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INVARIANTS WHEN THE TRANSFORMATION IS INFINITESIMAL, AND THEIR RELEVANCE IN BIO-MATHEMATICS AND IN THE THEORY OF TERRESTRIAL MAGNETISM

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§ I.

HISTORICAL INTRODUCTION

A theory of infinitesimal transformations, due to SOPHUS LIE (1), was based upon certain relations in three variables x_i and three parameters u_i, viz.,

(1) $$x_i' = X_i(x_1, x_2, x_3, u_i), (i = 1, 2, 3).$$

It was assumed that X_i can be expanded in a series of powers of u_i; that there will be no singularities in the way; that $u_i \doteq 0$, and that $X_i(x_1, x_2, x_3, 0) = x_i$. Then (1) takes the form,

(2) $$x_i' = x_i + u_i \xi_i(x_1, x_2, x_3) + \ldots, (i = 1, 2, 3).$$

It is implicit in LIE'S formulations that the second part of the relation

$$d x_i' = d x_i + u_i \left(\frac{\partial \xi_i}{\partial x_1} d x_1 + \frac{\partial \xi_i}{\partial x_2} d x_2 + \frac{\partial \xi_i}{\partial x_3} d x_3\right),$$

never is cancelled although each term may have two small factors. Much attention is claimed by the case where $x_1 = x$, $x_2 = y$ are cartesian plane coordinates and $x_3 = \pi$, the slope of a line-element of the point (x, y).

(1) Mathematische Annalen, Bände 5, 8, und 11; 1872-1878.

Here the particularization $u_1 = u_2 = u_3 = \delta t$ is adopted, and the invariance of $l = d\,y - \pi\,d\,x$, expressed by the relation,

$$d\,y' - \pi'\,d\,x' = \varrho\,(d\,y - \pi\,d\,x)\,,\,(\varrho = \varrho\,(x\,,y\,,\pi))\,,$$

is a total (necessary and sufficient) condition in order that (2) should be a contact transformation. In three-space the condition for an *elementverein* (connection) of surface elements is,

$$\pi\,d\,x + \lambda\,d\,y - d\,z = 0\,,$$

$\pi\,,\lambda$ being the coordinates of direction of the tangent plane at $(x\,,y\,,z)$. The total condition for a contact transformation, V, of surfaces, is the relation of invariance,

(3) $\pi'\,d\,x' + \lambda'\,d\,y' - d\,z' = \varrho\,(\pi\,d\,x + \lambda\,d\,y - d\,z)\,,\,(\varrho = \varrho\,(x\,,y\,,z\,,\pi\,,\lambda))\,,$

under V,

$$V:\ x' = X(x\,,y\,,z\,,\pi\,,\lambda)\,,\,y' = Y(x\,,y\,,z,\pi\,,\lambda)\,,\,z' = Z(x\,,y\,,z\,,\pi\,,\lambda)\,,$$
$$\pi' = \Pi(x\,,y\,,z\,,\pi\,,\lambda)\,,\,\lambda' = \Lambda\,(x\,,y\,,z\,,\pi\,,\lambda)\,.$$

Some transformations V will have a special infinitesimal form like (2), † and all form a group $\{V\}$.

We generalize by studying the invariance of an arbitrary quantic $H = H(\varphi\,,r\,,\theta\,,d\,\varphi\,,d\,r\,,d\,\theta)$, in $d\,\varphi\,,dr\,,d\,\theta$ as the facients in H, and adopt the usual relations between the cartesian and the polar coordinates, viz.,

$$x = r\cos\theta\cos\varphi\,,\,y = r\cos\theta\sin\varphi\,,\,z = r\sin\theta\,.$$

We consider both the direct and the inverse invariant problems, and make extensions, to n variables, which generalize the concepts and theories of connection and contact. This mathematics is then applied in the study of two problems which are dominated by orbital theory, one in bio-mathematics and the other in the theory of terrestrial magnetism (2).

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(2) Since the present paper was written I have seen the following two recent memoirs which are developed by means of the tensorial calculus: A. KAWAGUCHI, On the theory of non-linear connections: I. Introduction to the theory of general non-linear connections, Tensor, N. S., vol. 2, 1952, p. 123-142.

HLAVATÝ, Embedding theory of a W_m in a W_n, Rendiconti del Circolo Matematico di Palermo, S. II, t. 1, 1952, p. 403-438, (See especially, DEFINITION, p. 429).

† For a method of proof see LIE und SCHEFFERS, *Geom. der Berührungstr.*, 1896, S. 90.

The differential equation for invariants

The transformation on the plan of (2) is accordingly,

$$U:\ \varphi' = \varphi + u\,O(\varphi, r, \theta),\ r' = r + v\,P(\varphi, r, \theta),\ \theta' = \theta + w\,Q(\varphi, r, \theta),$$

in which $|u|, |v|, |w|$ are $\doteq 0$. Since any u, v, w may be either positive or negative, we ordinarily assume $O(\varphi, r, \theta), P(\varphi, r, \theta), Q(\varphi, r, \theta)$ to be positive functions, $\mp \varepsilon, \mp \sigma$; $\varepsilon \doteq 0, \sigma \doteq \infty$, on any continuous space-interval $[\varphi_1, r_1, \theta_1), (\varphi_2, r_2, \theta_2)]$ considered, and that H and its partial derivatives do not approach the infinite on the interval.

Substitution from U in H enables us to form an expansion, and the hypothesis,

$$H' (= H(\varphi', r', \theta', d\varphi', dr', d\theta')) = M\,H,$$

with M constant, leads to the equation,

$$\Omega H = \left\{ u\,O \frac{\partial}{\partial \varphi} + v\,P \frac{\partial}{\partial r} + w\,Q \frac{\partial}{\partial \theta} + u\,(d\,O) \frac{\partial}{\partial (d\,\varphi)} + \right.$$

$$\left. + v\,(d\,P) \frac{\partial}{\partial (d\,r)} + w\,(d\,Q) \frac{\partial}{\partial (d\theta)} + (1 - M) \right\} H = 0\,.$$

In the direct invariant problem the equation $\Omega H = 0$ is to be solved for H, transformation U being given. The auxiliary system of LAGRANGE is as follows:

$$\text{(4)} \qquad \frac{d\,\varphi}{u\,O} = \frac{d\,r}{v\,P} = \frac{d\,\theta}{w\,Q} = \frac{d\,(d\,\varphi)}{u\,(d\,O)} = \frac{d\,(d\,r)}{v\,(d\,P)} = \frac{d\,(d\,\theta)}{w\,(d\,Q)} = \frac{d\,H}{(M-1)\,H},$$

and three particular integrals, out of six requisite for a complete integral of $\Omega H = 0$, are readily found in the forms,

$$(d\,\varphi)/O = \gamma_1,\ (d\,r)/P = \gamma_2,\ (d\,\theta)/Q = \gamma_3,\ \text{(constants } \gamma_i \text{ arbitrary)}.$$

Three additional particular integrals can be found in various cases where the functions O, P, Q have a more special form. After H is found as the complete integral of $\Omega H = 0$, the problem of the integration of the MONGE equation $H = 0$ can be considered.

The following are the particularizations which we shall emphasize most:

$$O = o(\varphi) = a\,\varphi^{e-1} + b\,\varphi^{e-2} + \ldots + k,\ P = p(r) = \alpha\,r^{e-1} + \beta\,r^{e-2} + \ldots + \varkappa,$$

$$Q = q(\theta) = \aleph\,\theta^{e-1} + \beth\,\theta^{e-2} + \ldots + l,$$

in which the caefficients $a, b, .., \alpha, \beta, .., \aleph, \daleth, ..$ are real numbers and each variable φ, r, θ may vary only over its finite, continuous range (3). The tranformation U becomes $U_1\ (u, v, w)$; †

$$U_1\ (u, v, w):\ \varphi' = \varphi + u\, o\,(\varphi),\ r' = r + v\, p\,(r),\ \theta' = \theta + w\, q\,(\theta),$$

$$(|u|, |v|, |w| \doteq 0).$$

Here U_1 generates a group $\{U_1\}$ which satisfies the following symbolism:

$$(5)\quad U_1^2 = U_1\,(u_1, v_1, w_1)\ U_1\,(u_2, v_2, w_2) = U_1\,(u_1 + u_2, v_1 + v_2, w_1 + w_2).$$

The group $\{U_1\}$ will be finite and of order ν if, for some integer ν,

$$(6)\qquad V_\nu = u_1 + u_2 + .. + u_\nu = 0,\ W_\nu = v_1 + v_2 + .. + v_\nu = 0,$$

$$X_\nu = w_1 + w_2 + .. + w_\nu = 0,$$

and otherwise of infinite order, but V_m, W_m, X_m, as m increases, must all remain $\doteq 0$. Three additional particular integrals of (4) are now;

$$H\, e^{-I(\varphi)} = \gamma_4,\ H\, e^{-J(r)} = \gamma_5,\ H\, e^{-K(\theta)} = \gamma_6, \qquad (e = 2.71828\,..)$$

in which,

$$I\,(\varphi) = \int \frac{d\,\varphi}{i\, o\,(\varphi)},\ J\,(r) = \int \frac{d\, r}{j\, p\,(r)},\ K\,(\theta) = \int \frac{d\,\theta}{k\, q\,(\theta)}, \qquad (i, j, k \text{ const.})$$

Hence the complete integral here, whether H is a quantic or a more general function, is;

$$\Phi = \Phi\,(H\, e^{-I(\varphi)}, H\, e^{-J(r)}, H\, e^{-K(\theta)}, (d\,\varphi)/o\,(\varphi), (d\, r)/p(r), (d\,\theta)/q\,(\theta)) = 0,$$

Φ being an arbitrary function of its six arguments. If H is assumed to be a quantic, it is of the form,

$$H = \Psi\,(e^{I(\varphi)}, e^{J(r)}, e^{K(\theta)}, (d\,\varphi)/o\,(\varphi), (d\, r)/p\,(r), (d\,\theta)/q\,(\theta)),\ (\Psi \text{ arbitrary}).$$

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(3) That is, some use will be made of equations $y = q_1 x^{e-1} + q_2 x^{e-2} + .. + q_e$, to represent a function determined by e points on a plane continuous interval, $[(x_1, y_1), .., (x_2, y_2)]$, that contains no singularities of the function.

† As long as u, v, w are parameters, each is the real continuum, in the vicinity of the origin. Compare the discussion at § II, (16) and following.

It is then a MONGE expression in which the differential arguments,

$$(d\,\varphi)/o\,(\varphi)\,,\,(d\,r)/p\,(r)\,,\,(d\,\theta)/q\,(\theta)\,, \tag{7}$$

are absolute invariants of the group $\{U_1\}$. The functional arguments of the type $e^{I(\varphi)}$ are relative invariants of $\{U_1\}$ which satisfy the relations,

$$e^{I(\varphi')} = e^{\frac{u}{i}} e^{I(\varphi)}\,,\ e^{J(r')} = e^{\frac{v}{j}} e^{J(r)}\,,\ e^{K(\theta')} = e^{\frac{w}{k}} e^{K(\theta)}\,. \tag{8}$$

When M is constant, the expressions (7), (8) *constitute a fundamental system of* MONGE *invariants of the group* $\{U_1\}$.

However the group may reduce to a subgroup of $\{U_1\}$ because, if a typical coefficient in H' is a polynomial,

$$\sum_s a_s \left(e^{I(\varphi')}\right)^{m_{1s}} \left(e^{J(r')}\right)^{m_{2s}} \left(e^{K(\theta')}\right)^{m_{3s}}\,,$$

the relation $H' = M\,H$ gives,

$$m_{11}\frac{u}{i} + m_{21}\frac{v}{j} + m_{31}\frac{w}{k} = m_{12}\frac{u}{i} + m_{22}\frac{v}{j} + m_{32}\frac{w}{k} = \ldots, = \log_e M\,.$$

In particular cases this set of equations may restrict the polynomial, and it may restrict the group to one of the subgroups of $\{U_1\}$. It is not, however, necessary *a priori* that the typical coefficient in H should be a rational, integral polynomial in the relative invariants.

If H has constant coefficients it can be linearly factorable. If its order in the differentials is h this will imply the vanishing of $\frac{1}{2}\,h\,(h-1)$ non-linear expressions in its $\frac{1}{2}(h+1)(h+2)$ coefficients. This would leave H in a form that is a natural generalization of a binary quantic, which, theoretically, is always linearally factorable. If the factors of H, with constant coefficients, are,

$$I_l = A_l\frac{d\,\varphi}{o\,(\varphi)} + B_l\frac{d\,r}{p\,(r)} + C_l\frac{d\,\theta}{q\,(\theta)}\,,\ (l = 1\,,\ldots,h)\,,$$

the problem of integrating $H = 0$ reduces to that of the integration of $I_l = 0$. Here I_l is an exact differential and its integral curves are known to form the system of any and all analytic curves on the single surface [4],

$$L = A_l \int \frac{d\varphi}{o(\varphi)} + B_l \int \frac{dr}{p(r)} + C_l \int \frac{d\theta}{q(\theta)} + D_l = 0 .$$

With l chosen, since I_l is invariant, the integral curves of $I_l = 0$ are permuted by transformations $\{U_1\}$. Since they are all on a single surface, it, $(L = 0)$, is unaltered by U_1, whence,

$$A_l u + B_l v + C_l w = 0 .$$

The group under which the surface $L = 0$ is invariant is, therefore, a $\{U_2\}$ symbolized by the relation,

$$U_2\left(-\frac{B_l}{A_l}v_1 - \frac{C_l}{A_l}w_1 , v_1 , w_1\right) U_2\left(-\frac{B_l}{A_l}v_2 - \frac{C_l}{A_l}w_2 , v_2 , w_2\right) =$$

$$= U_2\left(-\frac{B_l}{A_l}(v_1 + v_2) - \frac{C_l}{A_l}(w_1 + w_2) , v_1 + v_2 , w_1 + w_2\right) .$$

Since, in this symbol, u_1 , u_2 are dependent infinitesimals, $\{U_2\}$ is a formal subgroup of $\{U_1\}$ as long as the v_i , w_i are arbitrary infinitesimals.

Theorem. The equation $\mathrm{H} = 0$ *represents* h *surfaces which are respectively invariant under* h *subgroups of* $\{\mathrm{U}_1\}$. *These subgroups may not all be mutually exclusive.* †

Invariant curves

With l and the coefficients $A_l , .. , D_l$ chosen, as stated above, the torus, based on any curve $N = 0$ in the (φ , r) plane, $(\theta = 0)$,

$$N = K_1 \int \frac{d\varphi}{o(\varphi)} + L_1 \int \frac{dr}{p(r)} = 0 ,$$

intersects the corresponding surface $L = 0$ in a curve invariant under a $\{U_2\}$ for which simultaneously,

$$A_l u + B_l v + C_l w = 0 , K_1 u + L_1 v = 0 .$$

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(4) FORSYTH, *Differential- Gleichungen,* 1889, S. 290.

† They may not if, for all consecutive pairs $(v_1 , w_1) , (v_2 , w_2)$ in $\{U_2\}$, the relation $v_1 / w_1 - v_2 / w_2 = 0$ is satisfied.

This curve is therefore invariant under the one-parameter sub-group $\{U_3\}$, of $\{U_2\}$, for which the symbol is,

$$U_3\left(-\frac{L_1}{K_1}v_1, v_1, \frac{A_l L_1 - B_l K_1}{C_l K_1}v_1\right)U_3\left(-\frac{L_1}{K_1}v_2, v_2, \frac{A_l L_1 - B_l K_1}{C_l K_1}v_2\right)=$$

$$= U_3\left(-\frac{L_1}{K_1}(v_1+v_2), v_1+v_2, \frac{A_l L_1 - B_l K_1}{C_l K_1}(v_1+v_2)\right),$$

a result of considerable generality, since l may assume any one of h values.

Secondly, if two of the sub-groups, mentioned in the above theorem, coincide, the curve of intersection of the two corresponding surfaces $L=0$ is an invariant of the common sub-group.

Third, there are three configurations which are asymptotic to surface L or nearly so, which would otherwise cut $L=0$ in invariant curves, viz., (1) any plane $\varphi - \varphi_i = 0$, φ_i being a root of u $o(\varphi)=0$; (2) any sphere $r - r_i = 0$, r_i being a root of vp $(r)=0$; and (3) any cone $\theta - \theta_i = 0$, θ_i being a root of wq $(\theta)=0$.

Fourth, however, we can identify other classes of curves, which are invariant in the plane (φ, r), which is the plane (x, y). When the polar transformation is,

$$(9) \qquad s\colon\ \varphi' = \varphi + u\, o(\varphi),\ r' = r + v\, p(r),\ (|u|, |v| \doteq 0),$$

symbolized by,

$$\sigma = s^2 = s(u_1, v_1)\, s(u_2, v_2) = s(u_1+u_2, v_1+v_2), \text{ [5]}$$

the solution for the quantic $H(\varphi, r, d\varphi, dr)$, universally invariant under $\{s\}$, by the method for the case of three variables, gives,

$$H^{(2)} = H(e^{I(\varphi)}, e^{J(r)}, (d\varphi)/o(\varphi), (dr)/p(r)), \quad (H \text{ binary in } d\varphi, dr).$$

Here the variables are polar or cartesian according to preference. A geometrical interpretation of any equation, $H=0$, in $(dx)/o(x), (dy)/p(y)$, with constant coefficients, may be stated thus: *There exist* h *curves in the plane* (x, y) *each invariant under a corresponding sub-group of* {s}.

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[5] The product $s(u,v)\,s(u,v)$ will be referred to as the simple square of s.

This is because H is, in theory, factorable into h factors of the type,

$$a_j \frac{d\,x}{o\,(x)} + b_j \frac{d\,y}{p\,(y)}, \qquad (j = 1, 2, \ldots, h;\ a_j, b_j \text{ const})$$

so that the equation,

$$K = a_j \int \frac{d\,x}{o\,(x)} + b_j \int \frac{d\,y}{p\,(y)} + c_j = 0\,, \tag{10}$$

contains all of the theory of the integrals of $H = 0$. A curve $K = 0$ is invariant under all transformations $\{s\}$ for which,

$$a_j\,u + b_j\,v = 0\,,$$

and therefore under the one parameter sub-group whose symbol is

$$s_1\left(u_1\,, -\frac{a_j}{b_j}\,u_1\right) s_1\left(u_2\,, -\frac{a_j}{b_j}\,u_2\right) = s_1\left(u_1 + u_2\,, -\frac{a_j}{b_j}(u_1 + u_2)\right). \tag{11}$$

When there are no further hypotheses the curves $K = 0$ are permuted at random by trasformations $\{s\}$ outside of $\{s_1\}$.

Generalizations

Referring to the tranformation U in the original theory, in which the variables may be cartesian, we consider a quantic H in the differentials $d\,x, d\,y, d\,z, d\,\pi, d\,\lambda$, having coefficients which are functions of x, y, z, π, λ where π, λ are the variables of direction of a surface element through the point (x, y, z). Let $\{U\}$ become $\{U, T\}$ by adjunction of T;

$$T:\ \pi' = \pi + \mu\,R\,,\ \lambda' = \lambda + \nu\,S\,, (|\,\mu\,|\,, |\,\nu\,| \doteq 0)\,,$$
$$(O\,, P\,, Q\,, R\,, S \text{ functions of } x\,, y\,, z\,, \pi\,, \lambda)\,.$$

With M constant the corresponding $\Omega\,H = 0$ is an equation having eleven terms.

Then, with functions O, P, Q particularized to polynomials as in $\{U_1\}$, the whole group may include U_1 and T_1, where

$$T_1:\ \pi' = \pi + \mu\,m\,(\pi)\,,\ \lambda' = \lambda + \nu\,n\,(\lambda)\,,$$

with,

$$m\,(\pi) = m_1\,\pi^{e-1} + m_2\,\pi^{e-2} + \ldots + m_e\,,\ n\,(\lambda) = n_1\,\lambda^{e-1} + n_2\,\lambda^{e-2} + \ldots + n_e\,,$$

(real polynomials).

The invariant MONGE expression, under $\{U_1, T_1\}$, is then of the form,

$$H^{(3)} = H\,(e^{I(x)}, e^{J(y)}, e^{K(z)}, e^{L(\pi)}, e^{N(\lambda)},$$

$$(d\,x)/o\,(x), (d\,y)/p\,(y), (d\,z)/q\,(z), (d\,\pi)/m\,(\pi), (d\,\lambda)/n\,(\lambda)),$$

where,

$$L\,(\pi) = \int \frac{d\,\pi}{l\,m\,(\pi)}, \; N\,(\lambda) = \int \frac{d\,\lambda}{g\,n\,(\lambda)};$$

with the other integrals, and exponentials, as in (8). It is to be understood that the exponential relative invariants occur in such functional combinations, in the coefficients of H, that these coefficients, and the quantic as a whole, are invariants with a relation holding in the form (12). Existence for $H^{(3)}$ is established by the product of an H with constant coefficients, by the five exponential invariants themselves:

(12)' $$H^{(3)\prime} = M\,H^{(3)}.$$

Now the equation $H^{(3)} = 0$ may be said to define a situation like connection in 3-space, since it is an equation that involves the coordinates x, y, z of a point with the partial slopes π, λ of the points typical surface element, and both of these sets with the differentials.

A more complete formulation is the following:

Definition. If the group is $\{U, T\}$, and H is a quantic in the differentials, satisfying $\Omega\,H = 0$, then $H = 0$ defines a generalized connection of surface-elements. The relation $H' = M\,H$ is a total condition in order that a transformation (U, T) should preserve a generalized type of contact between surfaces (6).

In this definition and subsequently we can allow M to be a function of the variables,

$$M = M\,(x, y, z, \pi, \lambda),$$

the other features of the above theory being left unaltered. This being as-

(6) We shall refer to the traditional concepts, connection, and contact transformation, respectively, as the elementary cases. They will be in contrast with whatever are represented geometrically, at the point (x, y, z), by $H = 0$, and by $H' = M\,H$. A particular case of the generalized contact is where two surfaces intersect in a closed spatial oval of infinitesimal dimensions.

sumed, five particular integrals of $\Omega H^{(3)} = 0$ will remain unchanged. These are,

$$(d\,x)/o\,(x), (d\,y)/p\,(y), (d\,z)/q\,(z), (d\,\pi)/m\,(\pi), (d\,\lambda)/n\,(\lambda).$$

Since the other five particular integrals may be obtained by solving five properly chosen integrals for H, the former are necessarily invariants V_i each of which will satisfy a relation of invariance of the form,

$$V_i' = V_i(x', y', z', \pi', \lambda') = M_i(x, y, z, \pi, \lambda)\; V_i(x, y, z, \pi, \lambda) = M_i\, V_i, \quad (i = 1, .., 5).$$

A quite general form of the quantic H is then;

$$H = \sum_{k=1}^{q} \left(\sum_{j=1}^{p} a_{jk}\; V_1^{n_{1jk}} \dots V_5^{n_{5jk}} \right) \left(\frac{d\,x}{o\,(x)} \right)^{m_{1k}} \left(\frac{d\,y}{p\,(y)} \right)^{m_{2k}} \dots \left(\frac{d\,\lambda}{n\,(\lambda)} \right)^{m_{5k}}.$$

The following theorem is subject to the postulate that is introduced in the proof;

Theorem. If $\zeta_k = \sum_{j=1}^{p} a_{jk}\, V_1^{n_{1jk}} .. V_5^{n_{5jk}}$ *is invariant for all* k, *the quantic* H *is term-wise invariant and its terms have a common modulus.*

From the relation of invariance of ζ_k, viz.,

$$\zeta_k' \equiv \sum_{j=1}^{p} a_{jk} V_1'^{n_{1jk}} \dots V_5'^{n_{5jk}} = M^{(k)} \left(\sum_{j=1}^{p} a_{jk}\; V_1^{n_{1jk}} \dots V_5^{n_{5jk}} \right) = M^{(k)}\, \zeta_k,$$

we subtract that of the last term of ζ_k, *idem est*,

$$t'_{pk} \equiv a_{pk} V_1'^{n_{1pk}} \dots V_5'^{n_{5pk}} = N_{pk} (a_{pk}\; V_1^{n_{1pk}} \dots\; V_5^{n_{5pk}}) = N_{pk}\, t_{pk}.$$

On the left the result is cancellation of the last term of ζ_k'. On the right we obtain,

$$D_1 = M^{(k)} (t_{1k} + t_{2k} + \dots + t_{(p-1)k}) + (M^{(k)} - N_{pk})\, t_{pk}.$$

We next subtract $t'_{(p-1)k} = N_{(p-1)k}\, t_{(p-1)k}$ from $(\zeta_k' - t_{pk}') = D_1$, with the result;

$$D_2 = M^{(k)} (t_{1k} + t_{2k} + \dots + t_{(p-2)k}) + (M^{(k)} - N_{(p-1)k})\, t_{(p-1)k} + \\ + (M^{(k)} - N_{pk})\, t_{pk}.$$

Repeating the process p times we reach $D_p = 0$, whence,

$$\sum_{j=1}^{p} (M^{(k)} - N_{jk})\, t_{jk} = 0 .$$

Postulate. In the invariant theory with which we are concerned there exists no linear relation between the moduli $M^{(k)}$, N_{jk} and the terms t_{jk}.

Therefore

$$M^{(k)} = N_{jk}, \; (j = 1, \ldots, p) .$$

If we now subtract, from the relation of invariance of H, viz.,

$$H' \equiv \sum_{k=1}^{q} \zeta'_k \left(\frac{d\,x'}{o\,(x')}\right)^{m_{1k}} \cdots \left(\frac{d\,\lambda'}{n\,(\lambda')}\right)^{m_{5k}} = M \sum_{k=1}^{q} \zeta_k \left(\frac{d\,x}{o\,(x)}\right)^{m_{1k}} \cdots \left(\frac{d\,\lambda}{n\,(\lambda)}\right)^{m_{5k}} = M\,H,$$

the relation for the last term h_q of H, *idem est,* $h'_q = N_{pq}\, h_q$, so as to cancel the last term of H', and repeat the process q times, we obtain, since the differential products,

$$\left(\frac{d\,x}{o\,(x)}\right)^{m_{1k}} \cdots \left(\frac{d\,\lambda}{n\,(\lambda)}\right)^{m_{5k}},$$

are not linearally connected with the expressions $(M - N_{pk})\, \zeta_k$,

$$N_{jk} = M, \; (j = 1, \ldots, p;\; k = 1, \ldots, q), \qquad \text{q. e. d.}$$

The expressions $dx/o\,(x), \ldots, d\lambda/n\,(\lambda)$, $V_i, (i = 1, \ldots, 5)$ constitute a fundamental system of MONGE *invariants of (U_1, T_1).*

If H is of order h in the differentials, and the conditions for its linear factorability are satisfied, its factors give as many Pfaffian equations of the type of $J = 0$, where,

$$J = c_1\, V_1\, (d\,x)/o\,(x) + c_2\, V_2\, (dy)/p\,(y) + c_3\, V_3\, (d\,z)/q\,(z) + \\ + c_4\, V_4\, (d\pi)/m\,(\pi) + c_5\, V_5\, (d\lambda)/n\,(\lambda),$$

and $J' = M\, J$, (c_i constant).

The inverse invariant theory

In an inverse problem the invariant is given, here by being a solution of $\Omega\, H = 0$, and it is required to determine, possibly in terms of functions in part arbitrary, the transformations. If these generate the sub-group of one infinitesimal parameter obtained by making $v = w = \mu = \nu = u$ in (U,T), then since u will occur in $M = M(x, y, z, \pi, \lambda)$, we have $M = M_0 + \sigma u$, whence $M_0 = 1$. Since we shall wish to generalize the number of variables, we write x_1, x_2, x_3 respectively for x, y, z; P_1, P_2, P_3 for O, P, Q; Q_1, Q_2 for R, S and λ_1, λ_2 for π, λ.

The tranformation in n variables $x_1, \ldots, x_n$ and $n-1$ partial slopes $\lambda_1, \ldots, \lambda_{n-1}$ is,

$$U:\ x_i' = x_i + u_{(i)} P_i(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n-1}),\ \lambda_j' = \lambda_j + n_{(n+j)} Q_j(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n-1}),$$

$(i = 1, \ldots, n;\ j = 1, \ldots, n-1;\ |u_{(k)}| \doteq 0)$. The corresponding operator is Ω'_n, which becomes Ω_n when all $u_{(k)} = u_{(1)}$ and $u_{(1)}$ is divided out of Ω'_n:

$$\Omega'_n H = \left[\sum_{i=1}^{n} u_{(i)} \left(P_i \frac{\partial}{\partial x_i} + (d P_i) \frac{\partial}{\partial (d x_i)}\right) + \right.$$
$$\left. + \sum_{j=1}^{n-1} u_{(n+j)} \left(Q_j \frac{\partial}{\partial \lambda_j} + (d Q_j) \frac{\partial}{\partial (d \lambda_j)}\right) - \sum_{k=1}^{2n-1} u_{(k)} \sigma_k \right] H,$$

where M, in $H' = M(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n-1}, u_{(1)}, \ldots, u_{(2n-1)}) H$, is expansible into

$$M = M_0 + \sum_k u_{(k)} \sigma_k,\ (M_0 = 1,\ \Sigma \sigma_k = \sigma).$$

When, with $n = 3$, we assign H and make $\Omega_3 H$ vanish identically, Q_1, Q_2 are determined and P_1, P_2, P_3 are subjected to conditions. In particular if H is either of the quantics,

$$\lambda_1 d x_1 - d x_2,\ \lambda_1 d x_1 + \lambda_2 d x_2 - d x_3,$$

the known conditions that $\{U, T\}$ be a group of contact tranformations are obtained, (LIE und SCHEFFERS, *loco citato*, *S*. 93 und *S*. 598). If we assume that H is quadratic, adopt the notations,

$$P_{ix_1} = \partial P_i / \partial x_1 \ et\ cetera,$$

$$(a_{i1} P_1) = a_{i1} P_1 + a_{i2} P_2 + a_{i3} P_3 + a_{i4} Q_1 + a_{i5} Q_2;$$
$$(a_{i1} x_1) = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + a_{i4} \lambda_1 + a_{i5} \lambda_2 + a_{i6},$$

and choose,

$$H_2 = (a_{11} x_1)\, dx_1^2 + (a_{21} x_1)\, dx_1\, dx_2 + (a_{31} x_1)\, dx_2^2 +$$
$$+ (a_{41} x_1)\, d x_1\, d x_3 + (a_{51} x_1)\, d x_2\, d x_3 + (a_{61} x_1)\, d x_3^2,$$

then the quadratic, (in $dx_i, d\lambda_j$), $\Omega_3 H_2$ has the expressions (13) below as coefficients. Note that if we eliminate σ from the first six equations (13), there will remain five non-homogeneous equations, linear in P_i, Q_j, from which each of P_i, Q_j is determined in terms of the partial derivatives of P_1, P_2, P_3 and of the coefficients of H_2, subject to the determinant condition for consistency.

$$
(13)\quad
\begin{aligned}
&2(a_{11}x_1)P_{1x_1}+(a_{21}x_1)P_{2x_1}+(a_{41}x_1)P_{3x_1}+(a_{11}P_1)-\sigma(a_{11}x_1)=0,\\
&2(a_{11}x_1)P_{1x_2}+(a_{21}x_1)P_{2x_2}+(a_{21}x_1)P_{1x_1}+2(a_{31}x_1)P_{2x_1}\\
&\quad+(a_{41}x_1)P_{3x_2}+(a_{51}x_1)P_{3x_1}+(a_{21}P_1)-\sigma(a_{21}x_1)=0,\\
&(a_{21}x_1)P_{1x_2}+2(a_{31}x_1)P_{2x_2}+(a_{51}x_1)P_{3x_2}+(a_{31}P_1)-\sigma(a_{31}x_1)=0,\\
&2(a_{11}x_1)P_{1x_3}+(a_{21}x_1)P_{2x_3}+(a_{41}x_1)P_{1x_1}+(a_{41}x_1)P_{3x_3}\\
&\quad+(a_{51}x_1)P_{2x_1}+2(a_{61}x_1)P_{3x_1}+(a_{41}P_1)-\sigma(a_{41}x_1)=0,\\
&(a_{21}x_1)P_{1x_3}+2(a_{31}x_1)P_{2x_3}+(a_{41}x_1)P_{1x_2}+(a_{51}x_1)P_{2x_2}\\
&\quad+(a_{51}x_1)P_{3x_3}+2(a_{61}x_1)P_{3x_2}+(a_{51}P_1)-\sigma(a_{51}x_1)=0,\\
&(a_{41}x_1)P_{1x_3}+(a_{51}x_1)P_{2x_3}+2(a_{61}x_1)P_{3x_3}+(a_{61}P_1)-\sigma(a_{61}x_1)=0,\\
&2(a_{11}x_1)P_{1\lambda_j}+(a_{21}x_1)P_{2\lambda_j}+(a_{41}x_1)P_{3\lambda_j}=0,\\
&(a_{21}x_1)P_{1\lambda_j}+2(a_{31}x_1)P_{2\lambda_j}+(a_{51}x_1)P_{3\lambda_j}=0,\\
&(a_{41}x_1)P_{1\lambda_j}+(a_{51}x_1)P_{2\lambda_j}+2(a_{61}x_1)P_{3\lambda_j}=0,\ (j=1,2).
\end{aligned}
$$

The last three of these equations give the factorability condition (discriminant),

$$
D_2=\begin{vmatrix} 2(a_{11}x_1) & (a_{21}x_1) & (a_{41}x_1)\\ (a_{21}x_1) & 2(a_{31}x_1) & (a_{51}x_1)\\ (a_{41}x_1) & (a_{51}x_1) & 2(a_{61}x_1)\end{vmatrix}=0,
$$

a condition on H_2. Existence is established by the case $(a_{11}x_1)=(a_{31}x_1)=$ $=(a_{61}x_1)=\frac{1}{2}$; $(a_{21}x_1)=(a_{41}x_1)=(a_{51}x_1)=1$, in which case $D_2=0$ and,

$$
H_2=\frac{1}{2}(dx_1+dx_2+dx_3)^2 .
$$

Another special instance is,

$$H_2 = (d\,x_1 + d\,x_2 + d\,x_3)\,(\lambda_1\,d\,x_1 + \lambda_2\,d\,x_2 - d\,x_3)\,.$$

We consider next the problem where H is of order m in n differentials. In the ARONHOLD symbolism we then have,

$$H_m = (\alpha_1\,d\,x_1 + \alpha_2\,d\,x_2 + \ldots + \alpha_n\,d\,x_n)^m = \alpha_{dx}^m = \beta_{dx}^m = \ldots$$

As was done in H_2, we assume that the terms of H_m are arranged in normal order, so the differences $\varrho_1 - \varrho_2, \sigma_1 - \sigma_2, \ldots, w_1 - w_2$ for any two consecutive terms,

$$x_n^{\varrho_1}\,x_{n-1}^{\sigma_1} \ldots x_1^{w_1}\,,\; x_n^{\varrho_2}\,x_{n-1}^{\sigma_2} \ldots x_1^{w_2}\,,$$

satisfy the rule that the first difference of the set, which is not zero, is negative. The respective coefficients in H_m can then be placed in serial correspondence with the succession of positive integers, $1, 2, \ldots$; that is, they are $C_i(x_1, x_2, \ldots, x_n, \lambda_1, \ldots, \lambda_{n-1})$, in the order $i = 1, 2, \ldots \zeta$, where ζ is the number of terms in an n-ary m-ic. We have,

(14)
$$\Omega_n\,\alpha_{dx}^m = m\alpha_{dx}^{m-1}\left[\sum_{j=1}^{n}\left(P_1\frac{\partial\,\alpha_j}{\partial\,x_1} + P_2\frac{\partial\,\alpha_j}{\partial\,x_2} + \ldots + P_n\frac{\partial\,\alpha_j}{\partial\,x_n} + Q_1\frac{\partial\,\alpha_j}{\partial\,\lambda_1} + \ldots + Q_{n-1}\frac{\partial\,\alpha_j}{\partial\,\lambda_{n-1}}\right)d\,x_j\right] + m\,\alpha_{dx}^{m-1}\,\alpha_{dP} - \sigma\,H_m = 0\,,$$

this relation being an identity in the variables.

The terms t_{ki} with the factor $d\,\lambda_i$ have respective coefficients A_{ki} which are linear and homogeneous in $P_{1\lambda i}, P_{2\lambda i}, \ldots, P_{n\lambda i}$. Whatever i is, the eliminant of these terms is a matrix D_m^n of n rows and η columns, η being the number of terms in an n — ary $(m - 1)$ — ic. The laws of the structure of this matrix are complicated but we can write it for the ternary m — ic $(m > 1)$, as follows:

$$D_m^3 = \left\| \begin{array}{cccccccc} mC_1 & (m-1)C_2 & (m-2)C_3 & \ldots & C_m & (m-1)C_{m+2} & (m-2)C_{m+3} & (m-3)C_{m+4}\cdots \\ C_2 & 2\,C_3 & 3\,C_4 & \ldots & mC_{m+1} & C_{m+3} & 2C_{m+4} & 3C_{m+5}\cdots \\ C_{m+2} & C_{m+3} & C_{m+4}\cdots & & C_{2m+1} & 2C_{2m+2} & 2C_{2m+3} & 2C_{2m+4}\cdots \\ & \cdots & C_{2m} & (m-2)C_{2m+2} & (m-3)C_{2m+3}\cdots & & C_{3m-1}\cdots & \\ & \cdots(m-1)C_{2m+1} & & C_{2m+3} & 2C_{2m+4}\cdots & & (m-2)C_{3m}\ \cdots & \\ & \cdots & 2C_{3m} & 3C_{3m+1} & 3C_{3m+2}\cdots & & 3C_{4m-2}\cdots & \end{array} \right\|\,.$$

Evidently D^3_m terminates by its own laws when the number of columns reaches $\xi = \frac{1}{2} m(m+1)$. The condition $D^3_m = 0$ is exacting since it means that $\xi - 2$ essential determinants of the third order must vanish, identically in $x_1, x_2, x_3\ \lambda_1, \lambda_2$, if H_m is to be an invariant.

When H_m is the general quadratic quantic in n differentials $dx_1, \ldots, dx_n$, D^n_2 is a determinant of order n, the natural generalization of the discriminant of the ternary quadratic. When $n > 3$, however, the vanishing of this determinant does not cause the quadratic to degenerate.

If $m > 2$, $n = 3$, the vanishing of the $\xi - 2$ essential determinants of D^3_m is not equivalent to the existence of the $\frac{1}{2} m(m-1)$ known conditions for the linear factorability of H_m.

The equations which involve σ, from the terms of (14) free from $d\lambda_i$, are non-homogeneous linear equations in $P_1, \ldots, P_n, Q_1, \ldots, Q_{n-1}$, and remain of that type when σ is eliminated. If $n = 3$, the condition for the consistency of this linear system, since,

$$m\,\alpha_1^{m-1}\frac{\partial \alpha_1}{\partial x_1} = \frac{\partial C_1}{\partial x_1},\ m(m-1)\alpha_1^{m-2}\,\alpha_2\frac{\partial \alpha_1}{\partial x_1} + m\,\alpha_1^{m-1}\frac{\partial \alpha_2}{\partial x_1} = \frac{\partial C_2}{\partial x_1},\ \textit{et cetera},$$

is readily found to be the non-vanishing of the matrix E^3_m:

$$E^3_m = \left\| \begin{matrix} C_1\frac{\partial C_2}{\partial x_1} - C_2\frac{\partial C_1}{\partial x_1},\ C_1\frac{\partial C_2}{\partial x_2} - C_2\frac{\partial C_1}{\partial x_2},\ C_1\frac{\partial C_2}{\partial x_3} - C_2\frac{\partial C_1}{\partial x_3},\ C_1\frac{\partial C_2}{\partial \lambda_1} - C_2\frac{\partial C_1}{\partial \lambda_1},\ C_1\frac{\partial C_2}{\partial \lambda_2} - C_2\frac{\partial C_1}{\partial \lambda_2} \\ C_1\frac{\partial C_3}{\partial x_1} - C_3\frac{\partial C_1}{\partial x_1},\ C_1\frac{\partial C_3}{\partial x_2} - C_3\frac{\partial C_1}{\partial x_2},\ C_1\frac{\partial C_3}{\partial x_3} - C_3\frac{\partial C_1}{\partial x_3},\ C_1\frac{\partial C_3}{\partial \lambda_1} - C_3\frac{\partial C_1}{\partial \lambda_1},\ C_1\frac{\partial C_3}{\partial \lambda_2} - C_3\frac{\partial C_1}{\partial \lambda_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ C_1\frac{\partial C_\varrho}{\partial x_1} - C_\varrho\frac{\partial C_1}{\partial x_1},\ C_1\frac{\partial C_\varrho}{\partial x_2} - C_\varrho\frac{\partial C_1}{\partial x_2},\ C_1\frac{\partial C_\varrho}{\partial x_3} - C_\varrho\frac{\partial C_1}{\partial x_3},\ C_1\frac{\partial C_\varrho}{\partial \lambda_1} - C_\varrho\frac{\partial C_1}{\partial \lambda_1},\ C_1\frac{\partial C_\varrho}{\partial \lambda_2} - C_\varrho\frac{\partial C_1}{\partial \lambda_2} \end{matrix} \right\|,$$

$(\varrho = \frac{1}{2}(m+1)(m+2))$. None of the fifth order determinants should vanish. The following conclusion for n — space is now evident:

Theorem. A total condition in order that $H_m = 0$ should define a generalized connection of surface-elements, and $H'_m = M H_m$ a species of generalized contact tranformation, consists of $E^n_m \neq 0$, and the vanishing of the set,

$$(A_{ki}, D^n_m), \ (k = 1, \ldots, \eta; i = 1, \ldots, n-1).$$

When these conditions are fulfilled, the functions $P_1, \ldots, P_n, Q_1, \ldots, Q_{n-1}$ are expressed linearaly in terms of the first partial derivatives of $P_1, \ldots, P_n$ taken with regard to $x_1, \ldots, x_n$, only, and rationally in terms of the coefficients of H_m, and of the first partial derivatives of the latter coefficients, taken with regard to $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{n-1}$.

This defines connection broadly. In some geometric problems it may be preferable to use a special case of the theorem, and some such cases will here be mentioned.

(*i*) If $m = 1$, so that D^n_m is non-existent, and if,

$$(a_{i1}\, x_1) = a_{i1}\, x_1 + a_{i2}\, x_2 + \ldots + a_{in}\, x_n + a_{in+1}\, \lambda_1 + \ldots + a_{i2n-1}\, \lambda_{n-1} + a_{i2n},$$

we may use H as,

$$H_1 = (a_{11}\, x_1)\, dx_1 + (a_{21}\, x_1)\, d\, x_2 + \ldots + (a_{n1}\, x_1)\, d\, x_n .$$

The set of equations which contain σ, now n in number, is,

$$(a_{11}\, x_1)\, P_{1x_i} + (a_{21}\, x_1)\, P_{2x_i} + \ldots + (a_{n1}\, x_1)\, P_{nx_i} + (a_{i1}\, P_1) - \sigma\, (a_{i1}\, x_1) = 0\,,$$

$(i = 1, \ldots, n)$, where,

$$(a_{i1}\, P_1) = a_{i1}\, P_1 + a_{i2}\, P_2 + \ldots + a_{in}\, P_n + a_{in+1}\, Q_1 + \ldots + a_{i2n-1}\, Q_{n-1}\,.$$

After eliminating σ, therefore, we can solve only for $Q_1, \ldots, Q_{n-1}$, as linear expressions in the P_j and $P_{jx_k}, (k = 1, \ldots, n)$, of the rational domain containing the $(a_{i1}\, x_1)$ and, separately, the a_{ik}. The condition $E^n_1 \neq 0$

is as follows :

$$E_1^n = \left\| \begin{matrix} a_{2,n+1}\,(a_{11}\,x_1) - a_{1,n+1}\,(a_{21}\,x_1)\,,\, a_{2,n+2}\,(a_{11}\,x_1) - a_{1,n+2}\,(a_{21}\,x_1)\,, \dots \\ \dots\,,\, a_{2,2n-1}\,(a_{11}\,x_1) - a_{1,2n-1}\,(a_{21}\,x_1) \\ a_{3,n+1}\,(a_{11}\,x_1) - a_{1,n+1}\,(a_{31}\,x_1)\,,\, a_{3,n+2}\,(a_{11}\,x_1) - a_{1,n+2}\,(a_{31}\,x_1)\,, \dots \\ \dots\,,\, a_{3,2n-1}\,(a_{11}\,x_1) - a_{1,2n-1}\,(a_{31}\,x_1) \\ \dots\dots\dots\dots\dots \\ a_{n,n+1}\,(a_{11}\,x_1) - a_{1,n+1}\,(a_{n1}\,x_1)\,,\, a_{n,n+2}\,(a_{11}\,x_1) - a_{1,n+2}\,(a_{n1}\,x_1)\,, \dots \\ \dots\,,\, a_{n,2n-1}\,(a_{11}\,x_1) - a_{1,2n-1}\,(a_{n1}\,x_1) \end{matrix} \right\|$$

and the coefficients A_{ki} in the n functions $P_{j\lambda_i}$, of which coefficients there is one for each i, are,

$$A_{ki} = (a_{11}\,x_1)\,P_{1\lambda_i} + (a_{21}\,x_1)\,P_{2\lambda_i} + \dots + (a_{n1}\,x_1)\,P_{n\lambda_i} = 0\,,\ (i = 1\,, \dots, n-1)\,.$$

(*ii*) The elementary case of this H_1, and therefore of its connection and invariance, is,

$$\eta_1 = \lambda_1\,d\,x_1 + \lambda_2\,d\,x_2 + \dots + \lambda_{n-1}\,d\,x_{n-1} - d\,x_n\,.$$

Here $E_1^n = 1$, and the set of equations, obtained by elimination of σ, is,

$$Q_i - P_{nx_i} + \lambda_1\,P_{1x_i} + \lambda_2\,P_{2x_i} + \dots + \lambda_{n-1}\,P_{n-1x_i} - \lambda_i\,P_{nx_n}$$
$$+ \lambda_i\,(\lambda_1\,P_{1x_n} + \lambda_2\,P_{2x_n} + \dots + \lambda_{n-1}\,P_{n-1x_n}) = 0\,,\ (i = 1\,, \dots, n-1).$$

From these equations the respective values of $Q_1\,, \dots, Q_{n-1}$, which the method determines, are evident. In this instance,

$$A_{ki} = \lambda_1\,P_{1\lambda_i} + \lambda_2\,P_{2\lambda_i} + \dots + \lambda_{n-1}\,P_{n-1\lambda_i} - P_{n\lambda_i} = 0\,,\ (i = 1\,, \dots, n-1)\,.$$

(*iii*) In the problem of H_2, where $D_2 = 0$, and H_2 is degenerate, as in the example,

$$H_2 = (d\,x_1 + d\,x_2 + d\,x_3)\,(\lambda_1\,d\,x_1 + \lambda_2\,d\,x_2 - d\,x_3)\,,$$

the defining relation of the connection, $H_2 = 0$, presents an alternative.

Either there is elementary connection in the vicinity of $(x_1\,, x_2\,, x_3)$ or the vector $d\,x_1 + d\,x_2 + d\,x_3$ is zero. But in the contact relation,

$$H_2' = (1 + \sigma\,u)\,H_2\,, \qquad (15)$$

U will not transform the two factors of H_2, one into the other. Hence (15) asserts that $\{U\}$ preserves elementary contact and leaves $d x_1 + d x_2 + d x_3$ invariant, both. This will particularize U some, but the idea can be generalized for quantics H_m which are degenerate. To show that the transformation will be both existent and particularized we assumed P_j, Q_k in their general linear form, they being not linear usually, and assumed both of the factors of H_2, above, invariant. It was found that $\{U\}$ is then generated by the following transformation:

$$U: \begin{cases} x_1' = x_1 + u(a_1 x_1 + b_1 x_2 + c_1)\,,\; x_2' = x_2 + u(a_2 x_1 + b_2 x_2 + c_2), \\ x_3' = x_3 - u\,[(a_1 + a_2)\,x_1 + (b_1 + b_2)\,x_2 + c_3]\,, \\ \lambda_1' = \lambda_1 - u\,[a_1 \lambda_1 + a_2 \lambda_2 + (a_1 + a_2)]\,, \lambda_2' = \lambda_2 - u\,[b_1 \lambda_1 + b_2 \lambda_2 + (b_1 + b_2)]. \end{cases}$$

§ II.

Bio-Mathematics

The influence of the physical environment on the variation and evolution of biological organisms has been studied much more than the influence of the mathematical environment has. Within natural limitations, however, mathematical requirements are exacting. This is because biological characters can be given an expressive graphical representation and thus become the basis of a mathematical system. Also a biological organism, plant or animal, is a vehicle for the manipulation of energy.

Let T represent the family tree of an organism of double parentage, b', b being the parents of a; c', c those of b', *et cetera.*

$\underline{T}$

a

$b' \qquad\qquad b$

$c' \quad c \qquad d' \quad d$

$e' \; e \; f' \; f \quad g' \; g \; h' \; h$

. . . .

A line drawn from a to include one primed (male) letter from each generation is a line of heredity. Drop all primes thereon and a female line is represented. Any specimen in T, represented by a_j of a line L_i, can be described approximately by a finite number of characters which are measure numbers of its respective parts, such as the volume of an eye, or the tensile power of a biceps muscle. To give a character a graphical representation, suppose

$$L_i = (\alpha_1 = a\,,\ \alpha_2\,,\ \ldots\,,\alpha_n)\,,$$

and let α_j be described by an ordered set C_j of characters,

$$C_j:\ (x_{j1}\,,\ x_{j2}\,,\ \ldots\,,\ x_{jm})\,.$$

Fixing attention upon one, as the first, of the x_{ji}, we can say that, for any value of the age-variable t, from the time of the appearance of the part in the embryo to the time of the demise of the organism, the characters x_{11}, $x_{21}\,,\ldots$, will have the explicit values, respectively, of the numbers in the set K_t:

$$K_t:\ (x_{11}^{(t)}\,,\ x_{21}^{(t)}\,,\ \ldots\,,\ x_{n1}^{(t)})\,.$$

Regarding $(x_{j1}\,,t)$ as polar coordinates expressed by convenient units, t being the uniformly varying time, the successive values of $x_{j1} = x_{j1}^{(t)}$ will plot into a curve k_j which represents the continuous variation of x_{j1} during an interval $(t_1 < t < t_2)$. Likewise the values of x_{j+11} will give a curve k_{j+1} which will show some variation from k_j, and L_i will give a field F of n curves of this first character x_{j1}. There will be m such fields, one for each x_{ji}, $(i = 1\,,\ldots\,,m)$, in C_j. (See $k_j\,,k_{j+1}$ in Fig. 1).

Whatever the organism, the character-curve k_j will be of a spiral form, coming outward from near the origin and ending in a segment nearly circular. This is because the part measured by the character is small in the embryo, and remains nearly constant in magnitude during the organism's old age. There is a comparatively small class of exceptions in which the character-curve is approximately circular throughout, an example being the curve of the blood temperature, in man.

A variation from a segment in the field F can be expressed by a transformation like s in (9), which, with $o\,(\varphi)$, $p\,(r)$, $u\,,v$ properly chosen, will carry a segment of k_j into a corresponding segment of k_{j+1}. The variables are here written as $(r\,,\varphi)$ instead of $(x_{j1}\,,t)$. The numerical infinitesimals $|\,u\,|\,,|\,v\,|$ will necessarily be small in such transformations of biological curves. Passing on to a segment of k_{j+2}, from k_{j+1}, will mean that s must be combined with a like transformation,

$$s':\ \varphi' = \varphi + u_1\,o_1\,(\varphi)\,,\ r' = r + v_1\,p_1\,(r)\,,(|\,u_1\,|\,,|\,v_1\,| \doteq 0)\,,$$

$$o_1 = (a + \delta_1\,a)\,\varphi^{e-1} + (b + \delta_2\,b)\,\varphi^{e-2} + \ldots + (k + \delta_e\,k)\,,$$

$$p_1 = (\alpha + \delta_1'\,\alpha)\,r^{e-1} + (\beta + \delta_2'\,\beta)\,r^{e-2} + \ldots + (\varkappa + \delta_e'\,\varkappa)\,.$$

Geometric considerations show that all of the increments δ_g, δ'_h will be at least as small numerically [7] as $|u_1|, |v_1|$. If there should be a mutation in the line L_i affecting the character under consideration, this might not be true of δ_g, δ'_h but, in the case of a mutation, we can begin L_i with the mutant α_μ, (DE VRIES).

The product ss' is as follows:

$$(16) \qquad ss' : \varphi' = \varphi + (u + u_1)\, o(\varphi) + u_1\, \Delta\, o(\varphi), \\ r' = r + (v + v_1)\, p(r) + v_1\, \Delta'\, p(r).$$

Since, however, the inverse of s is in $\{s\}$, and

$$\Delta\, o(\varphi) = (\delta_1\, a)\, \varphi^{e-1} + (\delta_2\, b)\, \varphi^{e-2} + \ldots + (\delta_e\, k),$$
$$\Delta'\, p(r) = (\delta'_1\, \alpha)\, r^{e-1} + (\delta'_2\, \beta)\, r^{e-2} + \ldots + (\delta'_e\, \varkappa),$$

and $u_1\, \Delta\, o(\varphi) = v_1\, \Delta'\, p(r) = 0$, the product is symbolized by σ of (9), *idem est,*

Theorem. The character-curves of the field F are permuted by the group $\{s\}$, *which is fitted to the field, not vice versa, which group thus becomes the expression of the variation.*

Since we have fitted the group to the field, the values of the parameters u, v in $\{s\}$ have been numerically determined. Since the equation (10), viz.,

$$K = a_j \int \frac{d\varphi}{o(\varphi)} + b_j \int \frac{dr}{p(r)} + c_j = 0,$$

has two independent coefficients, one could begin the study of variations of an organism of a definite species by choosing two points (φ, r) on one of the organism's character-curves, *idem est*, two points corresponding, respectively, to two early ages of the organism in L_i and solving for b_j / a_j, c_j / a_j. This will determine k_j within a finite region delimited according to a relation $\varphi_1 < \varphi < \varphi_2$ with φ_1, φ_2 determined, but evidently, since the u, v in $\{s\}$ have been determined, the symbol (11) represents a sub-group of $\{s\}$ only if $s\,(u_1, -\frac{a_j}{b_j} u_1)$ amounts to a choice from among the $s\,(u, v)$ of $\{s\}$.

A certain geometric advantage results from a construction in three dimensions based on the series $k_j, k_{j+1}, \ldots$ Let a segment of k_{j+1} be drawn

(7) GLENN, Annali della Scuola Normale Superiore di Pisa, S. II, vol. 2, 1933, p. 297.

in the (x, y) plane and a corresponding segment of k_j (in position) in the (x, z) plane. The right cylinders erected on these curves as bases, elements parallel

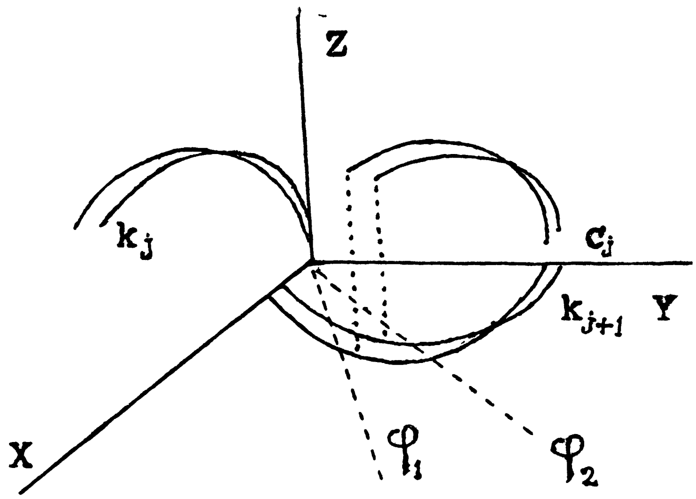

Figure 1

to the z and y axes, respectively, intersect in a twisted space-curve c_j. Next let k_{j+1} be drawn in the (x, z) plane and k_{j+2} in the (x, y) plane. The corresponding cylinders intersect in a space-curve c_{j+1}. Further repetitions give a field F_1 of which F is the orthogonal projection. Since k_j is projected into k_{j+1} through c_j, the c_j is an abstract form of the influence of the second parent in the process. It is known (MENDEL) that a unit character, at least, produces its succession in L_i without much recognizable influence intervening from other characters.

In polar coordinates a transformation that takes a segment of c_j into a corresponding segment of c_{j+1} is $U_1(u, v, w)$ of (5), φ being the time and u, v, w being determined, and we have seen that $c_{j+\varrho}$, under $\{U_1\}$, is likely to fall into coincidence with any one of many invariant curves in space, the transformations (variations) being, at the same time, captured by some subgroup of $\{U_1\}$.

Stabilization

A character of an organism becomes stabilized when its variations through L_i cause its curve to converge to coincidence with an invariant curve. Thus the curve becomes invariable in its line L_i. Its transformations may escape from the corresponding subgroup $\{s_1\}$ when hybridization produces a sufficient mutation. Stabilizations difficult for the organism to overcome have occurred among both plants and animals. As examples; the common gray squirrél, as known since 1492, seems to be a pretty stable organism. Other cases are; the white Embden goose (DARWIN), the morning glory vine (*Ipomaea purpurea*), and various animals represented in the Swiss Jurassic fossil beds (AGASSIZ).

All of these organisms have a period of rapid growth. Correspondingly, if an invariant curve in the plane (φ, r) is directed approximately toward the origin, through a considerable segment of its length, a curve k_j, coming, in L_i, into coincidence with this segment as an invariant, will have a period of rapid growth. Thus there is a connection between stabilization and rapidity of growth (8).

We would expect that some character-curves, varying through L_i, would get through the maze of invariant curves without falling into coincidence with any. This could produce uneven evolution, seen, in fact, in the giraffe; the horn-bill (*hydrocorax planicornis*), and toucan (*rhamphastos ariel*), among birds; and in the fennec fox (*canis zerda*), which has ears which are enormous in comparison with the rest of the animal.

Lamarck's first law

We consider the case of an organism in a definite line of heredity, whose characters which are essential to the process of evolution by natural selection do not become stabilized. For this species we prove LAMARCK'S first law, *idem est*, the following:

Theorem. Life, by its proper forces, continually tends to increase the size of the typical organism of any species, and of its parts, « up to a limit that it brings about ».

We first state formally a system of hypotheses.

(i) The equation (10), in polar coordinates, particularized as explained above for a specific k_j, we divide through by the numerically larger of the two numbers a_j, b_j. The result is,

$$a \int \frac{d\,r}{p\,(r)} + b \int \frac{d\,\varphi}{o\,(\varphi)} + c = 0\,, \tag{17}$$

(8) To define what is to be meant by rapid growth we take, for the rate of growth, the time (φ) rate J of change of the volume V of the organism or correlation of parts. Then $V = X(\varphi)$, $J = d\,V / d\,\varphi$. The average rate $\varDelta_R$ through a time-interval, is the volume at the close, (in cubic inches), minus the volume at the beginning, divided by the time (in months). When $\varDelta_R \geq 1$ the growth is rapid: Grass hopper, medium specimen, one day to 3 months $\varDelta_R = .041$; Cleome plant, 4 mo. to 5 mo., $\varDelta_R = 4.8$; Goose, 5 days to 3 mo., $\varDelta_R = 102$; Fossils in the Jurassic, (information inconclusive); Elephant, embryo to 25 yrs., $\varDelta_R = 952$.

where either a is $+1$ and $|b|$ a proper fraction, or a is a positive proper fraction and $|b| = +1$.

(*ii*) If $o(\varphi)$ does not vanish for the φ of any point within the open plane region I bounded laterally by lines $\varphi = \varphi'$, $\varphi = \varphi''$, then there is neither any invariant point of s, nor any intersection of any two non-invariant k_j, within I.

The latter conclusion follows because if two k_j intersect in I and if we transform one curve into the other by an s, the intersection, going into a point infinitesimally near on the latter curve, determines an invariant segment on a non-invariant curve; invariant, *idem est*, under the sub-group which consists of the simple powers of s.

(*iii*) If ϱ is a root of $p(r) = 0$, transformation s, in I, that involves $p(\varrho)$, gives a point the same distance ϱ from the origin. The circle $r = \varrho$ is not excluded as a possible k_{j+1} but it is invariant.

(*iiii*) By hypothesis, arcs of character-curves in I are arcs of virility of the organism.

If now we express each integral in (17) in polynomial form, for the interval $(\varphi' < \varphi < \varphi'')$, the equation of k_j in I becomes,

(18) $$g(r) \equiv a(p_1 r^{e-1} + p_2 r^{e-2} + \ldots + p_e) + b(o_1 \varphi^{e-1} + o_2 \varphi^{e-2} + \ldots + o_e) + c = 0,$$

abbreviated as,

$$g(r) \equiv m(r) + n(\varphi), [m(r) = a(p_1 r^{e-1} + p_2 r^{e-2} + \ldots + p_e)].$$

Hence,

(19) $$s^{\mathrm{i}}[m(r_1) + n(\varphi_1)] = m(r) + n(\varphi) + U'_{\mathrm{i}} + V'_{\mathrm{i}} = 0, \; (\mathrm{r}_1 = \mathrm{r}'),$$

where $U'_{\mathrm{i}} = u'_1 + u'_2 + \ldots + u'_{\mathrm{i}}$, $V'_{\mathrm{i}} = v'_1 + v'_2 + \ldots + v'_{\mathrm{i}}$; with U'_{i}, V'_{i} not both zero, and,

$$|u'_h| = |b\, u_h| \leq |u_h|, |v'_h| = |a\, v_h| \leq |v_h|.$$

Here r_1 is a positive root of $g = 0$; φ_1 being assigned in $(\varphi' < \varphi < \varphi'')$, and $(\mathrm{r}_1, \varphi_1)$ are connected through s with (r, φ). The u'_g, v'_h are positive or negative infinitesimals each at least as small numerically as the corresponding u_g, v_h of s^{i}.

Since no two k_j intersect in I, both $1/p(\mathrm{r})$, $1/o(\varphi)$ will be positive functions in I. Hence $m(\mathrm{r})/a$, $n(\varphi)/b$ increase as their respective variables increase. If we choose the unit of angle so $n(\varphi)$, as φ increases, varies by a numerically smaller amount than the corresponding variation of $m(\mathrm{r})$, then $g(\mathrm{r})$ will increase as r increases in the vicinity of any r_1 on k_j in I, and decrease as r decreases, $g(\mathrm{r})$ being, therefore, always an increasing function of r (9).

A decrease in r over a finite succession of arcs k_j in I, means recessive evolution, and atrophy if continued indefinitely. We exclude the possibility of atrophy, by hypothesis.

The following assumption appears to be a particularization at this stage of the argument. We assume that all sums $U_i' + V_i'$, $(\mathrm{i} = 1, 2, \ldots,$ indefinitely), are negative, *idem est*, that the infinitesimals u_k, v_k and therefore u_k', v_k' are such as to make $U_i' + V_i'$ negative. Since $g(\mathrm{r})$ is an increasing function, the addition, in (19), of $U_i' + V_i'$ to the absolute term of $g(\mathrm{r}) = 0$, has this effect: It increases root r in the vicinity $(\varphi' < \varphi < \varphi'')$, from r_1 outward, over the succession of infinitesimally spaced arcs given by $s^i g(\mathrm{r})$, $(i = 1, 2, \ldots)$.

The succession of spirals k_j in L_i, under the hypotheses, are therefore expanding as the time t increases, (with j).

As an additional hypothesis, we now make use of the fact of evolution by natural selection. The part with the character-curve k_j is by assumption relevant in the organism's struggle for existence. Hence there is a minimum position for the curve k_j in the area delimited by $(\varphi' < \varphi < \varphi'')$ and the corresponding radii. If k_j goes nearer to the origin than this absolute minimal, k_{a_0}, does, the organism's corresponding part becomes to weak for survival.

Then, there is also a series of relative minimals in the field F. If F, extended from k_{a_0}, not inclusive, as j is increased indefinitely but finitely, is examined, one curve $k_{a_0+a_1}$ will be found to have reached a position in the aforesaid area, nearer to the origin than the rest which lie beyond k_{a_0} chronologically. Next consider the portion of F lying beyond $k_{a_0+a_1}$. It contains a minimal $k_{a_0+a_1+a_2}$. The process can be repeated indefinitely, and gives us a sequence of infinitesimally spaced minimals chronologically ordered in F, each minimal lying outside of its chronological predecessor. This sequence may be written as:

$$M: \; k_{a_0}, \; k_{a_0+a_1}, \; k_{a_0+a_1+a_2}, \; \ldots, \; k_{a_0+a_1+\ldots+a_\zeta}, \; \ldots$$

---

(9) Also $n(\varphi)$ tends to be smaller than $m(\mathrm{r})$. Compare $f(\theta)$ with $g(\mathrm{r})$ in (30), § III.

Now, we can pass along the sequence M by operating on k_{a_0} by the successive powers of an s having the symbol,

$$s(x_1, y_1)\, s(x_2, y_2) \doteq s(x_1 + x_2, y_1 + y_2),$$

where the x_λ are sums Σu_μ and the y_λ are sums Σv_ν. Since M is an expanding sequence of spirals, if k_j is k_{a_0}, and,

$$X'_g = x'_1 + x'_2 + \ldots + x'_g, \; Y'_g = y'_1 + y'_2 + \ldots + y'_g$$

$$|x'_g| = |b\, x_g| \leq |x_g|, \; |y'_g| = |a\, y_g| \leq |y_g|,$$

the sum $X'_g + Y'_g$, replacing $U'_1 + V'_1$ in (19), is necessarily negative, satisfying here what we previously tentatively called a particularizing assumtion. We also now see by how much the sequence M expands from term to term, *idem est*, in terms of infinitesimals u'_g, v'_h.

Since M is chronologically an expanding sequence and all the rest of F, after any $k_{a_0+a_1+\ldots+a_\zeta}$, lies beyond $k_{a_0+a_1+\ldots+a_\zeta}$, in the area delimited by the rule $(\varphi' < \varphi < \varphi'')$, the field F is expanding, as a system of plane spirals. Hence the characters k_j in L_i, also, are increasing, on the average, in time, q. e. d.

Corollary. The analysis becomes indeterminate if the character r increases to a value as large as a real root of $p(r) = 0$.

We have noted that k_j is then converging to an invariant circle as a limit.

The space-curve c_{a_0}, of which the absolute minimal k_{a_0} is the projection, can readily be shown to be an extremal (calculus of variations) of an integral which represents the amount of work which the organism can do in a given time by means of a part with character-measure r. This fact is related to H. F. OSBORN'S systematization of the play of energy in evolutionary biology. †

† LAMARCK, *Histoire Naturelle des Animaux sans Vertèbres*, (1815), Introduction.
OSBORN, *The Origin and Evolution of Life*, (1916).
The mathematical problem is obviously related formally to the philosophy of the *elan vital* (vital force). *Vide* H. BERGSON, *L'Evolution Creatrice*, 1907.

§ III.

The earth's magnetic field of force, considered as a directional-central force

One can demonstrate without difficulty, by means of magnets symmetrically arranged, and permalloy filings, that a magnetic field can have a center of geometric symmetry. This will not prove that the force about a spherical magnet is central, in the usual meaning of the term, but it gives the following postulate a slight measure of reasonableness.

Hypothesis. The magnetic force of the earth is everywhere directional central, the force represented by the force-vector, which originates at the center of the earth, being a function, not only of the vector's length, (r), but also of its angles of direction, (φ , θ), in space.

The justification of this postulate will be that it is consistent with phenomena.

Many facts about the earth's magnetic field are contained in CARL STRØMER'S theory ([10]) on the orbits of cathode particles which have been pro-

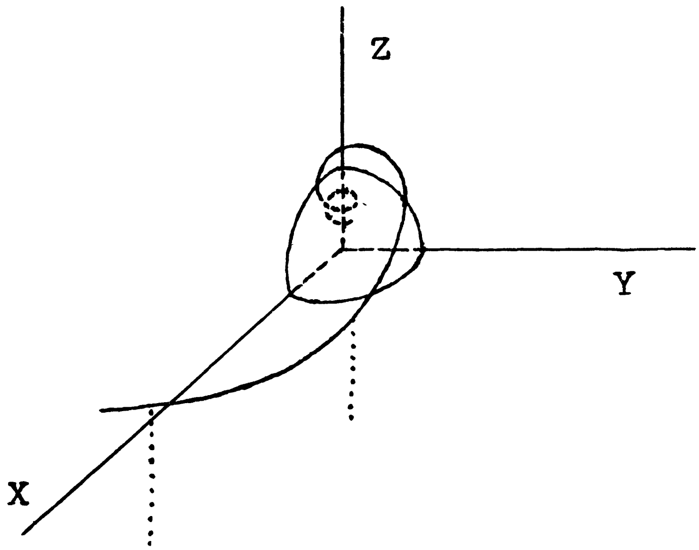

Figure 2

([10]) StØrmer, Modern Norwegian researches, *et cetera, Proc. Int. Math. Congress Toronto, 1924,* vol. I, p. 139-148.

jected to the earth's vicinity from the Sun. Their motions are ultimately dominated by a type of universal field of the earth, which field includes (non-additively), components both gravitational and magnetic. Their orbits are twisted space curves which approach the earth and dive beneath its surface some-where within the arctic circle (or antarctic), where there will often be such a concentration that the visible aurora borealis is produced.

Equations of the orbit derived from its property of Stability

Definition. A space curve upon which the cathode particle is constrained by the potential to move is a stable orbit.

The orbital equations are obtainable from the effect of perturbations, of a segment R of the orbit, by outside forces. These perturbations may be represented by transformation U_1 of (5), considered as an operator applied to the equation of R. It will be sufficiently general to use U_1 with $o(\varphi) = 0$, and the power of self-restitution which a stable orbit possesses will be represented by U_1 if it is further particularized to the case $W_\nu = 0$, $X_\nu = 0$ of (6). Then U_1 permutes ν perturbed orbits, including R, in a closed cycle. The determination of the equation of R then follows the steps of the determination of the factorable ternary H of (7). This H, though it is assumed to be factorable, will still contain arbitrary coefficients in number equal to,

$$\frac{1}{2}(h+1)(h+2) - \frac{1}{2}h(h-1) = 2h+1 .$$

Accordingly the orbit R is a curve on the surface,

$$J = \int \frac{dr}{p(r)} + K \int \frac{d\theta}{q(\theta)} + w(\varphi) = 0 , \qquad (w \text{ arbitrary})$$

and may be represented by the pair of equations,

$$(20) \qquad \int \frac{dr}{p(r)} = -A\, w(\varphi), \int \frac{d\theta}{q(\theta)} = B\, w(\varphi), \ (A - BK = 1;\ A, K \text{ arb. const.}).$$

The arbitrary function $w(\varphi)$ can be determined for R if we use the following equation which is derived independently in the next section:

$$(21) \qquad \frac{d\theta}{d\varphi}\cos\varphi - \frac{h_1}{g_1}\cos^2\theta + \sin\theta\cos\theta\sin\varphi = 0 .$$

From (20), on the arc R being used,

$$\frac{d\,\theta}{d\,\varphi} = B\,q\,(\theta)\,w'\,(\varphi)\,,\,(w' = \partial\,w/\partial\,\varphi)\,,$$

and by substitution in (21), since w is free from θ,

$$w\,(\varphi) = C\int\frac{d\,\varphi}{\cos\,\varphi} + D\,, \qquad (C\,,\,D \text{ arb. const.}) \tag{22}$$

$$= C\,\log_e(\sec\varphi + \tan\varphi) + D\,, \qquad (\mathrm{e} = 2.71828\ldots)$$

An algebraic form of the pair (20) is obtained by writing each integral in polynomial form, valid, to any approximation, by choice of n, for an interval $(\varphi_1 < \varphi < \varphi_2)$. Thus we obtain,

$$\begin{cases} g\,(r) \equiv \gamma_0\,r^n + \gamma_1\,r^{n-1} + \ldots + \gamma_n = A_1\,\log_e(\sec\varphi + \tan\varphi) + B_1\,, \\ f\,(\theta) \equiv \delta_0\,\theta^n + \delta_1\,\theta^{n-1} + \ldots + \delta_n = C_1\,\log_e(\sec\varphi + \tan\varphi) + D_1\,. \end{cases} \tag{23}$$

Equations of the orbit, derived from the acceleration

The force being central at any time t, the equations of motion of the particle c on its orbit of which R is a segment, are as follows:

$$\frac{d^2\,x}{d\,t^2} = -\,F\,(r\,,\,\theta\,,\,\varphi)\,\frac{x}{r}\,,\;\frac{d^2\,y}{d\,t^2} = -\,F\,(r\,,\,\theta\,,\,\varphi)\,\frac{y}{r}\,, \tag{24}$$

$$\frac{d^2\,z}{d\,t^2} = -\,F\,(r\,,\,\theta\,,\,\varphi)\,\frac{z}{r}\,,$$

the function $F\,(r\,,\,\theta\,,\,\varphi)$ representing the force, and $(r\,,\,\theta\,,\,\varphi)$ being the polar coordinates of c. The problem of finding the orbit is solved by finding two integrals of (24). We therefore eliminate $F\,(r\,,\,\theta\,,\,\varphi)$, algebraically, between the first and second of equations (24); also between the first and third. This gives,

$$y\,\frac{d\,x}{d\,t} - x\,\frac{d\,y}{d\,t} = g_1\,,\;z\,\frac{d\,x}{d\,t} - x\,\frac{d\,z}{d\,t} = h_1\,, \qquad (g_1\,,\,h_1 \text{ arbitrary}). \tag{25}$$

We then eliminate $d\,t$ and change to polar coordinates and obtain,

$$\frac{d\,\theta}{d\varphi}\cos\varphi - \frac{h_1}{g_1}\cos^2\theta + \sin\theta\cos\theta\sin\varphi = 0\,.$$

The combination of (24) and (25) leads to the relation,

$$\frac{d}{d\varphi}\left\{\frac{\left(\frac{dr}{d\varphi}\right)^2 + r^2\left(\frac{d\theta}{d\varphi}\right)^2 + r^2\cos^2\theta}{r^4\cos^4\theta}\right\} = -\frac{2}{g_1^2}\frac{dr}{d\varphi}F(r, \theta, \varphi). \tag{26}$$

The latter two equations, supported by the hypothesis of stability, are the equations of the whole orbit of c.

Avenues of approach for orbits

In equation (20), p (r) and $q(\theta)$ are known polynomials. Hence $\gamma_0, \ldots, \gamma_n$, $\delta_0, \ldots, \delta_n$ are known numbers. Only A_1, B_1, C_1, D_1 are arbitrary in equations (23), and two points on R determine these. But, under these conditions, (23) will not represent just any curve approaching the geographic pole from outer space, but only certain curves. However, if we have equations (23) representing a definite curve R, small changes may be made in A_1, B_1, C_1, D_1, and still (23) will represent an orbit consecutive to the original R. Hence there are bounded avenues through one of which an orbital R must approach the pole. These avenues were first discovered by K. BIRKELAND, experimentally [11].

The force-function

The formula (26), with the indicated differentiations performed, gives the equation,

$$\begin{aligned}
&\Big\{ r\cos\theta\left(\frac{dr}{d\varphi}\right)\left(\frac{d^2r}{d\varphi^2}\right) + r^3\cos\theta\left(\frac{d\theta}{d\varphi}\right)\left(\frac{d^2\theta}{d\varphi^2}\right) - r^2\cos\theta\left(\frac{dr}{d\varphi}\right)\left(\frac{d\theta}{d\varphi}\right)^2 \\
&+ 2r\sin\theta\left(\frac{dr}{d\varphi}\right)^2\left(\frac{d\theta}{d\varphi}\right) - 2\cos\theta\left(\frac{dr}{d\varphi}\right)^3 + 2r^3\sin\theta\left(\frac{d\theta}{d\varphi}\right)^3 \\
&+ r^3\cos^2\theta\sin\theta\left(\frac{d\theta}{d\varphi}\right) - r^2\cos^3\theta\left(\frac{dr}{d\varphi}\right)\Big\} / r^5\cos^5\theta \\
&= -\frac{1}{g_1^2}\frac{dr}{d\varphi}F(r, \theta, \varphi).
\end{aligned} \tag{27}$$

([11]) BIRKELAND, *The Norwegian aurora palaris expedition 1902-1903*, New York, (Longmans Green & Company.), 1913.

By differentiation of the relations (20) and substitution for $w'(\varphi)$ as obtained from (22), we find,

$$\frac{d\,r}{d\,\varphi} = -\,\xi\,p\,(r)/\cos\varphi\,,\ \frac{d\,\theta}{d\,\varphi} = \eta\,q\,(\theta)/\cos\varphi\,,\ (\xi = A\ C\,,\ \eta = B\ C)\,,$$

$$\frac{d^2\,r}{d\,\varphi^2} = [\xi^2\,p\,(r)\,p'\,(r) - \xi\,p\,(r)\sin\varphi]/\cos^2\varphi\,,$$

$$\frac{d^2\,\theta}{d\,\varphi^2} = [\eta^2\,q\,(\theta)\,q'\,(\theta) + \eta\,q\,(\theta)\sin\varphi]/\cos^2\varphi\,.$$

We substitute accordingly in (27) and solve for the force-function $F(r\,,\theta\,,\varphi)$. The result is as follows:

$$(28)\quad \begin{aligned} F(r\,,\theta\,,\varphi) = \Gamma\,\{\,&[2\,p\,(r)^2 - r\,p\,(r)\,p'\,(r) + \nu^2\,(\cos^2\theta\cos^2\varphi)\,r^2]\,p\,(r) \\ &+ \nu\,r\,p\,(r)^2\sin\varphi + \alpha^2\,\nu\,r^3\,q\,(\theta)^2\sin\varphi + \alpha^2\,r^2\,p\,(r)\,q\,(\theta)^2 \\ &+ \alpha^3\,r^3\,q\,(\theta)^2\,q'\,(\theta) + 2\,\alpha\,r\,p\,(r)^2\,q\,(\theta)\tan\theta \\ &+ 2\,\alpha^3\,r^3\,q\,(\theta)^3\tan\theta + \alpha\,\nu^2\,r^3\,q\,(\theta)\sin\theta\cos\theta\cos^2\varphi\,\} \\ &\div r^5\,p\,(r)\cos^4\theta\cos^2\varphi\,, \end{aligned}$$

in which formulary, $\Gamma = g_1^2\,\xi^2\,,\ \alpha = \eta/\xi\,,\ \nu = 1/\xi\,,\ p'\,(r) = \partial\,p/\partial\,r\,,\ q'\,(\theta) = \partial\,q/\partial\,\theta$.

As we have noted in another paper, the objective in such a theory is not a simple functional form of the force-function but a simple geometric form for the orbit, in view of the principle of least work by which it is determined. In (28), F is expressed in terms of known functions. The polynomials $p\,(r)$, $q\,(\theta)$ are yet to be calculated numerically, in a particular problem to be discussed in a later section. The expression,

$$G\,(r) = \Gamma\,[2\,p\,(r)^2 - r\,p\,(r)\,p'\,(r) + \nu^2\,(\cos^2\theta\cos^2\varphi)\,r^2]/r^5\,.$$

which, divided by $\cos^4\theta\,\cos^2\varphi$, is a part of the formula for $F(r\,,\theta\,,\varphi)$, is the known formula for the gravitational attraction on the cathode particle at the point $(r\,,\theta\,,\varphi)$, (GLENN, loco citato, p. 305). This shows how gravity enters the acting universal field of force as a component, although not as an additive component, strictly, since it is divided by a variable expression.

With $G(r)/\cos^4\theta\cos^2\varphi$ cancelled, $F(r, \theta, \varphi)$ becomes the magnetic force proper, and may be written as $F_1(r, \theta, \varphi)$, or, at the earth's surface,

$$(29) \qquad r^5 p(r)\cos^4\theta_i\cos^2\varphi_i\,\mu\,F_{1i} = \alpha^3 r^3 q(\theta_i)^2 q'(\theta_i) + \alpha^2 r^2 p(r)\, q(\theta_i)^2$$
$$+ 2\alpha^3 r^3 q(\theta_i)^3\tan\theta_i + 2\,\alpha\, r\, p(r)^2 q(\theta_i)\tan\theta_i + \nu\, r\, p(r)^2\sin\varphi_i$$
$$+ \alpha^2 \nu\, r^3 q(\theta_i)^2 \sin\varphi_i + \alpha\,\nu^2 r^3 q(\theta_i)\sin\theta_i\cos\theta_i\cos^2\varphi_i,$$

where (r, θ_i, φ_i) is the point on the surface of the earth, and $\mu = 1/\Gamma$, $F_{1i} = F_1(r, \theta_i, \varphi_i)$.

Concerning the magnetic force at the pole

We find the limit approached by $F_1(r, \theta, \varphi)$ at the geographic pole. This limit should be equivalent to a certain formula ([12]), due to Swann, representing the force at the pole, viz.,

$$H_Z = \varrho\,\sigma\,\tau^m r^{n+1} E(n),$$

where ϱ, σ, $E(n)$ are numerical, and τ is the earth's angular velocity. We note;

(*i*) Since R approaches the z-axis by spiralling around it, like a geodesic on a narrow conical surface with vertex at the origin, the set of points used to calculate $p(r)$, $q(\theta)$, should be chosen in part within and in part without the earth-sphere. We accordingly find,

$$L = \lim_{\substack{\theta \doteq \pi/2 \\ \varphi\to 2g\pi}} \cdot F_1(r, \theta, \varphi), \; (0 < g < 3;\; g \text{ an integer}).$$

(*ii*) The cyle of orbits, perturbations of R by $\{U_1\}$, can be chosen arbitrarily, in our choice of W_ν, X_ν and of $p(r)$, $q(\theta)$. In particular we can choose,

$$q(\theta) = (\theta - \pi/2)^5 I(\theta),$$

and assume,

$$\lim_{\theta\doteq\pi/2} I(\theta) = k < 0.$$

([12]) Swann, *Proc. Int. Math. Congress, Toronto, 1924,* vol. II, p. 257.

Analogously the $q(\theta)$ of (32) below, computed for an R that lies mostly without the earth-sphere, differs but little formally from $\beta(\theta) - .1335)^5$.

Since then,

$$\mathrm{Lim} \cdot [q(\theta)/\cos^5\theta] = k\lambda < 0, \ (\lambda \text{ numerical}),$$

we have,

$$L = [k\,\alpha\,\lambda\,p(r)]/r^4.$$

Since r is neither near zero nor very large, and the coefficients in $p(r)$ diminish toward the left, the significant part of L is a negative term near the middle of $p(r)$, multiplied by $k\alpha\lambda/r^4$, and this part is equivalent to the formula H_Z; (Compare $p(r)$ in (32)).

Another property of $F_1(r, \theta, \varphi)$ will lead us to choose the meridian of Fresno, California, as prime meridian, when we make certain numerical calculations. It is that if we use $\pi/2 + \gamma$ in place of φ, in F_1, the latter is unaltered by the change of sign of γ.

The parameters of the force

The three constants in the formula $F_1(r, \theta, \varphi)$ of the vertical force of the magnetic field of the earth, can be determined as soon as the functions $p(r)$, $q(\theta)$ have been numerically calculated. For then the observed values of the force at three respective points on the earth's surface, whose latitudes and longitudes are known, give by substitution a solvable set of three equations in μ, α, ν.

In order to calculate $p(r)$, $q(\theta)$ we reproduced an orbit of a cathode particle from one of STØRMER'S published photographs ([13]) of such curves. Our reproduction was a wire model of the curve, nine feet in length, showing its approach to an earth-sphere of radius $r = 8$ inches $(= .8)$, (as in Fig. 2). Since Størmer's equations were expressed in terms of intrinsic coordinates, we measured the polar coordinates of seven points, P_i on our wire curve, in order to use only polar coordinates in our formulae. These results are here tabulated. (Unit of distance 10 inches).

(30)

Point	r		θ		φ	
P_1	7.	in.	.60228	rad.	77°	38′
P_2	11.563	in.	.19470	rad.	67°	45.5′
P_3	16.667	in.	.0	rad.	59°	10′
P_4	23.813	in.	—.07407	rad.	49°	24′
P_5	30.216	in.	—.08500	rad.	39°	18.4
P_6	46.500	in.	.03100	rad.	18°	31′
P_7	77.000	in.	.10559	rad.	4°	53′

([13]) STØRMER, *loco citato*, p. 143 and accompanying Fig. 5.

The values of (r_i, φ_i) in this table, substituted successively in the first relation (23), give seven linear equations in $\gamma_i, (i = 0, \dots, 6), A_1, B_1$, which, when solved, lead to the results which follow.

$$g(r) \equiv r^6 - 27.86468 r^5 + 299.645 r^4 - 1581.05 r^3 + 4376.062 r^2$$
$$- 6627.904 r + 5979.501 = 1354.09 \log_e (\sec \varphi + \tan \varphi) . \dagger$$

In solving likewise for the $\delta_i, (i = 0, \dots, 5), C_1, D_1$ we used the (θ_i, φ_i) coordinates of six points P_i, omitting P_6. The results of the solution may be written as follows; (unit for θ, one radian):

$$f(\theta) \equiv \theta^5 - .955065\, \theta^4 + .229486\, \theta^3 - .007695\, \theta^2 - .0010823\, \theta$$
$$+ .0003767 = .0002924 \log_e (\sec \varphi + \tan \varphi) .$$

The relations $g(r), f(\theta)$ are the equations of our wire curve.

Next, by differentiating the relations,

$$\int \frac{d\,r}{p\,(r)} = g\,(r) , \int \frac{d\,\theta}{q\,(\theta)} = f(\theta) , (r_1 < r < r_2 , \theta_1 < \theta < \theta_2) ,$$

we obtain,

$$(31) \quad \begin{cases} p\,(r) \equiv (r^6 + \xi_1 r^5 + \xi_2 r^4 + \dots + \xi_6)/\zeta = 1/g'(r) , (g' = \partial g / \partial r) , \\ q\,(\theta) \equiv (\theta^5 + \eta_1 \theta^4 + \eta_2 \theta^3 + \dots + \eta_5)/\eta = 1 / f'(\theta) , (f' = \partial f / \partial \theta) . \end{cases}$$

With each substitution of an r-coordinate of a point P_i, the formula $1/g'(r)$ assumes a numerical value, whence the first equation (31) gives the right number of linear equations, enabling us to solve for $\xi_i, (i = 1, \dots, 6), \zeta$, and likewise, using the before-mentioned six values θ_i, we can solve for η_i, $(i = 1, \dots, 5), \eta$. The results for $p(r), q(\theta)$ are as follows:

$$(32) \quad \begin{cases} p\,(r) = (r^6 - 13.74014\, r^5 + 52.9725\, r^4 - 32.4913\, r^3 - 159.0318\, r^2 \\ \qquad + 198.0864\, r - 85.7738)/62775.36 , \\ q\,(\theta) = (\theta^5 - .668486\, \theta^4 + .016412\, \theta^3 + .013525\, \theta^2 \\ \qquad + .000389\, \theta - .000014)\, 10^{10}/151 . \end{cases}$$

† To keep decimal points properly placed in computations, we later multiplied this equation through by ζ. This has the effect of replacing ζ, in the first equation (31), by unity.

As we have mentioned previously, we chose for prime meridian the meridian that passes through Fresno. California, (Greenwich 120^0 W.), and through a point near Masqat, Arabia: For then, the symmetry of $F_1(r, \theta, \varphi)$, $(\varphi = \frac{1}{2}\pi + \gamma)$, under the change of sign of γ, corresponds to the fact that the known map of the isodynamic lines of vertical magnetic force, is symmetrical east and west of a meridian that lies 90^0 east of the meridian of Fresno.

We abbreviate $10^{10}/151$ as β.

Since now φ is longitude and θ is latitude, the respective vertical forces at three points on the earth's surface, taken simultaneously with the respective longitude-latitude coordinate pairs, suffice, in connection with (29), to give three equations in the three unknowns μ, α, ν. We chose these points as in the following scheme: (14).

City	Longitude	Latitude	Vertical force V
Kamloops, Br. Colum.	$\varphi_1 = 0^0$	$\theta_1 = 51^0\,38'$ N.	$V_1 = .6280$ (c.g.s.).
New Orleans, U. S. A.	$\varphi_2 = 30^0$ E.	$\theta_2 = 30^0$ N.	$V_2 = .45207$
Newton Kans. U. S. A.	$\varphi_3 = 23^0$ E.	$\theta_3 = 38^0$ N.	$V_3 = .48895$

In the limit formula in the preceding section, k, and therefore $p(r)$, are necessarily negative, and this led us to our choice of the radius of the earth-sphere as $r_1 = 8$ inches $(= .8)$. The system of data now to be used is tabulated below for the benefit of anyone who might wish to repeat the computations; $p(.8) = -28.2633$; (Six-place tables).

θ_1	θ_2	θ_3	$\sin\theta_1$	$\sin\theta_2$	$\sin\theta_3$
.901172 rad.	.523599	.663225	.78405	.5	.61566
$q(\theta_1)$	$q(\theta_2)$	$q(\theta_3)$	$\cos\theta_1$	$\cos\theta_2$	$\cos\theta_3$
$.176795\,\beta$	$-.004636\,\beta$	$.009964\,\beta$	.62069	.86603	.78801
$q'(\theta_1)$	$q'(\theta_2)$	$q'(\theta_3)$	$\tan\theta_1$	$\tan\theta_2$	$\tan\theta_3$
$1.405444\,\beta$	$.020019\,\beta$	$.227330\,\beta$	1.2632	.57735	.78129
$q''(\theta_1)$	$q''(\theta_2)$	$q''(\theta_3)$	$\cos^4\theta_1$	$\cos^4\theta_2$	$\cos^4\theta_3$
$8.238220\,\beta$	$.750333\,\beta$	$2.398434\,\beta$	.148423	.562512	.385592
φ_1	φ_2	φ_3	$\sin\varphi_1$	$\sin\varphi_2$	$\sin\varphi_3$
0. rad.	.523599	.401426	0.	.5	.39073
$\cos\varphi_1$	$\cos\varphi_2$	$\cos\varphi_3$	$\cos^2\varphi_1$	$\cos^2\varphi_2$	$\cos^2\varphi_3$
1.	.86603	.92050	1.	.750009	.847321

(14) At Kamloops there is an extrapolation, of φ, about 4^0 beyond the argest φ of (30).

After substitutions in (29), and numerical simplifications, the three equations take the following forms:

$$r^5 p(r)\mu = .241303(\alpha\beta)^3 - 6.06572(\alpha\beta)^2 + .076687(\alpha\beta)^3 + 3062.3(\alpha\beta) + .472604(\alpha\beta)\nu^2,$$

$$r^5 p(r)\mu = .0000012(\alpha\beta)^3 - .0020384(\alpha\beta)^2 - .0000003(\alpha\beta)^3 - 17.9368\,\alpha\beta) + 1675.34\nu + .0000288(\alpha\beta)^2\nu - .0040418(\alpha\beta)\nu^2,$$

$$r^5 p(r)\mu = .0000723(\alpha\beta)^3 - .0112416(\alpha\beta)^2 + .00000495(\alpha\beta)^3 + 62.2832(\alpha\beta) + 1563.04\,\nu + .0001243(\alpha\beta)^2\nu + .0131276(\alpha\beta)\nu^2.$$

We next subtract the second and third, of the latter equations, from the first, thus eliminating μ. The results are,

$$(33)\quad \begin{cases} a(\alpha\beta)^3 + b(\alpha\beta)^2 + c(\alpha\beta)^3 + d(\alpha\beta) + e\nu + f(\alpha\beta)^2\nu + g(\alpha\beta)\nu^2 = 0, \\ h(\alpha\beta)^3 + i(\alpha\beta)^2 + j(\alpha\beta)^3 + k(\alpha\beta) + l\nu + m(\alpha\beta)^2\nu + n(\alpha\beta)\nu^2 = 0, \\ (\alpha\beta) = \alpha\beta, \end{cases}$$

where the coefficients have the values, respectively, shown in the following list:

a	b	c	d	e	f	g
.241302	— 6.06368	.076687	3080.2368	— 1675.34	—.0000288	.476646
h	i	j	k	l	m	n
.241231	— 6.05448	.076682	3000.0168	— 1563.04	—.0001243	.459476

We solve the first equation (33) as a quadratic in ν, and substitute the resulting value of ν in the second, thus eliminating ν. The result is a sextic equation in $(\alpha\beta)$ with numerical coefficients, which equation is of interest also when $a, b \ldots, n$ are left arbitrary. It may be written thus:

$$(34)\quad (A^2 - FH^2)(\alpha\beta)^6 + (2AB + 4bgH^2)(\alpha\beta)^5 + (B^2 + 2AC - NH^2 + 2FHJ)(\alpha\beta)^4 + (2BC - 8bgHJ)(\alpha\beta)^3 + (C^2 + 2AD - e^2H^2 + 2HJN - FJ^2)(\alpha\beta)^2 + (2BD + 4bgJ^2)(\alpha\beta) + (2CD + 2e^2HJ - NJ^2) = 0,$$

wherein,

$$
(35)\quad \begin{cases} A = 2\,g\,[g\,(h+j) - n\,(a+c)] + f\,(f\,n - g\,m)\,,\ B = 2\,g\,(g\,i - b\,n)\,, \\ \qquad C = 2\,n\,(e\,f - d\,g) + g\,(g\,k - e\,m) + g\,(g\,k - f\,l)\,, \\ D = e\,(e\,n - g\,l)\,,\ F = f^2 - 4\,g\,(a+c)\,,\ H = f\,n - g\,m\,,\ J = g\,l - e\,n\,, \\ \qquad N = 2\,e\,f - 4\,d\,g\,. \end{cases}
$$

When the English letters in (34) are given their numerical values from (33) (35), the equation reciprocal to (34) takes the form,

$$
(36)\quad 4759279\,x^6 - 14975.93\,x^5 + 980.1582\,x^4 - 2.61354\,x^3 + .151259\,x^2 - .0009831\,x + .0000267 = 0\,.
$$

We have,

$$
(\alpha\,\beta) = 1/x\,,\ \nu = (-\,e - f\,(\alpha\,\beta)^2 \pm \sqrt{\Delta})/2\,g\,\alpha\,\beta\,,
$$

$$
\Delta = e^2 + N\,(\alpha\,\beta)^2 - 4\,b\,g\,(\alpha\,\beta)^3 + F\,(\alpha\,\beta)^4\,.
$$

Since N, b, and F are negative and Δ necessarily positive, $|\,\alpha\,\beta\,|$ is limited above. Also $(\alpha\,\beta)$ is shown to be negative by (33), and x is numerically small by (36). These conditions lead to the result $x = -.0486$ as an accurate determination. Hence, from the triad of numerical equations,

$$
(\alpha\,\beta) = -\,20.5762\,,\ \Delta = 111034.92\,,\ \nu = -\,102.4\,,\ \mu = 18389.9\,,
$$

the value of μ, as calculated from the third equation, falling short by about three percent of its true value.

There is a reason in phenomena for this latter fact. If the solution for μ, α, ν were subject to no inexactness, $F_1(r, \theta, \varphi)$ would have a fixed value at each point on the earth's surface, but, as is well known, V continually varies in the third decimal place and beyond. We note also that, due to the numerical form of the equation (36), x will remain invariant under these changes in V, and even under small errors in the table (30). In summary we have:

Theorem. The directional-central magnetic force of the earth, as this force engages a cathode particle moving on a segment R of its orbit, is,

$$
F_1(r, \theta, \varphi) = F(r, \theta, \varphi) - G(r)/\cos^4\theta\,\cos^2\varphi\,,
$$

as shown in (28), *and, for points on the earth's surface, as in* (29). *The con-*

stants in the formula F_1 have the following numerical values:

$$\alpha = -20.5762/\beta \, , \nu = -102.4 \, , \mu = 18389.9 \, .$$

The radius of the earth is taken to be $r = .8$, and angles are expressed in radians.

The formulary and the phenomena

Verification of $F_1(r, \theta, \varphi)$ as the true formula of the vertical magnetic force at the surface of the earth consists in showing that the (isodynamic) lines of equal vertical force on the earth's surface, when plotted from F_1, are the same as those obtained under the auspices of the British Admiralty, and of the American Coast and Geodetic Survey. These lines were discovered originally by means of instruments which were operated, in some cases, on ship-board. They were operated in very many places which were in fact chosen to represent all points of the surface of the earth. We plot these isodynamic lines as line-element connections according to the following principle. If the adjacent points (r, θ, φ), $(r, \theta + \delta\theta, \varphi + \delta\varphi)$ are both on the same isodynamic line, we have,

$$F_1(r, \theta + \delta\theta, \varphi + \delta\varphi) = F_1(r, \theta, \varphi) \, ,$$

whence,

$$\frac{\delta\theta}{\delta\varphi} = -\frac{\partial F_1(r, \theta, \varphi)}{\partial \varphi} \Big/ \frac{\partial F_1(r, \theta, \varphi)}{\partial \theta} \, .$$

We form the partial derivatives, from (29), and calculate their numerical values on the basis of $p(r)$, $q(\theta)$ as in (32), θ being held within the interval $(\theta_1 < \theta < \theta_2)$, represented by (30). These derivatives are found thus at each point where a parallel of latitude, say $\theta = \tau$, ($\tau = -5^0, 0^0, 12^0, 24^0, 36^0$), is intersected by a meridian $\varphi = \sigma$, (E), ($\sigma = 0^0, 12^0, 24^0, 36^0$, *et cetera*, at intervals of 12^0, to 180^0), the prime meridian being the meridian of Fresno. This process gives the slopes $\delta\theta/\delta\varphi$ at a sufficient number of points, enabling us to draw the isodynamic lines as element connections. For greater accuracy some interpolations may be introduced.

The lines obtained are purely mathematical, and they coincide with those obtained experimentally, in the manner above stated.

The partial derivatives are as follows:

$$\frac{\partial F_1(r,\theta,\varphi)}{\partial \varphi} = \{\nu\, r\, p(r)^2 \cos\varphi + \alpha^2 \nu\, r^3 q(\theta)^2 \cos\varphi + 2\,\alpha^3 r^3 q(\theta)^2 q'(\theta) \tan\varphi$$

$$+ 2\alpha^2 r^2 p(r)\, q(\theta)^2 \tan\varphi + 4\,\alpha^3 r^3 q(\theta)^3 \tan\theta \tan\varphi$$

$$+ 4\,\alpha\, r\, p(r)^2 q(\theta) \tan\theta \tan\varphi + 2\nu\, r\, p(r)^2 \sin\varphi \tan\varphi$$

$$+ 2\,\alpha^2 \nu\, r^3 q(\theta)^2 \sin\varphi \tan\varphi\} \div (r^5 p(r)\, \mu \cos^4\theta \cos^2\varphi),$$

$$\frac{\partial F_1(r,\theta,\varphi)}{\partial \theta} = \{2\alpha^3 r^3 q(\theta)\, q'(\theta)^2 + \alpha^3 r^3 q(\theta)^2 q''(\theta) + 2\,\alpha^2 r^2 p(r)\, q(\theta)\, q'(\theta)$$

$$+ 10\,\alpha^3 r^3 q(\theta)^2 q'(\theta) \tan\theta + 2\,\alpha^3 r^3 q(\theta)^3 \sec^2\theta + 2\,\alpha\, r\, p(r)^2 q'(\theta) \tan\theta$$

$$+ 2\,\alpha\, r\, p(r)^2 q(\theta) \sec^2\varphi + 2\,\alpha^2 \nu\, r^3 q(\theta)\, q'(\theta) \sin\theta$$

$$+ \frac{1}{2}\,\alpha\, \nu^2 r^3 q^1(\theta) \sin 2\theta \cos^2\theta + \alpha\, \nu^2 r^3 q(\theta) \cos 2\varphi \cos^2\varphi$$

$$+ 4\,\alpha^2 r^2 p(r)\, q(\theta)^2 \tan\theta + 8\,\alpha^3 r^3 q(\theta)^3 \tan^2\theta + 8\,\alpha\, r\, p(r)^2 q(\theta) \tan^2\theta$$

$$+ 4\,\nu\, r\, p(r)^2 \tan\theta \sin\varphi + 4\,\alpha^2 \nu\, r^3 q(\theta)^2 \tan\theta \sin\varphi$$

$$+ 4\,\alpha\, \nu^2 r^3 q(\theta) \sin^2\theta \cos^2\varphi\} \div (r^5 p(r)\, \mu \cos^4\theta \cos^2\varphi).$$

Some of the data for substitutions, not regularly available in tables, are here given. It will be remembered that $\alpha\, q(\theta) = (\alpha\,\beta)\, q_1(\theta)$ where $q_1(\theta)$ is the $q(\theta)$ of (32) without the latter's numerical factor $\beta = 10^{10}/151$. We use $r = .8$, $p(.8) = -28.2633$, $(\alpha\,\beta) = -20.5762$, $\nu = -102.4$, and the following table:

θ	$q_1(\theta)$	$q_1'(\theta)$	$q_1''(\theta)$
-5^0	.0000003	.00047	—.05593
0^0	—.000014	.000389	.02705
12^0	—.000072	—.006731	—.120463
24^0	—.003956	—.022235	.13072
36^0	.003381	.152824	1.88305
48^0	.102834	.948296	6.23893

Following is a table for five slopes, along a typical meridian, $\varphi = 48^0$ (E), (Fresno), corresponding, respectively, to $\theta = -5^0, 0^0, 12^0, 24^0, 36^0$. (Only numerators of $F_{1\theta}$, $F_{1\varphi}$ are given since the denominators cancel). We use the following abbreviations:

$$\partial F_1/\partial \varphi_{[r=.8,\,\theta=\theta_1,\,\varphi=\varphi_1]} = F_{1\varphi}(.8, \theta_1, \varphi_1).$$

The results for $\delta\,\theta/\delta\,\varphi$ are in all cases fairly regular. They begin with large values, near the equator, and decrease toward small values as θ approaches 36^0, φ being any angle between zero and 80^0:

$F_{1\varphi}$ (.8, − 5,°48°)	$F_{1\theta}$ (.8,—5°,48°)	$\delta\,\theta/\delta\,\varphi$
—151804.8	17021.7	8.92

$F_{1\varphi}$ (.8,0°,48°)	$F_{1\theta}$ (.8,0°,48°)	$\delta\,\theta/\delta\,\varphi$	$F_{1\varphi}$ (.8,12°,48°)	$F_{1\theta}$ (.8,12°,48°)	$\delta\,\theta/\delta\,\varphi$
—151804.8	.6924	219243.2	—151804.0	—41235.8	—3.68
$F_{1\varphi}$ (.8,24°,48°)	$F_{1\theta}$ (.8,24°,48°)	$\delta\,\theta/\delta\,\varphi$	$F_{1\varphi}$ (.8,36°,48°)	$F_{1\theta}$ (.8,36°,48°)	$\delta\,\theta/\delta\,\varphi$
—151703.0	—85472.4	—1.77	—151948.0	—148475.0	—1.02

It is obvious that the roles of the north and south poles are interchangable in the theory of the field. If we use Fresno as prime meridian, measure longitudes positive westward, and latitudes positive from the equator toward the south pole, the formula $F_1(r, \theta, \varphi)$, without change, applies in the eastern magnetic hemisphere, *idem est*, from Fresno westward to Masqat. The isodynamic system just described for the western hemisphere will be repeated, in inverted position, in the eastern. The drawings on later pages show the two systems in their respective positions with reference to the continents.

It will be noted that, with r constant and the range of θ, valid in $q(\theta)$, extending to approach $\pi/2$, the northern magnetic pole is the maximum point of the function $F_1(r, \theta, \varphi)$. Hence this pole is an intersection of the two curves,

$$\partial\, F_1(r, \theta, \varphi)/\partial\,\theta = 0\,,\ \partial\, F_1(r, \theta, \varphi)/\partial\,\varphi = 0\,,$$

Since, at a point on an isodynamic line l, for the vertical force V, we have $V = H \tan I$, H being the horizontal component and I the inclination, two points (H_1, I_1), (H_2, I_2) on l give an equation $a = b$, viz.,

$$\tan I_1/\tan I_2 = H_2/H_1\,.$$

To apply $a = b$ as a check on the accuracy of the lines l, the maps (tables) of H and I at points P, of the earth's surface, are used. Eight pairs of cities (P), each pair on an l, give the following cases of this verification, (check) of lines l, (Year of observations, 1922); through $a = b$.

Cities	*a*	*b*	*Cities*	*a*	*b*
Maracaibo & St. Isidore	1.32	1.04,	New Orleans & Okl'a City	1.08	.99
Norfolk & Spokane	1.00	1.04,	Baltimore & St. Paul	1.28	1.24
Bordeaux & Cologne	1.14	1.07,	Algiers & Ferrara	1.20	1.11
Tripoli & Kos	1.23	1.03,	Khartum & Jidda	1.28	1.04

If the following two drawings are placed side by side, two systems of closed curves l are seen, each being in a hemisphere between the semi-meridians of symmetry of the field. The existence of these systems is verified in part, by the known results of technology, which show two systems of analogously situated isodynamic lines of the horizontal force-component, also in the form of (irregular) ovals. Of course, if the values of V are the same along l, the values of $k V$, k being constant, will be the same along l, while $k V$, with k arbitrarily variable within narrow limits, will keep the same value along a curve l' which shows some variation from l. And k here may be cot I, with I limited below.

A source of small deviations from theoretical values of the magnetic force at a point P, is an irregular distribution of magnetic materials near the surface of the earth.

Lansdowne, Pa., October, 1953

O. E. Glenn

MAGNETIC HEMISPHERE, *W*
(Lines of equal vertical force)

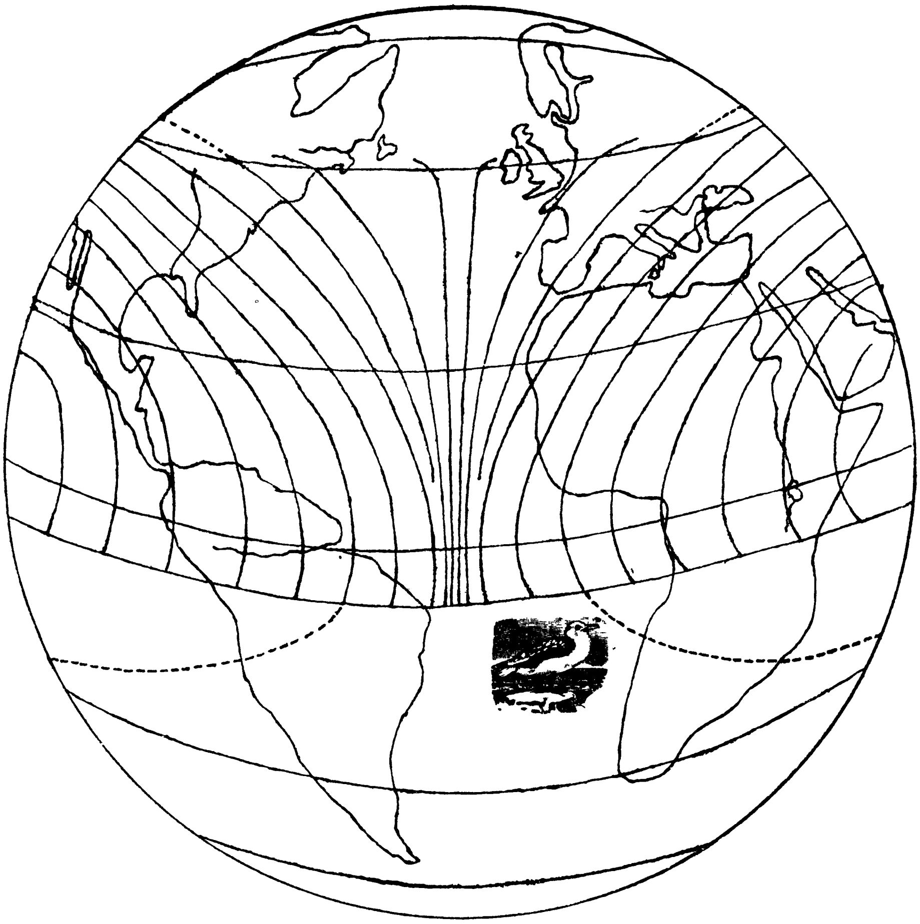

Figure 3

MAGNETIC HEMISPHERE, *E*.

(Lines of equal vertical force)

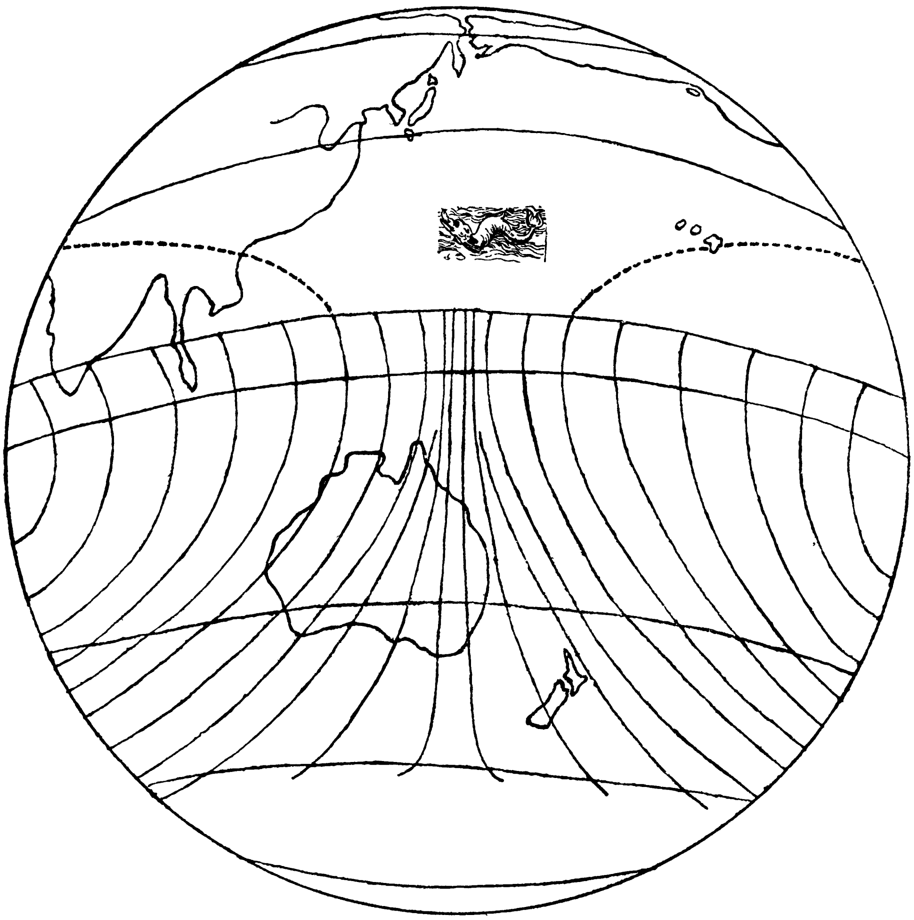

Figure 4