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RELATIONS BETWEEN LAWS OF LARGE NUMBERS AND ASYMPTOTIC

MARTINGALES IN BANACH SPACES

NGUYEN Van Hung - DINH Quang Luu

Resume : Dans cet article on donne quelques résultats sur des lois des grands nombres et établi quelques relations familières entre des martingales asymptotiques et des lois des grands nombres pour les martingales de difference.

Summary. In this note we give several results on laws of large numbers and establish some close relations between asymptotic martingales and laws of large numbers for martingale differences.

1. Introduction. The present paper concentrates on the Marcinkiewicz-Zygmund-Kolmogorov's and Brunk-Chung's laws of large numbers for martingale differences (D_k) with values in a separable Banach space E . Some general relations between asymptotic martingales and laws of large numbers in Banach spaces are established. Namely, according to the results of Pisier [10], Assouad [2], Szulga [12], Taglor [14], Acosta [1], Azlarov-Volodin [3], Woyczynski [16,17,18] and others, different geometric conditions should be imposed on E to proving the (weak) strong law of large numbers for (D_k) . In the vector-valued setting, our results show how these geometric conditions imposed on E can be replaced by other additional conditions imposed on (D_k) , for example, (D_k) is uniformly tight or the closed convex hull $\overline{\text{co}}(\{D_k(\omega), k \geq 1\})$ of (D_k) is compact, a.e.

Further, it is known that the Kronecker's lemma is in fact, a martingale method frequently used in proving the strong law

of large numbers in Banach spaces having the Radon-Nikodym property. On the other hand, according to Krengel-Sucheston [6] and Gut ([5], Example 3.15), the strong law of large numbers gives sometimes short proofs of some results on asymptotic martingales. This observation leads us to find out general relations between asymptotic martingales and laws of large numbers for martingale differences in Banach spaces which are presented at the end of this note.

Finally, the authors wish to express their thanks to Professor N.D. Tien for presenting the question of Woyczynski [18] which is the first starting point of this consideration.

2. Definitions and Preliminaries

Throughout this note let (Ω, \mathcal{A}, P) be a probability space, (\mathcal{A}_k) an increasing sequence of sub- σ -fields of \mathcal{A} , E a real separable Banach space and E^* the topological dual of E . By $L_{\mathcal{A}}^p$ we mean the usual Banach space of all E -valued \mathcal{A} -measurable Bochner integrable functions X defined on Ω such that

$$E \|X\|^p = \int_{\Omega} \|X(\omega)\|^p dP(\omega) < \infty, \quad (1 \leq p \leq 2).$$

A Banach space E is said to be p -smoothable ($1 \leq p \leq 2$) if (possibly after equivalent renorming), for $t \rightarrow 0$

$$\zeta(t) = \sup \left\{ \frac{\|x+ty\| + \|x-ty\|}{2} - 1, \|x\| = \|y\| = 1 \right\} = O(t^p).$$

E is superreflexive if E is p -smoothable for some $p > 1$ (cf. [10]). It is worth noting that the real line is p -smoothable for all p .

A sequence (D_k) in L^1_E is said to be adapted to (\mathcal{A}_k) if each D_k is \mathcal{A}_k -measurable. Unless otherwise stated, all sequences considered in this note are assumed to be taken from L^1_E and adapted to (\mathcal{A}_k) .

A sequence (D_k) is called a martingale difference (m.d.), if $E(D_k / \mathcal{A}_{k-1}) = 0$ for all $k \geq 1$ with $\mathcal{A}_0 = \{\emptyset, \Omega\}$, where given a sub- σ -field \mathcal{B} of \mathcal{A} , $E(. / \mathcal{B})$ denotes the \mathcal{B} -conditional expectation operator on L^1_E (cf. [9]). It is clear that (D_k) is a m.d. if and only if the sequence (M_n) , where $M_n = \sum_{k=1}^n D_k$, is a martingale. Throughout this note (D_k) and (M_n) should be understood in this connection.

In what follows we shall need the following result which follows from the martingale three series theorem of Szulga [12] and a characterization of p -smoothability of Assouad [2].

Proposition 2.1. (cf. [12]) If E is p -smoothable and $(\|D_k\|^r)$ is uniformly integrable then, for $n \rightarrow \infty$

$$E \|M_n\|^r = \begin{cases} o(n) & \text{if } 0 < r < p \\ O(n^{r/p}) & \text{if } r \geq p. \end{cases}$$

We shall use also Theorem 4.1 of Woyczynski [18] which shows how the identically distributed condition imposed on (D_k) by Elton [4] can be relaxed and how one can prove the a.e. convergence to zero of $(n^{-1/p} M_n)$ for $p \neq 1$.

Proposition 2.2. (cf. [18]). Let (D_k) be a m.d. in E .

a) If $(D_k) \ll X_0 \in L \text{Log} L$ and E is superreflexive then

$$M_n = o(n), \text{ a.e., as } n \rightarrow \infty,$$

where " $(D_k) < X_0$ " means that $\sup_k P(\|D_k\| > t) \leq C P(\|X_0\| > t)$ for some positive constant C and for all $t > 0$.

b) If $(D_k) < X_0 \in L^r$, $1 < r < 2$, and E is p -smoothable for some $p > r$ then, for $n \rightarrow \infty$

$$M_n = o(n^{1/r}), \text{ a.e.}$$

Further, a sequence (X_n) is called an L^1 -amart (cf. [8]), if

$$\lim_{n \rightarrow \infty} \sup_{m > n} E \|E(X_m / \mathcal{A}_n) - X_n\| = 0.$$

To establish a general relation between L^1 -amarts and the weak law of large numbers we shall need the following characterization of L^1 -amarts (cf. [7]) which has been recently extended by the second author even to the multivalued case (cf. [8]).

Proposition 2.3 (see [8]) A sequence (X_n) is an L^1 -amart if and only if (X_n) has a unique Riesz decomposition :

$X_n = g_n + p_n$, where (g_n) is a martingale and (p_n) is an L^1 -potential, i.e. $\lim_n E \|p_n\| = 0$. Moreover, if this occurs then the martingale (g_n) is given by

$$\lim_m E \|E(X_m / \mathcal{A}_n) - g_n\| = 0.$$

Finally, a sequence (X_n) is called a mil (martingale in the limit) if

$$\forall \epsilon > 0 \quad \exists p \quad \forall n > p \quad P \left(\sup_{p \leq k \leq n} \|E(X_n / \mathcal{A}_k) - X_k\| > \epsilon \right) < \epsilon.$$

The following elegant result of Talagrand ([13], Theorem 6) will be applied to establishing a general relation between the

class of mils and the strong law of large numbers for m.d.'s in general Banach spaces.

Proposition 2.4. (cf. [13]). Let (X_n) be a mil such that $\liminf_n E \|X_n\| < \infty$. Suppose that (X_n) converges scalarly to zero, a.e. that is, for each $f \in E^*$ the sequence $(\langle f, X_n \rangle)$ converges a.e. to zero. Then (X_n) converges (strongly) to zero, a.e.

For other related results on geometry of Banach spaces, laws of large numbers or on asymptotic martingales, the interested readers are referred correspondently to the Lecture Notes in Math. given by L. Schwartz [11], R.L. Taylor [14] and A. Gut and K.D. Schmidt [5].

3. Laws of large numbers for martingale differences.

As we have noted in the previous section, to establish the L_r -convergence of $(n^{-1}M_n)$, Szulga [12] needed the p -smoothability of E with $(1 \leq r < p)$. In the following result, this geometric condition imposed on E will be replaced by an additional condition imposed on the m.d. (D_k) as follows

Proposition 3.1. Let (D_k) be a m.d. such that $\sup_k E \|D_k\|^p < \infty$ for some $1 < p \leq 2$. Suppose more that (D_k) is uniformly tight, i.e. for each $\epsilon > 0$ there exists a compact convex subset K_ϵ of E such that $\sup_k P(D_k \in K_\epsilon) \geq 1 - \epsilon$. Then for all $1 \leq r < p$ we get

$$E \|M_n\|^r = o(n), \text{ as } n \rightarrow \infty.$$

Equivalently, (D_k) satisfies the weak law of large numbers, i.e. for $n \rightarrow \infty$

$$M_n = o(n), \text{ in probability.}$$

Proof. Let (D_k) be a uniformly tight m.d. such that $\sup_k E \|D_k\|^p < \infty$ for some $1 < p \leq 2$. Then by Lemma 5.22 [14] (see also [15], Lemma 2.3), it follows that for all $1 \leq r < p$, the m.d. (D_k) is uniformly bounded by some non-negative function $X_0 \in L^r$, i.e. $(D_k) < X_0$. Consequently, $(\|D_k\|^r)$ is uniformly integrable. Hence by virtue of Lemma 2.2 [15], the additional uniform tightness of (D_k) shows that (D_k) must be compactly uniformly r -th order integrable, i.e. for every $\epsilon > 0$ there exists a compact subset K_ϵ of E such that

$$\int_{\{D_k \in K_\epsilon^c\}} \|D_k\|^r dP < \epsilon .$$

Thus if we define the double array (a_{nk}) by

$$a_{nk} = \begin{cases} \frac{1}{n} & \text{if } k \leq n \\ 0 & \text{if } k > n , \end{cases}$$

then this array (a_{nk}) and the m.d. (D_k) satisfy just the assumptions of Corollary 3.9 [15] which implies that the following conditions are equivalent

- i) $E |\langle f, M_n \rangle|^r = o(n)$, for every $f \in E^*$,
- ii) $E \|M_n\|^r = o(n)$ as $n \rightarrow \infty$,
- iii) $M_n = o(n)$, in probability as $n \rightarrow \infty$.

But on the other hand, Proposition 2.1 shows that the condition (i) is satisfied. Then so do both conditions (ii-iii). It completes the proof of the proposition.

In the following remark, we borrow two examples of Taylor ([14], Example 4.11, p.80 and Example 5.22, p.127) to show that neither the uniform tight assumption nor the uniform L_p -bounded-

ness condition imposed on (D_k) in Proposition 3.1 cannot be, in general removed.

Remark 3.1. Let $E = \mathcal{L}_1$ and (e_n) be the usual basis for \mathcal{L}_1 .

a) Let (r_k) be the Rademacher independent sequence of random variables defined by $P(\{r_k = \pm 1\}) = 1/2$ and $D_k = r_k e_k$. It is clear that $\sup_k E \|D_k\|^p = 1$ for all p . But for $n \rightarrow \infty$ $E \|M_n\| \neq o(n)$ which shows that the uniform tight condition imposed on (D_k) in Proposition 3.1 cannot be, in general removed.

b) Now let (D_k) be the independent random elements (r.e.'s), given by

$$D_k = \begin{cases} \sqrt{n} e_n & \text{with probability } 1/2\sqrt{n} \\ 0 & \text{with probability } 1-1/\sqrt{n} \end{cases} .$$

Thus by the argument given in Example 5.22 [13], (D_k) is a uniformly tight m.d. with $E \|D_k\| = 1$. But for $n \rightarrow \infty$, $E \|M_n\| \neq o(n)$ which shows that the uniform L_p -boundedness assumption on (D_k) with $p > 1$ cannot be, in general removed.

It is worth noting also that by the Remark 3.1, it follows that the p -smoothability condition imposed on E in Proposition 2.2 is also essentially necessary to establish the strong law of large numbers for m.d.'s (D_k) . This observation leads us to the following general result.

Proposition 3.2. A sequence (X_k) of E -valued r.e.'s (which are not necessarily Bochner integrable) converges a.e. to some r.e. X if and only if (X_n) converges scalarly to X , a.e. and the set $\overline{\text{co}}(\{X_k(\omega), k \geq 1\})$ is convex compact, a.e.

Proof. Since the necessity condition is easy, so we give only a proof of the sufficiency condition. Indeed, let (X_k) be a sequence of E -valued r.e.'s which converges scalarly a.e. to some r.e. $X : \Omega \rightarrow E$. We shall show that (X_k) converges also strongly to X , a.e provided the set $\overline{\text{co}}(\{X_k(\omega), k \geq 1\})$ is convex compact, a.e. For this purpose, let T be a countable subset of E^* which is dense in the Mackey topology $\tau(E^*, E)$ of E^* . This with the first property of (D_k) implies that there exists a measurable subset Ω_1 of Ω with probability one such that the sequence $(\langle g, X_k(\omega) \rangle)$ converges to $\langle g, X(\omega) \rangle$ for all $g \in T$ and $\omega \in \Omega_1$.

Now let $\Omega_2 = \{\omega : \overline{\text{co}}(\{X_k(\omega), k \geq 1\}) \text{ is convex compact}\}$. Then by the second property of (D_k) , it follows that $P(\Omega_2) = 1$. Further, for each $\omega \in \Omega_1 \cap \Omega_2$, let $K(\omega)$ be the closed convex symmetric hull of the set $\bigcup_{k=1}^{\infty} \{X_k(\omega) - X(\omega)\}$. Then $K(\omega)$ is also compact since $K(\omega)$ is contained in the closed convex symmetric hull of the set $\overline{\text{co}}(\{X_k(\omega), k \geq 1\}) + \{-X(\omega)\}$ which is compact by the second condition imposed on (D_k) . Let $\tau_T(\omega)$ be the weakest topology on $K(\omega)$ which makes elements of T continuous. Then as in the proof of Theorem 5.27 [14], a sequence (x_n) of $K(\omega)$ converges to zero in the $\tau_T(\omega)$ -topology if and only if $(\|x_n\|)$ converges to zero. Hence, if $(\langle g, x_n \rangle)$ converges to zero for all $g \in T$ then $(\|x_n\|)$ converges also to zero. This conclusion follows also from the fact that if C is a convex weakly compact subset of E then a sequence (x_n) of C converges weakly to zero if and only if each $(\langle g, x_n \rangle)$ converges to zero ($g \in T$). Further, on each convex compact set the weak and the strong topologies

coincide which shows also the above conclusion.

Finally, note that for each $\omega \in \Omega_1 \cap \Omega_2$, the sequence $(X_k(\omega) - X(\omega))$ is contained in $K(\omega)$ and $(\langle g, X_k(\omega) - X(\omega) \rangle)$ converges to zero for all $g \in T$ and $P(\Omega_1 \cap \Omega_2) = 1$. Then the sequence $(X_k(\omega) - X(\omega))$ converges to zero, a.e. which completes the proof of the proposition. The following corollaries are easy consequences of Proposition 2.2 and Proposition 3.3.

Corollary 3.1. Let (D_k) be a m.d. in E .

a) If $p=1$ and $(D_k) \prec X_0 \in L\text{Log}L$ then, for $n \rightarrow \infty$ $M_n = o(n)$, a.e. if and only if the set $\overline{\text{co}}(\{\frac{1}{n} M_n(\omega), n \geq 1\})$ is convex compact, a.e.

b) for all $1 < p < 2$, $M_n = o(n^{1/p})$, a.e. as $n \rightarrow \infty$ if and only if the set $\overline{\text{co}}(\{n^{-1/p} M_n(\omega), n \geq 1\})$ is convex compact, a.e.

Corollary 3.2. Let (D_k) be a m.d. in E such that $(D_k) \prec X_0 \in L\text{Log}L$ and the set $\overline{\text{co}}(\{D_k(\omega), k \geq 1\})$ is convex compact, a.e. Then (D_k) satisfies the strong law of large numbers.

Remark 3.2. By Remark 3.1, it follows also that the compactness condition imposed on the sets $\overline{\text{co}}(\{D_k(\omega), k \geq 1\})$ in the above corollaries cannot be removed.

These above remarks lead us to the following general relations between asymptotic martingales and laws of large numbers for m.d.'s in general Banach spaces.

Proposition 3.3. Let (D_k) be a m.d. in E and (a_n)

an increasing sequence of real numbers such that $a_n \rightarrow \infty$, as $n \rightarrow \infty$.

a) $E \| M_n \| = o(a_n)$ if and only if $(a_n^{-1} M_n)$ is an L^1 -amart.

b) If $\liminf_n a_n^{-1} E \| M_n \| < \infty$ then $M_n = o(a_n)$, a.e. if and only if $(a_n^{-1} M_n)$ is a mil.

Proof. Let (D_k) be a m.d. in E and (a_n) an increasing sequence of real numbers such that $a_n \rightarrow \infty$, as $n \rightarrow \infty$.

(a) Suppose first that $E \| M_n \| = o(a_n)$ as $n \rightarrow \infty$. Then by definition, the sequence $(a_n^{-1} M_n)$ is an L^1 -potential. Hence by Proposition 2.3, $(a_n^{-1} M_n)$ is an L^1 -amart. Conversely, suppose that $(a_n^{-1} M_n)$ is an L^1 -amart. Then by Proposition 2.3, it follows that $(a_n^{-1} M_n)$ can be written in a unique form :

$$(3.1) \quad a_n^{-1} M_n = g_n + p_n \quad (n \in N),$$

where (g_n) is a martingale and (p_n) is an L^1 -potential, i.e. $\lim_n E \| p_n \| = 0$. Moreover, the martingale (g_n) is given by

$$(3.2) \quad \lim_{n < m \rightarrow \infty} E \| E(a_m^{-1} M_m / \mathcal{A}_n) - a_n^{-1} M_n \| = 0.$$

But

$$\lim_{n < m \rightarrow \infty} E \| E(a_m^{-1} M_m / \mathcal{A}_n) \| = \lim_{m \rightarrow \infty} a_m^{-1} E \| M_m \| = 0.$$

This with (3.2) yields $g_n = 0$, a.e. ($n \in N$). Consequently, by (3.1), $a_n^{-1} M_n = p_n$ ($n \in N$) which shows that $E \| M_n \| = o(a_n)$. It completes the proof of (a).

(b) Now, suppose that $M_n = o(a_n)$, a.e. as $n \rightarrow \infty$. We shall show that $(a_n^{-1} M_n)$ is a mil. For this purpose, let $\epsilon > 0$ be given. Since $M_n = o(a_n)$, a.e., there exists $p \in N$ such that for all $n > p$, one has

$$P \left(\sup_{p \leq k \leq n} a_k^{-1} \| M_k \| > \varepsilon \right) < \varepsilon .$$

Consequently,

$$\begin{aligned} & P \left(\sup_{p \leq k \leq n} \| E(a_n^{-1} M_n / \mathcal{A}_k) - a_k^{-1} M_k \| > \varepsilon \right) \\ & \leq P \left(\sup_{p \leq k \leq n} \| a_n^{-1} M_k - a_k^{-1} M_k \| > \varepsilon \right) \\ & \leq P \left(\sup_{p \leq k \leq n} a_k^{-1} \| M_k \| > \varepsilon \right) < \varepsilon . \end{aligned}$$

Thus by definition, $(a_n^{-1} M_n)$ is a mil.

Conversely, suppose that $(a_n^{-1} M_n)$ is a mil. We shall show first that $(a_n^{-1} M_n)$ converges to zero, in probability. Indeed, let $\varepsilon > 0$ be given. By definition, there exists $p \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, one has

$$P \left(\sup_{p \leq k \leq n} \| E(a_n^{-1} M_n / \mathcal{A}_k) - a_k^{-1} M_k \| > \varepsilon/2 \right) < \varepsilon/2 .$$

Let $k \geq p$ be any but fixed. Then for all $n \geq k$ we have

$$\begin{aligned} & P \left(\| a_k^{-1} M_k \| > \varepsilon \right) \\ & \leq P \left(\| E(a_n^{-1} M_n / \mathcal{A}_k) \| > \varepsilon/2 \right) + P \left(\| E(a_n^{-1} M_n / \mathcal{A}_k) - a_k^{-1} M_k \| > \varepsilon/2 \right) \\ & \leq P \left(\| a_n^{-1} M_k \| > \varepsilon/2 \right) + P \left(\sup_{p \leq q \leq n} \| E(a_n^{-1} M_n / \mathcal{A}_q) - a_q^{-1} M_q \| > \varepsilon/2 \right) \\ & \leq P \left(a_n^{-1} \| M_k \| > \varepsilon/2 \right) + \varepsilon/2 . \end{aligned}$$

Consequently,

$$P \left(\| a_k^{-1} M_k \| > \varepsilon \right) < \varepsilon , \text{ by letting } n \rightarrow \infty .$$

This shows that $(a_n^{-1} M_n)$ converges to zero in probability.

Further, let $f \in E^*$. It is clear that the sequence $(\langle f, a_n^{-1} M_n \rangle)$ is a real-valued mil with $\liminf_n E|\langle f, a_n^{-1} M_n \rangle| < \infty$. This with Theorem 4 [12] shows that $(\langle f, a_n^{-1} M_n \rangle)$ must converge, a.e. But on the other hand, we have shown that $(a_n^{-1} M_n)$ converges to zero in probability so $(\langle f, a_n^{-1} M_n \rangle)$ converges to zero, a.e.

Finally, since $(a_n^{-1} M_n)$ is a mil, by Proposition 2.4 it follows that $(a_n^{-1} M_n)$ converges strongly to zero a.e. This completes the proof of (b) and hence of the proposition.

Corollary 3.3. Let (D_k) be a m.d. in E .

- a) $E\|M_n\| = o(n)$ if and only if $(n^{-1} M_n)$ is an L^1 -amart.
- b) If $\sup_k E\|D_k\| < \infty$ then $M_n = o(n)$, a.e. if and only if $(n^{-1} M_n)$ is a mil.

Corollary 3.4. Let (D_k) be a m.d. of independent identically distributed r.e.'s in E . Then for all $1 < p < 2$, $(n^{-1/p} M_n)$ is a mil if and only if the Banach space E is of Rademacher type p (cf. [1]).

Remark 3.3. It is clear that in Corollary 3.4 we have used the results of A de Acosta [1] on the Marcinkiewicz-Zygmund's type strong law to establish an amart property of $(n^{-1/p} M_n)$. It is desirable to apply different results in amart theory [5,6] to prove such a type strong law of large numbers as suggested by A. Gut ([5], p.30).

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