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Self-duality and CP violation in gravity

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SELF-DUALITY AND CP VIOLATION IN GRAVITY

Abstract:

In analogy with source free Maxwell theory, it is shown that if Einstein's vacuum theory is submitted to duality invariance, a quadratic correction of the action (which takes into account the occurrence of string-like topological excitations and related U(1) degrees of freedom) can be introduced, which displays a CP violation on (self-dual) extrema of the action.

I - Duality rotation and topological excitations.

We first recall various results concerning duality rotation and show that this transformation generates string-like topological excitations of the gravitational field. We shall then take the view-point that such excitations feature in the phase-space of solutions to vacuum Einstein's equation via quadratic corrections to the action.

A stationary space-time will be denoted by (M, g_{ab}, ξ^a) where g_{ab} is the metric and ξ^a the stationary Killing vector field. Let $\pi : M \rightarrow \Sigma$ denote the projection map from M to the 3-manifold Σ of orbits of ξ^a. A stationary solution to Einstein's vacuum equation (1) can be characterized by a complex potential τ on Σ :

$$\tau = \omega + i\lambda$$

where λ and ω are respectively the norm and twist of ξ^a. If Σ is asymptotically flat, the solution can also be characterized by monopole moments corresponding to the Hansen potentials on Σ :

$\phi_M = \frac{1}{4}(\lambda^2 + \omega^2 - 1)$, the mass monopole

$\phi_J = \omega/2\lambda$, the dual mass monopole.

The duality rotation is thus introduced as follows. Given a stationary vacuum solution described by $\tau = \omega + i\lambda$, another solution $\tau' = \omega' + i\lambda'$ is generated by the transformation (1) :

$$\tau' = (\tau \cos\theta + \sin\theta) / (-\tau \sin\theta + \cos\theta)$$

Such a transformation was introduced by Geroch while reformulating previous results by Buchdahl, Harrison and Ehlers for the purpose of generating stationary solutions to the Einstein vacuum equation. As a result, each solution generates a U(1) family of exact solutions parametrized by θ.

Introducing the 2-plane (ϕ_M, ϕ_J) at each point of Σ, and related vectors $v_a = grad_a\, \phi_M$, $w_b = grad_b\, \phi_J$, the 2-flat $v_{[a} w_{b]}$ rotates according to :

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

Furthermore, there are specific values of θ for which Σ is topologically non trivial (non vanishing second cohomology class). This occurrs when $F_{ab} = \nabla_{[a}\lambda^{-1}\xi_{b]}$ (F_{ab} is the pull-back of $F_{ab} = \lambda^{-3/2}\omega^m \epsilon_{mab}$, a 2-form on Σ with $\omega^m = D^m\omega$) verifies

$$\int_{S_2} \nabla_{[a}\lambda^{-1}\xi_{b]}\, dS^{ab} \neq 0$$

The occurence of such a charge (which requires that S_2 is a non contractible 2-chain) has been displayed in (1) under the action of the duality rotation, and comes as follows.

Recall the existence of three real divergence-free vector fields on Σ :

$$V_1^a = (\tau - \tau)^{-2}\, h^{am} (D_m \tau + D_m \tau)$$

$$V_2^a = (\tau - \tau)^{-2}\, h^{am} (\tau D_m \tau + \tau D_m \tau)$$

$$V_3^a = (\tau - \tau)^{-2}\, h^{am} (\tau^{-2} D_m \tau + \tau^2 D_m \tau)$$

where $h^{am} = \lambda\, h^{am}$ is the rescaled projected metric on Σ.

Under the duality rotation, these vector fields are mapped into linear combinations of themselves :

$$f_\theta V_1^a = V_1^a \cos^2\theta - V_2^a \sin 2\theta + V_3^a \sin^2\theta$$

$$f_\theta V_2^a = \tfrac{1}{2} V_1^a \sin 2\theta + V_2^a \cos 2\theta - \tfrac{1}{2} V_3^a \sin 2\theta$$

$$f_\theta V_3^a = V_1^a \sin^2\theta + V_2^a \sin 2\theta + V_3^a \cos^2\theta$$

and the corresponding curl – free two forms $\epsilon_{abc} V_i^c = F^i_{ab}$ admit the following pull – backs :

$$F^1_{ab} = \nabla_{[a}\, \omega \lambda^{-1} \xi_{b]}$$

$$F^2_{ab} = \nabla_{[a}\, \omega \lambda^{-1} \xi_{b]} - \tfrac{1}{2} \epsilon_{abcd} \nabla^c \xi^d$$

$$F^3_{ab} = \nabla_{[a}\, \lambda^{-1} (\omega^2 + \lambda^2)\ \xi_{b]}) - 2\lambda \nabla_a \xi_b - \omega\, \epsilon_{abcd} \nabla^c \xi^d .$$

Clearly the expression of F^2_{ab} shows why the Komar mass $\tfrac{1}{2} \int_{S^2} \epsilon_{abcd} \nabla^c \xi^d\, d S^{ab}$ gets converted into the magnetic mass under duality rotation.

A typical example is provided by the NUT solution after a duality rotation (with $\theta = \pi/4$) has been performed on the Schwarzschild solution monopoles .

Here $\phi_M = \dfrac{1}{4\lambda} (\lambda^2 - 1)$, $\phi_J = 0$ implies $\phi'_M = 0$, $\phi'_J = - \phi_M$.

The NUT solution has been interpreted as a gravitational magnetic monopole where the cohomological charge is given by the NUT parameter.

The space-time topology is that of a non trivial U(1) bundle over S_2 ; this is related to the fact that Σ , the base space, is non contractible.

A (cohomologically) dual interpretation of the above situation is the following . Since F_{ab} is closed on Σ, there must exist a 1-form A_b such that $F_{ab} = D_{[a} A_{b]}$, where A_b is not globally defined on Σ. Since ϕ_M and ϕ_J depend on λ and ω only, A_b can be chosen within $v_{[a}\, w_{b]}$ .This 2-flat is consequently not globally defined on Σ, reflecting the fact that each space-like section of the space-time under consideration carries a non trivial topology (handle).

This result can be viewed as an extension to gravity of a result obtained in (2) which deals with the onset of string-like topological excitations of the Maxwell field and related overturning of the spin-plane.In gravity this overturning features as a global defect of the field potential : A_b . In the next section we consider the possibility that topological excitations could induce quadratic corrections to the Einstein-action.

II - Quadratic corrections to the action

The emergence, under duality rotation, of vacuum solutions with non trivial topologies (non trivial U(1) bundle structure over non contractible base space) suggests to incorporate such data to the phase-space of Relativity.We shall adopt the view-point that these excitations could induce quadratic corrections to the Einstein-action. Since these excitations feature as Maxwellian U (1) gauge connections, we shall follow an approach inspired from Weyl's 1918 Unified theory where the presence of an electromagnetic field was incorporated via a modification of the tensor parallel transport law. In analogy, we shall focuss on a deformation of the local parallel transport and twistor curvature in the twistor space associated to the space-times under consideration. Let us briefly summarize. Recall that the local parallel transport of a twistor

$$Z^{\alpha} = (\omega_A , \pi_{A'})$$

or its dual

$$Z_{\alpha} = (\omega^A, \pi^{A'})$$

is given by :

$$\nabla_a \omega^B = -i \epsilon_A{}^B \pi_{A'} \ , \ \nabla_a \omega^{B'} = i \epsilon_{A'}{}^{B'} \pi_A$$

$$\nabla_a \pi^{B'} = -i P_{AA'B}{}^{B'} \omega^B \ , \ \nabla_a \pi_B = i P_{AA'BB'} \omega^{B'} \ ,$$

where $P_{AA'BB'}$ can be identified with

$$P_{ab} = \frac{1}{2}\left(R_{ab} - \frac{R}{6} g_{ab}\right) .$$

Recall that the local twistor curvature is given by

$$\nabla_{[c} \nabla_{d]} Z^{\beta} = K_{cd\alpha}{}^{\beta} Z^{\alpha}$$

where

$$K_{cd\alpha}{}^{\beta} = \left(\begin{array}{c|c} i \epsilon_{C'D'} \Psi_{CDA}{}^{B} & \begin{array}{l} \epsilon_{CD} \nabla^{A'}{}_{A} \Psi_{B'C'D'A'} + \\ \epsilon_{C'D'} \nabla^{B}{}_{B'} \Psi_{ACDB} \end{array} \\ \hline 0 & -i \epsilon_{CD} \Psi_{C'D'}{}^{A'}{}_{B'} \end{array}\right)$$

(ψ is the Weyl spinor).

The availability of the Maxwellian gauge connection A_b and related U(1) bundle curvature 2-form F_{ab} suggests the following deformation of the local parallel transport law :

$$\nabla_a \pi_{B'} = -i(P_{AA'BB'} + \chi F_{AA'BB'})\,\omega^B$$
$$\nabla_a \pi_B = i(P_{AA'BB'} + \chi F_{AA'BB'})\,\omega^{B'} .$$

A natural extension of the Lagrangian proposed by Yang for gauge fields suggests to consider the deformed twistor curvature $K_{cd\,\alpha}{}^{\beta}$ as a gauge field, an SU(2,2) valued 2-form.

The expression of $Kcd\,\alpha^{\beta}$ is :

$$K_{CC'DD'A}{}^{B} = i(\epsilon_{C'D'}\,\psi_{CDA}{}^{B} + \chi(\epsilon_C{}^B F_{DD'C'A} - \epsilon_C{}^B F_{DC'D'A}))$$
$$K_{CC'DD'AB'} = \epsilon_{CD}\,\nabla^{A'}{}_A\,\psi_{B'C'D'A'} + \epsilon_{C'D'}\,\nabla_{B'}{}^{B}\,\psi_{ACDB}$$
$$+\ \chi(\nabla_{CC'}\,F_{DD'AB'} - \nabla_{DD'}\,F_{CC'AB'})$$
$$K_{CC'DD'}{}^{A'}{}_{B'} = -i[\epsilon_{CD}\,\psi_{C'D'}{}^{A'}{}_{B'} + \chi(\epsilon_{C'}{}^{A'}F_{DD'CB'} - \epsilon_{D'}{}^{A'}F_{CC'DB'})] .$$

At this stage we recall that the Einstein-Hilbert action for general relativity is given by

$$S = \int \sqrt{-g}\ {}^4R\,dv = \int \sqrt{-g}\ g^{ac}g^{bd}\,{}^4R_{abcd}\,dv$$

and can be expressed (3) via S U (2) valued connection 1-forms $A_{aA}{}^{B}$ and related field strength :

$${}^4F_{ab\,A}{}^{B} = 2\,\partial_{[a}\,{}^4A_{b]\,A}{}^{B} + [{}^4A_a,{}^4A_b]_A{}^{B} .$$

If $\sigma^a{}_{AA'}$ denotes the SU (2) soldering forms, one has

$$S = \int d^4x\ \sigma^a{}_A{}^{A'}\,\sigma^b{}_{BA'}\,{}^4F_{ab}{}^{AB} .$$

This expression suggests to introduce the following quadratic correction to the action :

$$\Delta S = \int d^4x\ K_{cd\,\alpha}{}^{\beta}\,K^{cd\,\alpha}{}_{\beta} .$$

A straight forward calculation leads to

$$\Delta S = \int d^4x\ (C_{abcd}\,C^{abcd} + 8\,\chi F_{ab}F^{ab}).$$

Note that the above expression is rather reminiscent of Yang-Mills lagrangians and can be written

$$\Delta S = \int d^4x\ (C_{abcd}\,({}^*C_{mn}{}^{cd})\,\epsilon^{mnab} + 8\,\chi\,F_{ab}\,({}^*F_{mn})\,\epsilon^{mnab}).$$

$$= \int d^4x\ [\mathrm{tr}\,(C_\wedge {}^*C) + 8\chi\ \mathrm{tr}(F_\wedge {}^*F)] .$$

Note that F_{ab} being an exact 2-form, the term $F_{ab}\,F^{ab}$ will contribute to the action via a total divergence and provide extrema on self adjoint fields i.e fields s.t.

$${}^*F_{ab} = \pm\, i\,F_{ab}.$$

If the horizontal component $q_c{}^m\,q_c{}^m\,K_{mn\,\alpha}{}^{\beta}$ is substituted to the twistor curvature, the quantity $C_{abcd}\,C^{abcd}$ can be replaced by

$$R_{ab\,A}{}^{B}\,(\sigma_{cM}{}^{A}\,\sigma^{d}{}_{B}{}^{M} - \sigma_{cB}{}^{M}\,\sigma^{d}{}_{M}{}^{A})\,R^{abc}{}_{d}\ .$$

where $\sigma_{cM}{}^{A}$ (etc...) are the SU (2) soldering forms on horizontal sections.

Further more $q^{m}_{a}\,q^{n}_{q}\,R_{mn\,A}{}^{B} = {}^{4}F_{abA}{}^{B}$ where ${}^{4}F_{ab\,A}{}^{B}$ is the curvature 2 – form related to the spinorial curvature $R_{ab\,A}{}^{B}$. ${}^{4}F_{ab\,A}\,B$ is therefore closed w.r.t. the spinorial connection (Bianchi identities) :

$$ {}^{4}F_{ab\,A}{}^{B} = 2\,\partial_{[a}A_{b]\,A}{}^{B} + [{}^{4}A_{a},{}^{4}A_{b}]_{A}{}^{B}$$

$$= {}^{4}D_{[a}A_{b]\,A}{}^{B}$$

This implies that the term

$q_{a}{}^{m}\,q_{b}{}^{n}\,C_{mncd}\,C^{abcd}$

will contribute to the action via a total divergence, and provides extrema on self - adjoint fields, i.e. such that

$^{*}C_{abcd} = \pm\, i\,C_{abcd}$.

The contribution of the quadratic correction to the action, to the field equation, is to be presented in forthcomming papers (Ref. 7 - 8). The correction turns out to be of the Yang-Mills type.

In this sense, the resulting Lagrangian theory is invariant under the action of duality rotation.

III - CP violation

We focuss here on the term

$$\int \mathrm{tr}\ F \wedge {}^{*}F$$

which has been added to the Lagrangian, and which has a topological origin.Since * is associated to the natural, metric independent, totally skew density of weight one, there is a change of sign if the orientation is reversed. In this sense the action is not invariant under parity, and also CP transformations. Such quadratic terms can be found in the litterature, and had been proposed to investigate CP violation especially in quantum gravity. The previous section can be viewed as a derivation which takes into account the availabillity of U(1) gauge degrees of freedom in presence of string - like topological excitations.

Remark :

Under duality rotation, solutions such as the NUT (which is acausal, with closed time-like Killing orbits) can be transformed into solutions such as the extended Schwarzschild solution with two distinc acausal asymptotic regions with opposite time orientation. In this transformation of solutions, the magnetic mass (e.g the topological NUT charge) is converted into the mass (e.g the Schwarzschild mass). In (9) it has been proposed to relate duality rotation to a conversion of an infinitesimal motion into energy. We have here the gravitational analogue of the situation. Introducing P(x), the 2-plane $v_{[a} w_{b]}$ and V the direction of the U(1) gauge orbits ($-t^a$, the Killing vector field in the stationary case), the couple (V, $e^{i\beta} v_{[a} w_{b]}$) is the gravitational analogue of the Maxwellian couple (V , $e^{i\beta} n_1 \wedge n_2$) where β is the " strange " angle of Yvon Takabayasi. (see Ref. 9 - 10) .

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