ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Probabilités et applications

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Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 90, série *Probabilités et applications*, nº 6 (1987), p. 25-41

<http://www.numdam.org/item?id=ASCFPA_1987__90_6_25_0>

UNIFORM RATES OF CONVERGENCE IN EXTREME-VALUE THEORY - NORMAL AND GAMMA MODELS

L. CANTO E CASTRO

Abstract: A class of distribution functions F(x), in the domain of attraction for maxima of Gumbel law $\Lambda(x)=\exp(-\exp(-x))$, $x\varepsilon R$, is considered in this paper, particular and relevant elements of this class being the normal and gamma distributions.
Applying a technique similar in spirit to the one used by Hall (1979), we develop uniform upper and lower bounds for

$$\sup_{x\varepsilon R}|F^n(a_nx+b_n) - \Lambda(x)|$$

for suitable choice of attraction coefficients $\{a_n\}_{n\geqslant 1}$, $(a_n>0)$ and $\{b_n\}_{n\geqslant 1}$. The bounds obtained in a normal context compare favourably with the ones obtained in Hall's paper. A few unsolved points related to gamma distribution with shape parameter smaller than one are emphasized.

1 - Introduction

Let $\{X_n\}_{n\geqslant 1}$ be a sequence of independent, identically distributed (i.i.d.) random variables (r.v.) with common parent distribution function (d.f.), F(x). Then the r.v. $M_n=\max(X_1,\ldots,X_n)$ has a well known d.f. given by $F_{M_n}(x)=F^n(x)$.

Limit behaviour of M_n, suitably normalized, is well expressed in the following result due to Gnedenko (1943): if there are attraction coefficients $a_n>0$, $b_n\varepsilon R$ and a non degenerated d.f. G, such that

$$F^n(a_nx+b_n) \underset{n\to\infty}{\longrightarrow} G(x)$$

then G is of one of the three possible forms:

Type I (Gumbel) : $G(x) \equiv \Lambda(x) = \exp(-\exp(-x))$, $x \in R$

Type II (Fréchet): $G(x) \equiv \Phi_\alpha(x) = \begin{cases} \exp(-x^{-\alpha}) & , x>0 \\ 0 & , x \leqslant 0 \end{cases}$ $(\alpha>0)$

Type III (Weibull): $G(x) \equiv \Psi_\alpha(x) = \begin{cases} \exp(-(-x)^{\alpha}) & , x \leqslant 0 \\ 0 & , x>0 \end{cases}$ $(\alpha>0)$

We will say, then, that the d.f. F belongs to the domain of attraction (for maxima) of G, and denote such fact by $F \in \mathfrak{D}(G)$.

A general description of limit laws and domains of attraction concerning maxima of i.i.d. r.v.'s may be found in Galambos (1978).

Each domain of attraction may be caractherized by necessary and suficient conditions on the tail behaviour of the d.f. F. We will present here the von Mises's suficient condition concerning Type I (Gumbel) domain of attraction:

If

(1) F is absolutely continuous with density function f

(2) $x^F = \sup\{x: F(x)<1\} \leqslant +\infty$

(3) $\exists x_0 : f'(x)<0$, $\forall x \in [x_0, x^F[$ (1.1)

(4) $\lim_{t \uparrow x^F} \frac{f'(t)(1-F(t))}{f^2(t)} = -1$

then $F \in \mathfrak{D}(\Lambda)$.

Our interest in this paper, is the rate of convergence, more precisely, the uniform rate of convergence, of $F^n(x)$, $F \in \mathfrak{D}(\Lambda)$, towards the limit law Λ. Related results on this subject, may be seen in Fisher and Tippet (1928), Uzgören (1954), Anderson (1971, 1976), Galambos (1978), Gomes (1978, 1984), Hall (1979) and Cohen (1982a, 1982b).

We refer a result of great interest to us (Anderson (1971)):

Let F be a absolutely continuous d.f., with right endpoint $x^F = +\infty$, and such that

$$\lim_{x\to\infty} \frac{d}{dx}\left(\frac{1}{k(x)}\right) = 0 \tag{1.2}$$

where

$$k(x) = \frac{-f(x)}{F(x)\log F(x)} . \tag{1.3}$$

Let a_n and b_n be defined by

$$\begin{aligned} F(b_n) &= \exp(-1/n) \\ a_n &= 1/k(b_n) \end{aligned} \tag{1.4}$$

Furthermore,suppose that $k'(x)$ has constant signal for large x and that

$$\lim_{x\to\infty} \frac{k''(x)}{k(x)\ k'(x)} = 0 \tag{1.5}$$

then

$$F^n(a_n x+b_n) - \Lambda(x) = \lambda(x)\ d_n(x)\ (1+o(1)) \tag{1.6}$$

where

$$d_n(x) = -\frac{1}{2}\, x^2 \left(\frac{1}{k}\right)'_{b_n} (1+o(1)) \tag{1.7}$$

uniformly for x in finite intervals.

2 - Distribution functions in a class $\mathcal{L}$

We shall work here with d.f.'s in the following class:

Definition (Gomes (1984))

$\mathcal{L}$ is the class of d.f.'s of the form

$$F(x) = \{1-A(\gamma x+\delta)^p \exp(-(\gamma x+\delta)^q)(1+\varepsilon(x))\} I_{[y_o,+\infty[}$$

where y_o is the greatest real solution of the equation $A(\gamma x+\delta)^p \exp(-(\gamma x+\delta)^q)(1+\varepsilon(x))=1$, and A chosen so that the solution does exist. Also $\delta, p\in R$, $q,\gamma\in R^+$ and $\lim_{x\to\infty} \varepsilon(x)=0$.

For $F\in\mathcal{L}$, $(p,q)\neq(0,1)$ and some limit conditions on $\varepsilon(x)$,$\varepsilon'(x)$ and $\varepsilon''(x)$, may be proved that $\lim_{x\to\infty} \frac{d}{dx}\left(\frac{1}{k(x)}\right)=0$, where $k(x)$ is given by (1.3), and as we have $\frac{d}{dx}\left(\frac{1}{k(x)}\right) = -\frac{f'(x)}{F(x)\log F(x)} + 1 + \log F(x)$, F is, by (1.1), in the domain of attraction of Gumbel law.

Moreover, it can be seen that

$$k(x) = q\gamma\ (\gamma x+\delta)^{q-1} - \frac{p\gamma}{\gamma x+\delta} - \frac{\varepsilon'(x)}{1+\varepsilon(x)} + o\left(\frac{1}{x^{\xi}}\right)\ ,\ \xi>0 \qquad (2.1)$$

and that conditions presented in Anderson's result hold for $F \in \mathcal{L}$, assuring uniform convergence on finite intervals.

The Normal and Gamma d.f.'s are obviously members of the class $\mathcal{L}$, and will be considered in the sequel.

Indeed, let $\Phi(x)$ be the Normal d.f. and $\phi(x)$ its derivative. It is well known that

$$1 - \Phi(x) = \frac{\phi(x)}{x}\left[1 - \frac{1}{x^2} + O\left(\frac{1}{x^4}\right)\right],\ x>0 \qquad (2.2)$$

and we can even write

$$\Phi(x) = 1 - \frac{1}{2\sqrt{\pi}}\left(\frac{x}{\sqrt{2}}\right)^{-1}\exp\left[-\left(\frac{x}{\sqrt{2}}\right)^2\right](1+\varepsilon(x))$$

with $\varepsilon(x) = -1/x^2 + 3/x^4 + O(1/x^6)$, so that $\varepsilon(x)\to 0$ as $x\to\infty$, i.e, $\Phi\varepsilon\mathcal{L}$ with $A=1/2\sqrt{\pi}$, $\gamma=1/\sqrt{2}$, $\delta=0$, $p=-1$ and $q=2$.

Computational results for $k(x) = -\dfrac{\phi(x)}{\Phi(x)\log\Phi(x)}$, and for $k'(x)$ are presented in the following tables:

x	k(x)	x	k(x)	x	k(x)	x	k(x)
-3.9	0.415284	-1.9	0.643597	0.0	1.151104	2.0	2.400740
-3.8	0.423126	-1.8	0.660614	0.1	1.192743	2.1	2.484415
-3.7	0.431239	-1.7	0.678398	0.2	1.236388	2.2	2.569455
-3.6	0.439636	-1.6	0.696995	0.3	1.282110	2.3	2.655738
-3.5	0.448330	-1.5	0.716451	0.4	1.327974	2.4	2.743133
-3.4	0.457339	-1.4	0.736814	0.5	1.380039	2.5	2.831553
-3.3	0.466676	-1.3	0.758137	0.6	1.432255	2.6	2.920868
-3.2	0.476359	-1.2	0.780473	0.7	1.486965	2.7	3.010756
-3.1	0.486406	-1.1	0.803880	0.8	1.543898	2.8	3.101760
-3.0	0.496836	-1.0	0.828418	0.9	1.603173	2.9	3.193210
-2.9	0.507668	-0.9	0.854150	1.0	1.664777	3.0	3.285106
-2.8	0.518926	-0.8	0.881143	1.1	1.728759	3.1	3.377531
-2.7	0.530630	-0.7	0.909464	1.2	1.795034	3.2	3.470562
-2.6	0.542305	-0.6	0.939134	1.3	1.863584	3.3	3.563916
-2.5	0.555477	-0.5	0.970377	1.4	1.934253	3.4	3.657063
-2.4	0.563673	-0.4	1.003118	1.5	2.007271	3.5	3.750742
-2.3	0.582423	-0.3	1.037484	1.6	2.082253	3.6	3.845057
-2.2	0.596756	-0.2	1.073552	1.7	2.159204	3.7	3.939754
-2.1	0.611707	-0.1	1.111400	1.8	2.238013	3.8	4.034791
-2.0	0.627303			1.9	2.318566	3.9	4.130069

Table of k(x) - normal parent

x	k'(x)	x	k'(x)	x	k'(x)	x	k'(x)
-3.9	0.07743	-1.9	0.16646	0.0	0.40659	2.0	0.82944
-3.8	0.08004	-1.8	0.17394	0.1	0.42630	2.1	0.84377
-3.7	0.08275	-1.7	0.18184	0.2	0.44672	2.2	0.85685
-3.6	0.08561	-1.6	0.19017	0.3	0.46783	2.3	0.86869
-3.5	0.08861	-1.5	0.19901	0.4	0.48955	2.4	0.87933
-3.4	0.09179	-1.4	0.20834	0.5	0.51183	2.5	0.88885
-3.3	0.09513	-1.3	0.21820	0.6	0.53457	2.6	0.89731
-3.2	0.09865	-1.2	0.22862	0.7	0.55767	2.7	0.90478
-3.1	0.10236	-1.1	0.23962	0.8	0.58102	2.8	0.91137
-3.0	0.10628	-1.0	0.25125	0.9	0.60450	2.9	0.91714
-2.9	0.11040	-0.9	0.26351	1.0	0.62796	3.0	0.92217
-2.8	0.11475	-0.8	0.27645	1.1	0.65124	3.1	0.92655
-2.7	0.11934	-0.7	0.29009	1.2	0.67421	3.2	0.93034
-2.6	0.12413	-0.6	0.30444	1.3	0.69670	3.3	0.93359
-2.5	0.12923	-0.5	0.31954	1.4	0.71856	3.4	0.93636
-2.4	0.13467	-0.4	0.33541	1.5	0.73964	3.5	0.93868
-2.3	0.14035	-0.3	0.35204	1.6	0.75982	3.6	0.94058
-2.2	0.14636	-0.2	0.36945	1.7	0.77897	3.7	0.94208
-2.1	0.15269	-0.1	0.38764	1.8	0.79700	3.8	0.94317
-2.0	0.15937			1.9	0.81384	3.9	0.94393

Table of k'(x) - normal parent

The corresponding figures are:

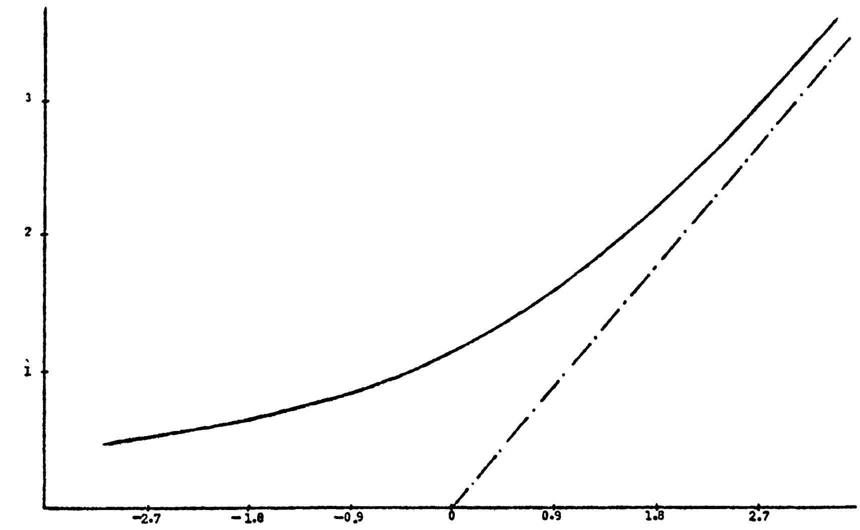

Fig 1 - Graphic representation of k(x)
- normal parent.

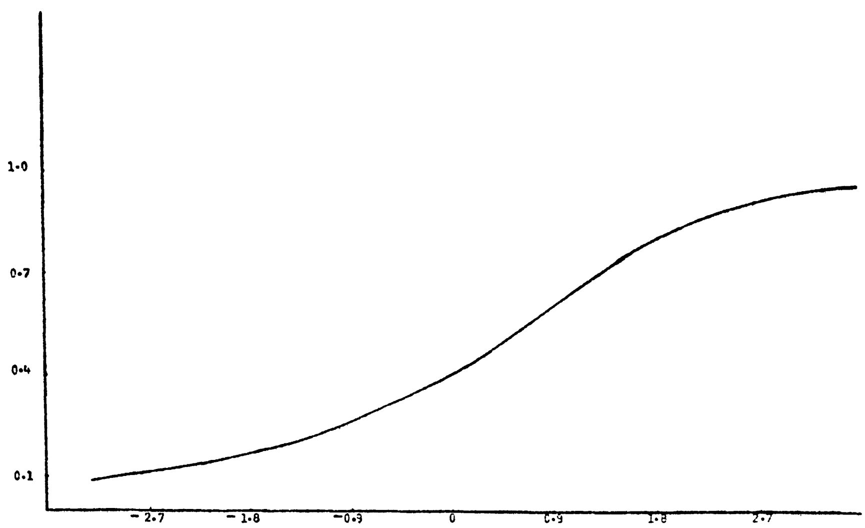

Fig 2 - Graphic representation of k'(x)
- normal parent.

Some properties of k(x) and k'(x), suggested to us by these figures, and needed later in section 3, are stated here without proof (for an analytical proof see L. Canto e Castro (1985)).

Proposition 2.1

For a normal parent d.f.

$$k(x) \text{ is strictly increasing}$$
$$\lim_{x \to +\infty} k(x) = +\infty \tag{2.3}$$
$$k(x) > x$$

Proposition 2.2

For a normal parent d.f.

$$k'(x) \text{ is strictly increasing}$$
$$\lim_{x\to+\infty} k'(x) = 1 \qquad (2.4)$$
$$k'(x) > 0 .$$

Let $G(\alpha,1)$ denote a gamma r.v., with density function

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x} I_{[0,+\infty[} .$$

Its d.f. has the following expansion

$$1 - F(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x} \left[1 + \frac{\alpha-1}{x} + \frac{(\alpha-1)(\alpha-2)}{x^2} + O\left(\frac{1}{x^3}\right)\right] \qquad (2.5)$$

$$= \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x} (1 + \varepsilon(x)) , \quad x>0$$

where

$$\varepsilon(x) = \frac{\alpha-1}{x} + \frac{(\alpha-1)(\alpha-2)}{x^2} + O\left(\frac{1}{x^3}\right) \to 0 , \text{ as } x\to\infty .$$

Thus $F\in\mathcal{C}$ with $A=1/\Gamma(\alpha)$, $\gamma=1$, $\delta=0$, $p=\alpha-1$, $q=1$.

It can be shown that the asymptotic expansions of $k(x)$ and $k'(x)$ are, here

$$k(x) = 1 - \frac{\alpha-1}{x} + \frac{\alpha-1}{x^2} + O\left(\frac{1}{x^3}\right)$$

and

$$k'(x) = \frac{\alpha-1}{x^2} + O\left(\frac{1}{x^3}\right) ,$$

so that their behaviour for large x is quite different from normal case, depending on having $\alpha>1$ or $\alpha<1$, as it is illustrated in the next graphic representations:

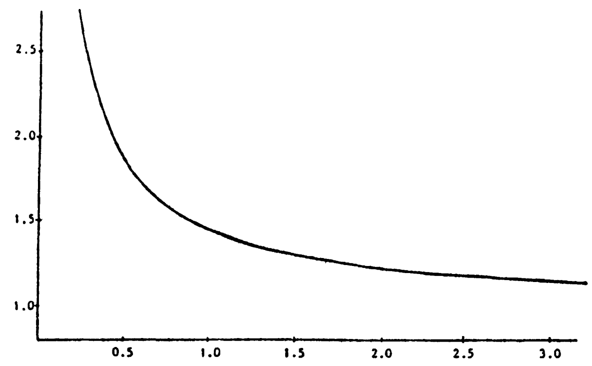

Fig 3 - k(x) for a G(0.5,1) r.v.

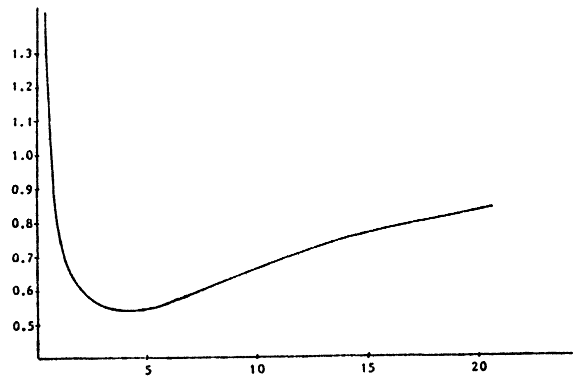

Fig 4 - k(x) for a G(5,1) r.v.

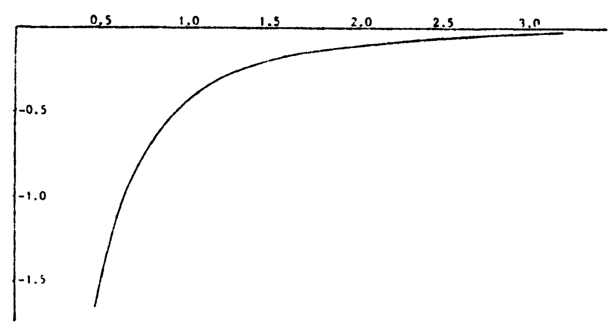

Fig 5- k'(x) for a G(0.5,1) r.v.

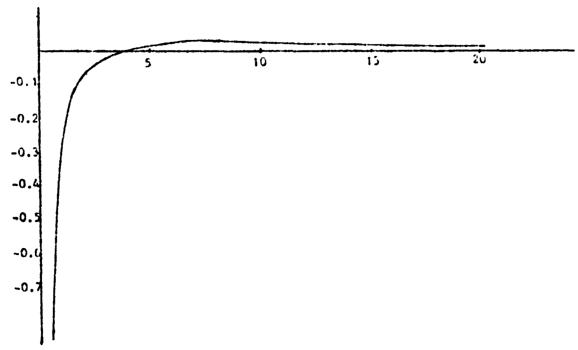

Fig 6- k'(x) for a G(5,1) r.v.

For $\alpha>1$ we have:

Lemma 2.1

If F is the d.f. of the r.v. $G(\alpha,1)$ and f its density function, then

$$\frac{1}{1+\frac{\alpha-1}{x}+\dots+\frac{(\alpha-1)\dots(\alpha-p-1)}{x^{p+1}}} < \frac{f(x)}{1-F(x)} < \frac{1}{1+\frac{\alpha-1}{x}+\dots+\frac{(\alpha-1)\dots(\alpha-p)}{x^{p}}}$$

for $p-1<\alpha<p$, and p integer.

Lemma 2.2

Let F be the d.f of G(α,1) with α>1. Then we have

$$k(x)\left(\frac{\alpha-1}{x^2} + R_2(\alpha,x) - \frac{f(x)}{2F(x)}\right) < k'(x) < k(x)\left(\frac{\alpha-1}{x^2} + R_1(\alpha,x)\right)$$

where $R_1(\alpha,x) = \frac{1}{x^2}\sum_{j=1}^{p}\frac{b_{1j}(\alpha)}{x^j}$ and $R_2(\alpha,x) = \frac{1}{x^2}\sum_{j=1}^{p+1}\frac{b_{2j}(\alpha)}{x^j}$

can be easily calculated for every value of α.

Proof:

Remarking that $f'(x) = f(x)\left(\frac{\alpha-1}{x} - 1\right)$ it follows that

$$k'(x) = k(x)\left(\frac{\alpha-1}{x} - 1 + k(x)(1 + \log F(x))\right).$$

Besides, it can be proved that, in a general case,

$$\frac{f(x)}{1-F(x)} + \frac{f(x)}{2} < k(x) < \frac{f(x)}{1-F(x)} + \frac{f(x)}{F(x)},$$

so that

$$k(x)\left(\frac{\alpha-1}{x} - 1 + \frac{f(x)}{1-F(x)} - \frac{f(x)}{2F(x)}\right) < k'(x) < k(x)\left(\frac{\alpha-1}{x} - 1 + \frac{f(x)}{1-F(x)}\right).$$

The result is a consequence of Lema 2.1 since we may write

$$1 - \frac{\alpha-1}{x} + \frac{\alpha-1}{x^2} + R_2(\alpha,x) < \frac{f(x)}{1-F(x)} < 1 - \frac{\alpha-1}{x} + \frac{\alpha-1}{x^2} + R_1(\alpha,x).$$

We may thus state the following results:

Proposition 2.3

For a Gamma(α,1) parent d.f.

- $\lim_{x\to+\infty} k(x) = 1$, $\lim_{x\to 0} k(x) = +\infty$ $(\alpha>0)$

- $k(x)$ is strictly increasing after some x_o, when $\alpha>1$.

Proposition 2.4

For a Gamma$(\alpha,1)$ parent d.f.

– $\lim_{x\to+\infty} k'(x) = 0$, $\lim_{x\to 0} k'(x) = -\infty$ $(\alpha>0)$

– $k'(x)>0$, after some x_o when $\alpha>1$.

3 - Uniform rate of convergence, for Normal and Gamma parents, as members of the class $\mathcal{L}$.

It is our interest now, to prove that convergence in (1.6) is uniform whenever F is a Normal or Gamma d.f. .

Considering that $\Delta_n(x) = F^n(a_n x+b_n) - \Lambda(x)$ may be writen in the form

$$\Delta_n(x) = d_n(x)\lambda(x+\psi_n(x)) \text{ , with } \psi_n(x) \text{ between 0 and } d_n(x), \quad (3.1)$$

$d_n(x)$ given by

$$d_n(x) = \frac{x^2}{2}\frac{k'(b_n + a_n\theta_n(x))}{(k(b_n))^2} \text{ , } \theta_n(x) \text{ between 0 and } x \text{ ,} \quad (3.2)$$

we have

Theorem 3.1

Let Φ be the Normal d.f., a_n and b_n given by (1.4), then

$$\frac{c_1}{\log n} < \sup_{x\in R} |\Phi^n(a_n x + b_n) - \Lambda(x)| < \frac{3}{\log n} \text{ , } \forall_n \quad (3.3)$$

for some constant c_1 .

To prove this theorem we will need the following results:

Lemma 3.1

For d.f.'s in the class $\mathcal{L}$ we have

$$\left(\frac{1}{k}\right)'_{b_n} = \frac{q-1}{q}\frac{1}{\log n}\left(1 - \frac{p}{q}\frac{\log\log n}{\log n}\right) + o\left(\frac{1}{(\log n)^2}\right)$$

(Proof in Gomes(1978)).

Lemma 3.2

For b_n given by (1.4) and for $n \geq n_o \geq 7$ we have

$$b_n^2 < 2\log n \tag{3.4}$$

and

$$b_n^2 > 2\log n - \log\log n - \log 4\pi + 2\log\left(1 - \frac{1}{b_{n_o}^2}\right) . \tag{3.5}$$

Proof:

From $\frac{\phi(x)}{x}\left(1 - \frac{1}{x^2}\right) < 1 - \Phi(x) < \frac{\phi(x)}{x}$, $x>0$,

follows that

$$\frac{1}{\sqrt{2\pi}\, b_n} e^{-b_n^2/2} > 1 - e^{-1/n} .$$

After some calculations, we get

$$b_n^2 < -2\log(1 - e^{-1/n}) - 2\log b_n - 2\log 2\pi < 2\log n - 2\log b_n + 2\log\frac{e}{\sqrt{2\pi}}$$

and as, for $n \geq 7$, $b_n \geq b_7 > 1.11 > e/\sqrt{2\pi}$, (3.4) follows immediately.

On the other hand, we will have

$$\frac{1}{\sqrt{2\pi}} e^{-b_n^2/2} < (1 - e^{-1/n})\, b_n \left(1 - \frac{1}{b_n^2}\right)^{-1}$$

giving, for $n \geq n_o$

$$b_n^2 > 2\log n - \log\log n - \log 4\pi + 2\log\left(1 - \frac{1}{b_{n_o}^2}\right)$$

what completes the demonstration.

We also need some numerical results concerning special functions. They are stated here without proof:

$$\sup_{x>0} e^{-x} \frac{x^2}{2} < 0.28 \tag{3.6}$$

$$b_n^2 > 0.5 \log n \text{ , for } n \geq 21 \tag{3.7}$$

$$\sup_{n\geq 21} \frac{\log\log b_n^2}{b_n} < 0.264 \qquad (3.8)$$

$$\sup_{x>0} \frac{(\log x^2)^\alpha}{x^\beta} = \left(\frac{2\alpha}{e\beta}\right)^\alpha, \qquad \alpha,\beta>0 \qquad (3.9)$$

Proof of Theorem 3.1

The lower bound in (3.3) is obvious from (1.6) , (1.7) and Lemma 3.1, since we conclude that $\Phi^n(a_nx+b_n)-\Lambda(x)$ is $O(1/\log n)$, as $n \to \infty$, so that its supreme will certainly be greater than $c_1/\log n$ for some c_1.

To show that the upper bound holds, let us consider, separately, the intervals $[0,+\infty[$, $]-c_n,0[$ and $]-\infty,-c_n]$, with $c_n=\log\log b_n^2$.

We note that, as $3/\log n>1$ for $n<21$, we will always consider $n\geq 21$.

From (3.1), (3.2), and considering the properties of $k'(x)$ in proposition 2.2, we have

$$0 \leq \Phi^n(a_nx+b_n) - \Lambda(x) = \frac{x^2}{2} \frac{k'(b_n+a_n\theta_n(x))}{\left(k(b_n)\right)^2} \lambda(x+\psi_n(x))$$

$$\leq \frac{x^2}{2\left(k(b_n)\right)^2} \lambda(x+\psi_n(x)) ,$$

with $\psi_n(x)$ between 0 and $d_n(x)$.

Then, for $x\in[0,+\infty[$, it follows, since $0<\psi_n(x)<d_n(x)$

$$|\Phi^n(a_nx+b_n) - \Lambda(x)| = \Phi^n(a_nx+b_n) - \Lambda(x)$$

$$\leq \frac{1}{\left(k(b_n)\right)^2} \frac{x^2}{2} \lambda(x)$$

$$\leq \frac{1}{\left(k(b_n)\right)^2} \frac{x^2}{2} e^{-x}$$

and as $\left(k(b_n)\right)^2 > b_n^2$ and $\frac{x^2}{2} e^{-x} < 0.28$ we will have, together with (3.5)

$$|\Phi^n(a_n x+b_n) - \Lambda(x)| < \frac{0.28}{b_n^2} < \frac{0.56}{\log n} \quad , \text{ for } x>0.$$

Suppose, now, that $x\varepsilon]-c_n,0[$. We get

$$|\Phi^n(a_n x+b_n) - \Lambda(x)| = d_n(x)\ \lambda(x+\psi_n(x))$$

$$< \frac{1}{(k(b_n))^2}\ \frac{x^2}{2}\ e^{-x}\ \Lambda\left(x + \frac{1}{(k(b_n))^2}\ \frac{x^2}{2}\right) .$$

After studying functions as $e^{-x}\ x^2/2a\ \Lambda(x + x^2/2a)$ for different values of a>0, we conclude that they are all strictly decreasing in $[-1.06\ ,\ 0]$ and have only one local maximum on the right of the only local minimum, for x in $]-\infty\ ,\ -1.06[$.

Hence, for $c_n \leqslant 1.06$ we have

$$|\Phi^n(a_n x+b_n)-\Lambda(x)| \leqslant \frac{1}{2(k(b_n))^2}(\log\log b_n^2)^2(\log b_n^2)\,\Lambda\left(-\log\log b_n^2+\frac{1}{2}\left(\frac{\log\log b_n^2}{k(b_n)}\right)^2\right)$$

$$< \frac{1}{2b_n^2}\ (\log b_n^2)^2\ \Lambda(-\log\log b_n^2 + \tfrac{1}{2}(0.264)^2)$$

using (3.8), and because $(\log\log x)^2 < \log x$ for x>1.65.

Now

$$\Lambda(-\log\log x + c) = \exp(-\exp(\log\log x - c)) = x^{-e^{-c}}$$

so that, using (3.9),

$$|\Phi^n(a_n x+b_n) - \Lambda(x)| \leqslant \frac{1}{2b_n^2}\ \frac{(\log b_n^2)^2}{b_n^{1.93}} < \frac{0.295}{b_n^2} < \frac{0.59}{\log n}$$

for $x\varepsilon[-c_n,0]$ and $c_n \leqslant 1.06$.

When $c_n>1.06$, we can use the fact that $k(x)<x$ to get

$$c_n = \log\log b_n^2 < k(b_n)\ \frac{\log\log b_n^2}{b_n} < 0.264\ k(b_n)$$

so that $c_n>1.06$ implies $k(b_n)>4$.

The study made on the functions mentioned above, shows that, for $a \geqslant 16$, its local maximum point will be in the interval $[-1.14,-1.06]$.

Hence

$$\frac{1}{(k(b_n))^2}\frac{x^2}{2}e^{-x}\Lambda\left(x+\frac{1}{(k(b_n))^2}\frac{x^2}{2}\right) < \frac{(-1.14)^2}{2(k(b_n))^2}e^{-1.14}\Lambda\left(-1.06+\frac{(1.14)^2}{32}\right)$$

$$< \frac{0.125}{b_n^2}, \quad \text{for } c_n > 1.06.$$

Therefore, for any c_n, we can conclude that, for $x \in [-c_n, 0[$

$$|\Phi^n(a_n x+b_n) - \Lambda(x)| < \frac{0.59}{\log n}.$$

Finally, for $x<-c_n$, the upper bound of $\Delta_n(x)$ follows almost immediately if we remember that $\Phi^n(a_n x+b_n)$ is greater than $\Lambda(x)$ and that both of them are positive and strictly increasing.

We get

$$\Phi^n(a_n x+b_n) - \Lambda(x) \leqslant \sup_{x\in[-c_n,0]} (\Phi^n(a_n x+b_n) - \Lambda(x)) + \Lambda(-c_n)$$

$$< \frac{0.59}{\log n} + \frac{1}{b_n^2}$$

$$< \frac{2.59}{\log n}.$$

Summarizing,

$$s_1 = \sup_{x\in[0,+\infty[} |\Delta_n(x)| < \frac{0.56}{\log n},$$

$$s_2 = \sup_{x\in[-c_n,0[} |\Delta_n(x)| < \frac{0.59}{\log n},$$

and

$$s_3 = \sup_{x\in]-\infty,-c_n[} |\Delta_n(x)| < \frac{2.59}{\log n}$$

what completes the proof.

Remark 1 - The upper bounds shown before may be significantly reduced for large values of n, and for $n \geq 10^6$, we will have, for instance

$$s_1 < \frac{0.175}{\log n}, \quad s_2 < \frac{0.18}{\log n}, \quad s_3 < \frac{0.31}{\log n}.$$

Remark 2 - The methodology used in the last proof has been inspired on a similar result due to Hall (1979), where he has also obtained a uniform rate of convergence of order $1/\log n$, considering the attraction coefficients α_n and β_n such that $f(\beta_n)/\beta_n = 1/n$ and $\alpha_n = 1/\beta_n$. Except for x in the interval $]-\infty, -c_n[$, where the upper bounds are the same, our upper bounds are smaller than the ones obtained in Hall's paper.

A uniform rate of convergence result for $G(\alpha,1)$, $\alpha>1$, is now stated:

Theorem 3.2

Let F be the d.f. of $G(\alpha,1)$ r.v., $\alpha>1$, a_n and b_n given by (1.4), then

$$\frac{c_1}{(\log n)^2} < \sup_{x\in R} |F^n(a_n x + b_n) - \Lambda(x)| < \frac{c_2}{(\log n)^2}$$

for some c_1 and c_2 positive constants.

Proof:

The lower bound follows from Lemma 3.1 since q=1.

Using a similar technique as before, the proof will be different for the intervals $]-\infty,-c_n[$, $[-c_n, 0[$ and $[0,+\infty[$.

Now $c_n = (\log\log b_n^2)/a_n$ and we consider $n > n_o$, where n_o is such that $k'(b_{n_o} - \log\log(b_{n_o})^2) > 0$.

For $x > -c_n$, we have, using (3.2), Lemma 2.2, and noting that $k'(x)>0$ implies $k(x) \leq 1$,

$$0 < d_n(x) = \frac{x^2}{2} \frac{k'(b_n + a_n\theta_n(x))}{\left(k(b_n)\right)^2} \quad , \ \theta_n(x) \text{ between } 0 \text{ and } x$$

$$\leqslant \frac{x^2}{2} \frac{k(b_n + a_n\theta_n(x)) \ A}{(b_n+a_n\theta_n(x))^2 \left(k(b_n)\right)^2}$$

$$\leqslant \frac{x^2}{2} \frac{A}{(b_n+a_n\theta_n(x))^2 \left(k(b_n)\right)^2}$$

for some $A>0$.

So, for $x>0$, it follows that

$$F^n(a_n x + b_n) - \Lambda(x) = d_n(x) \ \lambda(x+\psi_n(x)) \ , \quad 0 < \psi_n(x) < d_n(x)$$

$$\leqslant \frac{A}{b_n^2\left(k(b_n)\right)^2} \frac{x^2}{2} \ e^{-x}$$

$$\leqslant \frac{A_1}{(\log n)^2} \qquad , \quad (A_1>0)$$

since $\left(b_n^2\left(k(b_n)\right)^2\right)^{-1} = O\left((b_n^2)^{-1}\right) = O\left(1/(\log n)^2\right)$.

For $x\varepsilon\left[-c_n,0\right[$ we have

$$0 \leqslant F^n(a_n x+b_n)-\Lambda(x) \leqslant \frac{A\ x^2}{2(b_n-\log\log b_n^2)\left(k(b_n)\right)^2} \ e^{-x}\Lambda\left(x+\frac{A\ x^2}{2\left[(b_n-\log\log b_n^2)k(b_n)\right]^2}\right)$$

$$\leqslant \frac{a\ x^2}{2\ b_n^2} \ e^{-x} \ \Lambda\left(x + \frac{a\ x^2}{2\ b_n^2}\right) .$$

Analagously to the Normal case, this last function is strictly decreasing in $\left[-c_n,0\right[$ if $c_n<1.06$, and consequently

$$F^n(a_n x + b_n) - \Lambda(x) < \frac{a\ c_n^2}{2\ b_n^2} \ e^{c_n} \ \Lambda\left(c_n + \frac{a\ c_n^2}{2\ b_n^2}\right)$$

$$< \frac{a}{2\ b_n^2} \frac{(\log b_n^2)^2}{b_n^\beta} < \frac{A_2}{(\log n)^2}$$

Consequently, for $x<-c_n$, we have

$$|F^n(a_nx+b_n) - \Lambda(x)| < F^n(b_n-a_nc_n) - \Lambda(-c_n)$$

$$< \frac{B}{b_n^2} + \Lambda(-\log\log b_n^2)$$

$$< \frac{A_4}{(\log n)^2} \quad , \text{ with } A_4>0 \ .$$

Thus, we have shown that, for x in any of the three intervals, the convergence is uniform of order $1/(\log n)^2$, so that c_2 can be taken as the maximum of the $\{A_i\}$, i=1,2,3,4 , and ($\log n_o)^2$,(note that all these constants depend essncialy on the value of α).

When we tried to develop the upper bound in (3.10) for the case $\alpha<1$, a few problems were presented, related, fundamentally, to the fact of having k'(x) less than zero, as suggests the graphic representation in fig.5, (the analytical proof of the properties of k(x) and k'(x) for $\alpha<1$, has some unsolved points, as well).

It is our conviction,however, that the solution of these problems will be very important to establish a uniform rate of convergence result, for a large family of d.f.'s in the clas $\mathcal{L}$.

Aknowledgment: the author is grateful to I.N.I.C. for financial support to present this paper at "Ecole d'Eté de Calcul des Probabilités - St.-Flour".

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Reçu en Octobre 1986