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ANGUS E. TAYLOR

Analytic functions in general analysis

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ANALYTIC FUNCTIONS IN GENERAL ANALYSIS

by ANGUS E. TAYLOR (California - U. S. A.).

1. - Introduction. — Since the publication of FRÉCHET's thesis in 1906 the functional calculus, or abstract theory of functions, has made substantial progress in the abstraction of various portions of classical analysis. FRÉCHET seems to have been one of the first to develop the theory of polynomials from an abstract point of view, and GATEAUX combined the notion of differential and the notion of functional polynomial to generalize the CAUCHY-WEIERSTRASS theory of analytic functions. HILDEBRANDT and GRAVES have contributed to the generalization of TAYLOR's theorem and theorems on implicit functions and differential equations (1).

In this paper I propose to develop the theory of analytic functions along the lines indicated by GATEAUX (2); That his work is susceptible to thorough abstraction was pointed out by L. M. GRAVES (3). I have, however, some further results on generalizations of the CAUCHY-RIEMANN equations, and on the singularities of abstract analytic functions. RIEMANN's theorem concerning removable singularities may be generalized, and in certain cases functions may be characterized in terms of their poles by MITTAG-LEFFLER's theorem. Some interesting departures from classical theory are displayed by examples.

2. - Postulates and Definitions. - We shall use $E, E', \ldots,$ to denote vector spaces as defined by BANACH (4), and the norm of an element x of such a space will be written $\|x\|$. $E(R)$ denotes a *real* vector space, that is, one for which the multiplier domain is the real number system, and $E(C)$ denotes a complex space.

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(1) See the survey paper of L. M. GRAVES, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 641-662. Further references are given in this paper.

(2) R. GATEAUX, Bullétin de la Société Math. de France, vol. 47 (1919), pp. 70-97, and vol. 50 (1922), pp. 1-21.

(3) GRAVES, loc. cit., pp. 651-653. My own work on this subject was done indepéndently. starting from a quite different point of view which will be mentioned in a later paragraph, Valuable suggestions about connecting my work to that of GATEAUX were made to me by Professor A. D. MICHAL.

(4) S. BANACH, Fundamenta Mathematicae, vol. 3 (1922), pp. 133-181.

From a real space a complex space may be constructed as follows: $E(C)$ shall consist of elements $\{x, y\}$, where x and y are in $E(R)$. We define

$$\{x_1, y_1\} = \{x_2, y_2\} \text{ if and only if } x_1 = x_2,\ y_1 = y_2$$
$$\{x_1, y_1\} + \{x_2, y_2\} = \{x_1 + y_1, x_2 + y_2\}$$
$$(a + i \cdot b) \cdot \{x, y\} = \{a \cdot x - b \cdot y, b \cdot x + a \cdot y\}$$
$$\|\{x, y\}\| = (\|x\|^2 + \|y\|^2)^{1/2}$$

and with these definitions it is clear that $E(C)$ is a complex vector space. Since $\{x, y\} = \{x, 0\} + i \cdot \{y, 0\}$, and since the correspondence $x \rightarrow \{x, 0\}$ sets up a one to one isomorphism between $E(R)$ and a subclass of $E(C)$, we may for convenience write, as we do with complex numbers, $z = \{x, y\} = x + i \cdot y \cdot E(C)$ is complete if and only if $E(R)$ is complete, and a variable quantity in $E(C)$ will approach a limit if and only if its « real » and « imaginary » parts do likewise. We call $E(C)$ the couple-space associated with $E(R)$.

It is convenient to adopt the following standard definitions. *Domain* (5). - An open point set in E.

Region. - A domain plus some, all, or none of its boundary points.

Sphere (Open or closed). - A set of points defined by $\|x - x_0\| < r$ or $\|x - x_0\| \leq r$; x_0 is called the center of the sphere and r its radius.

Compact Set. - A set of points in E such that every infinite subset gives rise to at least one limit point in E.

3. - Preliminary Theorems. - We shall recall briefly some of the fundamental propositions pertaining to the FRÉCHET and GATEAUX concepts of differential. In this paragraph we shall also discuss a few properties of compact sets which we shall require further on in the paper.

Definition. - Let $f(x)$ be a function on E to E', defined in the neighborhood of a point x_0. If for each y in E the difference quotient

$$\frac{f(x_0 + \tau y) - f(x_0)}{\tau}$$

approaches a limit as the number τ tends to zero in any manner whatsoever, the limit is called the GATEAUX differential, with increment y, of $f(x)$ at x_0, and we denote it by $\delta f(x_0; y)$.

It is assumed that the reader is familiar with the notion of a FRÉCHET differential (6). The following important propositions are noted without proof.

(5) It is sometimes convenient to use domain to mean an open, *connected* point set, particularly when we are interested in analytic continuation of a function. However, it is unnecessary at this point to introduce the definition of connectedness.

(6) M. FRÉCHET, Annales de l'École Normale Supérieure, vol. 42 (1925) pp. 293-323. For a compact resumé of the FRÉCHET differential and its properties see pp. 649 (conditions D_2 and D_4), of the paper by GRAVES referred to above.

THEOREM 1. - *If $f(x)$ on E to E' is defined in the neighborhood of x_0 and has a Fréchet differential at x_0, then $f(x)$ is continuous there.*

THEOREM 2. - *If $\varphi(a)$ is a function of the numerical variable a, to the space E, such that $\varphi(a)$ has a derivative at a_0, and if $f(x)$ on E to E' admits a Fréchet differential at x_0, where $x_0=\varphi(a_0)$, then $f(\varphi(a))$ has a derivative at a_0, and*

$$\left\{\frac{\partial f(\varphi(a))}{\partial a}\right\}_{a=a_0}=df(x_0;\,\varphi'(a_0))$$

where $df(x_0; y)$ is the Fréchet differential.

THEOREM 3. - *If $f(x)$ admits a Fréchet differential at x_0 it admits a Gateaux differential there and the two are equal. Consequently the Fréchet differential of a given function is unique.*

THEOREM 4. - *Let $f(x)$ be defined and continuous in a closed set H of E, with values in E'. Then $f(x)$ is bounded and uniformly continuous in every compact set G extracted from H.*

Theorem 4 was enunciated by GATEAUX in the second memoir cited above. His proof is readily adapted to the abstract spaces with which we are concerned. Gateaux also makes use of the following theorem.

THEOREM 5. - *Let $\{f_n(x)\}$ be a sequence of functions defined and continuous in a domain D of E, with values in E'. Let $f(x)$ on D to E' be a function such that $\lim_{n\to\infty} f_n(x)=f(x)$, the convergence being uniform in every compact subset of D. Then $f(x)$ is continuous in D.*

Finally, to avoid repetitions of similar arguments, we demonstrate the proposition:

THEOREM 6. - *Let $f(x, y)$ be a function with values in a space E_3, defined for x in a domain D of a space E_1, and y in a closed set F of a space E_2. Then if x_0 is in D and G is a compact set in F, $f(x, y)$ is continuous at x_0, uniformly with respect to y in G.*

Proof: Suppose the theorem false. Then there will exist a number $\varepsilon>0$, elements x_n in D, and elements y_n in G such that the inequalities

$$||x_n-x_0||<1/n \qquad ||f(x_n, y_n)-f(x_0, y_n)||>\varepsilon$$

are valid when $n=1, 2,\dots$. Since G is compact and contained in the closed set F we may suppose that the points y_n converge to a point y_0 in F. Then the inequality

$$0<\varepsilon<||f(x_n, y_n)-f(x_0, y_n)||\leq||f(x_n, y_n)-f(x_0, y_0)||+||f(x_0, y_0)-f(x_0, y_n)||$$

holds for $n=1, 2,\dots$. But by the continuity of $f(x, y)$ the right member tends to zero with $1/n$, and we are led to a contradiction.

4. - Analytic Functions of a Complex Variable. — A substantial portion of the theory of functions of a complex variable remains valid when the function values are not necessarily numerical, but are assumed to lie in a complex, complete vector space.

A function $f(a)$ of the complex variable a, with values in a complex, complete space E is said to be analytic in a domain T of the complex plane if it has a derivative at each point of T. It is analytic at a point a_0 if it has a derivative at each point of some neighborhood of a_0. This is the usual definition; it must be born in mind that the derivative $f'(a)$ is calculated according to the definition of limit in the space E.

The fundamental tool for working with analytic functions is the complex line-integral, and it may be defined in the ordinary fashion. WIENER first pointed out that when this is done Cauchy's integral theorem remains valid (7). The essential point is that the space E is complete, so that a continuous function is integrable. From this basic theorem there is no difficulty in establishing a whole sequence of theorems pertaining to analytic functions, as they are developed, for instance, in the seventh chapter of OSGOOD's: *Lehrbuch der Funktionentheorie*, vol. I. Most important are the theorems regarding CAUCHY's integral formula, representation of analytic functions by infinite series and line-integrals, and the CAUCHY-TAYLOR series. The theorems of LIOUVILLE, MORERA, and LAURENT remain valid, as do the theorems regarding power series with coefficients in the space E. These proofs are given in my thesis, California Institute of Technology, 1936.

The first divergences from classical theory begin to appear when one examines the zeros and singular points of analytic functions. A point a_0 in the neighborhood of which $f(a)$ is analytic, without actually being analytic at the point, is called an isolated singularity of $f(a)$. If it is possible to re-define $f(a_0)$ so that $f(a)$ is analytic at a_0, the singularity is removable. If $\|f(a)\|$ becomes infinite as a approaches a_0, the singularity is called a pole. All other singularities will be lumped together under the name « essential ».

As in classical theory, RIEMANN's theorem for removable singularities is true (8). The behavior of functions near poles or essential singularities is not exactly as simple as before, however, as the following examples will show. Let E be the space whose elements are complex-valued functions of a complex variable, defined and continuous on the unit circle. If $\varphi(z)$ is such a function we define its norm to be

$$\|\varphi\| = \max_{|z|=1} |\varphi(z)|$$

(7) N. WIENER, Fundamenta Mathematicae, vol. 4 (1923), pp. 136-143. WIENER's observations are based on the treatment of CAUCHY's theorem in the Cambridge Tract n.° 15 (1914), by G. N. WATSON.

(8) The proof given in OSGOOD: *Lehrbuch der Funktionentheorie*, vol. I, 5th ed., pp. 325, requires no modification.

Suppose that $\psi(a)$ is a numerically-valued function of a, analytic at a point a_0. Then the function on the a -plane to E

$$f(a) = e^{\psi(a)z}$$

is analytic at a_0, with derivative

$$f'(a) = \psi'(a) z e^{\psi(a)z}$$

The singularities of $f(a)$ will be precisely the singularities of $\psi(a)$. Now

$$\| f(a) \| = \max_{|z|=1} | e^{\psi(a)z} | = \max_{x^2+y^2=1} e^{xR(\psi)-yI(\psi)}$$

where $R(\psi)$ = real part of $\psi(a)$,

$I(\psi)$ = imaginary part of $\psi(a)$

If $\psi(a) \neq 0$ we may choose $x = \frac{R(\psi)}{|\psi|}$, $y = -\frac{I(\psi)}{|\psi|}$, and thus see that

$$\| f(a) \| \geqq e^{|\psi(a)|} \geqq 1$$

an inequality which remains true when $\psi(a) = 0$.

From this we see that $a = a_0$ is a pole of $f(a)$ if and only if it is a pole of $\psi(a)$. If $\psi(a)$ has an essential singularity at a_0, so does $f(a)$, but $f(a)$ never takes on values within the unit sphere in E. Thus ~~the theorems of Weierstrass and Picard~~ are seen to be invalid in this case.

Other differences in behavior worthy of note are dispayed by the functions e^{az} and $e^{\frac{1}{a}z}$ in the above special instance. The first of them is an entire function of a, with a pole at infinity; it is nevertheless not a polynomial, contrary to the situation in classical theory. Also, $e^{\frac{1}{a}z}$ has a pole at $a = 0$, but the « principal part » of this pole is a non--terminating series, given in the LAURENT expansion:

$$e^{\frac{1}{a}z} = 1 + \frac{z}{a} + \frac{z^2}{a^2 2!} + \dots + \frac{z^n}{a^n n!} + \dots$$

Thus there are poles and zeros of infinite, as well as of finite order.

The notion of a rational function, as the quotient of two polynomials with *abstract* values, is denied us, since we have not postulated division in the space E. Of course a function such as

$$f(a) = \frac{P(a)}{p(a)} = \frac{a^n a_0 + a^{n-1} a_1 + \dots + a_n}{a^m \mu_0 + a^{m-1} \mu_1 + \dots + \mu_m}$$

where $P(a)$ is an abstract polynomial, and $p(a)$ is a numerical polynomial, is a sort of rational function. Its only singularities are poles of finite order, occurring at the roots of $p(a)$, and at $a = \infty$ in case $n > m$. It admits a representation by rational fractions, as we may prove in the usual may, by subtracting the principal parts of the function at the poles, and utilizing LIOUVILLE's theorem.

The problem of determining the nature of a function whose only singularities in the finite part of the plane are poles of finite order, is solved by the use of the theorem of MITTAG-LEFFLER.

THEOREM 7. - *If $f(a)$ is an analytic function which has in the finite plane no other singularities than poles of finite order, then it has the form*

$$f(a)=\sum_{n=1}^{\infty}\left[g\left(\frac{1}{a-a_n}\right)-\gamma_n(a)\right]+G(a)$$

where $g_n\left(\frac{1}{a-a_n}\right)$ is the principal part of $f(a)$ at the pole a_n, $\gamma_n(a)$ is a suitable abstract polynomial, and $G(a)$ is an entire function. In particular if the number of poles is finite, then

$$f(a)=\frac{P(a)}{p(a)}+G(a)$$

where $\frac{P(a)}{p(a)}$ is a «rational function» of the type discussed above.

MITTAG-LEFFLER's theorem, of which this is a corollary, may be proved as in OSGOOD, loc. cit., vol. I, pp. 565-566.

5. - Analytic Functions on a Vector Space. — Having seen in § 4 that we have at our disposal the methods and results of classical theory for functions of a complex variable, the extension to general analysis is achieved naturally through the medium of the GATEAUX differential.

Let E, E' be two complex vector spaces, and let E' be complete. A function $f(x)$ on a domain D of the space E, to the space E', is said to be analytic in D if it is continuous and has a GATEAUX differential at each point of D. A function is said to be analytic at a point x_0 if it is analytic in some neighborhood of the point. The fundamental theorem may be stated as follows:

THEOREM 8. - *If $f(x)$ on D to E' is continuous in D, a necessary and sufficient condition that it be analytic in D is that for each $n>0$, $f(a_1x_1+ .... +a_nx_n)$ be an analytic function of $(a_1,...., a_n)$, in the sense of § 4, for all a's and x's such that $a_1x_1+ .... +a_nx_n$ is in D.*

The proof is a simple matter of observing that when $a_1^0x_1+ .... +a_n^0x_n$ is in D, $a_1x_1+ .... +a_nx_n$ is also in D for $|a_i-a_i^0|<r_i$, where r_i is suitably small, and also that the partial derivatives of $f(a_1x_1+ .... +a_nx_n)$, as a function of $a_1,...., a_n$, are merely GATEAUX differentials of f at points of D.

As in the theory of functions of several complex variables we infer that the successive partial derivatives of $f(a_1x_1+ .... +a_nx_n)$ are continuous functions of the α's, and that the order of differentiation is immaterial ([9]). Therefore we have the following theorem.

THEOREM 9. - *If $f(x)$ is analytic in D it has Gateaux differentials of all orders there, and the nth differential is a completely symmetric function of the n increments.*

Two important questions present themselves regarding the differential $\delta f(x;\, y)$: is it an analytic function of x, for fixed y, and is it linear y? The answer to both these questions is affirmative, as we now show.

Suppose that x_0 is a point of D, and let y be an arbitrary, fixed point of E. We may choose positive numbers r, r' such that $x+\tau\cdot y$ is in D when $\|x-x_0\|<r'$ and $|\tau|\leqslant r$. With these restrictions $f(x+\tau\cdot y)$ is an analytic function of τ, and

$$\delta f(x;\, y)=\frac{1}{2\pi i}\int\limits_C \frac{f(x+\tau y)}{\tau^2}\,d\tau$$

C being a circle of radius r about the origin. Then

$$\|\delta f(x;\, y)-\delta f(x_0;\, y)\|\leqslant\frac{1}{2\pi}\int\limits_C \left\|\frac{f(x+\tau y)-f(x_0+\tau y)}{\tau^2}\right\|\,|d\tau|$$

From this it follows that $\delta f(x;\, y)$ is continuous at x_0, provided that we can, for a given $\varepsilon>0$, choose a δ such that $\|x-x_0\|<\delta$ implies the inequality

$$\|f(x+\tau y)-f(x_0+\tau y)\|<\varepsilon$$

for all τ on C. That we can actually do this is a consequence of Theorem 6, since C is a compact, closed set.

To show that $\delta f(x;\, y)$ is linear in y we first prove that it is additive and homogeneous of the first degree. We shall then prove that it is continuous at $y=0$.

Let x_0 be a point of D, and let α, β, y_1, y_2 be given arbitrarily. Then let $y=\alpha y_1+\beta y_2$, and consider $f(x_0+\tau y)$, which is an analytic function of τ at $\tau=0$. Accordingly it has the expansion

$$f(x_0+\tau y)=f(x_0)+\tau\left[\frac{\partial f(x_0+\tau y)}{\partial\tau}\right]_{\tau=0}+\ldots.$$

However, if we write $\xi=\tau\alpha$, $\eta=\tau\beta$, then $f(x_0+\xi y_1+\eta y_2)$ is an analytic function of ξ, η in the neighborhood of $\xi=\eta=0$. If in the expansion of this function we pick out the terms of first degree in τ, and equate them to the corresponding term in the above series, we obtain the relation

$$\left[\frac{\partial f(x_0+\tau y)}{\partial\tau}\right]_{\tau=0}=\alpha\left[\frac{\partial f(x_0+\xi\eta_1+\eta y_2)}{\partial\xi}\right]_{\xi=\eta=0}+\beta\left[\frac{\partial f(x_0+\xi y_1+\eta y_2)}{\partial\eta}\right]_{\xi=\eta=0}$$

([9]) OSGOOD, loc. cit., vol. II, 1 2nd ed. (1929), p. 21. The desired formula is

$$\delta^n f(x;\, y_1,\ldots, y_n)=\left\{\frac{\partial^n f(x+\alpha_1 y_1+\ldots+\alpha_n y_n)}{\partial\alpha_n\ldots\partial\alpha_1}\right\}_{(\alpha)=(0)}$$

This is precisely the result

$$\delta f(x_0\,;\,\alpha y_1+\beta y_2)=\alpha\delta f(x_0\,;\,y_1)+\beta\delta f(x_0\,;\,y_2).$$

Let $r>0$ be chosen so that when C is a circle of radius r about $\tau=0$,

$$\delta f(x_0\,;\,y)=\frac{1}{2\pi i}\int\limits_C \frac{f(x_0+\tau y)}{\tau^2}\,d\tau$$

for sufficiently small $\|y\|$. This may be written

$$\delta f(x_0\,;\,y)=\frac{1}{2\pi i}\int\limits_C \frac{f(x_0+\tau y)-f(x_0)}{\tau^2}\,d\tau$$

whence, since $f(x)$ is continuous at x_0, we easily conclude that $\delta f(x_0\,;\,y)$ is continuous at $y=0$.

THEOREM 10. - *If $f(x)$ is analytic in D, then for each n the differential $\delta^n f(x\,;\,y_1,\ldots,y_n)$ is an analytic function of x in D, when $y_1,\ldots,y_n$ are fixed. It is continuous in the set $(x, y_1,\ldots,y_n)$ at every point where it is defined if the space E is complete. Therefore it is, for each x, a symmetric multilinear function of $y_1,\ldots,y_n$. In particular, $\delta^n f(x;\,y,\ldots,y)$ is a continuous function of x and y, homogeneous of degree n in y.*

From the preceding discussion we know that $\delta^n f(x\,;\,y_1,\ldots,y_n)$ is analytic in x and linear in each y_i. Therefore, by a theorem of KERNER ([10]), it is continuous in the set $(x, y_1,\ldots,y_n)$. The rest of the assertions are clearly true. We write

$$\delta^n f(x\,;\,y,\ldots,y)=\delta^n f(x\,;\,y).$$

The generalization of the CAUCHY-TAYLOR expansion theorem is now readily proved.

THEOREM 11. - *If $f(x)$ is analytic in the region defined by $\|x-x_0\|<\varrho$, it may be expanded in the form*

$$f(x)=f(x_0)+\delta f(x_0\,;\,x-x_0)+\ldots+\frac{1}{n!}\delta^n f(x_0\,;\,x-x_0)+\ldots$$

This series converges uniformly in every compact set G extracted from the sphere $\|x-x_0\|\leq\theta\varrho$, where θ, $0<\theta<1$, is arbitrary. Moreover, the series

$$\|f(x_0)\|+\|\delta f(x_0\,;\,x-x_0)\|+\ldots$$

converges uniformly in G.

Proof: Let x be an arbitrarily chosen point such that $\|x-x_0\|<\varrho$, and

([10]) M. KERNER, Studia Mathematica, vol. 3 (1931), p. 159, and Annals of Mathematics, vol. 34 (1933), p. 548. The application of these theorems requires that E be complete.

choose $\varrho_1 > 0$ so that $\|x - x_0\| \leq \varrho_1 < \varrho$. Then choose r so that $1 < r < \varrho/\varrho_1$. The function

$$\psi(\tau) = f(x_0 + \tau(x - x_0))$$

is analytic inside a circle of radius r with center at $\tau = 0$, and so, by TAYLOR's theorem for functions of a numerical variable,

$$\psi(1) = \psi(0) + \psi'(0) + 1/2!\, \psi''(0) + \dots.$$

or

$$f(x) = f(x_0) + \delta f(x_0\,;\, x - x_0) + \dots.$$

To complete the proof we shall, for simplicity, assume $x_0 = 0$. For an arbitrary θ, choose r so that $1 < r < 1/\theta$. Then

$$\frac{1}{n!}\,\delta^n f(0\,;\, x) = \frac{1}{2\pi i}\int_C \frac{f(\tau x)}{\tau^{n+1}}\, d\tau$$

where C is a circle of radius r about $\tau = 0$, and $\|x\| \leq \theta\varrho$. Let G be a compact set subject to this latter restriction. Then $\|f(\tau x)\|$ is bounded when x is in G and τ is on C, as follows from Theorem 4 as soon as we establish the fact that the aggregate of such points τx is a compact set. Since both G and C are compact this is not difficult. If M is the bound in question

$$\left\| \frac{1}{n!}\,\delta^n f(0\,;\, x) \right\| \leq \frac{M}{r^n}$$

when x is in G. Since $r > 1$ the member on the right is the general term of a convergent series of constants, and the series converges « absolutely » and uniformly in G.

It is interesting and somewhat surprising to observe that the GATEAUX differentials with which we have been dealing are in fact FRÉCHET differentials. The truth of this relation depends essentially on the use of complex variables and the completeness of the space E'.

THEOREM 12. - *If $f(x)$ is analytic at a point x_0 it admits Fréchet differentials of all orders in the neighborhood of the point.*

Proof: Let $f(x)$ be analytic when $\|x - x_0\| < \varrho$, and for definiteness choose a number ϱ_1, $0 < \varrho_1 < \varrho$ so that $\|x - x_0\| \leq \varrho_1$ implies $\|f(x) - f(x_0)\| < 1/2$. Then let us agree that for an arbitrary $y \neq 0$ we shall choose r so that $r\|y\| = \varrho_1$. Then

$$\delta^n f(x_0\,;\, y) = \frac{n!}{2\pi i}\int_C \frac{f(x_0 + \tau y)}{\tau^{n+1}}\, d\tau$$

where C is a circle of radius r with center at $\tau = 0$. But, y being fixed, $\|f(x_0 + \tau y)\|$ is a continuous function of τ on C, and so has a maximum there:

$$\|f(x_0 + \tau y)\| \leq M(y).$$

This gives

$$\| \delta^n f(x_0\,;\, y) \| \leqslant \frac{n!\, M(y)}{r^n}.$$

But

$$\| f(x_0+\tau y) \| \leqslant \| f(x_0) \| + \| f(x_0+\tau y) - f(x_0) \|$$

so that

$$0 \leqslant M(y) \leqslant \| f(x_0) \| + \max_{|\tau|=r} \| f(x_0+\tau y) - f(x_0) \| \leqslant \| f(x_0) \| + 1/2 = G$$

where G is a constant. Thus we have, recalling that $r \| y \| = \varrho_1$,

$$\| \delta^n f(x_0\,;\, y) \| \leqslant \frac{n!\, G}{\varrho_1^{\,n}} \| y \|^n.$$

Applying this inequality to the series in Theorem 11 when $\| y \| < \varrho_1$ we obtain

$$\| f(x_0+y) - f(x_0) - \delta f(x_0\,;\, y) \| \leqslant \frac{G}{1 - \frac{\| y \|}{\varrho_1}} \frac{\| y \|^2}{\varrho_1^{\,2}}$$

and from this inequality it is evident that $\delta f(x_0\,;\, y)$ is the FRÉCHET differential at x_0. The reasoning applies to any point at which $f(x)$ is analytic, and so, in particular, to all points of a certain neighborhood of x_0.

6. - **Abstract Power Series.** — Theorem 11 suggests the natural alternative of developing a theory of analytic functions from a Weierstrassian point of view. There is a generalization of the theorem of WEIERSTRASS pertaining to functions defined by infinite series.

THEOREM 13. - *Let the terms of the series*

$$f(x) = u_1(x) + u_2(x) + \ldots.$$

be analytic in a domain D, with values in E', and let the series converge uniformly in every compact set extracted from an arbitrary closed sphere lying in D. Then the series converges and defines a function analytic in D. The differentials of $f(x)$ may be obtained by termwise differentiation of the series.

Proof: The series converges and defines a continuous function $f(x)$ in D, by Theorem 5. If x_0 is any point of D and y is arbitrary, but fixed, the function $f(x_0+\alpha y)$ is analytic at $\alpha=0$ ([11]), for the series converges uniformly in α in a closed neighborhood of $\alpha=0$. This is enough to complete the proof.

([11]) This is a consequence of the special case of Theorem 13 when the variable is a complex number. This theorem is proved as in classical theory. See OSGOOD, loc. cit., vol. I, p. 319.

We are interested in « power series », i. e. series of the form

$$f(x)=h_0(x)+h_1(x)+\dots+h_n(x)+\dots$$

where $h_n(x)$ is a homogeneous polynomial of degree n ([12]). By the radius of convergence ϱ of such a series we mean the largest positive number such that series converges uniformly in every compact set extracted from the sphere $\|x\|\leqslant\theta\varrho$, where $0<\theta<1$. Since a homogeneous polynomial is an analytic function, a power series defines an analytic function within its sphere convergence. It is readily established that if a power series vanishes for all values of its argument in an arbitrarily small neighborhood of $x=0$, then the individual terms vanish identically. Hence the power series expansion of an analytic function is unique, and in the above series

$$h_n(x)=\frac{1}{n!}\,\delta^n f(0,x).$$

THEOREM 14. - *Let*

$$f(x)=\sum_{n=0}^{\infty}h_n(x)$$

be a power series with radius of convergence ϱ, *and let it converge for the value* $x=x_0$, *where* $\|x_0\|=\varrho$. *Then*

$$\lim_{\lambda\to1}f(\lambda x_0)=\sum_{0}^{\infty}h_n(x_0)$$

when the complex number λ *approaches unity along a path included between two chords of the unit circle which pass through* $\lambda=1$.

The proof of this theorem may be carried through by the same general argument that is used in establishing the generalization of ABEL's theorem ([13]).

7. - Singularities. — In order to prove that RIEMANN's theorem on removable singularities may be generalized we must first establish a proposition about functions defined by integrals.

([12]) A function $p(x)$ on E to E' is called a polynomial if it is defined and continuous for each x in E, and if there exists an integer n such that for every x, y in E

$$p(x+ay)=p_0(x,y)+ap_1(x,y)+\dots+a^np_n(x,y).$$

The least integer n satisfying this condition is the degree of $p(x)$. The polynomial is homogeneous of degree n if $p(ax)=a^np(x)$.

Corresponding to a homogeneous polynomial $h(x)$ of degree n there is a unique symmetric multilinear function $h'(x_1,\dots,x_n)$ of n variables over E, to E', such that $h'(x,\dots,x_n)=h(x)$. Then $h(x)$ is analytic and $\delta h(x;y)=nh'(x,\dots,x,y)$. This resumé is based on the thesis of R. S. MARTIN, California Institute of Technology, 1932. For a recent, somenwhat different treatment see MAZUR and ORLICZ, Studia Mathematica, vol. V (1935).

([13]) See TITCHMARSH: *Theory of Functions* (1932), p. 229. Details are given in my thesis.

THEOREM 15. - *Let $f(x, a)$, with values in E', be defined for all values of x in a domain D of the space E, and a on a rectifiable Jordan curve C in the complex plane. Let it be analytic in D for each a on C, and continuous in both variables together. Then the integral*

$$F(x) = \int_C f(x, a)\,da$$

defines a function analytic in D, with the differential

$$\delta F(x\,;\, y) = \int_C \delta f(x, a\,;\, y)\,da.$$

Proof: By Theorem 6 $f(x, a)$ is continuous in x, uniformly in a on the curve. Hence $F(x)$ is continuous in D. If x_0 is any point D,

$$F(x_0 + \tau y) = \int_C f(x_0 + \tau y, a)\,da$$

is, for a fixed, arbitrary y, an analytic function of τ at $\tau = 0$, by the correspondent of Thorem 15 in the numerical case ([14]). The result then follows.

RIEMANN's theorem is the following:

THEOREM 16. - *If $f(x)$ is analytic $0 < \| x - x_0 \| < h$, and bounded in this range, then $\lim_{x \to x_0} f(x)$ exists, and if we define*

$$f(x_0) = \lim_{x \to x_0} f(x)$$

the function is analytic at x_0 also.

We have already observed in § 4 that this theorem is true when E is the space of complex numbers. For the abstract treatment the following modification is made.

For convenience denote by D the domain $0 < \| x - x_0 \| < h$. Choose a fixed y in E, with $\| y \| = 1$, and consider the function $\psi(\tau) = f(x + \tau y)$, where x is a fixed element in D for which $0 < \| x - x_0 \| < \frac{h}{2} \cdot \psi(\tau)$ is analytic for all values of τ such that $x + \tau y$ is in D, that is for $0 \leqslant |\tau| < \| x - x_0 \|$ and $\| x - x_0 \| < |\tau| < h - \| x - x_0 \|$. On the circle $|\tau| = \| x - x_0 \|$ $\psi(\tau)$ can have *at most* one singularity, which may occur when $x + \tau y = x_0$. But since we know that the theorem in question is true for $\psi(\tau)$ we have

$$\psi(0) = f(x) = \frac{1}{2\pi i} \int_C \frac{f(x + \tau y)}{\tau}\,d\tau$$

where C is a circle of radius $\frac{h}{2}$ about $\tau = 0$. This representation is valid when $0 < \| x - x_0 \| < \frac{h}{2}$. But in the range $0 \leqslant \| x - x_0 \| < h/2$ the integral

$$F(x) = \frac{1}{2\pi i} \int_C \frac{f(x + \tau y)}{\tau}\,d\tau$$

([14]) TITCHMARSH loc. cit., p. 99.

defines a function analytic without exception [15]. Since $F(x)=f(x)$ when $x\neq x_0$ we conclude that $\lim_{x\to x_0} f(x)=F(x_0)$ and the theorem is proved.

It is difficult to say much about other types of singularities. If E' is the complex number space, however, the usual theorem that near an isolated essential singularity a function comes arbitrarily near all values, remains true, in contrast to the case of abstract functions of a complex variable. The proof depends on Theorem 16 and on the fact that when $f(x)$ is a numerically-valued function which is analytic in a domain D and doesn't vanish there, $\frac{1}{f(x)}$ is also analytic in D. For details see OSGOOD, loc. cit., vol. I, p. 328.

8. - **The Cauchy-Riemann Equations.** — In § 2 we defined the complex couple-space associated with a real vector space. If $E(C)$ and $E'(C)$ are two such couple-spaces, associated with the real vector spaces $E(R)$ and $E'(R)$, respectively, a function $f(z)$ on $E(C)$ to $E'(C)$ has the form

$$f(z)=f_1(x,y)+i\cdot f_2(x,y)$$

where $f_1(x,y)$ and $f_2(x,y)$ are functions of two variables over $E(R)$, with values in $E'(R)$.

Let us now suppose that $E'(R)$ is complete, and that $f(z)$ is defined in a domain D of $E(C)$. Then we can discuss the analyticity of $f(z)$ in terms of the properties of the functions f_1 and f_2. The fundamental proposition, a generalization of the classical theorem pertaining to the CAUCHY-RIEMANN equations, is as follows:

THEOREM 17. - *In order that $f(z)$ be analytic in D it is necessary and sufficient that the functions $f_1(x,y)$, $f_2(x,y)$ be continuous and admit continuous first partial Gateaux differentials at all points of D, and that the equations*

$$\delta_x f_1(x,y;\,\xi)=\delta_y f_2(x,y;\,\xi)$$
$$\delta_y f_1(x,y;\,\xi)=-\delta_x f_2(x,y;\,\xi)$$

be satisfied in D for an arbitrary ξ in $E(R)$.

Proof: If $f(z)$ is analytic in D it is continuous there, and the differential $\delta f(z;\,\Delta z)$ is linear in Δz, and continuous in the pair z, Δz. But

$$\delta f(z;\,\Delta z)=\lim_{\tau\to 0}\frac{f(z+\tau\Delta z)-f(z)}{\tau}.$$

Hence in particular, taking $\Delta z=\Delta x+i\cdot 0$, $\tau=t$, where t is real,

$$\delta f(z;\,\Delta z)=\lim_{t\to 0}\frac{f_1(x+t\Delta x,y)-f_1(x,y)+i\cdot f_2(x+t\Delta x,y)-if_2(x,y)}{t}.$$

[15] It is easily seen that $f(x+\tau y)$ is an analytic function of x when τ is on C and $0\leq\|x-x_0\|<\frac{h}{2}$. We then use Theorem 15.

This limit will exist, however, only if the separate parts have limits. Therefore

$$\delta f(z;\, \Delta x) = \delta_x f_1(x, y;\, \Delta x) + i \cdot \delta_x f_2(x, y;\, \Delta x).$$

Similarly we obtain

$$\delta f(z;\, \Delta x) = \delta_y f_2(x, y;\, \Delta x) - i\delta_y f_1(x, y;\, \Delta x)$$

Since the left member of these equations is continuous, we see that the four terms on the right must be continuous in $(x, y, \Delta x)$ when $x + i \cdot y$ is in D and Δx is arbitrary in $E(R)$. On equating corresponding parts we obtain the generalized CAUCHY-RIEMANN equations given in the theorem. The continuity of f_1 and f_2 is a consequence of the continuity of $f(z)$.

To prove the sufficiency of the conditions suppose that $\Delta z = \Delta z + i \cdot \Delta y$ is an arbitrary element of $E(C)$, and consider the expression

$$\frac{f(z+\tau\Delta z) - f(z)}{\tau} = \frac{f_1(x + s\Delta x - t\Delta y,\, y + t\Delta x + s\Delta y) - f_1(x, y)}{s + i \cdot t} + \\ + i \cdot \frac{f_2(x + s\Delta x - t\Delta y,\, y + t\Delta x + s\Delta y) - f_2(x, y)}{s + i \cdot t}$$

where z is in D and $\tau = s + i \cdot t$ is a sufficiently small complex number. Next, consider the function

$$F_1(s, t, u, v) = f_1(x + s\Delta x - t\Delta y,\, y + u\Delta x + v\Delta y)$$

of four real variables, with values in $E'(R)$. This function is continuous and admits continuous first partial derivatives near $(0, 0, 0, 0)$. It is then not difficult to show that it admits a total differential ([16]) at $(0, 0, 0, 0)$, that is

$$F_1(s, t, u, v) - F_1(0, 0, 0, 0) = sF_{1s}(0, 0, 0, 0) + tF_{1t}(0, 0, 0, 0) + \\ + uF_{1u}(0, 0, 0, 0) + vF_{1v}(0, 0, 0, 0) + \varepsilon(s, t, u, v)$$

([16]) We demonstrate the theorem, for simplicity, using only two variables; the general case is proved in a similar manner. Writing

$$f(s, t) - f(0, 0) - sf_s(0, 0) - tf_t(0, 0) = \varepsilon(s, t)$$

we have

$$\frac{\|\varepsilon(s, t)\|}{|s| + |t|} \leqslant \frac{\|f(s, t) - f(s, 0) - tf_t(s, 0)\|}{|s| + |t|} + \frac{\|f(s, 0) - f(0, 0) - sf_s(0, 0)\|}{|s| + |t|} + t \cdot \frac{\|f_t(s, 0) - f_t(0, 0)\|}{|s| + |t|} \leqslant$$

$$\leqslant \frac{\left\| \int_0^t [f_t(s, u) - f_t(s, 0)]du \right\|}{|s| + |t|} + \frac{\|f(s, 0) - f(0, 0) - sf_s(0, 0)\|}{|s|} + \|f_t(s, 0) - f_t(0, 0)\|$$

and from this the result follows without difficulty as a result of the *uniform* continuity of $f_t(s, t)$.

where

$$\lim_{(s,\dots,v)\to(0,\dots,0)} \frac{\|\varepsilon(s, t, u, v)\|}{|s|+|t|+|u|+|v|}=0$$

Therefore, when expressed in terms of GATEAUX differentials, we have

$$f_1(x+s\Delta x-t\Delta y, y+t\Delta x+s\Delta y)-f_1(x, y)=s\delta_x f_1(x, y; \Delta x)- \\ -t\delta_x f_1(x, y; \Delta y)+t\delta_y f_1(x, y; \Delta x)+s\delta_y f_1(x, y; \Delta y)+\varepsilon(s, t, t, s).$$

There is a similar relation involving the function $f_2(x, y)$ and an infinitesimal $\eta(s, t, t, s)$. On making use of the CAUCHY-RIEMANN equations we find that

$$\frac{f(z+\tau\Delta z)-f(z)}{\tau}=[\{\delta_x f_1(x, y; \Delta x)-\delta_x f_2(x, y; \Delta y)\}+ \\ +i\{\delta_x f(x, y; \Delta y)+\delta_x f(x, y; \Delta x)\}]\frac{s+i\cdot t}{\tau}+\frac{\varepsilon+i\eta}{\tau}.$$

But $\|\varepsilon+i\eta\|\leq\|\varepsilon\|+\|\eta\|$, and $|\tau|\geq 1/\sqrt{2}\,(|s|+|t|)$. From this we conclude that

$$\lim_{\tau\to 0}\frac{\|\varepsilon+i\eta\|}{|\tau|}=0$$

and hence that $f(z)$ has the differential

$$\delta f(z; \Delta z)=\delta_x f_1(x, y; \Delta x)-\delta_x f_2(x, y; \Delta y)+i\{\delta_x f_1(x, y; \Delta y)+\delta_x f_2(x, y; \Delta x)\}.$$

Since f_1 and f_2 are continuous, so is $f(z)$, and $f(z)$ is analytic. This proves the theorem.

The known properties of $\delta f(z; \Delta z)$, as the differential of an analytic function, enable us to draw conclusions about the properties of the functions f_1, f_2, and their differentials. The various GATEAUX differentials are in fact partial FRÉCHET differentials; their linearity is evident. Furthermore, $f_1(x, y)$ and $f_2(x, y)$ admit *total* FRÉCHET differentials. That is,

$$df_1(x, y; \xi, \eta)=\delta_x f_1(x, y; \xi)+\delta_y f_1(x, y; \eta)$$

with a similar formula for $f_2(x, y)$.

9. - **Conclusion.** — LIOUVILLE's theorem holds for the general theory under consideration. Knowing that it holds for functions of a complex variable, we make the extension very easily as follows: Consider $f(x_0+\alpha(x_1-x_0))$ where x_0 and x_1 are any two points in E, and $f(x)$ is assumed to be analytic everywhere, and $\|f(x)\|$ is bounded. Then the foregoing function of α satisfies the hypothesis of LIOUVILLE's theorem, and so has the same value when $\alpha=0$ and $\alpha=1$.

On the basis of Theorems 2, 8, and 12 we can lay down the following alternative definition of an analytic function: A function $f(x)$ on E to E' will be called analytic in a domain D of E if

1°) it is continuous in D,

2°) whenever $\varphi(a)$ is an analytic function on C to E, and T is a domain of the plane such that $\varphi(a)$ is analytic in T and $\varphi(a)$ lies in D when a lies in T, then $f(\varphi(a))$ is analytic in T.

This definition is suggested by the work of FANTAPPIÉ on analytic functionals of analytic functions ([17]), and it formed the original starting point for my investigations of the subject.

A power series definition of analyticity similar to the one embodied in Theorem 11 was used by R. S. MARTIN in his thesis (see footnote ([12]), and has been used subsequently by Professor A. D. MICHAL and others ([18]). It is essentially equivalent to ours *locally*, i. e. it gives a series uniformly convergent in a sufficiently small neighborhood of every point, but the region of analyticity of the function as a whole may not be the same as in our theory. It should be remarked that the concept of radius of analyticity is not as important here as in the classical theory, for the region of convergence of an abstract power series is not necessarily that defined by an inequality $\|x\|<\varrho$.

([17]) L. FANTAPPIÉ, Memorie dei Lincei, vol. 3, fasc. 11 (1930).

([18]) A. D. MICHAL and A. H. CLIFFORD, Comptes Rendus, vol. 197 (1933), pp. 735-737. A. D. MICHAL and R. S. MARTIN, Journal de Mathématiques Pures et Appliquées, vol. 13 (1934), pp. 69-91.