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THE RIEMANN-ROCH THEOREM FOR ALGEBRAIC CURVES

by A. MATTUCK and A. MAYER (*)

The Riemann-Roch theorem for a divisor $\mathfrak{a}$ on an irreducible algebraic curve C is

$$\dim \mathfrak{a} = \deg \mathfrak{a} - g + i(\mathfrak{a}).$$

The easy part of this result is that if the degree of $\mathfrak{a}$ is sufficiently large, then

$$\dim \mathfrak{a} = \deg \mathfrak{a} - g.$$

This is called the Riemann theorem; it is just a special case of the Hilbert postulation formula for polynomial ideals and may be therefore considered to be a relatively elementary result. The full Riemann-Roch theorem is deeper because the « index of specialty » term $i(\mathfrak{a})$ must be introduced and interpreted, and proofs of the theorem can be classified according to how this is done.

In the older German-Italian proofs, it is connected with the adjoint curves to a certain type of plane model of C; in Andre Weil's well-known proof, it is the dimension of the dual of a certain space of « repartitions »; in the similar sheaf-theoretic proof, it is the dimension of a certain $H^1(C, L)$. Yet ultimately what one wants $i(\mathfrak{a})$ to be is the number of independent holomorphic differentials on C having zeros at the points of $\mathfrak{a}$ (if, say, $\mathfrak{a}$ is positive). So all of these proofs have a second part making this identification, and generally speaking, the less the work that has gone into the formula, the greater the labor of this second step.

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Of course, to get the full theory of curves one must have the Jacobian variety as well. The classical construction of the Jacobian, as well as the abstract construction by A. Weil, both used the Riemann-Roch theorem in an essential way. On the other hand, Chow subsequently gave a projective construction of the Jacobian which used only the Riemann theorem. This exhibited the basic character of the Jacobian in a new way. It was taken up by Matsusaka and later by Grothendieck in their work on the Picard variety, and by now it appears that in the study of linear systems of divisors on a variety, the theory develops most naturally and with the fewest artificialities if the construction of the Picard variety comes first.

As an illustration of this, we assume here Chow's construction of the Jacobian, and show how on this basis the Riemann-Roch theorem can be formulated and proved quite naturally. An outstanding virtue of our method, we feel, is that the $i(\mathfrak{a})$ is right from the beginning the dimension of a space of differentials, and not « differentials ». We lay emphasis also on the formulation here given of the theorem, which we believe reveals its geometric significance in a striking way.

We assume familiarity with the foundations, with the geometrical theory of linear systems, as well as with Chow's construction of the Jacobian and one or two other facts about it, which are summarized briefly in § 3.

We begin with a preliminary section giving a differential analogue to the Newton identities of classical algebra; this is then applied to a discussion of differentials on symmetric products. The needed facts about the Jacobian are summarized, and the Riemann-Roch theorem is stated. We next relate our statement of this theorem to the classical formulation given above, and finally go on to prove it.

1. Some classical algebra.

Let $t_1, \dots, t_n$ be n independent transcendentals, and let $\sigma_1, \dots, \sigma_n, \dots$ be the elementary symmetric functions of the t_i. We make here the convention that $\sigma_i = 0$ when $i > n$. The well-known Newton identities express the sums of powers

$$s_j = t_1^j + \dots + t_n^j$$

recursively in terms of the σ_i. These identities have integral coefficients, and are valid in all characteristics. But if you try to invert them in order to express the σ_i in terms of the s_j, you get denominators, and so this inversion is impossible in characteristic p.

It is amusing therefore that analogous identities expressing the differentials

(1) $$\tau_j = t_1^j \, dt_1 + \dots + t_n^j \, dt_n , \qquad j = 0, 1, 2, \dots$$

in terms of the elementary symmetric differentials $d\sigma_1 , \dots , d\sigma_n$ do turn out to be invertible in all characteristics. To get these identities, we start with the identities

(2) $$\sigma_k - \sigma_{k-1} t_i + \dots + (-1)^k t_i^k = \sigma_k (i).$$

Here $\sigma_k (i)$ is the k^{th} elementary symmetric function of the $n-1$ transcendentals $t_1 , \dots , t_{i-1} , t_{i+1} , \dots , t_n$, and of course by our convention, $\sigma_k (i) = 0$ automatically when $k \geqq n$. To prove (2), it is enough to divide each σ_k into the sum of terms containing t_i and the rest of the terms:

$$\sigma_{k-r} = \sigma_{k-r} (i) + t_i \, \sigma_{k-r-1} (i),$$

then multiply by $(-1)^r t_i^r$ and finally sum on r, so that the successive terms cancel out.

If one sums the identities (2) on i, one gets Newton's identities. But if before summing, they are multiplied by dt_i, summing then gives

(3) $$\sigma_k \tau_0 - \sigma_{k-1} \tau_1 + \dots + (-1)^k \tau_k = d\sigma_{k+1} .$$

These are the identities we had in mind: the $d\sigma_k$ occur with unit coefficients, and the two sets of differentials mutually determine each other in all characteristics.

2. Symmetric Products.

Let C be a complete nonsingular algebraic curve, defined over an algebraically closed field k of arbitrary characteristic. Our proof of the Riemann-Roch theorem will take place on $C(n)$, the n-fold symmetric product of C, so we need to say a little about this variety. $C(n)$ is a

(1) See [1]. Briefly, this is true when C is the affine line since $C(n)$ is then just affine n-space, according to the fundamental theorem espressing every symmetric polynomial in terms of the elementary symmetric polynomials. An arbitrary nonsingular C is locally analytically isomorphic to the line everywhere.

nonsingular [1] n-dimensional variety, the quotient of the n-fold Cartesian product $C[n]$ under the action of the symmetric group on n letters. The points of $C(n)$ represent naturally and one-to-one the positive divisors of degree n on C, and so in the sequel we will let $\mathfrak{a} = \mathfrak{p}_1 + \dots + \mathfrak{p}_n$ denote either a divisor or the corresponding point of $C(n)$.

To calculate on $C(n)$, we use the following local parameters. Let $\mathfrak{p}$ be a point of C, t a local parameter at $\mathfrak{p}$, and $t_1, \dots, t_n$ replicas of t; then we use the elementary symmetric functions $\sigma_1(t_1, \dots, t_n), \dots, \sigma_n(t_1, \dots, t_n)$ as local parameters at the point $n\mathfrak{p}$ on $C(n)$. More generally, the obvious natural map

$$s: C(n_1) \times \dots \times C(n_r) \longrightarrow C(n), \qquad n = n_1 + \dots + n_r \tag{4}$$

is an analytic isomorphism at a point $n_1 \mathfrak{p}_1 \times \dots \times n_r \mathfrak{p}_r$, $\mathfrak{p}_i \neq \mathfrak{p}_j$ if $i \neq j$. This is easily checked when C is a line A; the general case follows from this because $C(n)$ and $A(n)$ are locally analytically isomorphic. Using (4), we see that at the point $n_1 \mathfrak{p}_1 + \dots + n_r \mathfrak{p}_r$ on $C(n)$, a local calculation can be performed instead on $C(n_1) \times \dots \times C(n_r)$, so that we can use r sets of elementary symmetric functions as local parameters.

The Riemann-Roch theorem is concerned with 1-forms (differentials) on C; we look therefore at the relation between these and the 1-forms on $C(n)$.

If X is a variety, we denote by $\mathfrak{D}(X)$ the k-space of holomorphic 1-forms on X.

PROPOSITION. The spaces $\mathfrak{D}(C)$ and $\mathfrak{D}(C(n))$ are naturally isomorphic, and if φ and Φ are corresponding 1-forms under this isomorphism,

$$(\varphi) \geqq \mathfrak{a} \iff \Phi = 0 \text{ at } \mathfrak{a}.$$

PROOF. We will identify both spaces with the space $\mathfrak{D}_S(C[n])$ of *symmetric* 1-forms on the direct product $C[n]$.

Map $\mathfrak{D}(C) \longrightarrow \mathfrak{D}_S(C[n])$ by sending

$$\Phi \longrightarrow \Phi' = \Phi_1 + \dots + \Phi_n, \qquad \Phi_i = (\delta\, pr_i)\, \Phi.$$

The map is bijective, because a holomorphic 1-form on a product of projective varieties is the sum of holomorphic 1-forms coming from the factors.

Map $\mathfrak{D}(C(n)) \longrightarrow \mathfrak{D}_S(C[n])$ by lifting the 1-forms from $C(n)$ to the finite covering $C[n]$. This is injective since the covering is separable; to finish the proof, we have to show that it is surjective.

To this end, let $t_1, \dots, t_n$ be n copies of a separating variable for the function field $k(C)$, and $\sigma_1, \dots, \sigma_n$ the corresponding elementary symmetric functions. Since the latter form a separating transcendence base for $k(C(n))$

and $k(C[n])$, the 1-form $d\sigma_1, \dots, d\sigma_n$ form a base for the 1-forms on $C(n)$ and their liftings give a base for the 1-forms on $C[n]$. Now the identities (3) show that the 1-forms $\tau_0, \dots, \tau_{n-1}$ can be invertibly expressed in terms of the $d\sigma_i$, so that these can be viewed as 1-forms on $C(n)$ and they also form a base.

Getting on with the proof of surjectivity, suppose that $\varphi' = \Sigma \varphi_i$ is given on $C[n]$. By the preceding, we can write

$$\varphi' = f_0 \tau_0 + \dots + f_{n-1} \tau_{n-1} .$$

The f_i are in $k(C(n))$ because φ' and the τ_i are symmetric and the τ_i form a base. Thus φ' is the lifting of a φ''; what we have to show is that the f_i are holomorphic on $C(n)$, so that φ' will be the lifting of a *holomorphic* Φ on $C(n)$.

Suppose first that we are at a point $n\mathfrak{q}$ on $C(n)$, and that t has been selected to be a local parameter at $\mathfrak{q}$, which makes the σ_i local parameters at $n\mathfrak{q}$. We have, for some large m,

$$\varphi = (a_0 + a_1 t + \dots + a_{m-1} t^{m-1} + g(t))\, dt, \qquad \text{so}$$

$$\varphi' = a_0 \tau_0 + a_1 \tau_1 + \dots + a_{m-1} \tau_{m-1} + \Sigma g(t_i)\, dt_i$$

where $g(t_i) \in \mathfrak{m}^m$, $\mathfrak{m}$ being the maximal ideal in the local ring of the point $(\mathfrak{q}, \dots, \mathfrak{q})$ on $C[n]$ By using now the identities (3) recursively, the τ_i for $i \geqq n$ can be expressed in terms of the lower ones and the σ_i, so that from the above we get polynomials $P_i(\sigma_1, \dots, \sigma_n)$ such that

$$\varphi' = P_0 \tau_0 + \dots + P_{n-1} \tau_{n-1} + \Sigma g(t_i)\, dt_i .$$

Comparing this with the first expression for φ' shows that

$$(f_0 - P_0) \tau_0 + \dots + (f_{n-1} - P_{n-1}) \tau_{n-1} = \Sigma g(t_i)\, dt_i .$$

If now we equate the coefficients of dt_i on both sides, we get a system of linear equations for $f_i - P_i$, so that, by Cramer's rule, $f_i - P_i \in \mathfrak{m}^{m-m_0}$. Thus f_i is holomorphic on $C[n]$ and therefore on $C(n)$ as well.

Moreover, since $\sigma_k = 0$ at $n\mathfrak{q}$, the identities (3) show that $\tau_k = 0$ at $\mathfrak{a}$, for $k \geqq n$. Thus $\Phi = a_0 \tau_0 + \dots + a_{n-1} \tau_{n-1}$ at $\mathfrak{a}$, so that $\Phi = 0$ at $n\mathfrak{q}$ on $C(n)$ if and only if $(\varphi) \geqq n\mathfrak{q}$ on C, since both are equivalent to the vanishing of $a_0, \dots, a_{n-1}$.

All this was supposing $\mathfrak{a} = n\mathfrak{q}$. For the general case, suppose that $\mathfrak{a} = n_1 \mathfrak{q}_1 + \dots + n_r \mathfrak{q}_r$, $\mathfrak{q}_i \neq \mathfrak{q}_j$. Then φ on C corresponds by the preceding to the holomorphic 1-form Φ_k on $C(n_k)$, to Φ on $C(n)$, and

$$(\delta s)\varphi = \Phi_1 + \dots + \Phi_r$$

where s is the map (4) above. By the foregoing case, the Φ_k are holomorphic at $n_k \mathfrak{q}_k$, so that $(\delta s)\Phi$ is holomorphic, and such is also Φ since s is a local analytic isomorphism And in the same way, $(\Phi) \geqq n_k \mathfrak{q}_k$ $(k = 1, \dots, r)$

$$\Longleftrightarrow \Phi_k = 0 \text{ at } n_k \mathfrak{q}_k \Longleftrightarrow (\delta s)\Phi = 0 \text{ at } (n_1 \mathfrak{q}_1, \dots, n_r \mathfrak{q}_r)$$

$\Longleftrightarrow \Phi = 0$ at $\mathfrak{a}$, this last again since s is a local analytic isomorphism.

3. The Jacobian.

We fix once for all a reference point $\mathfrak{p}_0$ on our curve C. Then by using $\mathfrak{p}_0$ we get, for $m < n$, canonical injections

$$i_{m,n}: C(m) \longrightarrow C(n)$$

defined by $i_{m,n}(\mathfrak{a}) = \mathfrak{a} + (n - m)\mathfrak{p}_0$.

We require the following facts about the Jacobian variety J of the curve C. J is an abelian variety, and for each $n > 0$ there is a canonical map

$$\alpha_n: C(n) \longrightarrow J$$

which is holomorphic, which is compatible with the injections $i_{m,n}$ in the sense that $\alpha_n \circ i_{m,n} = \alpha_m$, and such that the fibers $\alpha_n^{-1}(x)$, as x runs over J, exactly represent the different complete linear systems of degree n on C in the sense that the points of a fiber represent the totality of divisors in a particular complete linear system. Furthermore, if another map $i'_{m,n}$: $C(m) \longrightarrow C(n)$ is defined just as above, except that a different point $\mathfrak{q}_0$ is used in place of $\mathfrak{p}_0$, then the two maps differ by a translation on J: $i'_{m,n} = t \circ i_{m,n}$, where t is a translation.

All of these facts may be established by using the Riemann theorem alone, and this is done in the first (and easy) part of [2]. Briefly, the linear equivalence relation on the divisors of degree n divides them up into the complete linear systems; these are represented on $C(n)$ by subvarieties, and according to the Riemann theorem, if n is large, these subvarieties are all of the same dimension $n - g$ (which defines g, the arithmetic genus (2)).

(2) If $g = 0$, one deduces immediately (cf. the proposition of § 7) the existence of a linear system whose dimension and degree are one. This maps C birationally onto the projective line, for which the Riemann-Roch theorem may be verified directly. We consider only the case $g \geqq 1$, henceforth.

One fixes some large n, and proves that the quotient variety J of $C(n)$ modulo this « fibration » by complete linear systems exists; the Chow coordinates are the technique here. Finally, the normalization of J (which actually is the same as J itself) is easily seen to be a group variety, since the set of complete linear systems is a coset space of the group of divisor classes of degree 0. Since J is projective, it is an abelian variety of dimension g, satisfying the above, taking α_n to be the natural map onto the quotient. If $m < n$, the corresponding map α_m may be defined by the relation $\alpha_n \circ i_{m,n} = \alpha_m$ given above; by elementary properties of the linear equivalence relation, it too satisfies the above. Values of $m > n$ need not detain us here since they are covered by the Riemann theorem. We may drop the subscript on α^n, occasionally.

We need two more facts. First, if $n = g$. then α_g is a birational map. This too is given in [2, p. 475]: if z is a generic point of J, one sees easily (compare the proposition of § 7) that the fiber $\alpha_g^{-1}(z)$ represents a linear system of dimension 0, hence consists of a single point; if $\mathfrak{a}$ and $\mathfrak{b}$ are two independent generic points of the fiber $\alpha_n^{-1}(z)$ for some large n, then the unique point in $\alpha_g^{-1}(z)$ is seen to be rational over $k(\mathfrak{a})$, therefore over $k(\mathfrak{a}) \cap k(\mathfrak{b}) = k(z)$.

Second, by an elementary foundational argument, the holomorphic 1-forms on an n-dimensional abelian variety are exactly the ones invariant under translation. Thus they form an n-dimensional vector space [see 4, p. 54].

Putting these last two results together, we see that (3)

$$\mathrm{Dim}\ \mathfrak{D}(J) = \dim J = g.$$

On the other hand, since we have seen that $C(g)$ and J are birationally equivalent, by a standard result we have $\mathfrak{D}(J) \cong \mathfrak{D}(C(g))$. Combining this with the proposition of section 2 shows that $\mathfrak{D}(J)$, $\mathfrak{D}(C(g))$, $\mathfrak{D}(C)$, and therefore $\mathfrak{D}(C(n))$ for any n, are all isomorphic g-dimensional spaces. Thus for a curve, the arithmetic genus (from the Hilbert postulation formula), irregularity ($= \dim J$), and geometric genus ($= \mathrm{Dim}\ \mathfrak{D}(C)$) coincide.

4. Statement of the theorem.

Let n be any positive integer, and let $\mathfrak{a}$ be a positive divisor of degree n, or equivalently, a point of $C(n)$. The fiber F on which $\mathfrak{a}$ lies represents the divisors of the linear system $|\mathfrak{a}|$ on C. The quantities entering into the Riemann-Roch theorem (as stated, for example in the beginning) are

(3) Here we use $\dim |\mathfrak{a}|$ for the geometric dimension, i. e., the dimension as a projective space; for the dimension of a vector space, we use Dim.

the same for all the divisors of $|\mathfrak{a}|$: thus we can assume that $\mathfrak{a}$ is a non singular point of F. $C(n)$ and J are everywhere nonsingular, so that the inclusion and projection maps i and α in the sequence

$$F \underset{i}{\longrightarrow} C(n) \underset{\alpha}{\longrightarrow} J$$

induce the usual differential maps on the tangent spaces at the points in question:

$$(5) \qquad T_{\mathfrak{a},F} \underset{di}{\longrightarrow} T_{\mathfrak{a},C(n)} \underset{d\alpha}{\longrightarrow} T_{\alpha(\mathfrak{a}),J} \,.$$

In this sequence, we see that $d\alpha \circ di = 0$, because $\alpha \circ i$ sends F to a point, and hence is zero on $T_{\mathfrak{a},F}$, and $d(\alpha \circ i) = d\alpha \circ di$.

The Riemann-Roch theorem for $\mathfrak{a}$ is now simply the assertion that exactness holds in the above sequence: no tangent vector to $C(n)$ at the point $\mathfrak{a}$ collapses under the mapping $d\alpha$ unless it lies in the direction of the fiber F. We will prove the theorem in this form. To see that it is the same as the theorem given at the beginning of the paper (at least for positive divisors; the extension to all divisors is elementary and will be done in § 7), we formulate it dually in terms of cotangent spaces — the spaces of local holomorphic 1-forms.

In fact, dualizing (5) above gives

$$(5') \qquad T^*_{\alpha(\mathfrak{a}),J} \underset{\delta\alpha}{\longrightarrow} T^*_{\mathfrak{a},C(n)} \underset{\delta i}{\longrightarrow} T^*_{\mathfrak{a},F}\,,$$

$T^*_{\mathfrak{p},X}$ denoting the cotangent space to the variety X at the point $\mathfrak{p}$. Now, by duality $\delta i \circ \delta\alpha = 0$, and δi is surjective (because di is injective); therefore the middle of this sequence is exact if and only if $\mathrm{Dim\,im}\,(\delta\alpha) = \mathrm{Dim\,ker}\,(\delta i)$, or

$$g - \mathrm{Dim\,ker}\,(\delta\alpha) = n - \dim F.$$

Since the holomorphic 1-form on J are the translation-invariant ones, there is a canonical isomorphism $T^*_{\alpha(\mathfrak{a}),J} \cong \mathfrak{D}(J)$. So in view of our previous isomorphisms, $\ker(\delta\alpha)$ is isomorphic to the space of holomorphic 1-forms on $C(n)$ which are zero at $\mathfrak{a}$; thus by the proposition, $\mathrm{Dim\,ker}\,(\delta\alpha) = i(\mathfrak{a})$, and the exactness of (5) or (5′) becomes the Riemann Roch theorem.

5. The proof, first step.

We know that the theorem is true when n is large. The general idea is to prove it for $m < n$ by viewing $C(m)$ as imbedded in $C(n)$ by means of the maps $i_{m,n}$ introduced earlier, and studying the behavior of the tangent vectors in this imbedding. Essentially what one has to know is that $C(m)$ is « transversal » to the fibers of α_n.

To establish this, we need to know more about α_n. Since one does not know in advance exactly how $C(m)$ is located inside $C(n)$ for example, it is necessary to know that the fibers of α_n are everywhere nonsingular (in our previous argument, it was enough to select *one* non-singular point of the fiber). Though this is known, and proved for example in [2], we will reprove it here, in keeping with the spirit of this paper, as it is not quite on the same plane as the facts we are assuming. The lemma 1 below is the essential step: it gives both the nonsingularity of the fibers as well as the needed transversality argument for the conclusion of the proof. Even if we didn't prove the nonsigularity, we would have to repeat most of the work of lemma 1 in the final stage of our proof.

The fibers of α_n are essentially projective spaces, and for the purposes of the proof (the passage from n to a smaller integer m) it turns out to be better to work with the projective spaces themselves, rather than with the fibers.

The Jacobian plays no role in this section: we are concerned only with studying the fibers on $C(n)$ more closely.

We fix a large value of n, and consider the complete linear system $|\mathfrak{a}|$ of degree n and dimension $n-g$. We suppose C to be imbedded in projective $n-g$ space P^{n-g} so that the divisors of $|\mathfrak{a}|$ are just the hyperplane sections; we also suppose that C is not contained in any proper subspace, so that distinct hyperplanes have to intersect C in distinct divisors of $|\mathfrak{a}|$.

Let L^{n-g} be the dual projective space, the points of which represent the hyperplanes H in P^{n-g}, and map

$$f: L \longrightarrow C(n)$$

by sending the hyperplane H into the point corresponding to the divisor $H \cdot C$. Then f is holomorphic and injective, and its image is the fiber F which contains the point $\mathfrak{a}$. If we can show that df is everywhere injective, it will follow that F is nonsigular, and this will be lemma 2 below. Lemma 1 however gives the essence of the argument.

Consider therefore a fixed divisor $\mathfrak{a}$ in our linear system $|\mathfrak{a}|$, and suppose

$$\mathfrak{a} = m_1 \mathfrak{p}_1 + m_2 \mathfrak{p}_2 + \dots + m_r \mathfrak{p}_r, \qquad \mathfrak{p}_i \neq \mathfrak{p}_j \quad \text{if} \quad i \neq j.$$

From (4) we derive the local algebroid factorization

$$s^{-1} \circ f = f_1 \times \dots \times f_r$$

where $f_i \colon L \longrightarrow C(m_i)$ is a local map [i. e., expressed in power series].

LEMMA I. With the above notations and assumptions (in particular, n sufficiently large), we have

$$\text{rank}\, df_1 = \dim |\mathfrak{a}| - \dim |\mathfrak{a} - m_1 \mathfrak{p}_1| . \tag{6}$$

PROOF. We introduce coordinates to describe the map f_1 and show explicitly that the matrices connected with the two sides of (6) have the same rank.

Choose affine coordinates $U_0, \dots, U_{n-g} = 1$ in L^{n-g} and $1 = X_0, \dots, X_{n-g}$ in P^{n-g} so that the general hyperplane in P is $u_0 X_0 + \dots + u_{n-g} X_{n-g} = 0$, where the u_i are independent transcendentals (but $u_{n-g} = 1$), and so that the particular hyperplane $X_{n-g} = 0$ intersects C in the divisor $\mathfrak{a}$.

Let $x_0, \dots, x_{n-g}$ be a generic point of the curve C, and let t be a local uniformizing parameter at the point $\mathfrak{p}_1$. Then at $\mathfrak{p}_1$ we get the power series expansions

$$x_i = \sum_{j=0}^{m_1-1} a_{ij} t^j + (\text{higher powers of } t), \qquad i = 0, \dots, n-g.$$

Let A be the matrix (a_{ij}), $i = 0, \dots, n-g-1$; $j = 0, \dots, m_1 - 1$. Here we have omitted the last row-it would be all zeros since by hypothesis $X_{n-g} = 0$ intersects C with multiplicity m_1 at $\mathfrak{p}_1$. This matrix A is to be thought of as a generalized coordinate matrix of the divisor $m_1 \mathfrak{p}_1$: in the usual case, $m_1 = 1$, it would simply be the coordinate vector (minus the last 0) of the point $\mathfrak{p}_1$, while for $m_1 > 1$, it includes the higher order information as well.

The rank of A is, we claim, the right hand side of (6): $n - g - \dim |\mathfrak{a} - m_1 \mathfrak{p}_1|$. Namely, the quantity $n - g - \text{rank}\, A$ is evidently the dimension of the solution space to the linear equations

$$\sum_{i=0}^{n-g-1} U_i a_{ij} = 0, \qquad j = 0, \dots, m_1 - 1.$$

But the solutions, $(u'_0, \dots, u'_{n-g-1})$, correspond exactly to the hyperplanes $\Sigma u'_i X_i + X_{n-g} = 0$ which intersect C with multiplicity $\geq m_1$ at $\mathfrak{p}_1$, so that this dimension is the same as $\dim |\mathfrak{a} - m_1 \mathfrak{p}_1|$.

We now calculate the left hand side of (6). At the point $(u) \times p_1$ in $L^{n-g} \times C$, the local algebroid equation for the correspondence $H \longrightarrow H \cdot C$ which defines the map f is

$$g(U_0, \dots, U_{n-g-1}; T) = U_0 + U_1 x_1(T) + \dots + x_{n-g}(T) = 0.$$

Now $g(0; T) = x_{n-g}(T)$ has, as we have seen, the leading exponent m_1. Thus the formal Weierstrass preparation theorem gives the *identity* in (U) and T (writing m for m_1),

$$(7) \qquad g(U; T) = h(U; T)(T^m + s_1(U) T^{m-1} + \dots + s_m(U)),$$

where $h(0; 0) \neq 0$, and $s_i(0) = 0$.

Since $h(U; T)$ is invertible, the polynomial on the right is also a local equation for the above correspondence, so that our algebroid map $f_1 : L \longrightarrow C(m)$ is given in the usual symmetric function coordinates at the point $m_1 \mathfrak{p}_1$ on $C(m_1)$ by (4)

$$(7') \qquad (-1)^j \sigma_j = s_j(u) = \Sigma u_i s_{ij} + (\text{higher powers of } u_i), \qquad j = 1, \dots, m_1.$$

Evidently rank $df_1 =$ rank S, where S is the matrix (s_{ij}), so our lemma will be proved if we show rank $A =$ rank S. But just compare the coefficients of the terms in the identity (7) which are linear in U_i; you get

$$\Sigma a_{ij} T^j = h(0; T) \cdot \Sigma s_{ij} T_j^{m-j}, \qquad i = 0, \dots, n-g-1.$$

(4) This is an « evident » foundational result for which it is hard to give a reference, since it is algebroid, rather than algebraic. The algebraic correspondence F in $L \times C$ defined by the map f associates with the point (0) in L the divisor $m_1 \mathfrak{p}_1 + \dots m_r \mathfrak{p}_r$. By what should be a form of Hensel's lemma, this means that the algebroid correspondence defined by F in the neighborhood of $(0) \times C$ splits into r components: $F = F_1 + \dots + F_r$. By using the symmetric functions as in (7′) above, each component defines a local algebroid map $f_i' : L \to C(m_i)$, and the point is that $f = s \circ (\Pi f_i')$, where s is the map (4). Since s is an analytic isomorphism, this shows that f_i' is the same as the map $f_i = pr_i \circ s^{-1} \circ f$ defined above; this is what we are asserting by the equation (7′).

To justify these statements, there is no trouble if C is the affine line. Then the equation for the algebraic correspondence F is the polynomial $g(u; t) = 0$; by the usual Hensel lemma, it splits over $k[[u]]$ into r factors, giving the decomposition $F = F_1 + \dots + F_r$, and the rest follows immediately by direct calculation with the symmetric functions which are the coefficients of these polynomials.

For an arbitrary curve C, one can deduce the same facts somewhat clumsily by choosing the local parameter t at $\mathfrak{p}$ so that $t(\mathfrak{p}_i) \neq t(\mathfrak{p}_j)$ if $i \neq j$, and then using t to project C onto a line C_1 and the correspondence F into $L \times C_1$. From the previous case, $f_1 = f_1'$ for the projected correspondence, and it follows easily then that $f_1 = f_1'$ for the original correspondence as well, since $C(m)$ and $C_1(m)$ are isomorphic at $m\mathfrak{p}$.

Reading this modulo T^m, one sees that the columns of A are linear combinations of the columns of S, and vice-versa as well since $h(0;T)$ is invertible. This completes the proof.

LEMMA 2. Rank $df = n - g$, so that df is everywhere injective.

PROOF. Let $A = A_1$ and $S = S_1$ be the two matrices introduced in connection with the point $\mathfrak{p}_1$ in the proof of lemma 1. Introduce in the same way, by picking local parameters $t_2, \dots t_r$ at the other points $\mathfrak{p}_2, \dots, \mathfrak{p}_r$ occurring in the divisor $\mathfrak{a}$, matrices $A_2, \dots, A_r$, and $S_2, \dots, S_r$. The proof now runs exactly parallel to the preceding one, except that wants not A_1 but rather the matrix of $n-g$ rows and n columns formed by putting the A_i side by side: $(A_1, A_2, \dots, A_r)$, and similarly with the S_i. Repeating the argument about solving linear equations, one gets

$$\operatorname{rank}(A_1, \dots, A_r) = \dim|\mathfrak{a}| - \dim|\mathfrak{a} - m_1\mathfrak{p}_1 - \dots - m_r\mathfrak{p}_r|$$
$$= n - g,$$

since $\mathfrak{a} - m_1\mathfrak{p}_1 - \dots - m_r\mathfrak{p}_r$ is the 0 divisor. On the other hand, the algebroid map f, being the product $f_1 \times \dots \times f_r$, has the matrix $(S_1, \dots, S_r)$, as matrix of coefficients of the terms linear in the u_i; thus

$$\operatorname{rank}(S_1, \dots, S_r) = \operatorname{rank} df.$$

Now, as we saw before for A_1 and S_1, the columns of A_i are invertible linear combinations of the columns of S_i for each i. Therefore rank $(A_1, \dots, A_r) = \operatorname{rank}(S_1, \dots, S_r)$, which completes the proof.

6. Conclusion of the proof.

We still assume that n is sufficiently large. The sequence of maps we are considering is then

$$L \xrightarrow[f]{} C(n) \xrightarrow[\alpha]{} J.$$

Let $\mathfrak{b}$ be a point of $C(n)$ lying in $f(L)$, and suppose $\mathfrak{b} = f(u')$. Then the associated sequence of tangent spaces is

$$0 \longrightarrow T_{u',L} \xrightarrow[df]{} T_{\mathfrak{b},C(n)} \xrightarrow[d\alpha]{} T_{\alpha(\mathfrak{b}),J} \longrightarrow 0. \tag{8}$$

We assert that this sequence is exact. In fact, we have just shown df to be injective everywhere. Also, $d\alpha \circ df = 0$ since $d\alpha \circ df = d(\alpha \circ f)$, but $\alpha \circ f$ is the zero map. We say that $d\alpha$ is surjective; for this it suffices to show that the dual map $\delta\alpha : T^*_{\alpha(\mathfrak{b}),J} \longrightarrow T^*_{\mathfrak{b},C(n)}$ is injective. But by the remarks in § 3, if a cotangent vector is mapped by $\delta\alpha$ into zero, this means that there is a global holomorphic 1-form on $C(n)$ which is 0 at $\mathfrak{b}$. This in turn by the proposition of section 2 means that there is a holomorphic 1-form on C whose divisor of zeros contains $\mathfrak{b}$, and this is impossible if the degree of $\mathfrak{b}$ is greater than the degree of the divisors of the differential class, i. e.. it is impossible if n is sufficiently large. We conclude finally that the middle is exact because, by the preceding, Dim ker $(d\alpha) - n - g$, but ker $(d\alpha)$ contains im (df), which is also of Dimension $n - g$; hence they are equal.

We wish now to prove the theorem when $\mathfrak{a}$ is a divisor of arbitrary positive degree m, which we may take to be less than n. The complete linear system $|\mathfrak{a}|$ is then represented by a subvariety of $C(m)$. We select some point $\mathfrak{q}_0$ not contained in $\mathfrak{a}$, and consider the imbedding

$$i_{m,n}' : C(m) \longrightarrow C(n)$$

defined by putting $i_{m,n}'(\mathfrak{c}) = \mathfrak{c} + (n - m)\,\mathfrak{q}_0$, where $\mathfrak{c}$ is an arbitrary divisor of degree m.

The geometric situation we wish to describe is summarized by the following diagram:

$$\begin{array}{ccccc} L^{n-g} & \xrightarrow{f} & C(n) & \xrightarrow{\alpha_n} & J \\ j\uparrow & & i'\uparrow & & \uparrow t \\ M^r & \xrightarrow{g} & C(m) & \xrightarrow{\alpha_m} & J \end{array}$$

The top line we have already described. In the right hand square, t is the translation on the Jacobian which makes the square commutative, which we referred to in § 3. M^r is a certain subspace of the projective space L, which parametrizes the complete linear system $|\mathfrak{a}|$ in the same way that L parametrizes the system $|\mathfrak{b}|$, where $\mathfrak{b} = i_{m,n}'(\mathfrak{a})$. The integer r here is the dimension of $|\mathfrak{a}|$. M^r is precisely described as the subspace of L consisting of all hyperplanes whose intersection with C is a divisor of the form $\mathfrak{c} + (n - m)\,\mathfrak{q}_0$, and g is the map which sends the hyperplane H into the point representing the divisor $\mathfrak{c}$. From elementary properties of complete linear systems, (the so-called « residue theorem »), it follows that $g(M)$ is exactly the fiber of α containing $\mathfrak{a}$, since $|\mathfrak{a}| = i'^{-1}(|b|)$.

Pass as before from the above diagram to the corresponding one for the tangent spaces at the points in question:

$$\begin{array}{ccccc} T_{j(u'),L} & \xrightarrow{df} & T_{\mathfrak{b},C(n)} & \xrightarrow{d\alpha_n} & T_{\alpha(\mathfrak{b}),J} \\ \uparrow dj & & \uparrow di' & & \uparrow dt \\ T_{u',M} & \xrightarrow{dg} & T_{\mathfrak{a},C(m)} & \xrightarrow{d\alpha_m} & T_{\alpha(\mathfrak{a}),J} \end{array}$$

From § 4 we recall that what must be shown to finish the proof of the Riemann-Roch theorem is that if $w \in T_{\mathfrak{a},C(m)}$ is a tangent vector such that $d\alpha_m(w) = 0$, then w is tangent to the fiber of α_m passing through $\mathfrak{a}$. Since this fiber is nothing but $g(M)$, it will suffice to show that actually $w \in dg(T_{u',M})$, which we now do.

The central thing to establish is a transversality statement:

$$df(T_{j(u'),L}) \cap di'(T_{\mathfrak{a},C(m)}) = d[i' \circ g](T_{u',M}). \tag{9}$$

For if $d\alpha_m(w) = 0$, the exactness of the top line of our diagram evidently implies that $di'(w)$ belongs to both spaces on the left-hand side of (9); thus, by (9), $di'(w) \in di'[dg(T_{u',M})]$, and since di' is injective, we conclude that $w \in dg(T_{u',M})$.

Turning our attention to (9), the right-hand side is clearly included in the left. The right-hand side has dimension r, because $d[i' \circ g]$ is injective: it equals $df \circ dj$, and both df and dj are injective. So we are done if we show that the left-hand side also has dimension r. This crucial statement follows now from lemma 1. In fact, since $\mathfrak{b} = (n-m)\mathfrak{q}_0 + \mathfrak{a}$, we get at $(n-m)\mathfrak{q}_0 \times \mathfrak{a}$ the analytic isomorphism

$$s: C(n-m) \times C(m) \to C(n),$$

and as in lemma 1, we have $s^{-1} \circ f = f_1 \times f_2$. Now since s restricted to the subvariety $(n-m)\mathfrak{q}_0 \times C(m)$ is essentially the map i', it follows that

$$\text{a tangent vector } v \in \ker(df_1) \Longleftrightarrow df(v) \in di'(T_{\mathfrak{a},C(m)}).$$

This shows that $df(\ker(df_1))$ is just the left-hand side of (9). Since df is injective, we need only show that $\operatorname{Dim}\ker(df_1) = r$ to complete the proof of (9); but by lemma 1, applied to f_1, we have

$$\operatorname{rank} df_1 = \dim|\mathfrak{b}| - \dim|\mathfrak{a}| = n - g - r,$$

and of course, $\operatorname{rank} df_1 + \ker df_1 = \operatorname{Dim} T_{j(u'),L} = n - g$.

7. Complements.

The Riemann-Roch theorem for non-positive divisors follows easily by standard arguments, which we reproduce for convenience. Strictly speaking, we have proved it assuming $\dim |\mathfrak{a}| \geqq 0$, $\deg \mathfrak{a} > 0$; if however $\deg \mathfrak{a} = 0$ (so that $\mathfrak{a}$ is a principal divisor), the theorem amounts to the assertion that $\dim |\mathfrak{k}| = g - 1$, where $\mathfrak{k}$ is a canonical divisor. This was proved however in § 3.

Apply the theorem to the canonical class and deduce $\deg \mathfrak{k} = 2g - 2$; this shows that the theorem is symmetric in $\mathfrak{a}$ and $\mathfrak{k} - \mathfrak{a}$, so if either has $\dim \geqq 0$, we are done. If both have dimension -1, however, the theorem claims that $\deg \mathfrak{a} = g - 1 = \deg \mathfrak{k} - \mathfrak{a}$, and again by using the symmetry, this follows from :

PROPOSITION. If $\deg \mathfrak{a} \geqq g$, then $\dim \mathfrak{a} \geqq 0$.

PROOF. For any big positive divisor $\mathfrak{b} = \mathfrak{p}_1 + \dots + \mathfrak{p}_N$, $\mathfrak{p}_i \neq \mathfrak{p}_j$, $\dim |\mathfrak{a} + \mathfrak{b}| = N + \deg \mathfrak{a} - g$, by the Riemann theorem. Now the divisors of the system $|\mathfrak{a}|$ are by the « residue theorem » exactly those divisors of $|\mathfrak{a} + \mathfrak{b}|$ which contain $\mathfrak{p}_1, \dots, \mathfrak{p}_N$, minus these N points. But the divisors of $|\mathfrak{a} + \mathfrak{b}|$ form a projective space, and the divisors in this system containing $\mathfrak{p}_i$ form a subspace H_i of codimension at most one. Thus the intersection of the H_i has codimension at most N, that is, dimension at least $\deg \mathfrak{a} - g$, which by hypothesis is non-negative. Thus $\dim |\mathfrak{a}| \geqq 0$.

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The foundational results used are mostly in [3]. For the material on differentials and group varieties, as well as the symmetric products, see also [4]. In [1] is an exposition of symmetric products. The construction of the Jacobian is given in [2]. The Hilbert postulation formula is proved in [5].

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