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NOTE ON CERTAIN EQUATIONS, CONNECTED WITH BATEMAN FUNCTIONS

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INTRODUCTION

The main object of the present investigation is to study certain characteristic properties of the three functional equations:

$$(n+1) f_{2n+2}(z) + 2(n-z) f_{2n}(z) + (n-1) f_{2n-2}(z) = 0 , \quad \ldots \quad \text{(I)}$$

$$(n-1) f_{2n-2}(z) + (n-z) f_{2n}(z) = z f'_{2n}(z) , \quad \ldots \quad \ldots \quad \text{(II)}$$

$$f'_{2n}(z) + f'_{2n+2}(z) = f_{2n}(z) - f_{2n+2}(z) , \quad \ldots \quad \ldots \quad \text{(III)}$$

and the differential equation:

$$z \omega'' + (2n - z) \omega = 0 , \qquad (\omega \equiv f_{2n}(z)) , \qquad \ldots \text{(A)}$$

it being understood that the parameter n, (except when otherwise stated) is an integer (≥ 1).

The paper consists of five articles, of which the first reckons with the mutual relations subsisting among the four equations, which will be termed « Bateman equations » in the present context. Next Art. 2 treats of the *typical common solution* of the four equations in terms of two linearly independent particular solutions. Then Art. 3 makes a short digression on a certain triad of equations, satisfied by the two kinds of « generalised » Laguerre functions $L_n^{(\alpha)}(z)$ and $l_n^{(\alpha)}(z)$, and incidentally works out the typical *common* solution of the three equations. The connecting link between the Laguerre equations (for $\alpha = 1$) and Bateman equations being traced in the earlier part of Art. 4, it has been felt necessary to introduce a new function $k_{2n}(z)$, which satisfies all the four Bateman equations in the same manner as the ordinary Bateman function (*) $K_{2n}(z)$ does. In view of its special importance, $k_{2n}(z)$ has been designated as the Bateman function of *the second kind* in contrast to $K_{2n}(z)$, which will be here designated as Bateman function of *the first kind.* The latter part of Art. 4 expresses the *typical common* solution of the four Bateman equations — obtained already in a *general* way in Art. 2 — in terms of the two kinds of Bateman functions. Finally Art. 5 deals mainly with the « generating » function of *any* enumerable sequence of analytic functions $\{f_{2n}(z)\}$, satisfying the functional equation (I), and then applies the result to particular sequence $\{K_{2n}(z)\}$.

Although at times we have felt constrained to touch on certain *known* results, still we believe that this paper embodies some amount of original matter.

ART. 1 — In order to take account of the inter-relations among the four Bateman equations, we have to pair them in all possible ways.

So there are altogether ${}^4C_2$ or 6 cases to consider.

Case I. — Firstly, supposing $f_{2n}(z)$ to satisfy (I) and (II), we have by simple subtraction

$$(z - n) f_{2n}(z) - (n+1) f_{2n+2}(z) = z f'_{2n}(z) . \qquad \ldots \quad \ldots (1)$$

If we now write $(n+1)$ for n in (II) and then add the resulting relation to (1) and drop the factor z, we at once get (III). Next (II) gives on differentiation

$$z f''_{2n}(z) + (z - n + 1) f'_{2n}(z) + f_{2n}(z) = (n-1) f'_{2n-2}(z) . \qquad \ldots (2)$$

(*) As is well-known, the function $K_{2n}(z)$, originally introduced by Bateman, is but a particular variety of the confluent hypergeometric function $W_{k,m}(z)$, and has in recent years engaged the attention of several mathematicians, notably Shabde, Shastri and Srivastava.

[[Refer to :
Bateman (1931) . . . Trans. American Math. Soc., XXXIII (Pp. 817-31);
Shabde (1932) . . . Bull. Cal. Math. Soc., XXIV (Pp. 109-34);
» » . . . Procedings of Benares Math. Soc., XIV (Pp. 24-30);
Shastri (1935) . . . Journal of the Indian Math. Soc., III (Pp. 14, 157);
Srivastava (1950) . . . Bull. Cal. Math. Soc., XLII (P. 82)]].

If n be now replaced by $(n-1)$ in (III), it becomes:

$$f'_{2n-2}(z) = -f'_{2n}(z) - f_{2n}(z) + f_{2n-2}(z). \qquad \ldots \ldots (3)$$

Eliminating $f'_{2n-2}(z)$ from (2) and (3), we get

$$z f''_{2n}(z) + z f'_{2n}(z) + n f_{2n}(z) = (n-1) f_{2n-2}(z). \qquad \ldots (4)$$

Further, eliminating $f_{2n-2}(z)$ from (II) and (4), we derive

$$z f''_{2n}(z) + (2n - z) f_{2n}(z) = 0\,,$$

showing that $f_{2n}(z)$ is a solution of (A).

Case II. — *Secondly*, supposing $f_{2n}(z)$ to satisfy (II) and (III), we find, on writing $(n+1)$ for n in (II),

$$z f'_{2n+2}(z) = n f_{2n}(z) + (n + 1 - z) f_{2n+2}(z). \qquad \ldots \ldots (5)$$

Elimination of $f'_{2n+2}(z)$ from (III) and (5) gives

$$z f'_{2n}(z) + (n - z) f_{2n}(z) + (n+1) f_{2n+2}(z) = 0. \qquad \ldots \ldots (6)$$

Further, eliminating $f'_{2n}(z)$ from (III) and (6), we easily get (I). Then the relation (A) follows at once from *Case I.*

Case III. — *Thirdly,* supposing $f_{2n}(z)$ to satisfy (I) and (III), we eliminate $f'_{2n+2}(z)$ from (III) and the relation, derived from (I) by differentiation, and then reduce the result so as to obtain

$$(n - 2z - 1) f'_{2n}(z) + (n-1) f_{2n}(z) - (n+1) f_{2n+2}(z) + (n-1) f'_{2n-2}(z) = 0. \ \ldots (7)$$

Then eliminating $f'_{2n-2}(z)$ from (III) and (7), we get

$$2z f'_{2n}(z) = (n-1) f_{2n-2}(z) - (n+1) f_{2n+2}(z). \qquad \ldots \ldots (8)$$

If we now eliminate $f_{2n+2}(z)$ from (I) and (8), we arrive at (II). Finally the relation (A) follows immediately from *Case I* or *II.*

Case IV. — *Fourthly*, supposing $f_{2n}(z)$ to satisfy (I) and (A), we differentiate (I), and then utilise the three equalities — which are inherent in (A) — viz.,

$$\left.\begin{aligned} f''_{2n}\ \ (z) &= \frac{z-2n}{z} f_{2n}(z)\,, \\ f''_{2n+2}(z) &= \frac{z-2n-2}{z} f_{2n+2}(z)\,, \\ f''_{2n-2}(z) &= \frac{z-2n+2}{z} f_{2n-2}(z)\,, \end{aligned}\right\} \qquad \ldots\ \ldots\ \ldots (9)$$

so as to derive the relation:

$$(n+1)(z-2n-2)f_{2n+2}(z) - 2(z-n)(z-2n)f_{2n}(z) + \\ + (n-1)(z-2n+2)f_{2n-2}(z) = 4zf'_{2n}(z)\,. \qquad \ldots\ \ldots (10)$$

Elimination of $f_{2n+2}(z)$ from (I) and (10) now easily leads to (II). The relation (III) then follows at once from *Case I.*

Case V. — *Fifthly,* supposing $f_{2n}(z)$ to satisfy (II) and (A), we have, on differentiating (II) and then utilising (A),

$$(n-1)f'_{2n-2}(z) + (n-z-1)f'_{2n}(z) = (z-2n+1)f_{2n}(z)\,.$$

Substituting herein the value of $f'_{2n}(z)$, as provided for by (II), we obtain after easy reductions

$$zf'_{2n-2}(z) + (n-z-1)f_{2n-2}(z) + nf_{2n}(z) = 0\,,$$

which can, on n being changed into $(n+1)$, be written as:

$$zf'_{2n}(z) + (n-z)f_{2n}(z) + (n+1)f_{2n+2}(z) = 0\,. \qquad \ldots\ \ldots (11)$$

Elimination of $f'_{2n}(z)$ from (II) and (11) now plainly leads to (I). Then (A) follows automatically from *Case I.*

Case VI. — *Sixthly,* supposing $f_{2n}(z)$ to satisfy (III) and (A), we differentiate (III), and in the derived result we insert the values of $f''_{2n}(z)$ and $f''_{2n+2}(z)$ as given by (9). We thus obtain

$$\frac{z-2n}{z} f_{2n}(z) + \frac{z-2n-2}{z} f_{2n+2}(z) = f'_{2n}(z) - f'_{2n+2}(z)\,. \qquad \ldots (12)$$

Eliminating $f'_{2n+2}(z)$ from (III) and (12) and reducing, we get

$$f'_{2n}(z) = \frac{z-n}{z} f_{2n}(z) - \frac{n+1}{z} \cdot f_{2n+2}(z), \qquad \dots \quad \dots (13)$$

i.e.,

$$f'_{2n+2}(z) = \frac{z-n-1}{z} f_{2n+2}(z) - \frac{n+2}{z} f_{2n+4}(z). \qquad \dots \quad \dots (14)$$

Elimination of $f'_{2n}(z)$ and $f'_{2n+2}(z)$ from (12), (13) and (14) and subsequent simplification now squarely lead to:

$$(n+2) f_{2n+4}(z) + 2(n+1-z) f_{2n+2}(z) + n f_{2n}(z) = 0,$$

which evidently coincides with (I) as soon as n is changed into $(n-1)$. Thus (I) being a consequence of the combination of (III) and (A), the remaining relation (II) follows at once from *Case III* or *Case IV*.

We may now amalgamate the results of all the six *Case s I-VI* and finalise our conclusions in the form of a lemma:

LEMMA. — *The four equations, viz. (I), (II), (III) and (A) are tantamount to only two affective equations. In other words, if a sequence of functions* $\{f_{2n}(z)\}$ *satisfy any two of the four equations, it must as a matter of course satisfy the other two.*

Art. 2 — The homogeneous difference equation (I) being of the first degree and second order, its *general* solution must be expressible in the form:

$$f_{2n}(z) = \alpha_{2n}(z)\, g_n(z) + \beta_{2n}(z)\, h_n(z), \qquad \dots \quad \dots \quad \dots (15)$$

where $\alpha_{2n}(z)$ and $\beta_{2n}(z)$ are two linearly independent particular solutions of (I), and $g_n(z)$ and $h_n(z)$ are two *arbitrary* functions of z, which are periodic in n *with unit period*. As only positive integral values of n are here contemplated, the periodic functions of n, *viz.* $g_n(z)$ and $h_n(z)$, which admit of a *unit* period, are practically *independent* of n and are, therefore, representable in the simple forms $g(z)$ and $h(z)$. So (15) assumes the form:

$$f_{2n}(z) = \alpha_{2n}(z)\, g(z) + \beta_{2n}(z)\, h(z), \qquad \dots \quad \dots \quad \dots (16)$$

where $g(z)$ and $h(z)$ are *arbitrary* functions of z.

If we now look for a *common* solution of (I) and (II), the straight forward method is to start with (16), subject to the restriction that $\alpha_{2n}(z)$ and $\beta_{2n}(z)$ are particular solutions not only of (I) *but also of* (II) and then to impose the extra condition that the resulting value of $f_{2n}(z)$, as given by

(16), may satisfy (II). As a matter of fact, when the values of $f_{2n}(z)$ and $f_{2n-2}(z)$, as provided for by (16), and the consequent value of $f'_{2n}(z)$ derived by differentiation, are inserted in (II), we obtain, on a re-shuffling of terms:

$$g(z) . \{ z \;\; \alpha'_{2n}(z) - (n-1)\, \alpha_{2n-2}(z) - (n-z)\, \alpha_{2n}(z) \}$$

$$+ h(z) . \{ z . \beta'_{2n}(z) - (n-1)\, \beta_{2n-2}(z) - (n-z)\, \beta_{2n}(z) \}$$

$$+ \alpha_{2n}(z)\, g'(z) + \beta_{2n}(z)\, h'(z) = 0 \, ,$$

which simplifies to

$$\alpha_{2n}(z)\, g'(z) + \beta_{2n}(z)\, h'(z) = 0 \, . \qquad \ldots \ldots \ldots (17)$$

Then repeating the line of argument, adopted in the under-mentioned paper [1], we conclude finally that the relation (17) can be an identity in z, if and only if

$$g'(z) = 0 \quad \text{and} \quad h'(z) = 0 \, , \quad \text{(identically)}$$

thus requiring both $g(z)$ and $h(z)$ to be constants (obviously independent of n).

Hence attending to the proved lemma of Art. 1, we arrive at the following proposition:

PROP. A. — *The four equations (I), (II), (III), and (A) are equivalent to anly two independent equations; and the most general solution, common to them all, can be presented in the form:*

$$f_{2n}(z) = a\, \alpha_{2n}(z) + b\, \beta_{2n}(z) \, , \qquad \ldots \ldots (18)$$

where $\alpha_{2n}(z)$ and $\beta_{2n}(z)$ are two independent particular solutions [2], *common to (I) ond (II), — and therefore also to (III) and (A), — and 'a', 'b' are arbitrary numerical constants, independent of 'n'.*

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(1) Refer to H. D. Bagchi and. P. C. Chatterjee; « *Note on certain functional equations, connected with Hermite and Weber functions* » [Vide: *Annali della Scuola Normale Superiore di Pisa (1952)*].

(2) It must be borne in mind that, when $\alpha_{2n}(z)$ and $\beta_{2n}(z)$ are selected as *perfectly arbitrary* solutions of the differential equation (A), they may not, and, in general, will not satisfy any of the two functional equations (I), (II), and so there is every chance of the relation (I) or (II) *not* being fulfilled. Thus, as will generally happen, when the functions $\alpha_{2n}(z)$, $\beta_{2n}(z)$ satisfy (A), but *not* (I) or (II), the consequent value of $f_{2n}(z)$, as given by (18), will satisfy (A) but *not* (I) or (II). This explains the cogency of the restrictions — noted as above — that $\alpha_{2n}(z)$ and $\beta_{2n}(z)$ must be two (linearly independent) particular solutions of both (I) and (II) and therefore also of (III) and (A).

Plainly one of the two functions, (say, $\alpha_{2n}(z)$) can at pleasure be taken to be $K_{2n}(z)$. A convenient choice of the other function $\beta_{2n}(z)$ will be made in the latter portion of Art. 4, when the common solution (18) will be presented in a *more concrete* form. In the meantime we propose to make a short digression on certain equations, connected with the « generalised » Laguerre function $L_n^{(\alpha)}(z)$.

ART. 3 — We know that, when $\alpha > -1$ and n is a positive integer, the two functional equations:

$$(n+1)\,\psi_{n+1}(z) - (2n + \alpha + 1 - z)\,\psi_n(z) + (n+\alpha)\,\psi_{n-1}(z) = 0\,, \quad \ldots \quad \text{(I)}'$$

and

$$n\,\psi_n(z) - (n+\alpha)\,\psi_{n-1}(z) = z\,\psi'_n(z)\,, \quad \ldots \quad \ldots \text{(II)}'$$

and the differential equation:

$$z\,v'' + (\alpha + 1 - z)\,v' + n\,v = 0\,, \quad (v \equiv \psi_n(z)) \quad \ldots \text{(A)}'$$

are equivalent to *only two* [3] independent equations and have for a common solution

$$\psi_n(z) = L_n^{(\alpha)}(z)\,.$$

For our present purpose we propose to call $L_n^{(\alpha)}(z)$ the « generalised » Laguerre function *of the first kind* and to use the epithet « generalised » Laguerre function of the second kind in respect of the function $l_n^{(\alpha)}(z)$, which has been defined by G. Palama [4] in the form:

$$l_n^{(\alpha)}(z) = -\frac{\Gamma(\alpha)}{\sqrt{2\pi}}\,z^{-\alpha}\,G(-\alpha - n\,,\ 1-\alpha\,,\ z)\,,$$

i.e.,

$$l_n^{(\alpha)}(z) = -\frac{\Gamma(\alpha+n+1)}{\sqrt{2\pi}}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha - m)\,z^{m-\alpha}}{\Gamma(\alpha+n-m+1)\,m\,!}\,.$$

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(3) For a *more general* result refer to H. D. BAGCHI & B. N. MUKHERJI: « *Note on 'generalised' LAGUERRE function and associated equations* » [Vide *Proceedings of the Indian Academy of Sciences* (Bangalore) (1952)]. In this paper, besides (I)′ and (II)′, two other functional equations have been considered and the equivalence of the four functional equations and the differential equation (A)′ to only *two* effective equations has been stated categorically.

(4) See Nota di GIUSEPPE PALAMA (Lecce) [Vide *Boll. dell'Un. Mat. It.* (1950)]. It is hardly necessary to mention that in the above context the symbol G stands for a Kummer function, definable as usual in the terms of a hyper-geometric function F by the relation:

$$G(\beta\,,\gamma\,,z) = \lim_{\epsilon \to 0} F\left(\beta\,,\frac{1}{\epsilon}\,;\gamma\,,\epsilon\,z\right).$$

As observed by G. Palama (*loc. cit.*), the function $l_n^{(\alpha)}(z)$ satisfies both (I)′ and (A)′; so it must satisfy also (II)′, seeing that these three equations are equivalent to *two* effective equations. If we now once again utilise the principle and chain of reasoning, adopted elsewhere [5] and made use of also in the foregoing article, we arrive at following proposition:

PROP. B — *The most geueral solution, common to any two — and therefore to all — of the three equations (I)′, (II)′ and (A)′, is that* [6] *given by*

$$\psi_n(z) = a\, L_n^{(\alpha)}(z) + b\, .\, l_n^{(\alpha)}(z)\,, \qquad \ldots \ldots (19)$$

where 'a' and 'b' are arbitrary constants, independent of 'n'.

ART. 4 — If we now put $\alpha = 1$, the equations (I)′, (II)′ and (A)′ of the previous article become

$$(n+1)\,\psi_{n+1}(z) - (2\,n + 2 - z)\,\psi_n(z) + (n+1)\,\psi_{n-1}(z) = 0\,, \qquad \ldots (\mathrm{I})''$$

$$n\,\psi_n(z) - (n+1)\,\psi_{n-1}(z) = z\,\psi_n'(z) \qquad \ldots \ldots (\mathrm{II})''$$

and

$$z\,v'' + (2 - z)\,v' + n\,v = 0\,, \qquad (v \equiv \psi_n(z))\,. \qquad \ldots (\mathrm{A})''$$

It is easy to verify by elementary analysis that the transforming scheme

$$f_{2n}(z) = \frac{2\,.\,(-1)^{n-1}}{n}\,.\,z\,e^{-z}\,\psi_{n-1}(2\,z)\,, \qquad \ldots \ldots (20)$$

which is the same as

$$\psi_n(2\,z) = \frac{(-1)^n\,.\,(n+1)}{2}\,.\,z^{-1}\,e^z f_{2n+2}(z)\,, \qquad \ldots \ldots (21)$$

converts the two equations (I)″ and (II)″ respectively into (I) and (II) and *vice versa*. Evidently, then, either of the two equivalent relations (20), (21) may be looked upon as the *connecting link* between the two pairs of functional equations, *viz* (I), (II) and (I)″, (II)″. That is to say, a definite homo-

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[5] Vide H. D. BAGCHI and B. N. MUKHERJI (*loc. cit.*).

[6] For obvious reasons the two functions $L_n^{(\alpha)}(z)$ and $l_n^{(\alpha)}(z)$, occurring in (19), are replacable by two other (linearly independent) functions $\alpha_n(z)$ and $\beta_n(z)$, provided that these latter satisfy not only (A)′ but also (I)′, (II)′. In case $\alpha_n(z)$ and $\beta_n(z)$ satisfy only (A)′ but *not* (I)′ or (II)′, then the *new* form of (19) — with $\alpha_n(z)$ and $\beta_n(z)$ in place of $L_n^{(\alpha)}(z)$ and $l_n^{(\alpha)}(z)$ — will no longer represent a common solution of the three equations.

graphic correspondence exists between a solution of (I) and that of (I)″, and a similar correspondence exists also between a solution of (II) and that of (II)″. Consequently, to every *common* solution of (I), (II), there must answer a *determinate common solution* of (I)″, (II)″ and conversely (7).

If we now make a special choice of the common solution of the two functional equations (I)″, (II)″, *viz.*

$$\psi_n(z) = L_n^{(1)}(z),$$

the corresponding function $f_{2n}(z)$, as defined by (20) or (21), is, by a well-known lemma, the same as Bateman function (of the first kind) *viz.*, $K_{2n}(z)$; in fact

$$K_{2n}(z) = \frac{2 \cdot (-1)^{n-1}}{n} \cdot z\, e^{-z}\, L_{n-1}^{(1)}(2\,z) \,. \qquad \ldots \quad \ldots (22)$$

If we now take another *common* solution of (I)″, (II)″, *viz.*

$$\psi_n(z) = l_n^{(1)}(z),$$

(7) Inasmuch as (I) and (II) together lead to (A), and (I)″ and (II)″ together lead to (A)″, it follows from the above discussion that the two differential equations, *viz.*, (A) and (A)″ must be convertible into each other by an appropriate change of variables, (provided, of course, that the parametric constant n is suitably modified at the same time). In point of fact, when $(n-1)$ is put for n, (A)″ becomes

$$z \frac{d^2 u}{d z^2} + (2 - z) \frac{d u}{d z} + (n - 1)\, u = 0\,, \qquad \ldots \quad \ldots (B)$$

of which a solution must be $u = \psi_{n-1}(z)$, where $\psi_n(z)$ is any solution of $(A)''$.

If now the variables (z, u) be subjected to the trasformation:

$$z = 2\,x \quad \text{and} \quad u = k\, z^{-1}\, e^{\frac{z}{2}}\, w\,, \quad \text{where} \quad k \equiv \frac{4 \cdot (-1)^{n-1}}{n}\,,$$

it can be shown without much trouble that the differential equation (B) is ultimately carried over into

$$x \frac{d^2 w}{d x^2} + (2\,n - x)\, w = 0 \qquad \ldots \quad \ldots (C)$$

Since the two differential equations (A) and (C) are *essentially the same*, differing from one another only in having a different symbolisation for their independent variables, the connection between the original differential equations (A)″ and (A) is clear on all hands.

we can, by means of (20), introduce a new function $k_{2n}(z)$, defined formally by

$$k_{2n}(z) = \frac{2 \cdot (-1)^{n-1}}{n} \cdot z\, e^{-z}\, l_{n-1}^{(1)}(2z) . \qquad \ldots \quad \ldots (23)$$

It is crystal-clear from the foregoing discussion that the function $k_{2n}(z)$ enjoys, in common with $K_{2n}(z)$, the distinctive property of being a *simultaneous* solution of all the four equations (I), (II), (III) and (A). As premised in the Introduction, $k_{2n}(z)$ will be called Bateman function *of the second kind.*

We may now revert to the topic of Art. 2 and give a *precise* form to Prop. A by substituting $K_{2n}(z)$ and $k_{2n}(z)$ for $\alpha_{2n}(z)$ and $\beta_{2n}(z)$. The proposition accordingly assumes the following definite form:

PROP. C — *The most general common solution of the four equations (I), (II), (III) and (A) can be put in the form :*

$$f_{2n}(z) = a\, K_{2n}(z) + b\, k_{2n}(z) , \qquad \ldots \quad \ldots (24)$$

where $K_{2n}(z)$ and $k_{2n}(z)$ are the two kinds of Bateman functions and 'a' and 'b' are two arbitrary constants, independent of 'n'.

ART. 5 — We shall now give a finishing touch to the subject under discussion by constructing the « generating » function V of *any* enumerable sequence of analytic functions $\{f_{2n}(z)\}$, which satisfy the functional equation (I).

To be precise, we set

$$V = \sum_{n=0}^{\infty} h^n f_{2n}(z) , \qquad (|h| < R) , \qquad \ldots \quad \ldots (25)$$

so that

$$\frac{\partial V}{\partial h} = \sum_{n=1}^{\infty} n\, h^{n-1} f_{2n}(z) , \qquad \ldots \quad \ldots (26)$$

it being implied that R is the radius of convergence [8] of the power-series in h, *viz.*, V.

Multiplying (I) by h^{n+1} and allowing n to run through the series of values $1, 2, 3, \ldots$ and then summing the resulting relations, we get

$$\sum_{n=1}^{\infty} (n+1)\, h^{n+1} f_{2n+2}(z) + 2h^2 \sum_{n=1}^{\infty} n\, h^{n-1} f_{2n}(z)$$

$$- 2zh \sum_{n=1}^{\infty} h^n f_{2n}(z) + h^2 \sum_{n=2}^{\infty} (n-1)\, h^{n-1} f_{2n-2}(z) = 0 .$$

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[8] Manifestly R is either a function of z or else a constant.

This can, by virtue of (25) and (26), be reduced to the symbolic form:

$$\frac{\partial V}{\partial h} + P V = Q , \qquad \ldots \quad \ldots (27)$$

where

$$\left. \begin{aligned} P &\equiv - \frac{2 z}{(1 + h)^2} , \\ Q &\equiv \frac{f_2 (z) - 2 z f_0 (z)}{(1 + h)^2} . \end{aligned} \right\} \qquad \ldots \quad \ldots (28)$$

and

If we now solve (27) as a *linear* differential equation, (wherein h and V are respectively the independent and dependent variables and z is a mere parameter), and finally utilise the values of P and Q, as given by (28), we readily obtain the primitive in the form:

$$V = \frac{\{f_2 (z) - 2 z f_0 (z)\} \int\limits_0^h \frac{e^{\frac{2z}{1+t}}}{(1 + t)^2} d t + \chi (z)}{e^{\frac{2z}{1+h}}} . \quad \ldots \quad \ldots (29)$$

To determine the constant of integration, — which, in the present *case*, is but a function of the parameter z, *viz.* $\chi (z)$, — we remark that the quantities

$$V \text{ and } \int\limits_0^h \frac{e^{\frac{2z}{1+t}}}{(1 + t)^2} . d t$$

reduce to $f_0 (z)$ and 0 respectively, as soon as h is put $= 0$.

So (29) gives at once: $\chi (z) = e^{2z} f_0 (z)$.

R being readily verified to be *unity*, we can re-state the above result in the form of a proposition, *viz.*

PROP. D — *If* $\{f_{2n} (z)\}$ *be an enumerable set of analytic solutions of the functional equation* (I), *then the associated* «*generating*» *function* (V) *is given by:*

$$V \equiv \sum_{n=0}^{\infty} h^n f_{2n} (z) = \frac{\{f_2 (z) - 2 z f_0 (z)\} \int\limits_0^h \frac{e^{\frac{2z}{1+t}}}{(1 + t)^2} d t + e^{2z} f_0 (z)}{e^{\frac{2z}{1+h}}} , \quad (|h| < 1), \qquad (30)$$

it being understood that the path of integration for the lineintegral, occurring in the numerator of the right side of (30) *is an arbitrary Jordan curve, joining the origin* 0 *to an interior point ' h ' of the unit circle* $| t | = 1$ *and not passing through any singularity of any function of the set* $\{f_{2n}(z)\}$.

To account for the presence of the terms $\{f_0(z)\}$ and $\{f_2(z)\}$ in the summation-formula (30), one has to remember that (I) being a difference-equation of the *second* order, its complete solution must involve *two* disposable functions. In fact, the entire sequence of functions $\{f_{2n}(z)\}$, compatible with (I), becomes perfectly determinate, when $f_0(z)$ and $f_2(z)$ have *preassigned* (functional) values.

Thus, for istance, when we set

$$f_0(z) = K_0(z) = e^{-z} \quad \text{and} \quad f_2(z) = K_2(z) = 2\, z\, e^{-z},$$

(so that, as a matter of course, $f_2(z) - z f_0(z) = 0$, (identically)), then $f_{2n}(z) = K_{2n}(z)$ for *every* positive integer n, and (30) reduces to the *known* result, *viz.*,

$$\sum_{n=0}^{\infty} h^n K_{2n}(z) = e^{-z \cdot \frac{1-h}{1+h}}.$$

ERRATA TO ART. 4, (p. 278)

We deeply regret to have to confess that the concluding portion of Art. 4 contains an *erroneous* result. The root cause of this discrepancy is that we have overlooked the fact that Palama's function $l_n^{(\alpha)}(z)$ doet *not* exist when $\alpha = 1$ or any other positive integer. As a conseguence, the definition of $k_{2n}(z)$, by means of the equation (23), is meaningless, involving, as it does, the *illusoy* function $l_{n-1}^{(1)}(2\,z)$. Fwrthermore Prop. C, which expresses the common solution of the four equations I-III and (A) *in a special form,* involving both $K_{n2}(z)$ and $k_{2n}(z)$, is invalid, although Prop. A, which expresses the common solution of the same four equations *in the most general form,* is quite right and unexeptionable. Inquisitive readers may propose to amend the *faulty* Prop. C by a convenient choice of a handy *second* function, which like $K_{2n}^{(1)}(z)$ satisfies the tetrap of equations referred to.