ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Mathématiques

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Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 97, série *Mathématiques*, nº 27 (1991), p. 93-106
<http://www.numdam.org/item?id=ASCFM_1991__97_27_93_0>

Ann. Sci. Univ. "Blaise Pascal", Clermont II, Sér. Math., Fasc. 27ème, pp. 93-106

BANACH POWER - ASSOCIATIVE ALGEBRAS : THE COMPLEX AND (OR) NON COMMUTATIVE CASES

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INTRODUCTION

In a preceeding paper [6], we defined the class of real Banach power-associative algebras (Bpa-algebras) and proved its equivalence with that of JB-algebras. The redundance of the Jordan condition $A.(B.A^2) = (A.B).A^2$ in the definition of JB-algebras was already noticed for the complex case and the more general setting of V-algebras in [10]. In [6], this fact allowed us to deduce the Jordan structure of a system of observables from more elementary and physical principles.

It is then natural to extend our investigations to the class of commutative and non commutative complex Bpa-algebras with involution. We will see that if the parallelism with JB^*-algebras remains in the commutative case, it disappears in the non commutative one.

In the sequel, the mention "non associative" (respectively : "non commutative") will mean : not necessarily associative (respectively : not necessarily commutative) : see [11]. Algebras $\mathcal{A}$ are called Banach algebras if there is a norm on $\mathcal{A}$ such that $\|A.B\| \leqslant \|A\| \, \|B\|$, A, B $\in \mathcal{A}$, and $\mathcal{A}$ is complete. For any A,B in an algebra $\mathcal{A}$, define $A \circ B = (A.B + B.A)/2$ and $U_A B = A.(B.A) + (B.A).A - B.A^2$. An involution will be an application, denoted $*$, from a complex vector space $\mathcal{A}$ into itself such that

$$(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^* \qquad A, B \in \mathcal{A} ; \alpha,\beta \in \mathbb{C}.$$

$$(A^*)^* = A$$

It $A = A^*$, A is called self-adjoint and the set of self-adjoint elements is denoted by $\mathcal{A}_{s.a.}$.

Finally, $\mathcal{A}'$ will be the dual of $\mathcal{A}$.

I. THE COMPLEX COMMUTATIVE CASE

The real commutative case beeing described in [6], we begin with some definitions corresponding to the complex commutative case.

Définition I.1.

A commutative Banach power-associative $*$-system (Bpa*-system) is a complex Banach space $\mathcal{A}$ equipped with

1°) a square map, i.e. an application from $\mathcal{A}$ into $\mathcal{A}$ denoted

$$A \in \mathcal{A} \longrightarrow A^2 \in \mathcal{A}$$

such that

(1) $$(iA)^2 = -A^2$$

inducing a "product"

$$A.B = [(A+B)^2 - A^2 - B^2]/2 \qquad A,B \in \mathcal{A}$$

and a power operation

$$A^n = A^{n-1}.A \qquad n > 2$$

If there exists some element $\mathbb{1} \in \mathcal{A}$ such that

$$A.\mathbb{1} = A, \qquad A \in \mathcal{A}$$

it will be called a unit of $\mathcal{A}$.
If $A,B,\ldots \in \mathcal{A}$, $\mathcal{P}(A,B,\ldots)$ will be the subspace of $\mathcal{A}$ generated by $A,B,\ldots$, and $\mathbb{1}$ if it exists, through linear combinations, powers and products, and $C(A,B,\ldots)$ its closure.

2) an involution * such that

(2) $\mathbb{1}^* = \mathbb{1}$ if $\mathbb{1}$ exists

$$\left.\begin{array}{ll}(3) & A^2.A^2 = A^4 \\ (4) & A^m.(iA^n) = i(A^m.a^n) \\ (5) & (A^2)^* = A^2\end{array}\right\} \quad \text{if } A = A^*$$

(6) $\|A^*A\| = \|A\|^2$

$$\left.\begin{array}{ll}(7) & B.(iC) = i(B.C) \quad B,C \in \mathcal{P}(A,A^*) \\ (8) & \text{the square is continuous on } C(A,A^*)\end{array}\right\} \quad \begin{array}{l}\text{if the product is associative} \\ \text{on } \mathcal{P}(A,A^*)\end{array}$$

A subsystem $\mathcal{B} \subset \mathcal{A}$ will be a subspace of $\mathcal{A}$ stable under squaring and involution.

Proposition I.2. Let $\mathcal{A}$ be a commutative Bpa**-system. Then $0^*=0$, $\mathbb{1}$ is unique if it exists, $0^2 = 0$, $A.0 = 0$, $(-A)^2 = A^2$, $\mathcal{A}_{s.a.}$ is stable under product.

If $\mathcal{B} \subset \mathcal{A}$ is a subsystem of $\mathcal{A}$ such that $A.(iB) = i(A.B)$, $A,B \in \mathcal{B}$, the product on $\mathcal{B}$ is distributive and $\mathbb{C}_Q$-bilinear with $A.A = A^2$ where $\mathbb{C}_Q$ denotes the complex numbers with rationnal real and imaginary parts, the involution on $\mathcal{B}$ is multiplicative and (2) is redundant if $\mathbb{1} \in \mathcal{B}$.

If $A \in \mathcal{A}_{s.a.}$, $\|A^2\| = \|A\|^2$ and the subsystem $C(A)$ is an associative commutative Banach algebra. In particular $\|\mathbb{1}\| = 1$.

Proof. The first assertions are obvious thanks to (1) and (5) as $0^2 = (i0)^2 = -0^2 = 0$. If $\mathcal{B}$ is a subsystem as quoted above, then

A.(-B) = -(A.B) on $\mathcal{B}$ and the claimed properties can be proved as in ([6], Proposition I.4). If $A = A^*$, $\mathcal{P}(A)$ is such a subsystem thanks to (4). As Albert's proof of the fact that (3) implies power-associativity relies on the use of rational numbers only [1], $\mathcal{P}(A)$ is associative and $\|A^2\| = \|A\|^2$ on $\mathcal{A}_{s.a.}$ by (6). If B, C $\in \mathcal{P}(A)$, then $\|B.C\| \leq 2\|B.C\|$ as in ([6], Proposition I.4) so that bilinearity on $\mathcal{P}(A)$ extends to the complexes by density. Moreover $\|B.C\| \leq \|B\| \|C\|$ thanks to (6) and the associativity as in ([6], Corollary I.6). Finally the product on C(A) beeing continuous on C(A) thanks to (8), all these properties extends to the subsystem C(A).

If $\mathcal{A}$ has a unit, we will define the numerical range of $A \in \mathcal{A}$ as in the context of associative algebras according to $v_{\mathcal{A}}(A) = \{\varphi(A) \ ; \ \varphi \in D_{\mathcal{A}}(\mathbb{1})\}$ where $D_{\mathcal{A}}(\mathbb{1}) = \{\varphi \in \mathcal{A}' \ ; \ \varphi(\mathbb{1}) = \|\varphi\| = 1\}$. The significance of this definition relies on the Hahn-Banach theorem, which also insures that $V_{\mathcal{B}}(A) = V_{\mathcal{A}}(A)$ if $A \in \mathcal{B} \subset \mathcal{A}$ with $\mathbb{1} \in \mathcal{B}$. The number $v_{\mathcal{A}}(A) = \sup \{|\lambda| \ ; \ \lambda \in V_{\mathcal{A}}(A)\}$ will be called the numerical radius of A, and A will be said hermitian if $V_{\mathcal{A}}(A) \subseteq \mathbb{R}$. The set of hermitian elements will be denoted by $H(\mathcal{A})$ and is obviously a real Banach space. It is evident from the definition that $V_{\mathcal{A}}(\lambda_{\mathbb{1}} + \mu A) = \lambda + \mu V_{\mathcal{A}}(A)$, $V_{\mathcal{A}}(A+B) \subset V_{\mathcal{A}}(A) + V_{\mathcal{A}}(B)$ and $v_{\mathcal{A}}(A) \leq \|A\|$ where $\lambda,\mu \in \mathbb{C}$ and $A,B \in \mathcal{A}$. If $A = A^*$, the following results, valid in associative Banach algebras, are still true in $\mathcal{A}$ by restriction to $C(\mathcal{A})$:

(9) $V_{\mathcal{A}}(A)$ is a non empty compact convex subset of $\mathbb{C}$;
(10) If ρ is the spectral radius, then $\rho_{C(A)}(A) = v_{\mathcal{A}}(A) = \|A\|$.

Proposition I.3. Let $\mathcal{A}$ be a commutative Bpa*-system with unit. Then $\mathcal{A}_{s.a.} = H(\mathcal{A})$; $\mathcal{A} = H(\mathcal{A}) + iH(\mathcal{A})$ and $\|A^*\| \leq 2\|A\|$.

Proof. If $A = A^*$ and $\lambda \in \mathbb{R}$, $\|\mathbb{1} + \lambda^2A^2\| = \|(\mathbb{1} + i\lambda A)(\mathbb{1}-i\lambda A)\| = \|\mathbb{1}+i\lambda A\|^2$ and $\lim_{\lambda\to 0} \frac{1}{\lambda} (\|\mathbb{1}+i\lambda A\| -1) = \lim_{\lambda\to 0} \frac{1}{\lambda} (\|\mathbb{1} + \lambda^2A^2\|^{1/2} - 1) = 0$. Hence $A \in H(\mathcal{A})$ by ([4], lemma 5.2) applied to C(A). Conversely, let $A \in H(\mathcal{A})$ and $A = A_1+iA_2$ where $A_1 = (A+A^*)/2$ and $A_2 = (A-A^*)/2i$. Then A_1 and A_2 beeing self-adjoint are hermitian, and so is $iA_2 = A-A_1$. So if $\varphi \in D_{\mathcal{A}}(\mathbb{1})$, then $\varphi(A_2) \in \mathbb{R}$, $i\varphi(A_2) \in \mathbb{R}$ which implies successively that $\varphi(A_2) = 0$, $v(A_2) = 0$ and $A_2=0$ by (10). Hence $A = A_1 \in \mathcal{A}_{s.a.}$. Let now $A = A_1+iA_2 \in H(\mathcal{A}) \oplus iH(\mathcal{A}) = \mathcal{A}_{s.a.} \otimes i\mathcal{A}_{s.a.}$. For any $\varphi \in D_{\mathcal{A}}(\mathbb{1})$, $|\varphi(A^*)| = |\varphi(A_1)-i\varphi(A_2)| = |\varphi(A_1) + i\varphi(A_2)|$

$= |\varphi(A)|$ and $v_{\mathcal{A}}(A) = v_{\mathcal{A}}(A^*)$. By (10), $\|A+A^*\| = v_{\mathcal{A}}(A+A^*) \leq v_{\mathcal{A}}(A) + v_{\mathcal{A}}(A^*) = 2v_{\mathcal{A}}(A) \leq 2\ \|A\|$. Changing A into iA, $\|A-A^*\| \leq 2\|A\|$ and $\|A^*\| = 1/2\ \|A+A^* - (A-A^*)\| \leq 2\|A\|$.

Proposition I.4. Let $\mathcal{A}$ be a commutative Bpa*-system with unit and $a \in \mathcal{A}$ be such that $\mathcal{P}(A,A^*)$ carries an associative product. Then $C(A,A^*)$ is an associative commutative C^*-algebra. Moreover if $A = A^*$ then $\mathcal{P}(A,A^*)$ is associative, $C(A) \cap \mathcal{A}_{s.a.}$ is a real JB-algebra and $H(\mathcal{A})$ is a real Bpa-system.

Proof. By (7) and Proposition I.2, $\mathcal{P}(A,A^*)$ has a distributive, $\mathbb{C}_Q$-bilinear and associative product. If $B,C \in \mathcal{P}(A,A^*)$ with $B = B_1+iB_2$, $C = C_1+iC_2$ and $B_i,C_i \in \mathcal{A}_{s.a.}$ for $i = 1,2$, then $\|B.C\| \leq 2 \sum_{i,j=1}^{2} \|B_i\|\ \|C_j\|$. Hence bilinearity extends to the complexes because if $\lambda_r = \lambda_{r_1} + i\ \lambda_{r_2} \in \mathbb{C}_Q$ tends to $\lambda \in \mathbb{C}$, then $\|B.\lambda C - \lambda B.C\| = \|B.\lambda C - B\lambda_r C + \lambda_r B.C - \lambda B.C\| \leq 2 \sum_{i,j,k=1}^{2} |(\lambda-\lambda_r)|\ \|B_j\|\ \|C_k\| + |\lambda-\lambda_r|\ \|B.C\|$ tends to zero. Moreover $(B.C)^* = B^*.C^*$, $\mathcal{P}(B.C,(B.C)^*)$ is associative and $\|B.C\|^2 = \|B^*.B.C^*.C\| \leq 2\ \|B^*.B\|\ \|C^*.C\| = 2\|B\|^2\ \|C\|^2$, whence $\|B.C\| \leq \|B\|\ \|C\|$ by induction. In particular $*$ is isometric on $\mathcal{P}(A,A^*)$, and also on $C(A,A^*)$ (by Proposition I.3) which is then an associative commutative C^*-algebra. If $A = A^*$, $C(A)_{s.a.}$ is a real JB-algebra, and $H(\mathcal{A})$ a real Bpa-system by [6], Corollary II.4).

Definition I.5. A commutative Bpa*- algebra is a commutative Bpa*- system $\mathcal{A}$ such that

(1) $\quad A.(iB) = i(A.B) \qquad A,B \in \mathcal{A}$.

Hence the involution is multiplicative on $\mathcal{A}$ and (2) is redundant (see Proposition 1.2).

Definition I.6. A JB*- algebra is a complex Banach space $\mathcal{A}$ which is a complex Jordan algebra with involution such that

$$\|A.B\| \leq \|A\|\ \|B\|$$

$$(A.B)^* = A^*.B^*$$
$$\|U_A A^*\| = \|A\|^3$$

It has been noted in [10] that, in this definition, the multiplicability of the involution and the Jordan identity could be replaced by the weaker condition $\mathbb{1}^* = \mathbb{1}$ in the case of a unital JB^*-algebra.

Definition I.7. A commutative V-algebra is a commutative and non associative Banach algebra $\mathcal{A}$ with unit such that $\mathcal{A} = H(\mathcal{A}) \oplus iH(\mathcal{A})$. If $A = A_1 + iA_2$ with $A_i \in H(\mathcal{A})$, $i = 1,2$, then $A^* = A_1 - iA_2$ defines a natural continuous involution on $\mathcal{A}$.

It is proved in ([10], theorem 12) that the class of unital commutative JB^*- algebras coïncides with the one of commutative V-algebras with their natural involution and consequently is made of complex unital Jordan algebras with multiplicative and isometric involution.

Theorem I.8. A commutative JB^*- algebra is a commutative Bpa^*-algebra. Conversely, if $\mathcal{A}$ is a unital commutative Bpa^*-algebra, then it is *-isomorphic and homeomorphic to a JB^*-algebra with respect to a norm $\| \ \|_1$ such that $\|A\| \leqslant \|A\|_1$.

Proof. Let $\mathcal{A}$ be a commutative JB^*-algebra. The involution beeing multiplicative, is also isometric ([13], lemma 3) so that $\|A^*A\| = \|A\|^2$ if A and A^* generate an associative subalgebra as in ([12], definition and remarks, p. 291-292). The other points are obvious as $\mathcal{A}$ is a Jordan algebra. Conversely, let $\mathcal{A}$ be a unital commutative Bpa^*-algebra. By Proposition 1.4, $\mathcal{A}_{s.a.} = H(\mathcal{A})$ is a real Bpa-algebra, that is to say a JB-algebra ([6], Theorem V.1). By ([12], Theorem 2.8), there exists a norm $\| \ \|_1$ on $\mathcal{A}$ such that $(\mathcal{A}, \| \ \|_1)$ is a JB^*-algebra and
$\|A\|_1 = \inf\{\lambda \ ; \ A \in \lambda \text{ conv } \{e^{iB} ; B = B^*\}\}$. Hence

$$A/\|A\|_1 = \sum_{i=1}^{n} \lambda_i \, e^{iA_i}, \ \sum_i \lambda_i = 1, \ \lambda_i \geqslant 0 \text{ and}$$

$\|A\|/\|A\|_1 \leqslant \Sigma \lambda_i \, \|e^{iA_i}\| = \Sigma \lambda_i \left\| e^{iA_i} \left(e^{iA_i}\right)^* \right\|^{1/2} = \Sigma \lambda_i = 1$, or $\|A\| \leqslant \|A\|_1$, $A \in \mathcal{A}$. It is easy to check that $\|A\| = \|A_1\|$ if $A \in \mathcal{A}_{s.a.}$ ([12], lemma 2.3). Hence, if $A, B \in \mathcal{A}_{s.a.}$, $\|A\| = \frac{1}{2} \|A+iB + A-iB\|_1 \leqslant \|A+iB\|_1$ and $\mathrm{Max}\{\|A\|, \|B\|\} \leqslant \|A+iB\|_1 \leqslant \|A\| + \|B\|$. As * is continuous, the two norms are equivalent.

Corollary I.9. **Let $\mathcal{A}$ be a unital Bpa*- algebra. Then the following are equivalent :**

1) $\mathcal{A}$ **is a JB*-algebra**
2) $\|A.B\| \leqslant \|A\| \; \|B\| \qquad A,B \in \mathcal{A}$;
3) $\|U_A \; A^*\| \leqslant \|A\|^3 \qquad A \in \mathcal{A}$.

Proof. 1) $\to$ 2) and 3) are obvious by definition. 2) $\to$ 1) by Proposition 1.3, which means that $\mathcal{A}$ is a V-algebra, and ([10], Theorem 12). 3) $\to$ 1) because then $\|A\|_1 \leqslant \|A\|$ as in ([12], lemma 1.1). In fact, if $\|A\| < \|A_1\| = 1$, let B_n the sequence defined by $B_0 = A$, $B_n = U_{B_{n-1}}B^*_{n-1}$. Then $\|B_n\| = \left\|U_{B_{n-1}}B^*_{n-1}\right\| \leqslant \|B_{n-1}\|^3 \leqslant \|B_{n-2}\|^{3^2} \leqslant \ldots \leqslant \|A\|^{3^n}$ tends to zero as n tends to infinity. By equivalence of the norms, the same would be true for $\|B_n\|_1$. But this is impossible because $\|B_n\|_1 = \|A\|_1^{3^n} = 1$.

Corollary I.10. **Let $\mathcal{A}$ be a commutative Bpa*-system with unit. In Definition 1.1, the condition (6) is equivalent to $\|U_A A^*\| = \|A\|^3$ if $\mathcal{P}(A,A^*)$ is associative.**

Proof. By Proposition 1.4, (6) implies the new condition. Conversely, assume the new definition. Let A, A^*, $\mathbb{1}$ generate an associative subalgebra $\mathcal{P}(A,A^*)$, and $B,C,D \in \mathcal{P}(A,A^*)$: the involution is multiplicative on $\mathcal{P}(A,A^*)$. The algebraic identity $B.C^*.D = \frac{1}{16} \sum_{\epsilon^4=1=\eta^2} \epsilon\eta \; U_{B+\epsilon C+\eta D} \; (B+\epsilon C+\eta D)^*$ allows to write that

$\|B.C^*.D\| \leqslant \frac{8}{16} \left\|U_{B+\epsilon C+\eta D} \; (B+\epsilon C+\eta D)^*\right\| = \frac{1}{2} \|B+\epsilon C+\eta D\|^3 \leqslant \frac{1}{2} (\|B\| + \|C\| + \|D\|)^3$.

Thus $\|B.C^*.D\| \leqslant \frac{27}{2} \|B\| \; \|C\| \; \|D\|$ and replacing C or B and D by $\mathbb{1}$, product and involution are continuous on $\mathcal{P}(A,A^*)$: $\|B.C\| \leqslant k\|B\| \; \|C\|$ and $\|C^*\| \leqslant k\|C\|$ with $k > 1$. Consequently $\|B\|^3 = \|B.B^*.B\| \leqslant k\|B.B^*\| \; \|B\|$ and $\|B\|^2 \leqslant k\|B.B^*\|$. So

$$\|B\|^6 = \|B.B^*.B\|^2 \leqslant k\|(B.B^*.B).(B.B^*.B)^*\| = k\|(B.B^*)^3\|$$
$$= k\left\|U_{B.B^*} \; (B.B^*)^*\right\| = k\|B.B^*\|^3 ,$$

and $\|B\|^2 \leq k^{1/3}\|B.B^*\|$. By induction, $\|B\|^2 \leq \|B.B^*\|$. On the other hand, $\|B\|^3 = \|U_B B^*\| \leq k\|U_{B^*}B\| = k\|B^*\|^3$ and $\|B\| \leq k^{1/3}\|B^*\|$, so $\|B\| \leq \|B^*\|$ by induction, and $\|B\| = \|B^*\|$ by symmetry. Using the same trick, we get now $\|B.B^*\|^3 = \|(B.B^*)^3\| = \|(B.B^*.B).(B.B^*.B)^*\| \leq k\|B.B^*.B\|^2 = k\|B\|^6$. Thus $\|B.B^*\| \leq k^{1/3}\|B\|^2$ and by induction $\|B.B^*\| \leq \|B\|^2$. So in particular $\|A.A^*\| = \|A\|^2$ and (6) is verified.

The above proof is an adaptation of ([2], Theorem 1.1). Hence Definition 1.5 is a weakened definition of Alvermann's commutative F^*-algebras. In fact Alvermann's proofs do not use the Jordan identity but only the power-associativity. But on the other hand he defines commutative JB- and JB^*-algebras as Jordan algebras, which is redundant.

II. THE REAL NON COMMUTATIVE CASE

By analogy with the real commutative case, we introduce the following definition.

Definition II.1. A non commutative Bpa-algebra is a real Banach space $\mathcal{A}$ equipped with a non commutative non associative bilinear product such that

$$\begin{aligned} &(A.B).A = A.(B.A) \qquad \text{(flexibility)} \\ &A^2.A^2 = A^4 \\ &\|A^2\| = \|A\|^2 \qquad\qquad A,B \in \mathcal{A} \\ &\|A^2-B^2\| \leq \operatorname{Max}\{\|A\|^2, \|B\|^2\} \end{aligned}$$

The following concepts are standard (see [3], [2]) :

Definition II.2. An F-algebra is a real non commutative unital Jordan algebra complete with respect to a norm such that

$$\begin{aligned} &\|A^2\| = \|A\|^2 \\ &\|A^2\| \leq \|A^2 + B^2\| \end{aligned}$$

Definition II.3. A non commutative unital JB-algebra $\mathcal{A}$ is an F-algebra such that

$$\|A.B\| \leqslant \|A\| \; \|B\| \qquad A,B \in \mathcal{A}$$
(i.e. $\mathcal{A}$ is a Banach algebra).

Let us recall that in a non associative unital algebra $\mathcal{A}$, the Jordan condition $(A.B).A^2 = A.(B.A^2)$ implies the flexibility and the equivalence of the Jordan condition with either one or the other of the following ones : $(A^2.B).A = A^2.(B.A)$; $A.(A^2.B) = A^2.(A.B)$; $(B.A^2).A = (B.A).A^2$ so that $\mathcal{A}^+$ (the symmetrized algebra) is a Jordan algebra. Conversely, if $\mathcal{A}$ is flexible and $\mathcal{A}^+$ is Jordan, then $\mathcal{A}$ is Jordan ([11], p. 141-142). From these remarks it is easy to conclude that a non commutative non associative real unital algebra $\mathcal{A}$ is an F-algebra if and only if it is flexible and $\mathcal{A}^+$ is a commutative unital JB-algebra ([2], Corollary 2.3). Similarly, a non commutative non associative real Banach algebra $\mathcal{A}$ is a non commutative non associative unital JB-algebra if and only if it is flexible and $\mathcal{A}^+$ is a commutative unital JB-algebra.

The following lemma is then obvious thanks to ([6], Corollary II.3) as powers coïncide in $\mathcal{A}$ and $\mathcal{A}^+$.

Lemma II.4. The condition $\|A^2\| \leqslant \|A^2+B^2\|$ in Definitions II.2 and II.3 is equivalent to $\|A^2-B^2\| \leqslant \mathrm{Max}\{\|A\|^2, \|B\|^2\}$. A non commutative unital JB-algebra is an F-algebra which is in turn a non commutative unital Bpa-algebra.

But conversely one has :

Theorem II.5. Let $\mathcal{A}$ be a non commutative non associative real algebra which is also a Banach space. Then
1) $\mathcal{A}$ is a non commutative unital Bpa-algebra if and only if $\mathcal{A}$ is flexible and $\mathcal{A}^+$ is a commutative unital Bpa-algebra, and is then an F-algebra ;
2) The condition $\|A^2-B^2\| \leqslant \mathrm{Max}\{\|A\|^2, \|B\|^2\}$ in Definition II.1 is equivalent to $\|A^2\| \leqslant \|A^2+B^2\|$ if $\mathcal{A}$ has a unit ;
3) Among the non commutative unital Bpa-algebras, the non commutative unital JB-algebras are those for which $\|A.B\| \leqslant \|A\| \; \|B\|$ and form a non trivial subclass made of necessarily commutative unital JB-algebras.

Proof. The flexibility implies $A^2.A = A.A^2$ which, together with $A^2.A^2 = A^2.A^2 = A^4$, is equivalent to the power-associativity condition $A^{m+n} = A^m.A^n$ [1], ([11], p. 130). Hence $\mathcal{A}^+$ is a commutative Bpa-algebra, or else a commutative JB-algebra by ([6], Theorem V.1), the converse beeing obvious under the flexibility hypothesis. So $\mathcal{A}$ is an F-algebra according to the remarks following Definition II.3, and the equivalence between the two metric conditions follows from ([6], Corollary II.3) as powers in $\mathcal{A}$ and $\mathcal{A}^+$ coïncide. The same remarks joined to ([3], theorem 7.4) and ([2], example 3.1) assert the last claim.

Hence Definition II.1 is a weakened definition of F-algebras. If we add the condition $\|A.B\| \leqslant \|A\| \, \|B\|$ we get a weakened definition of non-commutative (and hence commutative by [3]) unital JB-algebras.

III. THE COMPLEX NON COMMUTATIVE CASE

By combination of the preceeding cases, it is then natural to begin with the following definition.

Definition III.1. A non commutative Bpa*-algebra is a complex Banach space $\mathcal{A}$ equipped with a non commutative non associative bilinear product such that

$$(A.B).A = A.(B.A) \qquad A,B \in \mathcal{A}$$

and with an involution such that

$$(A.B)^* = B^*.A^*$$
$$A^2.A^2 = A^4 \text{ if } A = A^* \qquad A,B \in \mathcal{A}$$
$$\|A^* \circ A\| = \|A\|^2 \text{ if } A \text{ and } A^* \text{ generate an associative subalgebra with respect to the product } \circ \text{ .}$$

As above it will be interesting to compare this class of algebras with the next ones.

Definition III.2. A non associative unital JB*-algebra is a non commutative non associative complex Banach algebra $\mathcal{A}$ equipped with an involution such that

$$\mathbb{1}^* = \mathbb{1}$$
$$\|U_A A^*\| = \|A\|^3$$

As indicated after Definition I.6, it is then a non commutative complex Jordan algebra with multiplicative and isometric involution ([10], Theorem

12).

Notice that the remarks following Definition II.3 remain valid for non commutative JB^*-algebras (see [8], Proposition 1.2 and the fact that the involution is necessarily multiplicative on JB^*-algebras) : a non commutative non associative complex Banach algebra $\mathcal{A}$ is a non commutative unital JB^*- algebra if and only if it is flexible and $\mathcal{A}^+$ is a commutative unital JB^*-algebra.

Definition III.3. A non commutative V-algebra is the non commutative version of the commutative V-algebra of Definition I.7.

According to the different forms of the Vidav-Palmer theorem, one has the following identifications between the above classes of algebras :

- {associative non commutative C^*-algebra} ≡ { associative non commutative V-algebras} ([7], Theorem 3.1) ;
- {non associative non commutative C^*-algebra} ≡ { alternative non commutative V-algebras} ≡ {non commutative V-algebras such that $\|A^*.A\| = \|A\|^2$} ([10], section 2) ;
- {non commutative unital JB^*-algebras} ≡ { non commutative V-algebras} ([10], Theorem 12).

Finally, as in Section II, one can introduce the class of F^*-algebras.

Definition III.4. An F^*-algebra is a non commutative complex unital Jordan algebra complete with respect to a norm and equipped with an involution such that

$$(A.B)^* = B^*.A^*$$
$$\|U_A A^*\| = \|A\|^3.$$

We are indebted to Professor A. Rodriguez-Palacios for noticing to us that the condition $\|A^*\| = \|A\|$ in Alverman's original definition of F^*-algebras is redundant by the same argument as in ([5], lemma (2.13)).

The remark following Definition II.3 remains valid for F^*-algebras ([2], Corollary 2.3) : a non commutative non associative complex algebra $\mathcal{A}$ is an F^*- algebra if and only if it is flexible and $\mathcal{A}^+$ is a commutative unital JB^*-algebra whose involution is multiplicative on $\mathcal{A}$.

The following lemma is then obvious.

Lemma III.5. A non commutative unital JB^*-algebra is an F^*-algebra which is in turn a non commutative unital Bpa^*-algebra.

But conversely,

Theorem III.6. Let $\mathcal{A}$ be a non commutative non associative complex algebra which is a Banach space. Then
1) $\mathcal{A}$ is a non commutative unital Bpa^*-algebra if and only if $\mathcal{A}$ is flexible and $\mathcal{A}^+$ is a commutative unital Bpa^*-algebra whose involution is multiplicative on $\mathcal{A}$. In particular it is an F^*-algebra if moreover $\|U_A A^*\| \leqslant \|A\|^3$ or $\|A \circ B\| \leqslant \|A\| \, \|B\|$.
2) Among the non commutative unital Bpa^*-algebras, the Banach algebras are the non commutative unital JB^*-algebras.

Proof. The flexibility condition giving that $U_A = U_A^+$ where U_A^+ is defined with respect to $\circ$, all these assertions are obvious thanks to Corollary I.9 and the remarks following Definitions III.2 and III.4.

Corollary III.7. The following classes of algebras are identical :
{non commutative unital JB^*-algebras} ≡ {non commutative unital Bpa^*-algebras such that $\|A.B\| \leqslant \|A\| \, \|B\|$} ≡ {non commutative V-algebras}.

Corollary III.8. In Definition III.1, the following subset of hypothesis

$$(A.B).A = A.(B.A)$$
$$A^2.A^2 = A^4$$
$$\|U_A A^*\| \leqslant \|A\|^3$$
$$\|A^* \circ A\| = \|A\|^2 \text{ if } A \text{ and } A^* \text{ generate an associative subalgebra with respect to the product } \circ$$

can be replaced by the following one

$$(A.B).A^2 = A.(B.A^2)$$
$$\|U_A A^*\| = \|A\|^3$$

if there is a unit. Moreover, in the case of Banach algebras, the Jordan condition is redundant in the second set of hypothesis.

Hence, Theorem III.6 provides with a weakened definition of F^*-algebras and of non commutative unital JB^*-algebras.

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Corrigendum

◇ Reference [6], Proposition II.2, Proof : the sentence : "Conversely if $\|A\| \leqslant 1 \ldots \|C^2-D^2\| \leqslant \max\{\|C^2\|, \|D^2\|\}$" is useless and should be replaced by : "As $\mathcal{A}$ is an order unit-space and as squares are positive, then $\|C^2-D^2\| \leqslant \max\{\|C^2\|, \|D^2\|\}$.

◇ Reference [6], Corollary V.2 should be read : "The class of JB-algebras coïncides with the class of real Banach spaces with a square map inducing a product such that ..."

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