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Theory of planetary systems

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THEORY OF PLANETARY SYSTEMS

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The principle involved in the ancient law of BODE (1) for the distances of the planets from the Sun, has been rejected by some astronomers because no mathematical proof of this law was ever given. In this paper we prove BODE's principle and generalize both the law and the mathematics of which it is a consequence. A mathematical theory of N. BOHR's law of stationary states in atoms is here developed. In the last section suggestions are made toward the verification of the law of frequencies of M. PLANCK.

I. - Relations between integral curves and algebraic covariants.

A field of curves, τ, to which reference will be made, is usually a narrow ensemble of some length, of curvilinear segments. As originally considered in this series of articles, τ is a field composed of the segments into which an orbit can be perturbed without losing it's stability.

We consider a differential equation,

$$(1) \qquad g(d^l r/d\theta^l,\ d^{l-1}r/d\theta^{l-1}, \ldots,\ dr/d\theta,\ r)=0,$$

the function g being assumed rational and integral in r and in the derivatives and otherwise numerical and real. Within a chosen field τ an integral curve may be approximately expressed by a parabolic equation,

$$(2) \qquad \theta = \alpha r_1^{n-1} + \beta r_1^{n-2} r_2 + \ldots + \varkappa r_2^{n-1}, \qquad (= q(r_1, r_2)).$$

From $q(r, 1)=\theta$, each θ gives $n-1$ values of r, and it follows that (2) can represent $n-1$ integral curves, each in it's field τ. Assume $\varphi(r_1, r_2)$ to be a covariant of order h of $q(r_1, r_2)$ under the transformations (real),

$$T: \quad r_1 = \mu_1 r_1' + \nu_1 r_2', \quad r_2 = \mu_2 r_1' + \nu_2 r_2', \quad ((\mu\nu) \neq 0).$$

(1) JOHANN E. BODE (1747-1826), predicted the existence of a planet (Diane) on the basis of the law in question. KEPLER and also TITUS of Wittenberg had perceived the principle in the special form in which it was used by BODE.

LEMMA I. - *The equation* (1) *can be transformed rationally so* $w=\varphi(r, 1)$ *becomes the dependent variable.*

In proof we have, generally,

$$d^k r/d\theta^k = E(\varphi^{(k)},\ldots, \varphi', w^{(k)},\ldots, w'), \qquad (\varphi^{(k)}=\partial^k\varphi/\partial r^k),$$

E being rational and $w^{(k)}=d^k w/d\theta^k$. Substitution in (1) gives an equation in which the derivatives are $w^{(k)}$, $(k=1,\ldots, l)$, with coefficients rational and integral in r, viz.,

$$A(w^{(l)}, w^{(l-1)},\ldots, w', r)=0. \tag{3}$$

SYLVESTER's dialytic eliminant, formed between (3) and the equation $\varphi(r, 1)-w=0$, to eliminate r, gives a rational equation in w as dependent variable with integral polynomials in w as coefficients, i. e., q. e. d.,

$$B(w^{(l)}, w^{(l-1)},\ldots, w', w)=0. \tag{4}$$

Since, in B, w is $\varphi(r, 1)$ instead of $\varphi(r_1, r_2)$, absolute invariance of w is for the special case, only,

$$T_1: \quad r=\alpha_1 r'+\beta_1.$$

LEMMA II. - *Integral curves of* $g=0$, *as represented by* (2), *are transformed into integral curves by* T_1.

In proof, $B=0$ is only a transformed form of $g=0$. When $T_1 q(r, 1)$ induces a transformation upon $w=\varphi(r, 1)$, both $g=0$ and $B=0$ are unaltered; but $q(r, 1)$ is transformed, hence $\theta=T_1 q(r, 1)$ represents integral curves.

LEMMA III. - *If* χ *is the numerical w-coordinate of a point* P *of an integral curve* σ *of* $B=0$, *the equation,*

$$\varphi(r, 1)-\chi=0, \tag{5}$$

is invariant.

The invariance, evidently, is for transformation of $\theta=q(r, 1)$ by T_1.

The roots of (5) are the points where h integral curves of $g=0$ cross OP or OP produced. Let P trace σ, as delimited by a τ. Then these roots s_i, $(i=1,\ldots, h)$, trace h integral curves a_i of $g=0$. When $\theta=q(r, 1)$ is transformed by T_1, σ remains fixed, or it may go into another integral curve (σ) of it's equation, $B=0$. In the first alternative the curves a_i are permuted among themselves by the operation $T_1 q(r, 1)$. In the second the curves respectively are sent forward to coincidence with those of another a_i-set, in general in a permuted order, (cf. (30) *et seq.*).

The differential equation next to be considered, a special case of $g=0$, is the known equation of central plane orbits, the force at the center $O(0, 0)$ being the arbitrary $F(r)$, $(=F_1(r)/m)$, r being the distance and m the mass of the planet N, viz.,

(6) $$r\frac{d^2r}{d\theta^2}-2\left(\frac{dr}{d\theta}\right)^2-r^2=-r^5F(r)/\gamma^2.$$

Any integral curve of (6) will satisfy KEPLER's law of areas, but the curve is a stable orbit only if

(7) $$F(r)=C[2p(r)^2-rp(r)p'(r)+r^2/\lambda^2]/r^5, \qquad (p'=\partial p/\partial r),$$

$p(r)=ar^{n-1}+br^{n-2}+\dots+k$, $C=\gamma^2\lambda^2$, where all letters excepting r are constants. The general integral of (6) is known to be expressible as

(8) $$\theta+c=\varDelta(r, m, s, v, \beta), \qquad (R_3, \text{ p. } 30) \quad [2],$$

c being arbitrary and (s, v, β) a dependent set where s is the distance from O to an initial position I of N, v the initial velocity and β the angle between OI and the vector v.

The covariant in the special case is $F(r)$, $(=w=\varphi(r, 1))$. It is shown to be invariant for the particular case of T_1, S_z': $r=r'/z$, (Cf. (24)). If we expand $F(r)$ to a polynomial of order $2n-2$ in r, the invariant equation (5) becomes (9) below.

Suppose a to be a curve (8) in τ and σ the corresponding uniquely determined curve of the force (an integral curve of $B=0$). As N passes a point $Q(r, \theta)$ of a, the corresponding point P (on OQ produced) of σ, is such that w of P is the force which O exerts upon N at the instant when N is passing through Q. Theorem 1 now follows. The curves C_i are curves a_i of Lemma III and χ has it's former interpretation.

THEOREM 1. - *If* $g=0$ *is the equation* (6) *and* σ *a uniquely selected integral curve of the corresponding equation* $B=0$, *there will be as many curves* C_i *appropriate for orbits of* N *and having* σ *as their common force-curve, as there are real positive roots of the invariant equation* $F(r)=\chi$, *viz.*,

(9) $$A_0r^{2n-2}+A_1r^{2n-3}+\dots+(A_{2n-7}+C^{-1}\chi)r^5-Lr^3-Mr^2-Ur-V=0.$$

THEOREM 2. - *If* m *is fixed, all curves* C_i *of Theorem 1 are separated, no two are consecutive geometrically.*

In proof, a relation between an orbit C_i in a τ, and any curve K_i into which C_i has been perturbed, can be written in the form,

$$T': \quad \theta'=\theta, \quad r'=r+s_0p(r), \quad (s_0 \doteq 0), \quad (\text{Cf. } p, \text{ in (7), and } R_1, \text{ p. } 301).$$

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[2] We refer (in the text) to R_1, R_2, R_3, the following memoirs respectively: GLENN, Annali della R. Scuola Normale Superiore di Pisa, ser. 2, vol. 2 (1933-XI), p. 297; vol. 4 (1935-XIII), p. 241; vol. 6 (1937-XV), p. 29.

We form the time-derivatives in T', and multiply through by the mass m and obtain,

$$(10) \qquad F(r') = (1 + s_0 p'(r))F(r) + 2s_0 p''(r)S(r), \qquad (p' = \partial p/\partial r),$$

where $S(r)$ is the formula of instantaneous energy-potential, or vis viva, of the planet N,

$$S(r) = \frac{m}{2}(dr/dt)^2, \qquad (t = \text{time}).$$

The direction of the work dS is along r. The force which O exerts upon m at a point (r', θ') of K_i is greater or less than that exerted upon m at the corresponding point (r, θ) on C_i, according as

$$(11) \qquad G \equiv s_0 p'(r) + 2s_0 p''(r)R > 0, \quad \text{or} \quad G < 0, \qquad (R = S(r)/F(r)),$$

s_0 being taken positive when K_i is on the side opposite C_i from the origin. If $G > 0$ (3), and K_i is outside of C_i, the orbit is stable if the increase, in $F(r)$ along r, from C $(= C_i)$ to K_i, exceeds the increase, from C to K_i of $\Psi =$ (momental force + centrifugal force). If the perturbation of C is inward (s_0 negative), the decrease in $F(r)$ must exceed the decrease in Ψ. These are the conditions for the property of self-restitution characteristic of a stable orbit.

If $G \leqslant 0$, so $F(r') \leqslant F(r)$ when K_i is outside of C, the property of stability is that the decrease in $F(r)$, along r from C to K_i should be less than the decrease of Ψ from C to K_i. If K_i is inside of C the increase in $F(r)$ from C to K_i must be less than the increase of Ψ from C to K_i.

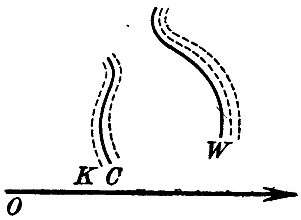

FIG. 1.

The pair of cases $G > 0$, $G \leqslant 0$ may be included in a single statement as follows: Let W (called the effective force) be the component, along the radius, of $F(r)$, the momental force, and the centrifugal force. Then if a stable orbit C is perturbed, inward or outward, the curve of W is always perturbed outward (Fig. 1) (4). However, q. e. d., this property would not hold if, for C, we take a K_i.

Another type of conclusion is obtained by solving $G < 0$ for r. If $n = 4$ the result is

$$(12) \qquad r < \{-(b + 6aR) \pm \sqrt{[(b^2 - 3ac) + 36a^2R^2]}\}/3a.$$

Assume R to be very small in absolute value. Then the limiting value of r for stability of motion of m upon C, in the case $G < 0$, approximates the smallest number of the pair,

$$[-b \pm \sqrt{(b^2 - 3ac)}]/3a.$$

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(3) When $G > 0$ it is not to be assumed that it is very much larger, since $s_0 \doteq 0$.

(4) This fact is related to a property of metals which become harder when deformed by pressure.

Either sign in the latter formula may give a very small limit. Hence,

LEMMA IV. - *When the ratio of the central force to the formula of vis viva, $S(r)$, of the rotating body is very large, a stabilizing condition, for example $G<0$, may compel the radial distance r of N to be, as a compensation, exceedingly small. This is the case within an atom.*

The equation of the orbit of the electron in the atom of hydrogen, near aphelion, is

(13) $$r=v\tan(e\theta+\beta)-u,$$

(14) $$\left\{ \begin{array}{l} [p=ar^2+br+c,\quad n=3,\quad e=m_1/2\tau_0,\quad m_1=\sqrt{(4ac-b^2)},\quad u=b/2a, \\ \qquad\qquad v=m_1/2a,\quad \beta=-e\delta_0], \end{array} \right.$$

this equation being the integrated form of

$$\theta=\tau_0\int dr/p+\delta_0.$$

The number $-u$ is positive and small, and v negative, and $|v|$ very small in comparison with $|u|$.

With $n=3$ the relation $G<0$ gives

(15) $$r<-b/2a-4R+\varepsilon^2(\theta),\qquad (\varepsilon \doteq 0,\ a \text{ positive}).$$

Thus, a necessary and sufficient condition in order that r of (15) should be appropriate to the hydrogen atom is that $1/R$ should be very large in absolute value.

II. - Energy, and a hypothesis of Laplace.

The historical, but hypothetical, concept of an ether undoubtedly allows « properties » of ether which are inexplicable (not merely unexplained). Any conclusion derived by experiment upon such a « substance » is subject to question as to it's validity, on philosophical grounds. Accordingly, in our reasoning about energy, we shall not speak of waves in the ether. Energy will be considered only in it's association with mass.

We enlarge the concept of the field Σ of potential of a material body b which is at rest within a region S, to include all forms of energy which exist in Σ because of the presence of b. If b begins to move in S, there will be a transformation of Σ since an energy M (kinetic) of momentum of b in S then intervenes. But M is the capacity for doing work of b's momentum $m(ds/dt)$, in S, (s = distance), measured by

(16) $$M=\frac{d}{dt}[m(ds/dt)],$$

and is the instantaneous rate of change of the momentum. At the instant, momental energy M is a component [5] of the field of b, which is still a field Σ, to be designated by Σ_M and regarded as the transformed of the original field Σ of b. From this point of view there is but one type of energy, that of a field Σ associated with a mass b at an instant of time.

A mechanical system (automatic machine) is to mean here any set of related objects which, within itself, is capable of doing work in response to an associated energy E, and if E is all confined within the system, the system is said to embody a level of energy. Least action and least work are both involved in energy-levels. If a mechanical system at rest has energy introduced and acts thereafter as a level and an automatism, the general motion to which it converges, under the principle of least work, is that in which the energy-field Σ_i associated with any object in the system, remains as near constant as the structure and action of the system permits. It follows from this principle alone that planetary motions in a solar system, with the lapse of time, will tend toward the circular form [6]. Conditions in the environment of a mechanical system may be variable, and such that the system will converge to different ultimate motion-types at different times. Most energy-levels in nature lose energy by leakage, for instance in the form of radiation, but atoms of elements normally are exceptions.

The essential fact of LAPLACE's hypothesis is that a solar system evolves by the concentration to a center O, along infinite spiral paths, of nebular material, small particles, (R_3, p. 34). It is as important to note that much of the energy which will be associated with the system, flows to O in association with the nebular masses. When a sufficiently large concentration has reached O, a force function of the form $F(r)$ becomes dominant, particles at a sufficient distance retreat because O has become repellent for particles and the mechanical system becomes a level of energy. The normal situation then is that planetary nebular masses of large sizes will have formed and will be in rotation around a central mass or sun. A good example of this stage of the evolution is the circular nebula Messier ($M5$), in the constellation *Canes Venatici*.

Within a field τ, the orbit of a typical planetary nebular mass will be of the form, (R_3, p. 30),

$$\theta = \sum_{i=1,\dots,n} (\nu_{1i} m^{q-1} + \nu_{2i} m^{q-2} + \dots + \nu_{qi}) r^{n-i}. \tag{17}$$

In order that (17) should represent a stable orbit it is necessary and sufficient that $F(r)$, as determined by substituting in (6) from (17), should reduce to the functional form (7) and it will if certain $n-2$ rational relations exist between

(5) Cf. EDDINGTON: *The Mathematical Theory of Relativity* (1923), p. 135.

(6) KRALL: *État limite, etc.* Proc. Internat. Math. Congress, Zürich (1932), vol. 2, p. 258.

the coefficients $\nu(m)$, $\xi(m), \ldots, \varrho(m)$, of the respective powers of r in (17). For example, when $n=5$, the relations are,

$$(18)\quad \begin{cases} C_{15} \equiv 4(16O^4\nu - 36\nu\xi O^2\pi + 16\nu^2 O\pi^2 + 9\nu\xi^2\pi^2)/\pi^5\lambda = 0, \\ C_{25} \equiv (48O^4\xi - 108\xi^2 O^2\pi + 96\nu\xi O\pi^2 + 27\xi^3\pi^2 - 32\nu O^3\pi - 16\nu^2\pi^3)/\pi^5\lambda = 0, \\ C_{35} \equiv (32O^5 - 96\xi O^3\pi + 48\nu O^2\pi^2 + 54\xi^2 O\pi^2 - 24\nu\xi\pi^3)/\pi^5\lambda = 0. \end{cases}$$

When $n=4$ they are,

$$C_{14} \equiv [C_{15}/\nu]_{\nu=0} = 0, \qquad C_{24} \equiv (OC_{15}/2\nu - C_{35})_{\nu=0} = 0.$$

The relations exist within the degree of approximation of the coincidence of

$$\int dr/p(r) = \lambda\theta + \mu,$$

(in the form (17)), with the orbit's segment in τ.

III. - The law of Bode and it's generalization.

By theory of the equations (2), (17), the $(n-1)$ inner orbits of a solar system, each within a specific perturbational field τ, have a universal equation,

$$(19)\qquad \theta = \nu r_1^{n-1} + \xi r_1^{n-2} r_2 + \ldots + \sigma r_2^{n-1}, \qquad (= f(r_1, r_2)).$$

When it's coefficients (numerical) are substituted in known relations which connect $\nu, \ldots, \varrho$ with the respective $a, \ldots, k$, then $F(r)$ becomes a numerical function of r. When $n=5$ the relations in question are,

$$(20)\quad \begin{cases} a = (16O^4 - 36\xi O^2\pi + 16\nu O\pi^2 + 9\xi^2\pi^2)/\pi^5\lambda, \\ b = (-8O^3 + 12\xi O\pi - 4\nu\pi^2)/\pi^4\lambda, \\ c = (4O^2 - 3\xi\pi)/\pi^3\lambda, \\ d = -2O/\pi^2\lambda, \qquad e = 1/\pi\lambda. \end{cases}$$

To obtain the equations for $n=4$ we change a, b, c, d to $0, a, b, c$, respectively, and ν, ξ, O, π to $0, \nu, \xi, O$, in the last four relations for $n=5$, (R_3, p. 32).

In order that $F(r)$, thus evaluated, may be an attraction r must not be too large and $|\varrho|$, (the coefficient next to the last in (19)), not too small.

A set of generators of T is the following:

$$(21)\quad \begin{cases} T_y: & r_1 = \eta + y\zeta, \quad r_2 = \zeta; \\ S_z: & r_1 = \eta, \quad r_2 = z\zeta, \quad (z \neq 0); \\ W: & r_1 = -\zeta, \quad r_2 = \eta. \end{cases}$$

We also employ the nonhomogeneous transformation,

$$S_z' : \quad r = r_1/r_2 = \eta/z\zeta = r'/z.$$

We have,

$$S_z' f(r, 1) = f'(r', 1),$$

where

$$\nu' = \nu/z^{n-1}, \quad \xi' = \xi/z^{n-2}, \ldots, \quad \varrho' = \varrho/z, \quad \sigma' = \sigma, \quad r' = zr,$$

hence, for n general, the relations corresponding to (20) give

$$(22) \qquad r' = zr, \quad a' = a/z^{n-2}, \quad b' = b/z^{n-3}, \ldots, \quad i' = i/z, \quad j' = j, \quad k' = zk.$$

THEOREM 3. - *The force-function $w = F(r)$ of (7) is a relative covariant when $\theta = f(r, 1)$ is transformed by S_z'.*

The statement of the theorem is readily verified by substituting from (22) in,

$$(23) \qquad F'(r') = C(2p_1^2 - r'p_1p_1' + r'^2/\lambda^2)/r'^5, \qquad (p_1' = \partial p_1/\partial r'),$$
$$(p_1 = a'r'^{n-1} + b'r'^{n-2} + \ldots + k').$$

The result is,

$$(24) \qquad F'(r') = F(r)/z^3.$$

The formula which defines force is a function of two variables, mass and acceleration. Hence we can make the relation of invariancy (24) absolute whenever it is permitted to alter independently either the unit of mass or the unit of acceleration.

The transformation of $\theta = f(r, 1)$ by $S_z'^k$ multiplies the radial coordinate w by $1/z^{3k}$, $(k = 1, 2, \ldots)$, while we have, in (9), the equations, $(k = 1)$,

$$(25) \qquad \begin{cases} r' = zr, \quad \chi' = \chi/z^3, \quad A_i' = A_i/z^{2n-i-4}, & (i = 0, \ldots, 2n-7), \\ L' = L/z, \quad M' = M, \quad U' = Uz, \quad V' = Vz^2. \end{cases}$$

The first relation (25) shows that the roots of (9) are all multiplied by z. The rest of (25) shows that (9) is an equation of such a special nature that it remains invariant when $\theta = f(r, 1)$ is transformed by S_z'. Hence the roots of (9) are permuted by this transformation. Iterating S_z' we therefore obtain only a finite number of permutations of the $2n-2$ roots by means of a transformation of infinite order. This is a paradox. It is explained however by noting that (9) does not terminate at the left with any specific A_α. It would become an infinite series with the indefinite increase of n, except that the upper coefficients converge to zero, within the approximations, thus terminating the series before n reaches a very large value.

If s_1 is the least positive root, and zs_1 the second in magnitude,

$$(26) \qquad \Sigma(s_1, z): \quad s_1, \quad zs_1, \quad z^2 s_1, \dots, \quad z^g s_1, \dots,$$

is a sequence of roots. Let s_{j+1} be the least positive root not in $\Sigma(s_j, z)$, $(j=1, 2, \dots)$. Then all numbers,

$$\Sigma(s_{j+1}, z): \quad s_{j+1}, \quad zs_{j+1}, \dots, \quad z^h s_{j+1}, \dots,$$

are roots and the sequences are mutually exclusive if z is chosen so no z^β equals a quotient s_i/s_j. All positive roots of (9) are represented by $\Sigma(s_j, z)$, $(j=1, 2, \dots)$.

Since the roots of (9) are intersections of integral curves of (6) with a chosen radial line, it follows, q. e. d.,

THEOREM 4. - *If s_1 is the mean distance from the sun of the innermost planet of a solar system and z is selected so zs_1 is the mean distance to the second planet, theoretically there exist planets at all distances $z^k s_1$, $(k=0, 1, 2, \dots)$, until the outer edge of the energy-level is reached.*

This is a general form of BODE's law. We next prove that if any stable orbits having σ as their force-curve intersect OP in respective points s_j' exterior to the set of points representing the positive roots of (9), these exterior points are also represented by numbers in $\Sigma(s_j, z)$ when the latter is regarded as an algebraic formula in two arbitrary parameters. For example when the roots $\Sigma(s_1, z)$ are the mean distances of the planets in our solar system, the distances of the asteroids may give exterior points.

Assume the roots of $f(r, 1)=0$ to be s_1, $zs_1, \dots, z^{n-2}s_1$. Since all are roots of (9), $f(r, 1)$ will be termed an inside quantic. Let Σ' be any set of $n-1$ values of r corresponding to intersections with OP of stable orbits for σ, which includes at least one point s_j'. Then Σ' is the set of roots of a quantic $f_1(r, 1)$ which we designate as an outside quantic. Let $\Gamma_{in}=0$, $(i=1, \dots, n-2)$, be the conditions for stability of $\theta=f_1(r, 1)$, Γ_{in} being the same function of the coefficients of f_1 that C_{in} is of the coefficients of f. Then the set Γ_{in} vanishes because the set C_{in} vanishes, (both groups of curves, $\theta=f$, $\theta=f_1$ being orbits of $F(r)$). However C_{i+1n} and C_{in} are of different weights, and if we begin with $n=3$ where there is only one C_{in}, and Γ_{in}, and proceed inductively, it is seen that, respectively, $\Gamma_{in}=0$ because $C_{in}=0$, $(i=1, \dots, n-2)$, hence,

$$(27) \qquad \Gamma_{in}=\Omega_i C_{in}, \qquad (i=1, \dots, n-2),$$

Ω_i being undetermined. Therefore $\theta=f_1(r, 1)$ is the transformed of $\theta=f(r, 1)$ by a linear transformation which leaves the conditions for stability invariant, as in (27). Of the generators of T, however, only S_z (here S_ξ'), induces a transformation which leaves the C_{in} invariant. Hence $f_1(r, 1)=f(r/\xi, 1)$, and is an inside quantic, contrary to hypothesis, if ξs_1 is a root of (9); otherwise the roots of $f_1(r, 1)$ are in $\Sigma(\xi s_1, z)$, i. e. in $\Sigma(s_j, z)$ as an algebraical formula, q. e. d.

If the unit of length is chosen as s_1, (26) is

$$\Sigma(1, z): \quad 1, \quad z, \quad z^2, \ldots, \quad z^k, \ldots$$

Then, in restoring the arbitrariness of the unit, we may adjoin the transformation which expands the point-circle, $(0, \theta)$, into the concentric circle of radius $-Q_0/P_0$, thus obtaining, (Cf. Lemma V),

$$r' = P_0 r + Q_0, \qquad (P_0, \ Q_0 \text{ constant}, \ r' = z^k).$$

Rules for practical use of BODE's principle are now as follows. [1] Measure in any convenient unit the respective mean distances q_1, q_2, q_3 of the three innermost orbits of a solar system. [2] With $q_2/q_1 = z$ we then have BODE's formula in the form,

$$(28) \qquad P_0 r = (q_2/q_1)^k - Q_0, \qquad (k = 0, 1, \ldots, N_0).$$

[3] Substitute in the latter equation, successively, $(k=0, r=q_2)$, $(k=1, r=q_3)$, determine P_0, Q_0, and this gives a numerical formula for r in a numerical case and the following in the general case,

$$(29) \qquad r = \frac{q_1(q_3 - q_2)}{q_2 - q_1}(q_2/q_1)^k + \frac{q_2^2 - q_1 q_3}{q_2 - q_1}, \qquad (k = 0, 1, \ldots, N_0).$$

Each value of k gives the position of a planetary orbit. Since the three q_i in (29) remain arbitrary, any three (inner) curves known to exist as orbits of respective planets, determine a bodeian cycle. Thus our zone of asteroids probably constitutes a bodeian cycle. This can be verified when the various mean distances have been found by astronomical measurement.

For planets of our sun the formula (28) was identified by BODE in the form $z = q_2/q_1 = 2$, $P_0 = 1/3$, $Q_0 = -4/3$. It is shown in Table I that the present method improves these values.

Relations to the nebular hypothesis. - LAPLACE assumed that the planets originally were nebular and diffuse. We have assumed that the motions of these nebular masses were stable and therefore that the central force (in any system) was a definite $F(r)$ (Cf. (7)). Both hypotheses are here verified since the tables below show that all satellites of Sun, Saturn and Jupiter, (except for some disarrangement of two of Jupiter's outer moons), are such that the positions now occupied by discovered satellites, after the latter have undergone condensation and altered their attractions to the newtonian formula of inverse squares, are the proper positions, under the general bodeian principle, for the nebular planets corresponding to $F(r)$, the approximations being all close. The satellites of the Sun, of Saturn, and Jupiter have been able to maintain these positions of their orbits, while condensing to their present forms. We therefore have important elements of proof that the nebular hypothesis of LAPLACE is true.

The following tables give the respective mean distances of the satellites of our Sun, of Saturn, and Jupiter, as computed from the formula (28), and as compared with the distances obtained by astronomical measurement.

We have completed the list of names of these satellites. This seemed desirable although perhaps the numerical notation will be the one adopted eventually for the asteroids.

The variations from the accurate bodeian positions, shown by the moons Dirce and Eurydice, of Jupiter, are probably due to the disturbing influence which was introduced when the adventitious « eighth » moon was captured by Jupiter's gravitational field. Both Phoebe in Saturn's system and Ptolemaios in Jupiter's, have retrograde motion. In our theory of a finite field, τ, the direction of motion of a planet, in it's orbit, is immaterial.

The planets of our solar system.

Unit of distance, one tenth of the earth's mean distance from the sun. $z=q_2/q_1=7.2/3.9$; (Revised to $z=1.93$). Formula,

$$.332143r=(1.93)^k+1.391429, \qquad (N_0=8).$$

The mean distance from the sun, of the newly-discovered planet Pluto is equivalent to 5,419,415,000 miles.

TABLE I

Name of planet	Measured mean distance	Distance by formula
Mercury	3.9	3.73
Venus	7.2	7.2
Earth	10.0	10.0
Mars	15.2	15.4
Diane	26.5	25.8
Jupiter	52.0	46.0
Saturn	95.4	84.8
Uranus	191.8	159.8
Neptune	300.5	304.5
Pluto	—	583.8

The moons of Saturn.

Unit of distance, the equatorial radius of Saturn, (37500 miles). $z=3.94/3.07$; (Revised to $z=1.3$). Formula,

$$.32258r=(1.3)^k+.27097, \qquad (N_0=16).$$

PICKERING announced that he had found a tenth moon, Themis. The discovery has not been verified.

TABLE II

Name of moon	Measured mean distance	Distance by formula
Mimas	3.07	3.03
Enceladus . . .	3.94	3.94
Tethys	4.87	4.87
Dione.	6.25	6.08
Rhea	8.73	7.65
Lucretia	—	9.69
Clio.	—	12.35
Minerva	—	15.80
Titan	20.22	20.29
Hyperion. . . .	24.49	26.13
Terpsichore . .	—	33.71
Faustina	—	43.58
Japetus.	58.91	56.39
Clymene	—	73.06
Kala	—	94.73
Theresa	—	122.89
Tyche	—	159.51
Phoebe	214.24	207.12

The moons of Jupiter.

Unit of distance, the mean radius of Jupiter, (43250 miles). $z=5.93/2.55$; (Revised to $z=1.7$). Formula,

$$.19943r=(1.7)^k+.18262, \qquad (N_0=10).$$

The exceptional « eighth » moon of Jupiter, (considered to be a captured asteroid), has eccentricity .4, inclination greater than 30°, retrograde motion, mean distance 337.58.

TABLE III

Name of moon	Measured mean distance	Distance by formula
Giulia.	2.55	3.49
Io	5.93	5.93
Europa	9.44	9.44
Ganymede . .	15.06	15.41
Callisto	26.49	25.55
Daphne.	—	42.80
Dauphine. . . .	—	72.11
Dirce	164.16	121.94
Eurydice	168.79	206.67
Ptolemaios . . .	346.82	350.70
Xeipe	—	595.55
Krishna	—	1011.79

IV. - The law of N. Bohr ([7]).

When, under appropriate circumstances, a PLÜCKER tube is attached to an induction coil, the central force (Cf. (7)), $F(r)$, within an atom of the gas in the

([7]) NIELS H. D. BOHR: *The Theory of spectra and atomic constitution*. (Transl. Camb., 1922).

tube, is altered by the occurrence of small variations of the numbers ν_{ki}, λ (Cf. (17) and R_3, p. 34). The physical counterpart of these variations will be perturbations of any specific electronic orbit in the atom in question. If the exciting force is so far increased that $C_{in} \neq 0$, for some i, this will be because of a decrease of the denominator $[\varrho(m)]^n$ of C_{in} to the same or a higher order of smallness than that of the numerator. Then, catastrophically, the electron becomes unstable upon it's original orbit q and seeks a new orbit q' upon which the conditions for stability, $C_{in}=0$, are automatically restored. To determine events, note that, in view of the algebraic uniqueness of $\Sigma(s_j, z)$, (Cf. (26) *et seq.*), as the formula for planetary distances, any orbit to which the electron can go and remain stable, is a transform, in $\theta=f(r, 1)$, of q by an $S'_{\xi z^e}$, which transformation leaves the C_{in} invariant.

Hence the new orbit is one of a bodeian set, which may contain q, or not, sufficiently distant from O that, in the invariant relations,

$$(30) \qquad C'_{in}=C_{in}/(\xi z^e)^{e_i}, \quad (i=1,\ldots, n-2;\ e,\ e_i \text{ positive integers}),$$

corresponding to the transformation of $\theta=f(r, 1)$ *by* $S'_{\xi z^e}$, *the* $(\xi z^e)^{e_i}$ *in the denominators annul all functions* C_{in} *which were not already zero at the instant of instability. This proves the principle of stationary states.*

Moreover, the electron, in it's transition from q to q', can remain on a stable path. This results from the following argument.

LEMMA V. - *With* a_0 *limited above, we can transform the orbits in* τ *by* D: $r=r'+a_0$ *without destroying the property of stability.*

Since the stable orbits in τ are given by the equation,

$$(31) \qquad \int dr/p(r)=\lambda\theta+\mu, \qquad (\lambda, \mu \text{ constant}),$$

the transformation D leaves the form of their equation invariant, q. e. d.; however, if a_0 is too large, T' would not be suitable, after the transformation D, to carry an orbit into a perturbed orbit, and this would cause a contradiction of the general theory of $F(r)$.

We observe next that, if $F(r)$ approximates closely to the formula of the inverse cube, H/r^3, the general integral of (6) is

$$r=1/(Je^{\delta\theta}+Ke^{-\delta\theta}), \qquad (=X(\theta)).$$

The doubly spiral integral curve is all in the finite plane with the $(u-1)$-st circuit (measured from $\theta=0$) intersecting the u-th, if

$$K/J=e^{(4u-2)\delta\pi}, \qquad (e=2.71828\ldots,\ \pi=3.14159\ldots).$$

It follows that $r = X(\theta) + a_0$, the spiral curve in Figure 2, is stable, with $F(r)$ equal to a particular form $F_1(r)$ considerably different from H/r^3. Assume that the exciting force, in the example of the PLÜCKER tube, causes the original force, $F(r)$, to assume the form $F_1(r)$ while passing to the ultimate functional form, of the force, in the new level of energy.

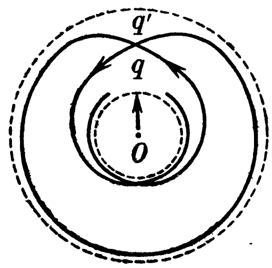

FIG. 2.

Then the electron on q which originally approximated to the inner dotted circle of radius a_0, retreats to it's new orbit q' in approximate coincidence with the outer dotted circle, but it makes the transition along the stable spiral path. Our interim spiral is a special, but typical case.

With the restoration of the original energy-level, the electron may return along the same spiral but usually the hour-hand (polar axis) will have rotated between the two catastrophic instants.

Remark concerning Planck's law of frequency. - Any astronomical planet or solid sun has a gravitational field such that concentric spaced layers of gaseous and other particles will form around it, (KENNELLY and HEAVISIDE), (R_3, p. 36). The successful hypothesis that a solar system is a model of an atom of a chemical element therefore gives a logical basis for assuming that an electron, in rotation within an atom, is surrounded by HEAVISIDE layers. On the basis of such an inner mechanism conclusions about radiation follow; however perhaps experimentation should precede the development of a mathematical theory. We only mention two facts.

[1]. If an electron, surrounded by layers, is shot through a gold foil it would necessarily leave an impression like interference bands. This effect has already been identified, first in the laboratory of J. J. THOMSON.

[2]. The formula $W_2 - W_1$, (W_2, W_1 being the numerical measures of the energy-field Σ of the electron, as Σ exists upon two respective BOHR orbits C_2, C_1), can be evaluated in terms of the sum of all numerically represented quants of energy which, as the electron falls from C_2 to C_1, are projected outward from the Heaviside layers in association with material particles. It can be shown that frequencies develop in the projected energy. These considerations give an approach to a proof of PLANCK's law.