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ON A NON SYMMETRIC OPERATION FOR TWO-PARAMETER MARTINGALES

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ABSTRACT: In this paper we define a martingale $M*N$ such that by symmetrization provides the martingale $\widetilde{MN}$ which takes part in the multi-dimensional Itô formula for continuous two-parameter martingales.

0. INTRODUCTION

In the compact version of Itô formula (see [3], [8]) for a continuous two-parameter L^4-martingale M, a new martingale $\tilde{M}$ is involved. By polarization, we can define $\widetilde{MN}$, and this martingale takes part in the multi-dimensional Itô formula (see [8]). In this work, we define a martingale $M*N$ (M and N continuous L^p-martingales, $p \geq 2$). Roughly speaking, $M*N$ is the limit of sums like $\sum_{i,j} M(\Delta^1_{ij})N(\Delta^2_{ij})$. Then,

$$\widetilde{MN} = \frac{1}{2} M*N + \frac{1}{2} N*M .$$

We prove a convergence in $\underline{\underline{H}}^{p/2}$ for $M*N$ which is very useful to compute $M*N$ and $\tilde{M}$ in several cases. We also compute the quadratic variation of $M*N$.

We should point out that the martingale $\tilde{M}$, written J_M, for a continuous, strong L^4-martingale, was defined by Cairoli-Walsh [2]. The martingale $M*N$, written J_{MN}, for two continuous L^4-martingales appeared in Guyon-Prum [4], where it was defined by a double stochastic integral of a corner function.

1. NOTATIONS AND DEFINITIONS

We consider on $\mathbb{R}_+^2$ the usual partial ordering $(s,t) \leq (s',t')$ if $s \leq s'$ and $t \leq t'$; we will write $(s,t) < (s',t')$ if $s < s'$ and $t < t'$. For $z, z' \in \mathbb{R}_+^2$, $z < z'$, $]z,z']$ will be the set $\{\zeta \in \mathbb{R}_+^2 : z < \zeta \leq z'\}$, and similarly we define $[z,z']$. Put $R_z =]0,z]$.

Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space and let $\{\underline{\underline{F}}_z , z \in \mathbb{R}_+^2\}$ be an increasing, complete, right-continuous family of sub-σ-fields of $\underline{\underline{F}}$; we also assume that $\{\underline{\underline{F}}_z , z \in \mathbb{R}_+^2\}$ satisfies the condition (F4) of Cairoli-Walsh [2]: If we define $\underline{\underline{F}}_{s\infty} = \bigvee_{v \geq 0} \underline{\underline{F}}_{sv}$ and $\underline{\underline{F}}_{\infty t} = \bigvee_{u \geq 0} \underline{\underline{F}}_{ut}$, then $\underline{\underline{F}}_{s\infty}$ and $\underline{\underline{F}}_{\infty t}$ are conditionally independent given $\underline{\underline{F}}_{st}$.

A stochastic process $M = \{M_z , z \in \mathbb{R}_+^2\}$ adapted to $\{\underline{\underline{F}}_z , z \in \mathbb{R}_+^2\}$ and integrable is said to be a martingale if for each $z \leq z'$, $E[M_{z'} | \underline{\underline{F}}_z] = M_z$. We will denote for $\underline{\underline{M}}_c^p$ $(p \geq 1)$ the set of all sample continuous L^p-martingales $M = \{M_z , z \in \mathbb{R}_+^2\}$ (that is, $E[|M_z|^p] < \infty$, for all $z \in \mathbb{R}_+^2$); and for $\underline{\underline{M}}_c^p(z_o)$ the set of all sample continuous martingales $M = \{M_z , z \in [0,z_o]\}$ with $E[|M_{z_o}|^p] < \infty$.

For a process $X = \{X_z , z \in \mathbb{R}_+^2\}$ the increment over $]z,z']$ $(z=(s,t), z'=(s',t'))$ is $X(]z,z']) = X_{z'} - X_{st'} - X_{s't} + X_z$.

A process $A = \{A_z , z \in \mathbb{R}_+^2\}$ is said to be increasing if it is right-continuous, $A_z = 0$ on the axes, and $A(]z,z']) \geq 0$ for all rectangle $]z,z']$. Given a martingale M of $\underline{\underline{M}}_c^2$, we will denote by $<M> = \{<M>_z , z \in \mathbb{R}_+^2\}$ a continuous version of the quadratic variation of M (see [7]). This process $<M>$ is increasing.

Let $z=(s,t)$ be a point of $\mathbb{R}_+^2$. A grid over $[0,z]$ will be a finite sub-set $\Gamma = \Gamma^1 \times \Gamma^2$ of $[0,z[$, $\Gamma^1 = \{s_1,\dots,s_p\}$, $0=s_1<s_2<\dots<s_p<s$, $\Gamma^2 = \{t_1,\dots,t_q\}$, $0=t_1<t_2<\dots<t_q<t$. For $z' \in]0,z]$, $\Gamma_{z'}$ will be the set $\{z'' \in \Gamma : z'' < z'\}$. If $u=(s_i,t_j)$ is a point of the grid Γ, then, we will write $\Delta_u =]s_i,s_{i+1}]\times]t_j,t_{j+1}]$, $\Delta_u^1 =]s_i,s_{i+1}] \times]0,t_j]$ and $\Delta_u^2 =]0,s_i]\times]t_j,t_{j+1}]$, with the convention $s_{p+1}=s$ and $t_{q+1}=t$. The norm of the grid is the number

$$|\Gamma| = \max_{\substack{i=1,\dots,p \\ j=1,\dots,q}} \{|s_{i+1}-s_i| + |t_{j+1}-t_j|\} .$$

Let $\{\Gamma^n , n \geq 1\}$ be a sequence of grids over $[0,z]$. $\{\Gamma^n , n \geq 1\}$ is said to be a standard one if Γ^{n+1} is a refinement of Γ^n and $\lim_{n\to\infty} |\Gamma^n| = 0$.

If M is a martingale of $\underline{\underline{M}}_c^p$ $(p \geq 2)$, then there exists a martingale $\widetilde{M}$ of $\underline{\underline{M}}_c^{p/2}$ (see [7]) such that for all z_o and for all standard sequence $\{\Gamma^n , n \geq 1\}$ of grids over $[0,z_o]$,

$$\lim_{n\to\infty} \sup_{z\in[0,z_o]} E[|\sum_{u\in\Gamma_z^n} M(\Delta_u^1)M(\Delta_u^2) - \widetilde{M}_z|^{p/2}] = 0 .$$

The next result about one-parameter martingales will be needed (cf. lemma 2.1 of Nualart [7]).

<u>LEMMA 1.1</u> (Nualart): <u>Let</u> $M = \{M_t, t \in \mathbb{R}_+\}$ <u>be a square integrable continuous martingale with respect to an increasing</u>

family of σ-fields $\{\underline{\underline{F}}_t , t \in \mathbb{R}_+\}$ satisfying the usual conditions. Suppose $M_0 = 0$. Fix t_0 and denote by $\Lambda = \{s_1,\dots,s_n\}$, $0=s_1<s_2<\dots<s_n<t_0$ a finite set of points of $[0,t_0]$. Consider another finite set $\Lambda' \supset \Lambda$, whose points can always be written as σ_k^i , $i=1,\dots,n;\ k=1,\dots,r_i$, in such a way that $s_i=\sigma_1^i<\sigma_2^i<\dots<\sigma_{r_i}^i<s_{i+1}$ for all i . Set $|\Lambda| = \max_{i=1,\dots,n} \{|s_{i+1}-s_i|\}$, where $s_{n+1}=t_0$. Then,

$$\lim_{|\Lambda|\downarrow 0} \sup_{\Lambda'\supset\Lambda} E[\sup_i \sum_{k=1}^{r_i} (M(\sigma_{k+1}^i) - M(\sigma_k^i))^2] = 0 ,$$

where by convention, we put $\sigma_{r_i+1}^i=s_{i+1}$

2. THE MARTINGALE M*N

THEOREM 2.1: Let M and N be martingales of $\underline{\underline{M}}_c^p(z_0)$, $p \geq 2$. Then there exists a continuous martingale M*N of $\underline{\underline{M}}_c^{p/2}(z_0)$ such that for every standard sequence of grids $\{\Gamma^n , n \geq 1\}$ over $[0,z_0]$, if we define the martingales S^n as

$$S_z^n = \sum_{u\in\Gamma_z^n} M(\Delta_u^1)N(\Delta_u^2) \quad , \quad z \leq z_0 ,$$

then

$$\text{(i)} \lim_{n\to\infty} \sup_{z\in[0,z_0]} E[|S_z^n - M{*}N_z|^{p/2}] = 0 \qquad (2.1)$$

(ii) For any n , the martingales $S_{t_0}^n = \{S_{st_0}^n, \underline{\underline{F}}_{st_0}, s\leq s_0\}$ and $S_{s_0}^n = \{S_{s_0t}^n, \underline{\underline{F}}_{s_0t}, t\leq t_0\}$ are in $\underline{\underline{H}}^{p/2}$, and $\lim_{n\to\infty} S_{t_0}^n = M{*}N_{t_0}$ and $\lim_{n\to\infty} S_{s_0}^n = M{*}N_{s_0}$ in the convergence of $\underline{\underline{H}}^{p/2}$, that is,

$$\lim_{n\to\infty} E[\sup_{s\leq s_0} |S_{st_0}^n - M{*}N_{st_0}|^{p/2}] = 0 \qquad (2.2)$$

<u>and</u>

$$\lim_{n\to\infty} E[\sup_{t\leq t_o} |S^n_{s_o t} - M*N_{s_o t}|^{p/2}] = 0 .$$

PROOF

Without loss of generality, we can suppose M and N are zero on the axes.

The part (i) of the theorem will be proved adapting the proof of lemma 3.2 of Nualart [7]. The part (ii) for $p > 2$ is an obvious consequence of maximal Doob's inequality; for $p = 2$ it is proved using a modification of that lemma. We detail the main steps of this proof.

For simplicity, $z_o=(1,1)$ is supposed.

a) Let $p > 2$. We consider a grid Γ^n over $[0,1]^2$ and we denote its points by $u=(s_i,t_j)$, $i=1,\dots,p_n$; $j=1,\dots,q_n$, $0=s_1<s_2<\dots<s_{p_n}<1$, $0=t_1<t_2<\dots<t_{q_n}<1$ (we put $(s_{p_n+1},t_{q_n+1})=(1,1)$). By ${}^1\Gamma^n$ we will indicate a grid over $[0,1]^2$ such that has the same projection on the "t" axes as Γ^n. The points of ${}^1\Gamma^n$ will be denoted by $u'=(\sigma_{i'},t_j)$, $i'=1,\dots,p'_n$, $j=1,\dots,q_n$. Let I_i the set $\{i' : \sigma_{i'} \in [s_i,s_{i+1}[\}$. We define

$$\bar{S}^n_z = \sum_{u'\in {}^1\Gamma^n_z} M(\Delta^1_{u'})N(\Delta^2_{u'}) , \quad z \leq (1,1) .$$

Then

$$\lim_{n\to\infty} \sup_{{}^1\Gamma^n} E[|\bar{S}^n_{1,1} - S^n_{1,1}|^{p/2}] = 0 . \qquad (2.3)$$

In fact, this convergence would follow the same argument used in the proof of (3.6) in [7].

Similarly, we denote by ${}^2\Gamma^n$ a grid over $[0,1]^2$ which contains Γ^n and has the same projection on the "s" axes. The points of ${}^{2}\Gamma^{n}$ will be written by $u'=(s_i,\tau_{j'})$, $i=1,\dots,p_n$; $j'=1,\dots,q'_n$, and let J_j be the set $\{j' : \tau_{j'} \in [t_j, t_{j+1}[\}$. We define

$$\bar{\bar{S}}^n_z = \sum_{u' \in {}^{2}\Gamma^{n}} M(\Delta^1_{u'})N(\Delta^2_{u'}) .$$

By symmetry we obtain

$$\lim_{n\to\infty} \sup_{{}^{2}\Gamma^{n}} E[\,|\bar{\bar{S}}^n_{1,1} - S^n_{1,1}|^{p/2}\,] = 0 . \tag{2.4}$$

By the conjunction of (2.3) and (2.4) we obtain (i). By Cairoli-Doob's inequality, there exists a continuous version of $M*N$. The part (ii) is an immediate consequence of the maximal Doob's inequality.

b) Let p=2. With the same notation as above,

$$\lim_{n\to\infty} \sup_{{}^{1}\Gamma^{n}} E[\sup_{s\leq 1}|\bar{S}^n_{s,1} - S^n_{s,1}|] = 0 . \tag{2.5}$$

This can be shown as (3.8) of [7].

By symmetry,

$$\lim_{n\to\infty} \sup_{{}^{2}\Gamma^{n}} E[\sup_{t\leq 1}|\bar{\bar{S}}^n_{1,t} - S^n_{1,t}|] = 0 . \tag{2.6}$$

(2.5) and (2.6) imply (i). The continuity of $M*N$ is proved like [7]. It remains to show (ii) for p=2. We claim that

$$\lim_{n\to\infty} \sup_{{}^{2}\Gamma^{n}} E[\sup_{s\leq 1}|\bar{\bar{S}}^n_{s,1} - S^n_{s,1}|] = 0. \tag{2.7}$$

In fact,

$$E[\sup_{s\leq 1}|\bar{\bar{S}}^n_{s,1} - S^n_{s,1}|] =$$

$$= E[\sup_{s\leq 1} | \sum_{u=(s_i,t_j)\in\Gamma^n} \sum_{j'\in J_j} N(\Delta^2_{u'})M(](s_i\wedge s,t_j),(s_{i+1}\wedge s,\tau_{j'})])|] ,$$

where $\Delta^2_{u'} =]0,s_i]\times]\tau_{j'},\tau_{j'+1}]$, and $N(\Delta^2_{u'})$ does not depend on s. For all i,

$$\sum_j \sum_{j'\in J_j} N(\Delta^2_{u'})M(](s_i\wedge s,t_j),(s_{i+1}\wedge s,\tau_{j'})])$$

is a martingale in s with respect to $\{ \underline{\underline{F}}_{s1} , s\leq 1\}$. If $i\neq i'$, these martingales are orthogonal. Indeed, let $\{\xi_1,\dots,\xi_k\}$, $0=\xi_1<\xi_2<\dots<\xi_k<1$, be a partition of $[0,1]$ which is a refinement of $0=s_1<s_2<\dots<s_{p_n}<1$. Then,

$$\big(M(](s_i\wedge\xi_{k+1},t_j),(s_{i+1}\wedge\xi_{k+1},\tau_{j'})]) - M(](s_i\wedge\xi_k,t_j),(s_{i+1}\wedge\xi_k,\tau_{j'})])\big) \cdot$$

$$\cdot\big(M(](s_{i'}\wedge\xi_{k+1},\bar t_j),(s_{i'+1}\wedge\xi_{k+1},\bar\tau_{j'})]) - M(](s_{i'}\wedge\xi_k,\bar t_j),(s_{i'+1}\wedge\xi_k,\bar\tau_{j'})])\big) = 0,$$

if $i\neq i'$,for all t_j , $\tau_{j'}$, $\bar t_j$, $\bar\tau_{j'}$, because one of the two factors is always zero. Then,

$$\langle\sum_j\sum_{j'}N(\Delta^2_{u'})M(](s_i\wedge\cdot,t_j),(s_{i+1}\wedge\cdot,\tau_{j'})]),\sum_j\sum_{j'}N(\Delta^2_{u'})M(](s_{i'}\wedge\cdot,t_j),(s_{i'+1}\wedge\cdot,\tau_{j'})])\rangle_1 =$$
$$= 0 .$$

By Davis inequality,

$$E[\sup_{s\leq 1} |\bar{\bar S}^n_{s,1} - S^n_{s,1}|] \leq$$

$$\leq C\, E[|\sum_i \langle\sum_j \sum_{j'} N(\Delta^2_{u'})M(](s_i\wedge\cdot,t_j),(s_{i+1}\wedge\cdot,\tau_{j'})])\rangle_1|^{1/2}] .$$

For each i, let $\Lambda_i = \{s^i_1,s^i_2,\dots,s^i_{r_i}\}$ be a finite partition of $[s_i,s_{i+1}[$: $s_i=s^i_1<s^i_2<\dots<s^i_{r_i}<s_{i+1}$. By Fatou's inequality,

$$E[\sup_{s\leq 1} |\bar{\bar S}^n_{s,1} - S^n_{s,1}|] \leq$$

$$\leq C\ E\Big[\Big|\sum_i \lim_{|\Lambda_i|\downarrow 0} \sum_{s_k^i\in\Lambda_i} \Big(\sum_{j,j'} N(\Delta^2_{u'})\big[M(](s_i,t_j),(s^i_{k+1},\tau_{j'})]) - M(](s_i,t_j),(s^i_k,\tau_{j'})])\big]\Big)^2\Big|^{1/2}\Big] =$$

$$= C\ E\Big[\Big|\sum_i \lim_{|\Lambda_i|\downarrow 0} \sum_k \Big(\sum_{j,j'} N(\Delta^2_{u'})M(\Delta^{ij}_{kj'})\Big)^2\Big|^{1/2}\Big] \leq$$

$$\leq C \sup_\Lambda E\Big[\Big|\sum_{i,k}\Big(\sum_{j,j'} N(\Delta^2_{u'})M(\Delta^{ij}_{kj'})\Big)^2\Big|^{1/2}\Big],$$

where $\Delta^{ij}_{kj'} =]s^i_k, s^i_{k+1}] \times]t_j,\tau_{j'}]$, and $\Lambda = \{s^i_k,\ i=1,\dots,p_n,\ k=1,\dots,r_i\}$ is a finite partition of $[0,1]$ which is a refinement of $\{s_1,\dots,s_{p_n+1}\}$. Let $\{f_{ik},\ i=1,\dots,p_n, k=1,\dots,r_i\}$ a family of Rademacher functions over $[0,1]$. By Khintchine and Davis inequalities

$$E\Big[\sup_{s\leq 1} |\bar{\bar{S}}^n_{s,1} - S^n_{s,1}|\Big] \leq$$

$$\leq C \sup_\Lambda E\Big[\int_0^1 \Big|\sum_{i,k,j,j'} N(\Delta^2_{u'})M(\Delta^{ij}_{kj'})f_{ik}(t)\Big|\, dt\Big] =$$

$$= C \sup_\Lambda \int_0^1 E\Big[\Big|\sum_{i,k,j,j'} N(\Delta^2_{u'})M(\Delta^{ij}_{kj'})f_{ik}(t)\Big|\Big]\, dt \leq$$

$$\leq C \sup_\Lambda \int_0^1 E\Big[\Big|\sum_{i,k,j,j'} N(\Delta^2_{u'})^2M(\Delta^{ij}_{kj'})^2\Big|^{1/2}\Big]\, dt \leq$$

$$\leq C\Big(E\Big[\sup_i \sup_j \sum_{j'} N(\Delta^2_{u'})^2\Big]\cdot \sup_\Lambda E\Big[\sum_{i,j,k} \sup_{j'} M(\Delta^{ij}_{kj'})^2\Big]\Big)^{1/2}. \quad (2.8)$$

To bound the first factor of (2.8), let $\{f_{j'},\ j'\in J_j\}$ be a family of Rademacher functions over $[0,1]$. By Khintchine inequality,

$$E\Big[\sup_i \sup_j \sum_{j'} N(\Delta^2_{u'})^2\Big] \leq$$

$$\leq C\ E\Big[\sup_i \Big(\sup_j \int_0^1 \Big|\sum_{j'\in J_j} N(\Delta^2_{u'})f_{j'}(t)\Big|dt\Big)^2\Big] \leq$$

$$\underset{(\text{Doob})}{\leq} C\, E[\sup_j (\int_0^1 | \sum_{j'} (N(1,\tau_{j'+1})-N(1,\tau_{j'}))f_{j'}(t)|dt)^2] \leq$$

$$\leq C\, E[\sup_j \sum_{j'} (N(1,\tau_{j'+1})-N(1,\tau_{j'}))^2] .$$

By maximal Doob's inequality, we can bound the second factor of (2.8):

$$E[\sum_{i,j,k} \sup_{j'} M(\Delta_{kj'}^{ij})^2] =$$

$$= \sum_{i,j,k} E[\sup_{j'} M(\Delta_{kj'}^{ij})^2] \leq C \sum_{i,j,k} E[M(\Delta_{kj}^{i})^2] = C\, E[M_{1,1}^2] ,$$

where $\Delta_{kj}^{i} =]s_k^i, s_{k+1}^i] \times]t_j, t_{j+1}]$.

By (2.5) and (2.8) we have

$$\lim_{n,m\to\infty} E[\sup_{s\leq 1} |S_{s,1}^n - S_{s,1}^m|] = 0. \qquad (2.9)$$

In fact, for n,m, let Γ^{nm} be a grid over $[0,1]$ which has the same projection on the "t" axes as Γ^n , and on the "s" axes as Γ^m . We define

$$S_z^{nm} = \sum_{u\in\Gamma_z^{nm}} M(\Delta_u^1)N(\Delta_u^2) .$$

Then,

$$E[\sup_{s\leq 1} |S_{s,1}^n - S_{s,1}^m|] \leq$$

$$\leq \sup_{m\geq n} E[\sup_{s\leq 1} |S_{s,1}^n - S_{s,1}^{nm}|] + \sup_{n\geq m} E[\sup_{s\leq 1} |S_{s,1}^m - S_{s,1}^{nm}|] ,$$

and taken $n,m \to \infty$, (2.9) holds.

Finally, the convergence in (2.9) is the convergence in $\underline{\underline{H}}^1$, and since the space $\underline{\underline{H}}^1$ is complete, there exists a martingale $S_{s,1}$ of $\underline{\underline{H}}^1$ such that $\lim_{n\to\infty} E[\sup_{s\leq 1} |S_{s,1}^n - S_{s,1}|] = 0$. By (2.1) $\lim_{n\to\infty} \sup_{s\leq 1} E[|S_{s,1}^n - M{*}N_{s,1}|] = 0$. It follows $S_{s,1} = M{*}N_{s,1}$. ■

REMARKS

1) The operation $*$ is not commutative, but it is distributive with respect to the sum either from the rigth or from the left.

2) $\widetilde{M} = M*M$ and $\widetilde{MN} = \frac{1}{2} M*N + \frac{1}{2} N*M$.

3) $\widetilde{M+N} = \widetilde{M} + \widetilde{N} + M*N + N*M$.

This last remark allows to compute $\widetilde{M}$ when M is a sum of factors. Specifically,

COROLLARY 2.2: Let $M_1,\ldots,M_n$ be martingales of $\underline{\underline{M}}_c^p(z_o)$, $p \geqq 2$. Then

$$\widetilde{\sum_{i=1}^{n} M_i} = \sum_{i=1}^{n} \widetilde{M}_i + 2 \sum_{i \neq j} \widetilde{M_i M_j} .$$

3. AN EXAMPLE: THE FILTRATION PRODUCT OF FILTRATIONS GENERATED BY MULTI-DIMENSIONAL BROWNIAN MOTIONS

On the complete probability space $(\Omega,\underline{\underline{F}},P)$ we consider two independent multi-dimensional brownian motions $W = \{(W_s^1,\ldots,W_s^n) , s \in \mathbb{R}_+\}$ and $\hat{W} = \{(\hat{W}_t^1,\ldots,\hat{W}_t^m), t \in \mathbb{R}_+\}$. We will denote by $\{\underline{\underline{F}}_s^1 , s \in \mathbb{R}_+\}$ and $\{\underline{\underline{F}}_t^2 , t \in \mathbb{R}_+\}$ the completed filtrations generated by W and $\hat{W}$ respectively. Set $\bigvee_{s\geqq 0} \underline{\underline{F}}_s^1 = \underline{\underline{F}}_\infty^1$ and $\bigvee_{t\geqq 0} \underline{\underline{F}}_t^2 = \underline{\underline{F}}_\infty^2$. (We might suppose $\underline{\underline{F}} = \underline{\underline{F}}_\infty^1 \vee \underline{\underline{F}}_\infty^2$). We define the product filtration $\{\underline{\underline{F}}_z , z \in \mathbb{R}_+^2\}$ by $\underline{\underline{F}}_{st} = \underline{\underline{F}}_s^1 \vee \underline{\underline{F}}_t^2$. It is known that this filtration is right-continuous and satisfies (F4).

We define the bi-brownian process $W^{ij} = \{W_z^{ij}, z \in \mathbb{R}_+^2\}$ by

$$W_{st}^{ij} (\omega) = W_s^i(\omega)\hat{W}_t^j(\omega) .$$

Let $L_1^2(\mathbb{R}_+\times\Omega)$ be the set of equivalence classes of measurable processes $g:\mathbb{R}_+\times\Omega \longrightarrow \mathbb{R}$ adapted to $\{\underline{\underline{F}}_s^1 , s\in\mathbb{R}_+\}$ and such that $E\int_0^s g^2(x)dx < \infty$, for all s. Similarly, we define $L_2^2(\mathbb{R}_+\times\Omega)$. We will denote by $L^2(\mathbb{R}_+^2\times\Omega)$ the set of equivalence classes of measurable processes $f:\mathbb{R}_+^2\times\Omega \longrightarrow \mathbb{R}$ adapted to $\{\underline{\underline{F}}_z , z\in\mathbb{R}_+^2\}$ and such that $E\int_{R_z} f^2(\zeta)d\zeta < \infty$, for all z.

The results of Brossard-Chevalier ([1]) are extended without difficulty to the multi-dimensional case and we obtain

<u>PROPOSITION 3.1</u>: <u>Let</u> $M = \{M_z , \underline{\underline{F}}_z , z\in\mathbb{R}_+^2\}$ <u>be a square integrable martingale</u>. <u>Then there exists unique functions</u> $g_1,\dots,g_n \in L_1^2(\mathbb{R}_+\times\Omega)$, $h_1,\dots,h_m \in L_2^2(\mathbb{R}_+\times\Omega)$, $f_{11},\dots,f_{nm} \in L^2(\mathbb{R}_+^2\times\Omega)$ <u>such that</u>

$$M_{st} = M_{00} + \sum_{i=1}^{n} \int_0^s g_i(x)dW^i(x) + \sum_{j=1}^{m} \int_0^t h_j(y)d\hat{W}^j(y) + \sum_{i=1}^{n}\sum_{j=1}^{m} \int_{R_{st}} f_{ij}(x,y)dW^{ij}(x,y) .$$

If $M^1_{st} = \int_{R_{st}} f(z)dW^{ij}(z)$ and $M^2_{st} = \int_{R_{st}} \bar{f}(z)dW^{dk}(z)$, by means of the Itô formula we can compute $\tilde{M}^1$ (see [3]) and by means of the multi-dimensional version, we can compute $\widetilde{M^1M^2}$ Exactly,

$$\tilde{M}^1_{st} = \int_{R_{st}} \left(\int_0^{t'} f(s',y)d\hat{W}^j(y)\right)\left(\int_0^{s'} f(x,t')dW^i(x)\right)dW^{ij}(s',t')$$

and

$$\widetilde{M^1M^2}_{st} = \frac{1}{2}\int_{R_{st}} \left(\int_0^{t'} f(s',y)d\hat{W}^j(y)\right)\left(\int_0^{s'} \bar{f}(x,t')dW^d(x)\right)dW^{ik}(s',t') +$$

$$+ \frac{1}{2}\int_{R_{st}} \left(\int_0^{t'} \bar{f}(s',y)d\hat{W}^k(y)\right)\left(\int_0^{s'} f(x,t')dW^i(x)\right)dW^{dj}(s',t') .$$

REMARK: The computation of $M^1 * M^2$ cannot be reached by means of Itô formula, because this martingale does not appear in this formula. The expression of $M^1 * M^2$ can be deduced calculating the limit of (2.1). In order to obtain the explicit formula for $M^1 * M^2$ we need the convergence in $\underline{\underline{H}}^1$ given by (2.2). The result is

$$M^1 * M^2_{st} = \int_{R_{st}} \left(\int_0^{t'} f(s',y)d\hat{W}^j(y)\right)\left(\int_0^{s'} \bar{f}(x,t')dW^d(x)\right) dW^{ik}(s',t').$$

We introduce some notation. We restrict our study to the martingales vanishing on the axes: If M is a L^2-martingale, zero on the axes, with representation

$$M_{st} = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_{st}} f_{ij}(x,y)dW^{ij}(x,y) \quad ,$$

we define

$$Y^j(s,t) = \sum_{i=1}^{n} \int_0^s f_{ij}(x,t)dW^i(x) \quad , \; j=1,\dots,m \quad ;$$

$$\hat{Y}^i(s,t) = \sum_{j=1}^{m} \int_0^t f_{ij}(s,y)d\hat{W}^j(y) \quad , \; i=1,\dots,n \quad .$$

A Fubini theorem for bi-brownian stochastic integrals allows us to write

$$M_{st} = \sum_{j=1}^{m} \int_0^t Y^j(s,y)d\hat{W}^j(y) = \sum_{i=1}^{n} \int_0^s \hat{Y}^i(x,t)dW^i(x) \; .$$

The expression of $\tilde{M}^1$ and $\widetilde{M^1 M^2}$ and the corollary 2.2 give:

<u>PROPOSITION 3.2</u>: <u>Let</u> M <u>be a L^2-martingale</u>, <u>zero on the axes</u>. <u>With the preceding notations</u>,

$$\tilde{M}_{st} = \sum_{i=1}^{n} \sum_{j=1}^{m} \int_{R_{st}} \hat{Y}^i(x,y)Y^j(x,y)dW^{ij}(x,y) \; .$$

4. THE QUADRATIC VARIATION OF $M*N$

Some definitions are required: A stochastic process $\{M_z , z \in \mathbb{R}_+^2\}$ adapted to $\{\underline{\underline{F}}_z , z \in \mathbb{R}_+^2\}$ and integrable is said to be a 1-martingale if for any fixed t, the process $\{M_{st} , \underline{\underline{F}}_{st} , s \in \mathbb{R}_+\}$ is a martingale. Similarly, we define 2-martingales. Because of (F4), we have that M is a martingale if and only if it is a 1 and 2-martingale. For a L^2 1-martingale, we denote by $\langle M\rangle^1_{st}$ the proces $\langle M_{.t}\rangle_s$, that is, the quadratic variation of the one parameter martingale $\{M_{st} , s \in \mathbb{R}_+\}$.

Let M be a 1-martingale. M is said to have 1-orthogonal increments if for any couple of disjoints rectangles $]z_1,z_1']$ and $]z_2,z_2']$ we have

$$E[M(]z_1,z_1'])M(]z_2,z_2'])|F_{s_1\wedge s_2,\infty}] = 0 ,$$

where $z_i=(s_i,t_i)$, $z_i'=(s_i',t_i')$, $i=1,2$.

Similarly we define 2-martingales with 2-orthogonals increments. A martingale is said to have orthogonal increments if it has 1 and 2-orthogonal increments.

If M is a i-martingale with i-orthogonal increments, then the process $\langle M\rangle^i_z$ is increasing, i=1,2 (see [6]).

<u>LEMMA 4.1</u> : <u>Let</u> M <u>and</u> N <u>be two martingales of</u> $\underline{\underline{M}}_c^4(z_o)$ <u>zero on the axes</u>, <u>and let</u> Γ <u>a grid over</u> $[0,z_o]$. <u>We consider the martingale</u>

$$S_z = \sum_{u\in\Gamma_z} M(\Delta_u^1)N(\Delta_u^2) , \quad z \leq z_o .$$

<u>Then</u>

$$\langle S\rangle_z = \sum_{u\in\Gamma_z} \langle M\rangle^1(\Delta_u^1)\langle N\rangle^2(\Delta_u^2) .$$

PROOF

We consider two points of $[0,z_o]$: $z=(s,t)$ and $z'=(s',t')$, $z<z'$. Let $\bar{\Delta}^1_{z,z'} =]s,s']\times]0,t']$ and $\bar{\Delta}^2_{z,z'} =]0,s']\times]t,t']$. then

$$S_z = \sum_{u\in\Gamma} M(\Delta^1_u \cap R_z) N(\Delta^2_u \cap R_z) ,$$

and

$$S(]z,z']) = \sum_{u\in\Gamma} \big(M(\Delta^1_u \cap R.) N(\Delta^2_u \cap R.)\big)(]z,z']) .$$

By considering differents cases it can be proved that

$$S(]z,z']) = \sum_{u\in\Gamma_{z'}} M(\Delta^1_u \cap \bar{\Delta}^1_{z,z'}) N(\Delta^2_u \cap \bar{\Delta}^2_{z,z'}) .$$

We denote by A_z the process $\sum_{u\in\Gamma_z} \langle M\rangle^1(\Delta^1_u)\langle N\rangle^2(\Delta^2_u)$,which has the following properties:

a) A is increasing: Just as before,

$$A(]z,z']) = \sum_{u\in\Gamma_{z'}} \langle M\rangle^1(\Delta^1_u \cap \bar{\Delta}^1_{z,z'})\langle N\rangle^2(\Delta^2_u \cap \bar{\Delta}^2_{z,z'}) \geqq 0 .$$

b) A is continuous and adapted. So, it is predictable.

c) $S^2 - A$ is a weak martingale. That means, we have to show that

$$E[S^2(]z,z'])|\underline{\underline{F}}_z] = E[A(]z,z'])|\underline{\underline{F}}_z] .$$

In fact,

$$E[S^2(]z,z'])|\underline{\underline{F}}_z] = E[(S(]z,z']))^2|\underline{\underline{F}}_z] =$$

$$= \sum_{u\in\Gamma_{z'}} E[M(\Delta^1_u \cap \bar{\Delta}^1_{z,z'})^2 N(\Delta^2_u \cap \bar{\Delta}^2_{z,z'})^2|\underline{\underline{F}}_z] +$$

$$+ 2 \sum_{\substack{u,u'\in\Gamma_{z'}\\ u\neq u'}} E[M(\Delta^1_u \cap \bar{\Delta}^1_{z,z'}) N(\Delta^2_u \cap \bar{\Delta}^2_{z,z'}) M(\Delta^1_{u'} \cap \bar{\Delta}^1_{z,z'}) N(\Delta^2_{u'} \cap \bar{\Delta}^2_{z,z'})|\underline{\underline{F}}_z] .$$

The second term is zero: If $u,u'\in[z,z']$, $u=(u_1,u_2)$,

$u'=(u'_1,u'_2)$, $u_1 > u'_1$, $u_2 < u'_2$, then,

$$E[M(\Delta^1_u)N(\Delta^2_u)M(\Delta^1_{u'})N(\Delta^2_{u'})|\underline{\underline{F}}_z] =$$

$$= E[M(\Delta^1_u)N(\Delta^2_u)M(\Delta^1_{u'})E[N(\Delta^2_{u'})|\underline{\underline{F}}^2_{u'}]|\underline{\underline{F}}_z] = 0 .$$

We similarly calcule the other posibilities for u and u' .

For the first term, we suppose $u \in [z,z'[$ (the other cases are equally computed). Using the conditional independence we obtain

$$E[M(\Delta^1_u)^2N(\Delta^2_u)^2|\underline{\underline{F}}_z] = E[M(\Delta^1_u)^2N(\Delta^2_u)^2|\underline{\underline{F}}_u|\underline{\underline{F}}_z] =$$

$$= E[E[M(\Delta^1_u)^2|\underline{\underline{F}}_u]E[N(\Delta^2_u)^2|\underline{\underline{F}}_u]|\underline{\underline{F}}_z] =$$

$$= E[\langle M\rangle^1(\Delta^1_u)\langle N\rangle^2(\Delta^2_u)|\underline{\underline{F}}_z] .$$

By the unicity of the quadratic variation of S , we obtain $\langle S\rangle = A$.■

<u>LEMMA 4.2</u>: <u>Let</u> $\{M^n , n \geq 1\}$ <u>be a sequence of martingales of</u> $M^2_c(z_o)$ <u>such that</u>

$$\lim_{n\to\infty} E[|M^n_{z_o} - M_{z_o}|^2] = 0 .$$

<u>Then</u>

$$\lim_{n\to\infty} E[|\langle M^n\rangle_{z_o} - \langle M\rangle_{z_o}|] = 0 .$$

PROOF

By the Kunita-Watanabe inequality we have

$$\langle M\rangle^{1/2} - \langle N\rangle^{1/2} \leq \langle M-N\rangle^{1/2} .$$

Then, by the Burkholder inequalities for the continuous two-parameter martingales (see [9])and by Cairoli-Doob inequality,

$$E[|\langle M^n\rangle_{z_o} - \langle M\rangle_{z_o}|] =$$

$$= E[|\langle M^n\rangle_{z_o}^{1/2} + \langle M\rangle_{z_o}^{1/2}|.|\langle M^n\rangle_{z_o}^{1/2} - \langle M\rangle_{z_o}^{1/2}|] \leqq$$

$$\leqq \{E[(\langle M^n\rangle_{z_o}^{1/2} + \langle M\rangle_{z_o}^{1/2})^2].E[(\langle M^n\rangle_{z_o}^{1/2} - \langle M\rangle_{z_o}^{1/2})^2]\}^{1/2} \leqq$$

$$\leqq \{(2E[\langle M^n\rangle_{z_o}] + 2E[\langle M\rangle_{z_o}]).E[\langle M^n-M\rangle_{z_o}]\}^{1/2} \leqq$$

$$\leqq \{(2E[\sup_{z\leqq z_o} |M_z^n|^2] + 2E[\sup_{z\leqq z_o} |M_z|^2]).E[\sup_{z\leqq z_o} |M_z^n-M_z|^2]\}^{1/2} \leqq$$

$$\leqq C\{(E[|M_{z_o}^n|^2] + E[|M_{z_o}|^2])E[|M_{z_o}^n - M_{z_o}|^2]\}^{1/2} \leqq$$

$$\leqq C\{E[|M_{z_o}^n - M_{z_o}|^2]\}^{1/2} . \blacksquare$$

<u>THEOREM 4.3</u>: <u>Let</u> M <u>and</u> N <u>be two martingales of</u> $\underline{\underline{M}}_c^4(z_o)$, <u>zero on the axes</u>. <u>For any standard sequence of grids</u> $\{\Gamma^n,\ n \geqq 1\}$ <u>over</u> $[0,z_o]$ <u>we have</u>

$$\langle M*N\rangle_z = \lim_{n\to\infty} \sum_{u\in\Gamma_z^n} \langle M\rangle^1(\Delta_u^1)\langle N\rangle^2(\Delta_u^2) \quad \text{in } L^1 ,$$

<u>and if</u> M <u>has 1-Orthogonal increments and</u> N <u>has 2-orthogonal increments</u>, <u>then</u>

$$\langle M*N\rangle_z = \int\int_{R_z\times R_z} \Psi(\zeta,\zeta')\ d\langle M\rangle^1(\zeta)\ d\langle N\rangle^2(\zeta') ,$$

<u>where</u> $\Psi: \mathbb{R}_+^2 \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ <u>is the deterministe corner function</u>

$$\Psi(z,z') = \begin{cases} 1 & \text{if } s>s' \text{ and } t<t' \\ 0 & \text{otherwise.} \end{cases}$$

($z=(s,t)$ and $z'=(s',t')$).

PROOF

The first part is a consequence of the preceding lemmas. The second part holds because the functions

$$\Psi_n = \sum_{u \in \Gamma^n} 1_{\Delta_u^1 \times \Delta_u^2}$$

converge pointwise to Ψ . And then, by the dominated convergence theorem, they converge to Ψ in the norm

$$||\varphi|| = \left(\iint_{R_{z_o} \times R_{z_o}} \varphi^2(z,z')\, d\langle M\rangle^1(z) d\langle N\rangle^2(z')\right)^{1/2} . \blacksquare$$

REMARK: The martingale $M*N$ can always be written as

$$M*N_z = \iint_{R_z \times R_z} \Psi(\zeta,\zeta')\, dM_\zeta dN_{\zeta'}$$

where this integral must be understood as a double stochastic integral of a corner function (see [4], [10]). On the contrary, $\langle M*N\rangle$ cannot, generaly, be expressed as an integral with respect to $\langle M\rangle_z^1$ and $\langle N\rangle_z^2$, because these processes are not increasing in general.

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