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MONOGRAPH ON CAUSE AND EFFECT

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HISTORICAL NOTE

The best that has been said, perhaps, about cause and effect, was said the most simply in statements like the following by ETIENNE GILSON; « Where there is efficient causality, something new, which we call effect, is brought into existence by efficacy of its cause ». There has been considerable discussion on the question whether this something is always unique. Since any answer to the question will involve logic, and cause proceeds from some reality as a beginning, and effect predicates some end, we first consider the nature of some possible beginnings of thought concerning causation, and certain possibilities for classification of realities.

Our attention will be required to a type of nullity called non-effect, and some illustrations of the null may well be mentioned first. In a proof of any formula by mathematical induction we begin with a special case in which a special value of each parameter in the formula is in the proper place for that parameter, so the law of the formula can be abstracted. Thus to prove,

$$(1) \qquad (a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i ,$$

we may start with the case $(a+b)^1$, or $(a+b)^2$, but not with the null case, $(a+b)^0$. The latter expression has to be arbitrarily defined and then is out of conformity with the binomial law.

The representation of the positive integer M as a polynomial in an arbitrary integer k as a base, is possible uniquely, for if we could have,

$$(2) \qquad M = \alpha k^m + \beta k^{m-1} + \ldots + \varkappa \equiv \alpha' k^m + \beta' k^{m-1} + \ldots + \varkappa',$$

$$(\alpha, \ldots, \varkappa; \alpha', \ldots, \varkappa' < k),$$

$\varkappa' - \varkappa$ would be divisible by k, whence $\varkappa' = \varkappa$. Dividing out k from what is left in (2), $\iota' = \iota, .., \alpha' = \alpha$, which is a proof. Hence we write,

$$M = \alpha \beta \cdots \iota \varkappa (k),$$

but if some term, as βk^{m-1}, is not actually present, in the case of the M chosen, a symbol O is invented having the character of a positive integer and the property that it nullifies any term of M that has it as coefficiént. If no other term is missing we then have M unchanged, that is,

$$M = \alpha \, 0 \, \gamma \cdots \iota \varkappa (k),$$

but the existence of the zero is a postulate; also the simplest beginning is not $k = 1$: it is $k = 2, M = 1$.

To make the algebra of clásses logically complete, (1) it was found necessary to introduce the null (empty) class, whereupon it became expedient to revise Aristotelian logic. But the null class cannot figure as the beginning of an induction.

The ordered system Σ of types of reality.

KANT mentioned the fact that the weight of on object is something that remains outside the object, while standing in some connection with it. The judgment that the object has the weight is therefore, in the usual terminology, synthetic, and this principle has been shown, by the writer (2) to establish a classification of the totality of existent realities. For, in the case of any two realities, the judgment that A has B is synthetic if B is without A (and analytic if B is covertly contained in A). If we rule that B lies above A when the judgment is synthetic, the types of reality are ordered into a *scalea* Σ of degrees in which the realities essential to any degree are syntheses from respective realities of the preceding degree. By this rule alone Σ can be ordered in the form of a succession of ten degrees of rank n extending from O to 9, space and time forming the realities of zero degree. We here reproduce Σ, from our earlier paper; also, for illustration, formulations like a syllogism connecting one degree with the next higher, that is, the step from (ϵ) to (ζ), $n = 5$. There are realities without synthetic connections is the sense of Σ, but these same realities will have analytic connections in Σ.

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(1) R. OSBORN, Mathematics Magazine, vol. 26, (1953), p. 175.

(2) GLENN, loco citato, vol. 30, (1957), p. 117.

We follow KANT in regarding a reality as intuitive representation, (abstraction), in a cognition, the latter consisting of a concept plus its representation. We avoid fundamentalism in discussion of the a priori, (3). This province of the mind was built up evolutionally before the dawn of history, and it is therefore pre-history that maintains it now. Pre-historic man did not know, of course, what an objective reality was. It was he who first said « Ich verstehe nicht einmal was ein Ding ist ». Nevertheless he could represent a concept of a reality in his sensuous intuition. This we still do, but the mind has not stopped there. We are finding out something about things in themselves, about freedom, life itself, and so on. This historic something is not yet engrossed, evolutionally, on the scroll of the a priori. It remains as empirical generalization, intuitive excepting what has been made conceptual by actual inductive proof. During historical times wave-intuition has developed perhaps more than any other function of the mind. However, reality, which is pretty generally a manifestation of energy, has been found to be inseperable from the thought concerning it.

The scalea Σ

($\varkappa$) Perception of the living: (*A* complete knowledge of life and its processes has not been attained but, if assumed to be possible, some of its introspective cognitive realities would be syntheses of rank superior to the creative).

(ι) Creative cognitives: (Logic, wave-intuition, mathematical process et cetera).

(θ) Libertarian realities: (Phases of moral action. The invariable elements are the true, the beautiful, the useful, and the good (CROCE)).

(η) The humane realities: (Altruistic endeavors, writing of poetry, and the like. Social action is a mass-cognitive of this degree).

(ζ) Sensory realities: (Examples are,-the sense of beautiful objective form; the sense of social form; the sense of musical harmony; etc.)

(ε) Phases of subliminal (subconscious) mentality: (Those controlling animal skills, for example).

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(3) REIDEMEISTER, *Die Unsachlichkeit der Existenzphilosophie,* (1954), S. 28.

(δ) Subsistential realities: (All phases of self-perpetuation. The simplest case is taken to be the activity by which a virus, or other elementary phenomenon, maintains existence in its environment).

(γ) Phenomena: (In the meaning of occurrence).

(β) Objective realities: (Corporeal objects).

(α) Space-time realities: (Extensions as concepts. Their representations in (α) are null. Space extension is arbitrary as to direction, time not.

For the synthesis (ϵ, ζ), taken as a typical case, we readily intuit that an organism possessing some realities (ϵ) of subliminal mentality will have also some sensory abilities. However, a reality of (ζ), as implied by the phrase «a sense of», while it will be connected with a subliminal mental reality [ϵ], it will not be a part of [ϵ]. It is an unproved theorem that the connection will be unique. The judgment that [ϵ] has [ζ] is synthetic and verifies that (ζ) lies above (ϵ) in Σ.

The reader will wish to consult the author's previous paper for corresponding verifications of the other eight syntheses, and perhaps HEGEL'S theory of synthesis also. The theorem on the uniqueness of connection has been proved only for the synthesis (α, β), where it takes the form of a known theorem in the calculus of variations, viz. A unique (straight) line is the shortest distance between two points in space. Here this really means two points in (β). Any degree in Σ may contain any reality from a lower degree in Σ, but none that is essential to any higher degree.

In will be noted that, within (α), since space affords us no cognitions (KANT), we are unable to represent any extension $[AB]$ from a spatial point A to a spatial point B. To use a geometric line AB would be to depart from degree (α) to degree (β) (CLIFFORD). The former conclusion is consistent with intuition. Measure is relative to motion of the extension measured, and the earth has two kinds of motion. The solar system as a whole is in motion. On the other hand, relativitists question whether there is any such thing as absolute motion. Hence, any attempt to represent $[AB]$ in (α) becomes naturally illusory, that is, null.

Non-effect

Non-effect is a state of a combination of realities when no law is operating upon it to effect another reality.

As one can verify by a simple experiment, a large objective reality, falling in a vacuum toward the center of the earth, has no weight while it is falling. It is a combination of object with the phenomenon of falling, and is a state of non-effect. ARISTOTLE said that the planets in their orbits had no attraction toward the center (sun). FRANKLIN suggested that it might some time be possible to deprive an object of its weight and move it about easily from place to place. (4)

A planetary body P, in its regular plane orbit, is in a state of non-effect, dynamically. It tries to fall into the sun but got started obliquely. The negative of the vector PV of the radial component of the tangential inertial force, equals the vector PW of the central force $F(r)$, (See Figure I). The equation,

$$PV + PW = 0\,, \tag{3}$$

is the condition for orbital stability. But it is known that, (5)

$$P\,V = [d^2\,r/d\,t^2 - r\,(d\,\theta/d\,t)^2]\,m\,.$$

Eliminating t between PV and the differential form of KEPLER'S law of areas, $r^2\,d\,\theta = h\,d\,t$, the equation (3) becomes,

$$r\,d^2\,r/d\,\theta^2 - 2\,(d\,r/d\,\theta)^2 - r^2 = -\,r^5\,F(r)/h^2\,m\,, \qquad (m = \text{mass})\,,$$

which is the known equation for central orbits.

An inward perturbation of the orbit, at P, is equivalent to an increase of PW. Stability then requires an increase in $-PV$, which is to say that a force of reversion arises. (The principle applies to spatial orbits also).

In the case of the unperturbed orbit there is neither weight toward the sun nor force along the tangential direction. The idea that there must be an intervening medium of great tensile strength, between planet and sun, seems therefore to be false.

Apparently any systematic motion in nature is subject to a force of reversion toward some state of non-effect. An atom, considered as a system involving electrons in motion like planets, is a state of non-effect that will continue for an indefinite time unless an external force intervenes. The seed of a plant is a state of non-effect with a tendency to be perpetual. Under some conditions a grain of wheat may retain its power to germinate for a thousand years. When it does germinate it is only to return to another grain, which will be the same state of non-effect, after a fixed cycle of events.

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(4) BENJAMIN FRANKLIN, *Letter to Joseph Priestley*, (1780).

(5) A. ZIWET, *Theoretical Mechanics*, Part I, p. 83, (1893).

The cycle of events after an orbit is perturbed

If a planet runs into a comparatively small alteration of its usual field of force, its subsequent motion will include a cycle of diminishing vibrations transverse to its original orbit. Its path settles into a state of non effect which, according to circumstances, may or may not coincide with its original orbit.

For historical reasons mainly we mention also a contemporaneous theory of vibrations, also transverse to the orbit, but borne by a hypothetical aether, and, in some unspecified manner, keeping pace with the motion of the planet or corpuscle. These vibrations or waves have been represented by means of a pair of vectors D, E drawn from the moving body considered as a point and satisfying four equations which have been obscurely said to represent the «state of the aether». The equations were readily found to be,

$$(4) \qquad \begin{cases} \operatorname{div} D = 0\,, \ \operatorname{div} E = 0\,, \\ \operatorname{rot} D = -\dfrac{1}{c}\,\partial E/\partial t\,, \ \operatorname{rot} E = +\dfrac{1}{c}\,\partial D/\partial t\,, (t = \text{time}), \end{cases}$$

where c can be a costant, the velocity of light, or a variable function, and rot D is the vector whose components are, respectively,

$$\frac{\partial D_z}{\partial y} - \frac{\partial D_y}{\partial z}, \quad \frac{\partial D_x}{\partial z} - \frac{\partial D_z}{\partial x}, \quad \frac{\partial D_y}{\partial x} - \frac{\partial D_x}{\partial y},$$

while,

$$\operatorname{div} D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}.$$

When the formulary is taken to refer to the electro-magnetic theory (MAXWELL), the electric force D (current, orbit), can be regarded as the main phenomenon and the accompanying magnetic force E as a causal law of secular perturbations of the current. (6)

The meaning of equations (4) will be clear if they are considered to be a part of an invariant theory of the known transformations of rectangular space-coordinates which move the axes about an origin O fixed, say,

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(6) A. G. WEBSTER, *Partial Differential Equations of Mathematical Physics*, (1927), p. 17, 48.

H. BATEMAN, *Electrical and optical Wave-motion*, (1914), p. 1.

on the orbit, as a pivot. We would want to duplicate the wave motion by a chosen oscillation of the axes. Hence the equations to characterize the vectors D, E should be the simplest invariant equations involving them and the essentialities of their properties. But it is known that the simplest invariants are $\operatorname{div} D$ and $|\operatorname{rot} D|$ and, when equations (4) have been derived, the goal of characterizing D, E is reached. The equations express that the rate of change of E is proportional to $-c$, and the rate of change of D is proportional to $+c$, under the aforesaid oscillation of the axes; facts which are in accord with intuition.

However, the equations (4) do not very naturally represent the circumstances above mentioned, about a cycle of vibrations intermediate between two states of non-effect. The equations $\operatorname{div} D = 0$, $\operatorname{div} E = 0$, may be regarded as a particularization of the arbitrary alteration that we assume in the original field of force associated with P on an orbit R.

Mathematics appropriate for our purpose of describing the action of the force of reversion, after a spatial orbit R has been perturbed, is obtained from theory due to SOPHUS LIE, on infinitesimal transformations. If

$$o(r) = l\,r^s + m\,r^{s-1} + \ldots + x\,,\quad p(\theta) = \lambda\,\theta^s + \mu\,\theta^{s-1} + \ldots + \xi\,,$$

$$q(\varphi) = L\,\varphi^s + M\,\varphi^{s-1} + \ldots + X\,,$$

are real polynomials in their arguments, and the equations of R are given in polar coordinates, a considerable arc of the curve,

$$R : [R_1(r', \theta', \varphi') = 0\,,\ R_2(r', \theta', \varphi') = 0]\,,$$

goes into a neighboring (perturbed) arc, to a close approximation, by means of the following transformation T, $(= T(u, v, w))$, and, by proper alterations of u, v, w, into any segment of the field of perturbed orbits:

$$T:\ r' = r + u\,o(r)\,,\quad \theta' = \theta + v\,p(\theta)\,,\quad \varphi' = \varphi + w\,q(\varphi)\,,\quad (u, v, w \doteq 0)\,.$$

Here each variable has a limited range, $r_1 < r < r_2$, $\theta_1 < \theta < \theta_2$, $\varphi_1 < \varphi < \varphi_2$, and,

$$T(u, v, w)\,T(\pi, \varrho, \sigma) = T(u + \pi, v + \varrho, w + \sigma)\,.$$

There results what is primarily a theory of orbital segments. The curve TR is a perturbed segment, but T itself may be regarded as the operation of perturbation.

The effect of the perturbation of R, which is opposed by the force of reversion, in this: The alteration in the field of force when the moving body is P on R, may be depicted by a vector e drawn from P. The component of e along the line of inertia i increases the speed of P while the component of e along the normal n at P, vibrates R transversely. The vibration is really in the action of the force f of reversion. Note that the period of the vibration depends upon f and not upon e; that the wave accompanies P, and that it has a regularity somewhat like the motion of a pendulum.

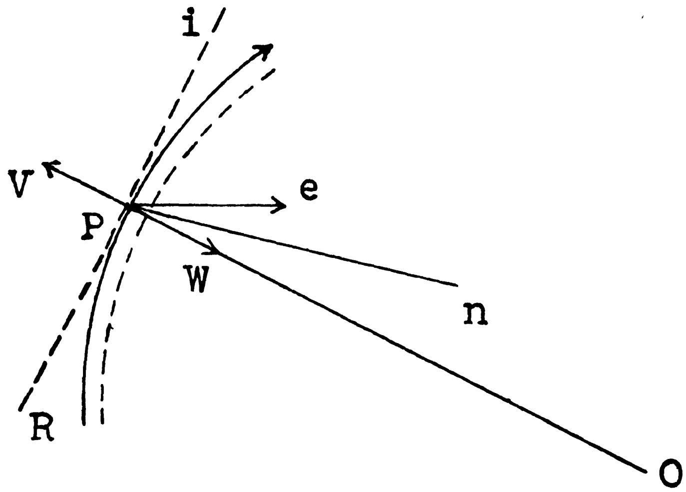

Figure 1

Let $u, v, w, u^1, ..$ be infinitesimal functions of t, the signs of $u(t)$, $u^1(t)$ being opposite. Suppose,

$$T(u(t_1),\ v(t_1),\ w(t_1)), \tag{5}$$

to be the operation of maximum vibration from R toward O, and

$$T(u^1(t_2),\ v^1(t_2),\ w^1(t_2)), \tag{6}$$

the next (reverse) maximum vibration, the t_j under a function being always the time when the vibration reaches its maximum. The succession of (5) and (6), their product, is equivalent to the operation of a vibration from R to the terminus of the reverse vibration, and is,

$$T(u(t_1)+u^1(t_2),\ v(t_1)+v^1(t_2),\ w(t_1)+w^1(t_2)). \tag{7}$$

The succession (5), (6) will be followed, under f, by the operation,

$$T(u(t_3)\,,\ v(t_3)\,,\ w(t_3))\,,$$

that corresponds to the next to greatest maximun vibration toward O. Next will follow,

$$T\ (u^1(t_4)\,,\ v^1(t_4)\,,\ w^1(t_4))\,,$$

extending the vibrations to a terminus nearer to R than that corresponding to (7). The program will evidently continue indefinitely, and the termini of the vibrations will approach a limit,

$$T(a\,,b\,,c)\,R\,,\ (a\,,b\,,c \text{ fixed and} \doteq 0)\,,$$

where we find,

$$\begin{aligned} &u(t_1) + u^1(t_2) + u(t_3) + u^1(t_4) + \ldots \doteq a\,, \\ &v(t_1) + v^1(t_2) + v(t_3) + v^1(t_4) + \ldots \doteq b\,, \\ &w(t_1) + w^1(t_2) + w(t_3) + w^1(t_4) + \ldots \doteq c\,. \end{aligned} \tag{8}$$

In this formulary P can be any point of the finite arc delimited by T on R.

We can transform the regime of formulas by (9),

$$t'_j = t_j + \delta_j\, t_j\,,\ (j = 1\,,2\,,\ldots;\ \delta_j\, t_j \doteq 0)\,. \tag{9}$$

Then the expression,

$$T(u(t'_1) + u^1(t'_2) + \ldots, v(t'_1) + v^1(t'_2) + \ldots,\ w(t'_1) + w^1(t'_2) + \ldots)\,,$$

becomes the product (10), $(u'(t) = \partial\, u(t)/\partial\, t)$,

$$\begin{aligned} &T(a\,,b\,,c)\,T(u'(t_1)\,\delta_1\,t_1 + u^{1\prime}(t_2)\,\delta_2\,t_2 + \ldots, \\ &v'(t_1)\,\delta_1\,t_1 + v^{1\prime}(t_2)\,\delta_2\,t_2 + \ldots,\ w'(t_1)\,\delta_1\,t_1 + w^{1\prime}(t_2)\,\delta_2\,t_2 + \ldots)\,. \end{aligned} \tag{10}$$

This shows how, if we take P a little farther along R, in the revised field of force, the corresponding transformations of amplitudes fail to alter the limit. The convergence of the vibrations will be to $T(a\,,b\,,c)\,R$ again, while the second factor of (10) leaves R invariant, and is $T(O\,,O\,,O)$. In any case of the theory, if the perturbing force $[e]$ approaches zero, the original R is restored, $(a=b=c=0)$. The theory includes that of the straight orbit R, and therefore the case of the motion of a corpuscle of light.

Now it is known that any curved orbital segment, for any astral body, can be considered as a central orbit, (7) within acceptable approximations, and so will have an associated force of reversion. A straight orbit of a light corpuscle, considered as the boundary or limiting case between two fields of plane orbits, slightly curved away from the boundary (or considered as the central filament of a tube of orbits all of which are slightly convex toward the filament), is an unstable situation. The corpuscle runs into perturbations which make the path curve, one way or another, thus introducing a center of orbital force, a force of reversion, and transverse vibrations, in the regular mode as above described for orbits of non-effect.

In a flight of light corpuscles, therefore, each particle follows its individual orbit of non-effect, except for perturbations, that is, transverse vibrations in the force of reversion. Its motion is, in the theoretical sense, determinate, predictable, and in unique conformity with the usual law of cause and effect. (8)

In all of the connections here studied, the hypothesis, that an aether exists, is unnecessary.

Non-linear transformations in orbit theory.

If we adjoin, to T, the transformation,

$$T_1 : r' = r'' + u_1\, o_1(r''), \quad \theta' = \theta'' + v_1\, p_1(\theta''), \quad \varphi' = \varphi'' + w_1\, q_1(\varphi''),$$

where we have,

$$o_1(r'') = l_1 r''^s + m_1 r''^{s-1} + \ldots + x_1, \quad p_1(\theta'') = \lambda_1 \theta''^s + \mu_1 \theta''^{s-1} + \ldots + \xi_1,$$

$$q_1(\varphi'') = L_1 \varphi''^s + M_1 \varphi''^{s-1} + \ldots + X_1,$$

a transformation of coordinates in this theory is furnished by the relations,

$$U : r + u\, o(r) = r'' + u_1\, o_1(r''), \quad \theta + v\, p(\theta) = \theta'' + v_1\, p_1(\theta''),$$

$$\varphi + w\, q(\varphi) = \varphi'' + w_1\, q_1(\varphi''),$$

(7) GLENN, Annali della Scuola Normale Superiore di Pisa; Ser. 3, vol. 8, (1954).

(8) DE BROGLIE, *Matière et Lumière* (1937), § IV.

$(u_1, v_1, w_1 \doteq 0)$. In terms of a notation that displays all variables which occur in T, transformation U combines the relations of,

$$(11) \qquad T(u, v, w) \equiv T(u, v, w; r', \vartheta', \varphi'; r, \theta, \varphi),$$

with the relations of,

$$(12) \qquad T_1(u_1, v_1, w_1) \equiv T_1(u_1, v_1, w_1; r', \theta', \varphi'; r'', \theta'', \varphi'').$$

For convergence to $T(a, b, c) R$ the parameters of T satisfy the regime (8). The transformations of (9) change the amplitudes of the vibrations but they do not change the limit $T(a, b, c) R$ approached. More generally we can arbitrarily choose the three series in (8) as long as the terms chosen are infinitesimal and there is convergence to a, b, c, respectively. Once such a choice is made, there is implied a correspondingly erratic variation of the field [e], but not a random variation.

We show how the transformation,

$$T(u, v, w), \ (u = u(t_1), \ v = v(t_1), \ldots),$$

repeated as implied by (8), gives the limit curve $A : T(a, b, c) R$, while also,

$$T_1(u_1, v_1, w_1), \ (u_1 = u_1(\tau_1), \ v_1 = v_1(\tau_1), \ldots),$$

repeated for an analogous regime, can give a corresponding limit curve, $A_2 : T_1(a_2, b_2, c_2) S$. Formula (11) is an operator involving $r', \theta', \varphi'; r, \theta, \varphi$, and (12) is an operator involving $r', \theta', \varphi'; r'', \theta'', \varphi''$, six of these variables being connected by the relations of U. With R and S expressed by the respective pairs of polar equations,

$$R : [R_1(r', \theta', \varphi') = 0, \ R_2(r', \theta', \varphi') = 0],$$

$$S : [S_1(r', \theta', \varphi') = 0, \ S_2(r', \theta', \varphi') = 0],$$

we note especially that S may be regarded as arbitrarily chosen but R can only be one of a determinate finite set of segments. For when we transform S by $T_1(u_1, v_1, w_1)$ and R by $T(u, v, w)$, the first transformed curve is transformed into the second by U, but not uniquely, since U carries the r'' of a point, on $T_1(n_1, v_1, w_1) S$, simultaneously into s values of r on s respective curves each of which could figure as $T(u, v, w) R$. Choose one of these s curves, assumed real, as the one that is carried into R by $T^{-1}(u, v, w)$. Then $T(u, v, w)$ merely perturbs R, and $T_1(u_1, v_1, w_1)$ merely perturbs S, while U carries $T_1(u_1, v_1, w_1) S$ into $T(u, v, w) R$, which facts are the essentials of the geometric situation.

Using then the latter perturbed segments as originals, we can repeat the process, transforming, in fact, S by

$$T_1(u_1(\tau_1) + u_1^1(\tau_2),\ v_1(\tau_1) + v_1^1(\tau_2),\ w_1(\tau_1) + w_1^1(\tau_2)),$$

and the chosen R by

$$T(u(t_1) + u^1(t_2),\ v(t_1) + v^1(t_2),\ w(t_1) + w^1(t_2)),$$

and choosing, retroactively, for R, a specific real branch given by the related U. Repetitions give a series of perturbed segments converging to A, and a series converging to A_2.

The last U transformation is the following:

$$U_0: r + a\,o(r) = r'' + a_2\,o_1(r''),\ \theta + b\,p(\theta) = \theta'' + b_2\,p_1(\theta''),$$

$$\varphi + c\,q(\varphi) = \varphi'' + c_2\,q_1(\varphi''),$$

which U_0 transformation accordingly represents the causality involved when one BOHR orbit shifts to another, under the well-known theory.

We derive conditions that A should be infinitesimally near to A_2, by an induction based on the quadratic case of U where we have, for $s = 2$, the formulas previously given for $o(r)$, $p(\theta), \ldots, o_1(r''), \ldots$ Here, in the solution of the first quadratic equation of U, for r, the radicand expression may be written as,

$$[u\,m + 1 + 2\,u\,l\,r'' + f(r'')]^2,$$

if we take,

$$f(r'') = -(2\,u\,l\,r'' + u\,m + 1) \pm \sqrt{\Delta_1},$$

with,

$$\Delta_1 = (u\,m + 1)^2 - 4\,u\,l\,(u\,n - u_1\,n_1) + 4\,u\,l\,(u_1\,m_1 + 1)\,r'' + 4\,u\,u_1\,l l_1\,r''^2,$$

Hence,

$$(13)\qquad r = r'' + f(r'')/2\,u\,l,\ \text{or},\ r = -r'' - \left(m + \frac{1}{u}\right)/l - f(r'')/2\,u\,l.$$

Likewise by solving the other equations of U and abbreviating as follows:

$$\Delta_2 = (v\,\mu + 1)^2 - 4\,v\,\lambda\,(v\,\nu - v_1\,\nu_1) + 4\,v\,\lambda\,(v_1\,\mu_1 + 1)\,\theta'' + 4v\,v_1\,\lambda\,\lambda_1\,\theta''^2,$$

$$\Delta_3 = (wM + 1)^2 - 4wL\,(w\,N - w_1 N_1) + 4wL\,(w_1 M_1 + 1)\,\varphi'' + 4w\,w_1\,L\,L_1\,\varphi''^2,$$

$$g(\theta'') = -(2\,v\,\lambda\,\theta'' + v\,\mu + 1) \pm \sqrt{\Delta_2}\,,$$

$$h(\varphi'') = -(2\,w\,L\,\varphi'' + w\,M + 1) \pm \sqrt{\Delta_3}\,,$$

we obtain the following values which are associated with the relation (13):

$$\theta = \theta'' + g(\theta'')/2\,v\,\lambda\,, \text{ or, } \theta = -\theta'' - \left(\mu + \frac{1}{v}\right)/\lambda - g(\theta'')/2\,v\,\lambda\,,$$

$$\varphi = \varphi'' + h(\varphi'')/2\,w\,L\,, \text{ or, } \varphi = -\varphi'' - \left(M + \frac{1}{w}\right)/L - h(\varphi'')/2\,w\,L\,.$$

It follows that, if each of $f(r'')$, $g(\theta'')$, $h(\varphi'')$ is close to zero in absolute value, as must be the case if the transformation is by U_0, from a point $(r'', \theta'', \varphi'')$ of A_2 to a point (r, θ, φ) of A, (A being assumed near to A_2), then we find,

$$r \doteq r'', \text{ or, } r \doteq -r'' - \left(m + \frac{1}{a}\right)/l\,,$$

$$\theta \doteq \theta'', \text{ or, } \theta \doteq -\theta'' - \left(\mu + \frac{1}{b}\right)/\lambda\,,$$

$$\varphi \doteq \varphi'', \text{ or, } \varphi \doteq -\varphi'' - \left(M + \frac{1}{c}\right)/L\,.$$

In the second value of r the absolute value of $-1/\,a\,l$ outranks the value of the rest of the terms combined, and similarly in the corresponding results for θ and for φ. Therefore U_0 will here transform A_2 either into a segment A infinitesimally near to A_2 or into an orbit of non-effect C, widely spaced from A_2 but belonging to the same regime as A.

The spaced type of orbit is brought into existence if a perturbation too forceful for the state of energy, of the electron orbits within an atom is applied and the electrons recede to respective orbits of a new level of energy, according to principles discovered by NIELS BOHR, although the second orbit referred to above might be merely another in the original level.

We can readily extend the mathematics, which relates to the transformation U, to the case of general orders by an induction that begins with the quadratic case just discussed. In the equation (from U),

(14) $$r + u\,o(r) = r'' + u_1\,o_1(r'')\,,$$

with the assumption that the orders are s, and,

$$o(r) = l\,r^s + m\,r^{s-1} + \ldots + x\,,$$

the coefficient of r^s is $u\,l$, and is small. Let the whole equation (14) be,

$$y = F(r) = \lambda'\,r^s + \mu'\,r^{s-1} + \ldots + \sigma', (= 0)\,,$$

the graph of $y = F(r)$ being as here shown. If we increase any order $s > 2$ by

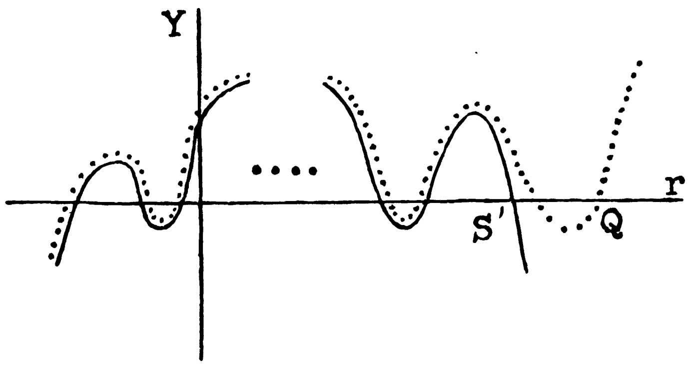

Figure 2

adding $\alpha\,r^{s+1}$ on the left side of $F(r)$, then, since α is small, and r may be assumed not large, the quantity added is small and the graph of the equation,

$$y = \alpha\,r^{s+1} + F(r)\,,$$

is near that of $y = F(r)$ up to its last intersection S'. Beyond S' the expression $\alpha\,r^{s+1}$ becomes large and the second curve no longer stays close to the first. With the sign of α properly assumed, the new curve departs from the first and eventually turns back and reaches Q, corresponding to root number $s + 1$.

The distance $\xi = S'\,Q$ is one of the roots of an equation whose coefficients are polynomials in α, λ', μ', .. σ'; σ' being a polynomial in r'', (Cf. 14). Thus $S'\,Q$ is determined non-uniquely by U.

In proof of this proposition, the abscissa of S' satisfies $F(r) = 0$, while that of Q, which is $r + \xi$, gives

$$\alpha\,(r + \xi)^{s+1} + F(r + \xi) = 0.$$

We rearrange the latter equation as $G(r) = 0$, in which the coefficients are polynomials in ξ, in the coefficients of $F(r)$, and in α. The r-eliminant between $F(r)$ and $G(r)$, equated to zero, is an equation $H_1(\xi) = 0$, such as is described in the lemma.

There is a like equation $H_2(\eta) = 0$, for the corresponding interval η beyond the greatest root θ resulting from the second equation of U, and an equation $H_3(\zeta) = 0$, for the interval ζ beyond the greatest root φ from the third equation of U. The general conclusions of the theory are seen to hold for any order $s > 1$.

From the point of view of mathematical culture it is fundamental that we can have an operator U which figures as a causal law and the effect of which is non-unique, but when this law combines with its relevant objective reality (a planet in its state of non-effect), the operator picks up a sufficient revision or fulfillment to make the effect one thing or the other but not both, that is to say, makes it either a secular perturbation of the original orbit or a finite shift to a different orbit. In this and the preceding section we have shown how two physical phenomena, relating to orbits, conform to the usual law of cause and effect. The inference had been questioned.

Concerning the problem of the frequency

An effect may be compound, with one effect occurring as a concomitant of another. What is described above as a vibration, in the force of reversion, contains the effect called frequency. Even if the field (e) is of the erratic type, the frequency of the i th vibration is,

$$\nu = t_{i+1} - t_i, \quad (i = 1, 2, \ldots),$$

so that, if a constancy of the frequency is a part of the hypothesis, then

$$(15) \qquad t_i - t_{i-j} = j\nu, \quad (j = 1, 2, \ldots, i-1).$$

Therefore, in the relation (8), all t_i, excepting t_1 can be eliminated. These become relations in t_1 and ν. Elimination of t_1 from the first two of the latter relations then gives an equation [1] in ν, a, and b, and elimination of t_1 from the first and third gives an equation [2] in ν, a, and c. Then [1] determines ν and [2] imposes a condition. The integers j thus brought into the equations [1], [2] need not exceed an integer j' for which the terms in (8) have become negligible.

It follows that the frequency ν is determined from the amplitude functions u, u^1, v, v^1, w, w^1, considered, not as continuous quantities, but as having numerical values at the times t_i. If the vibrations were pendular the equations [1], [2] would necessarily be identities in ν since the frequency and the amplitude of a pendulum are known to be functionally independent. The possibility that the mass m might divide out of equations [1] and [2] would make ν independent of the mass, which would be untenable mechanics. The non-pendular amplitudes depend upon the mass m and the velocity z of the celestial body in its orbit, because m and z are determinants of the orbit of non-effect into which the orbit is settling after perturbation. That is, at $P, u, u^1, v, v^1, w, w^1$, are functions of both m and z, and z is constant through the interval delimited by T on R.

This conclusion is consistent with results in the author's paper of 1937 in the Annali of the Pisan Scuola Normale Superiore, (vol. 6, p. 29). But, in the case of a stable orbit, secularly perturbed, the force of reversion which, along r, is initially merely the opposite of the force of perturbation, is evidently independent of r. Hence the amplitudes, and their frequencies, are also independent of r.

It follows that the solution of [1] for ν will be a function of m and z, and, in any interval $(m_1 < m < m_2\,;\, \nu_1 < \nu < \nu_2\,;\, z_1 < z < z_2)$, where this function is free from singularities, we will have, (9)

$$(16) \qquad \nu = (G_0 z^\mu + G_1 z^{\mu-1} + \ldots + G_\mu) m^\lambda + (H_0 z^\mu + H_1 z^{\mu-1} + \ldots + H_\mu) m^{\lambda-1} + \ldots + (M_0 z^\mu + M_1 z^{\mu-1} + \ldots + M_\mu),$$

the accuracy of the (numerical) coefficients, as is known, being controllable by choices of λ, μ.

Elements of the philosophy of cause and effect.

Some of the newer points of view in the science of physics unquestionably give new insights in the philosophy of reality. The conclusion that some materials are forms of energy, or nearly that, gives significant unity to our *scalea* of realities Σ. For it is evident that the realities in any degree above (β), including concomitant forms of consciousness, are also

(9) R. R. RAMSEY *Proc. Indiana (U. S. A.) Academy of Science* vol. 40. (1931), p. 38. (The article contains a discourse on PLANCK'S law of frequency).

forms of energy. This leaves the space-time order (α) as the only one where the specific realities are not expressible in terms of energy, unless it can be shown that space is a cache for latent energy, which seems improbable. For a more empirical reference, in one case, a simple subsistential reality (δ) may be considered. Its primary function was to aid self-perpetuation of a phenomenon, but the only thing that could bear directly upon a phenomenon, considered as an occurrence and therefore as a manifestation of energy, would be another manifestation of energy. Hence the assumed reality in (δ) would be descriptable in terms of energy.

In a cognition of a reality the intuitive representation, of the concept of the reality, must appertain to the same degree of Σ as does the concept itself. The representation could be in a lower degree but a given degree can include realities of any lower degree. A vector is a representation in (β) of a reality in (γ). A reality of one degree can however be depicted on one of the next higher degree. An objectivity, in (β), has a depiction upon the phenomenon (γ) of weight, viz. a number that expresses the weight of the object. An element of subliminal mentality can be depicted upon the sense of social form, and so on.

However not all realities of a given degree can be depicted on a chosen reality of the next higher degree. One could choose a specific humane reality (η) an then find that a given reality (ζ), as the sense of musical harmony, had nothing to do with it. By analogy we could state that not all spatial extensions, that is, realities of the degree (α), can be depicted on a given plane, or on any object in (β).

Mathematical laws play a leading part in the rise of many if not of all causes. We here mean deterministic and not probabilistic mathematical laws. KANT has written that it is not manifest, a priori, why reality, in general, should present the case of a specific reality A, as cause, as being connected with a second reality B, as effect, by law. It is probably true that no specific reality, of itself, causes a second reality. If so the second reality most likely would have piled up on the earth. A reality becomes involved in a cause when it is its fortune to combine with a relevant law of combination, and we can go a long way with the hypothesis that this law is always, in its essentials, mathematical.

An element that is necessarily conjectural may well be introduced here, in the formulation. We assume that intuition functions like a mathematical law associated with the atomism of the brain, and acts on occasion in combination with a reality to produce a cause, in the manner characteristic of mathematical laws which produce causes by aligning themselves with respective relevant realities.

Light effects, as we have noticed, exist in conformity with the rule stated, as effects of combination of specific reality with relevant mathematical law. BERTHOUD has identified the fundamentals in chemical change, under the principle of the conservation of the chemical elements, with perturbations of the electronic orbits within the atoms, and these perturbations follow mathematical laws. An electron may be in a given vicinity and a proper nucleus may be incipient there too, but only the mathematical law of central orbital motion can cause the electron to produce an atom as its effect. The process in general may be stated formally: The law L combines with the reality a to produce a cause the effect of which is a reality b. The reality a may be a combination of realities, [10] in a state of non-effect. Also it seems to be true always that transition from cause to effect is itself a reality (γ), that is, a phenomenon.

An object (β) in the form of a drop of water, does not, of itself, cause the phenomenon of wetness on a pane of glass. The drop becomes the cause only by the intervention of the law of adhesion, which can be mathematically expressed. The transition from drop to wetness is a phenomenon, in (γ).

A leaf, largely non-conductive as to heat, combined with a degree of coldness just below that of the temperature of saturation of the immediately surrounding air, is a state of non-effect until the leaf cools the air to the saturation point, giving a cause. The effect is dew. The law of saturation can be mathematically expressed. The transitory phenomenon is the precipitation of the dew from the air, onto the leaf.

An objectivity, water, combined with a phenomenon of coldness below the temperature 32^0 f, is a state of non-effect until the mathematical law of crystalization intervenes to produce a cause. The effect is ice. Here, although the law is unique, the original reality is a combination. The transitional phenomenon is the process of freezing. There may follow, in the effect, an additional phenomenon, viz. the arrangement of ice crystals in configurations.

A block of marble, combined with a sculptor's intuition of a statue, is a cause of which the statue is the effect, in (θ). There may be several intuitions but they can be rated as one. The transitory phenomenon is all the work done by the sculptor in order to realize his intuiton. Here an active agent in the phenomenon of transition is free will.

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[10] Often a combination of mathematical laws is equivalent to a single mathematical law.

One humane reality, say the writing of a piece of poetry, does not cause a libertarian reality (θ), as the activity of teaching, without the intervention of some very special intuition like the idea that a child should have the poetic as a part of his mental complex. We refer to teaching as libertarian because proper teaching is the truth living, also usefulness, goodness, and beauty, but always living. Here the transition from cause to effect is a compound phenomenon, of all the intervening preparation for teaching.

A cherry is not a subsistential reality, but, combined with the bird's intuition about it, becomes the cause of such a reality viz. eating, an effect in (δ). The transitional reality is the getting in position to eat and ranks as a phenomenon.

One original thought (ι) does not cause another such thought without the intervention of the wave-intuition as a law. The transitory phenomenon is fleeting but definite.

The number of mathematical laws in the universe which can function in the aforesaid manner to produce causes may be finite, in which eventuality it is a question of a set of laws always seeking appropriate *scalea* realities Σ as companions in causes, like PIRANDELLO'S play: Six Characters in Search of an Author. The number of incidences, law with relevant reality, has been and is comparatively small. Biological organisms have evolved respective method-processes for controlling the incidence of law with relevant specific reality, for the problem of reproduction of their type, as effect, or rather, as a complex of effects, and one result has in fact been a piling up and a consequent struggle for existence. Essentials of the laws of this struggle can be expressed mathematically.

The possibility of the transition, reality A to reality B is not recognizable a priori, as KANT insisted, because of the fortuitous and rare nature of the incidence of the law with the relevant reality, an incidence which must be null thousands of times to one success.

The transition A to B is not always a synthesis. If B is in a degree of Σ above the degree of A the transition is synthetical, as when the mathematical law of gravity combines with an object of (β) and causes weight, in (γ). But a sense of beautiful objective form might combine with the mathematics of the musical scale and effect a sense of musical harmony, both realities here being in (ζ). Hence the transition cause to effect is here not synthetic but analytic.

Because there is no basis, a priori, for any knowledge of the nature of the connection between a spatial reality (α) and an objectivity (β), this connection presents an antinomy which may indeed be avoided occasionally by some exercise of synthetic judgment but which will nevertheless remain, at least in the form of the before-mentioned theorem on the uniqueness of that connection. The same assertion, *mutatis mutandis*, applies to the connection between two respective realities one in each of any two consecutive degrees of Σ. We do not really know what the nature of the connection is between an object in (β) and its weight, or between a self-perpetuating phenomenon and its principle of subsistence in (δ). Nor do we know the nature of the connection between a definite humane reality in the work of an artist and any specific reality of (θ), in the sphere of liberty, or of the connection between any specific libertarian reality and any creative principle of the mind, but intuition prompts us to regard each respective connection as unique.