 + O(x)$, while the term containing the principal value integral is equal to $-B (\sin k\pi/2) A_1 + O[(-x)^\beta]$. Hence the right side of the equation (4.4) is $O[(-x)^\beta]$, $x \rightarrow 0^-$. From these remarks we may conclude that $\varphi_\sigma(x, 0)$ is integrable in the neighborhood of the origin, that is as $x \rightarrow 0^-$.

We are now left with the task of describing $\varphi_\sigma(x, 0)$ when $x \rightarrow -\infty$. In order to do this, we note that

$$\int_{-\infty}^0 \varphi_\sigma(t, 0) \frac{K_{k/2} [\lambda (x - t)]}{(x - t)^{k/2}} dt$$

is asymptotic to

$$D x^{\beta-1} \exp(-\lambda x) \int_{-\infty}^0 \exp(\lambda t) \varphi_\sigma(t) dt$$

when $x \rightarrow \infty$ and D is a known constant. The x dependence agrees with that of $F_1(x)$ when $x \rightarrow \infty$, so that we are left with the implication that the integral

$$\Phi_\sigma(\lambda) = \int_{-\infty}^0 \exp(\lambda t) \varphi_\sigma(t, 0) dt$$

exists. Thus $\varphi_\sigma(t, 0)$ cannot have an exponential growth which exceeds $\exp(-\lambda t)$, $t \rightarrow -\infty$ and indeed may be less than this. If we now replace x by $-x$ and t by $-t$ in equation (4.4), it will assume a more conventional form from which we can read off some of the properties of $\Phi_\sigma(\lambda)$.

Let us observe that the unilateral Laplace transform of $t^{-k/2} I_{k/2}(\lambda t)$, that is

$$\int_0^{\infty} t^{-k/2} I_{k/2}(\lambda t) \exp(-st) dt$$

is analytic in the right-half plane $\operatorname{Re} s > \lambda > 0$ of the complex s plane, while the first and second terms on the right side of the equation have unilateral Laplace transforms which are analytic in the right half planes $\operatorname{Re} s > 0$ and $\operatorname{Re} s > \lambda$ respectively. Hence the unilateral transform of $\varphi_{\alpha}(-t, 0)$ is analytic in the right half-plane $\operatorname{Re} s > \lambda$. This transform in turn can be represented as the quotient of two unilateral Laplace transforms and is $O(s^{\beta-1})$ for $\operatorname{Re} s > \lambda$ and $|s| \rightarrow \infty$. Since $\beta < 1$, this quotient vanishes at infinity in this half plane.

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*This work was supported in part by the Air Force Office
of Scientific Research.*

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