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AN APPROACH TO JORDAN-BANACH ALGEBRAS FROM THE THEORY OF NONASSOCIATIVE COMPLETE NORMED ALGEBRAS

Angel RODRIGUEZ PALACIOS

INTRODUCTION

The aim of this paper is not to make a survey of all the results which are known on Jordan-Banach algebras, but only of those results which can be obtained, as a specialization for Jordan-Banach algebras, from general results of the theory of nonassociative complete normed algebras. Also we will study some sides of the theory of nonassociative complete normed algebras which, surprisingly, lead to Jordan algebras. For clearness we also consider the case of noncommutative Jordan algebras, for they include (commutative) Jordan algebras and associative (even alternative) algebras. Thus, in each case, we will recall, from the theory of (associative) Banach algebras, the result which suggests the problem under consideration, we will state its nonassociative extension and, to conclude, we will see its particularization for noncommutative Jordan-Banach algebras. In this way we will re-encounter the associative result of departure.

A systematic study of general nonassociative normed algebras has been made only very recently. However there are some important classical precedents which we also include in this survey. Also we consider some interesting problems on nonassociative complete normed algebras, with indication of the answer (when it is known) in the case of non-commutative Jordan-Banach algebras. This paper contains also several new results and some new proofs of known results.

This paper is an improved version of my talks at the "Colloque sur les algèbres de Jordan" (Montpellier, October 1-4, 1985).

1. C^*-ALGEBRAS

There are many concepts in order to obtain Jordan structures which are the "analogous" of associative C^*-algebras. Thus we have the JB-algebras, JB^*-triples, JB^*-algebras and noncommutative JB^*-algebras. We give the first reference for each one of these concepts ([6,43,80,58], respectively) and the reader is referred to the books of Iochum [36], Hanche Olsen-Störmer [33] and Upmeier [74] for a complete study and a wide list of references.

But, since our purpose is "nonassociative", we look for those non-associative algebras which are the analogous ones of associative C^*-algebras. To this end we will recall the Vidav-Palmer theorem, for it gives a characterization of C^*-algebras which does not involve the product of the algebras but only its Banach space and its unit.

To state the Vidav-Palmer theorem we need some concepts which we give here for the nonassociative case in order to avoid repetition. Let A be a unital nonassociative normed algebra and let A' denote its dual Banach space. For any a in A we define the *numerical range* of a (V(a,a) or V(a) when confusion is not possible) by $V(a) = \{f(a) : f \in A', \|f\| = f(I) = 1\}$, where I denotes the unit of A. If A is a complex algebra and $V(a) \subset \mathbb{R}$ we will say that a is an *hermitian element* of A and H(A) will denote the closed real subspace of all hermitian elements of A. By definition, a *V-algebra* is a unital non-associative complete normed algebra A satisfying $A = H(A) + iH(A)$. It is not difficult to see that if A is a V-algebra then the mapping

$h+ik \longrightarrow h-ik$ (with h and k in H(A)) is a continuous involution on the Banach space of A which will be called the *natural involution* of the V-algebra A.

Now we can state the Vidav-Palmer theorem (see [16; Theorem 6.9] for example).

Theorem 1.1 (Vidav-Palmer). Every unital associative C^*-algebra is an (associative) V-algebra. Conversely, every associative V-algebra with its natural involution is a unital associative C^*-algebra.

From our point of view this theorem shows that nonassociative V-algebras are the natural nonassociative parallel of unital associative C^*- algebras. The question is : how far a theory of nonassociative V-algebras can be developped ? Surprisingly the answer is much better that it could be expected at first sight. The fact is that we have the following first result which is crucial for our purposes.

Theorem 1.2 [63]. The natural involution of every nonassociative V-algebra is an algebra involution.

This theorem was first proved by Martinez [49] for the particular case of Jordan algebras. We recall that an involution * on the vector space of a complex algebra is called an <u>algebra involution</u> if $(ab)^* = b^*a^*$ for every a and b in the algebra.

The second significant result on nonassociative V-algebras was proved by Kaidi, Martinez and Rodriguez under the assumption that the natural involution is an algebra involution. But, by Theorem 1.2, this assumption is superfluous. So we have:

Theorem 1.3 [42]. Every nonassociative V-algebra is a noncommutative Jordan algebra.

We recall that a <u>noncommutative Jordan algebra</u> is a nonassociative algebra A satisfying $(ab)a = a(ba)$ "flexibility" and $(a^2b)a = a^2(ba)$ "Jordan axiom" for all a and b in A.

To state the third and concluding result on nonassociative V-algebras we need to recall the concept of noncommutative JB^*-algebra. A <u>noncommutative JB^*-algebra</u> is a noncommutative Jordan complete normed complex algebra A with algebra involution satisfying $\|U_a(a^*)\| = \|a\|^3$ for all a in A (where $U_a(b) = a(ba) + (ba)a - ba^2$). The concept of JB*- algebra was introduced by Kaplansky in the unital commutative case and was studied by Wright [80] in this case, by Youngson [86] in the non-unital commutative case and by Paya, Pérez and Rodriguez [58] in the non-unital noncommutative general case. From the results in [19,59,7] one may describe all noncommutative JB*-algebras, for they are the closed selfadjoint subalgebras of ℓ^∞-products of noncommutative JB*-factors and the noncommutative JB*-factors are well known.

Now, as a consequence of Theorem 1.2 and 1.3 and a result of Yougson [84,85] and Rodriguez [62] on Jordan V-algebras, we have the following theorem which says that studying nonassociative V-algebras is just the same as studying unital noncommutative JB*-algebras.

Theorem 1.4 (nonassociative Vidav-Palmer theorem). The class of nonassociative V-algebras agrees with the class of unital noncommutative JB*-algebras.

An important consequence of Theorem 1.2 and 1.3 is the following corollary (see the proof of [63; Theorem 12] for details).

Corollary 1.1 Under the assumption of existence of a unit I, the algebraic axioms (flexibility and Jordan axiom) are superfluous in the definition of noncommutative JB*-algebras and the assumption that the involution is an algebra involution can be weakened to the assumption $I^* = I$.

For the non-unital case we have also a new result weakening the axioms of noncommutative JB*-algebras. This result is also a consequence of the ones above and of a theorem of Youngson [86] on the bidual of a non-unital (commutative) JB*-algebra. Following [7] a nonassociative complete normed complex algebra A is called <u>admissible</u> if the algebra A^+, obtained by symmetrization of the product of A, is a JB*- algebra for the given norm on A and for a suitable involution (notice that this involution is not assumed to be an algebra involution on A but only on A^+). It is clear that every noncommutative JB*-algebra is admissible. To state the converse we must prove that the algebra is flexible and that the given involution is an algebra involution on A. These facts are true as shows the following.

Corollary 1.2. Every nonassociative complete normed complex admissible algebra is actually a noncommutative JB^*-algebra.

<u>Proof</u>. Let A denote our admissible algebra. As in the proof of [58, Theorem 1.7] (where we assume A to be a noncommutative JB^*-algebra but we only use that A is admissible) we have that the bidual A" of A, with the Arens product, is a nonassociative V-algebra, the natural involution of which extends the given involution on A^+. Thus, by Theorem 1.4, A" (and so A) is a noncommutative JB^*-algebra.

It must be noted that Theorem 1.4 contains the associative Vidav-Palmer theorem and even its extension for alternative algebras. By definition, an *alternative C^*-algebra* is an alternative complete normed complex algebra A with algebra involution satisfying $\|a^*a\| = \|a\|^2$ for all a in A. Alternative C^*- algebras were studied in [58,20]. From the identity $U_a(b) = aba$, which is true for alternative algebras, it is not difficult to prove that, for an alternative normed complex algebra with algebra involution, the axioms $\|U_a(a^*)\| = \|a\|^3$ and $\|a^*a\| = \|a\|^2$ are equivalent. Thus the alternative C^*-algebras are just those noncommutative JB^*-algebras which are alternative. Now, as a direct consequence of Theorem 1.4, we have

Corollary 1.3 [62,20]. **The class of alternative V-algebras agrees with the class of unital alternative C^*-algebras.**

It is surprising that the algebras in the above corollary are the only unital nonassociative complete normed complex algebras with involution satisfying the Gelfand-Naimark axiom $\|a^*a\| = \|a\|^2$. The proof of this result uses Theorem 1.2 and 1.3 and a theorem of Wright-Yougson [81] on isometries of JB-algebras.

Theorem 1.5 [62]. **Let A be a unital nonassociative complete normed complex algebra with involution * satisfying $I^*=I$ (where I denotes the unit of A) and $\|a^*a\| = \|a\|^2$ for all a in A. Then A is alternative and * is an algebra involution on A (that is : A is an alternative C^*-algebra).**

In Theorem 1.3 and 1.5 we have obtained from geometric assumptions on a general nonassociative complete normed algebra some algebraic identities. In the same way we can even give a characterization of associative and commutative C^*-algebras among nonassociative complete normed algebras. To state this result we recall that the *numerical radius* $v(a)$ of an element a of a unital nonassociative normed algebras A is defined by $v(a) = \sup\{|z| : z \in V(a)\}$, and the *numerical index* $n(A)$ of A is defined by $n(A) = \inf\{v(a) : a \in A, \|a\| = 1\}$. Now we have :

Theorem 1.6 [62]. The class of nonassociative V-algebras with numerical radius equal to one agrees with the class of unital associative and commutative C^*-algebras.

We do not know a geometric characterization of JB-algebras, among nonassociative complete normed algebras, analogous to the one given in Theorem 1.4 for noncommutative JB^*-algebras . We recall that JB-algebras (which are the Jordan real analogous of associative C^*-algebras) are defined as those (commutative) Jordan-Banach real algebras A satisfying $\|a^2\| = \|a\|^2$ and $\|a^2-b^2\| \leqslant \max\{\|a^2\|,\|b^2\|\}$ for all a and b in A. In the absence of a geometric nonassociative characterization of JB-algebras we have, by putting together results of Alvermann-Janssen and Iochum-Loupias, the following remarkable theorem.

Theorem 1.7 [7,37]. Let A be a flexible power-associative complete normed real algebra satisfying $\|a^2\| = \|a\|^2$ and $\|a^2-b^2\| \leqslant \{\|a^2\|,\|b^2\|\}$ for all a and b in A. Then A is a JB-algebra.

Proof. From the power-associativity of A it follows that A^+ is a power-associative algebra. By [37; Theorem V.1] A^+ is a Jordan algebra. This fact and the flexibility of A implies that A is a noncommutative Jordan algebra as it is well known. But a noncommutative Jordan complete normed real algebra satisfying the usual geometric axioms of JB-algebras is actually commutative by [7; Theorem 7.4].

To conclude this section we state a theorem of Kadison which gives a nonassociative characterization of the (associative and commutative) Banach algebras of real valued continuous functions on a Hausdorff compact space (which are just the associative JB-algebras and also, by the commutative Gelfand-Naimark theorem, the selfadjoint parts of unital associative and commutative C^*-algebras). Thus a complex analogous of this Kadison's theorem is Theorem 1.6. We need to recall the concept of a *complete order unit space* which, in a form equivalent to the original one [4; page 69] but more close to our approach, can be formulated as follows : a complete order unit space is a real Banach space X with a norm-one element u (the *order unit* of X) such that for every x in X we have that $\|x\| = \sup\{|f(x)| : f \in X', \|f\| = f(u)=1\}$ (in terms of numerical ranges in Banach spaces, see Section 3, X is a real numerical range space with numerical index equal to one). An element x of X is said to be *positive* if $f(x) \geqslant 0$ for every f in X' with $\|f\| = f(u) = 1$. Now we have :

Theorem 1.8 [41]. Let A be a nonassociative real algebra which is also a complete order unit space. Assume that the order unit of A is also a multiplicative unit for A and that the product of positive elements of A is also positive. Then A is algebricaly and isometrically isomorphic to the algebra of all real valued continuous functions on a suitable Hausdorff compact space.

2. SMOOTH NORMED ALGEBRAS

By definition a *smooth normed algebra* is a unital nonassociative normed algebra A with the property that the normed space of A is smooth at the unit I of A (that is : there is a unique element f in A' such that $\|f\| = f(I) = 1$, or, equivalently, the numerical range of any element in A contains a unique number). From Bohnenblust-Karlin theorem (see [16; Theorem 4.1]), which is also true for the nonassociative case via [62; Corollary 2(a)], it follows that $\mathbb{C}$ is the unique (nonassociative) smooth normed complex algebra. But for associative real algebras the following result is well known (see [16; Theorem 6.16]) :

Theorem 2.1 (Bonsall-Duncan). $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (the algebra of real quaternions), with its usual modulus as norm, are the only associative smooth normed real algebras.

This theorem shows that the assumption of smoothness for normed algebras is very restrictive. Thus it seems interesting to study this assumption in the general nonassociative case. In this direction there are some particular results of Strzelecki [73] for power-associative algebras and

Nieto [56] for alternative algebras.

We give here a description of nonassociative smooth normed algebras, a consequence of which is that they are flexible quadratic algebras (so non-commutative Jordan algebras) and that the normed space of each one of them is an inner product space.

Our construction begins with an anticommutative normed real algebra E the normed space of which is an inner product space satisfying $(x \wedge y \mid z) = (x \mid y \wedge z)$ for all x,y and z in E (where $\wedge$ stands for the anticommutative product of E). Such an algebra is called a *pre-H-algebra*. As an example, which should not be forgotten, every inner product real space with zero product is a pre-H-algebra. Let E be a pre-H-algebra and consider the vector space $\mathbb{R} \times E$ with product $(z,x)(w,y) := (zw - (x|t), zy+wx+x \wedge y)$ and norm $\|(z,x)\|^2 := z^2+\|x\|^2$. It is not difficult to see ([63; Proposition 24]) that in this way we have obtained a smooth normed algebra which (being obviously a flexible quadratic algebra) is called the *flexible quadratic algebra of the pre-H-algebra E*. Now we can state the main result in this section.

Theorem 2.2 [63]. **Every nonassociative smooth normed algebra is the flexible quadratic algebra of a suitable pre-H-algebra.**

The proof of this theorem given in [63] uses the main results in the preceding section. We give here a new proof of Theorem 2.2 which does not involve the results in Section 1 and, consequently, we think that it is more simple. Previously we prove two lemmas.

Lemma 2.1. Let a be in H(A), where A denotes a unital nonassociative complete normed complex algebra. Then the number $\|a\|^2$ belongs to the numerical range of a^2.

Proof. We may assume $\|a\| = 1$ and also, by the nonassociative Sinclair theorem (see the comments before [63; Lemma 3]), that $1 \in V(a)$. Let g be in A′ such that $\|g\| = g(I) = g(a) = 1$ and, for any F in the unital Banach algebra BL(A) of all continuous linear operators on A, write $\hat{g}(F) := g(F(I))$. Thus $\hat{g}$ belongs to $(BL(A))'$ and satisfies $\hat{g}(L_a) = \hat{g}(I_A) = \|\hat{g}\| = \|L_a\| = 1$, where L_a (the operator of left multiplication by a) belongs to H(BL(A)) by [62; Corollary 2]. Applying [17; Corollary 26.10] we obtain that $g(a^2) = \hat{g}(L_a^2) = 1$. Therefore $1 \in V(a^2)$ as required.

Lemma 2.2. Let A be a unital nonassociative complete normed real algebra and let E denote the closed subspace of all elements x in A with $V(x)=\{0\}$. Then we have :

i) $xy - yx \in E$ **for x and y in E.**
ii) $(xy)z - x(yz) \in E$ **for x,y and z in E.**
iii) $-\|x\|^2 \in V(x^2)$ **for x in E.**
iv) $\|zI + x\|^2 = z^2 + \|x\|^2$ **for x in E and z in $\mathbb{R}$.**

Proof. As in the associative case [18; Section 13] we can consider the complete normed complexification $A_{\mathbb{C}}$ of A. Let a be in A, then $V(A,a) = \operatorname{Re}V(A_{\mathbb{C}},a)$ [63; Lemma 25] and, in particular, $a \in E$ if and only if $ia \in H(A^{\mathbb{C}})$. Now i) and ii) follows from [42; Lemma 3], iii) from the preceding lemma, and iv) from [63; Lemma 3(b)].

Proof of Theorem 2.2. Let A denote our nonassociative smooth normed algebra, which can be assumed complete, and let f be the unique element in A' with $\|f\| = f(I) = 1$. We have $V(a) = \{f(a)\}$ for every a in A and, in particular, $E = \operatorname{Ker}(f)$ (where E has the same sense that in the preceding lemma). From Lemma 2.2 i) and iii) it follows that, if for x,y in E we write $(x|y) := -f(xy)$, then $(\mid)$ is a symmetric bilinear form on E satisfying $(x|x) = \|x\|^2$. Therefore E is an Hilbert space. From Lemma 2.2 i) we have that E is closed under the anticommutative product given by $x \wedge y := (1/2(xy-yx)$. Since clearly we have $\|x \wedge y\| \leqslant \|x\| \|y\|$ and the equality $(x \wedge y|x) = (x|y \wedge z)$ is deduced from Lemma 2.2.i) and ii), it follows that E is an *H-algebra* (complete pre-H-algebra). For x in E we have $x^2 = wI+y$ for suitable w in $\mathbb{R}$ and y in E. But, by Lemma 2.2 iii), $w = -\|x\|^2$ and so, by Lemma 2.2 iv), $\|x\|^4 + \|y^2\|^2 \leqslant \|x\|^4$, which implies $y=0$. Thus we have proved that $x^2 = -\|x\|^2 I$, so $(1/2)(xy+yx) = -(x|y)I$ and so $xy = -(x|y)I + x\wedge y$ for all x,y in E. From the last equality it is clear that the mapping $zI+x \to (z,x)$ from $A(= \mathbb{R}I \oplus E)$ onto the flexible quadratic algebra of the H-algebra E is an isomorphism. This isomorphism is an isometry by Lemma 2.2. iv), which concludes the proof.

Remark 2.1. Theorem 2.2 gives a complete description of commutative smooth normed algebras, since the flexible quadratic algebra of a pre-H-algebra is commutative if and only if the given pre-H-algebra has zero product, that is : the pre-H-algebra is an arbitrary inner product space equipped with the zero product. To get a similar full description of noncommutative smooth (even complete) normed algebras we need to know the H-algebras with non zero product. In Section 6 we will find some information about this problem.

Remark 2.2. For a unital normed algebra A define $\partial(A)$ as the diameter of the set $\{f \in A' : \|f\| = f(I) = 1\}$. The smooth normed algebras are then characterized as those unital normed algebras A satisfying $\partial(A) = 0$. Lummer [47] proved the following extension of Theorem 2.1 : **There is a positive number k such that, if a unital associative normed algebra A satisfies $\partial(A) < k$, then A is isomorphic (not necessarily isometric) to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.** It would be interesting to find a nonassociative extension of this result. From Lummer theorem it follows easily that if A is a unital power-associative normed algebra with $\partial(A) < k$, then A is a quadratic algebra with the property that every element is invertible in the (associative) subalgebra generated by it. In particular if A is alternative, then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ (the algebra of real octonions).

We conclude this section with a result in which we enclose the theorem of Albert-Urbanic-Wright [2,3,79,75] on absolute valued algebras with unit and the theorem of Nieto [56] on alternative smooth normed algebras. This result can be obtained from Theorem 2.2 with more or less difficulties (see [56,63] for details). We recall that, by definition, an absolute valued algebra is a real or complex algebra with a norm satisfying $\|ab\| = \|a\|\|b\|$ for all a,b in the algebra. $\mathbb{C}$ is the unique unital (nonassociative) absolute valued complex algebra. But, for the real case, we have :

Theorem 2.3 [50,75]. **Let A be a real algebra. Then the following statements are equivalent :**

i) A is a unital nonassociative absolute valued algebra

ii) A is an alternative smooth normed algebra

iii) A = $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ with its usual modulus as norm.

3. REMARKS ON NUMERICAL RANGES

In the preceding sections we have seen the more attractive results on numerical ranges in general unital nonassociative complete normed algebras, with the remarkable consequence that, if such an algebra has a "good behaviour" for numerical ranges, then it is a noncommutative Jordan algebra. It was first noted in [49,84] that most of the theory of numerical ranges in (associative) Banach algebras [16,17] can be extended to Jordan-Banach algebras, and in [42,62,63] one may find some ingenuous (but powerful) techniques in order to transfer results on associative numerical ranges to the general nonassociative case. The proof of Lemma 2.1 is an example of this procedure. Thus one is tempted to develop a theory of nonassociative numerical ranges by reproving directly in the nonassociative case the results which are known for the associative one. Actually this is possible and the author has given at the University of Granada a postgraduate courses on this topic. However, in absence of power-associativity, the arguments in the proofs are sometimes unnecessarily intrincate and no more instructive than the associative ones. In conclusion it is preferable to prove the result in the associative case first and later, making use of the above mentioned techniques, extend it (if possible) to the nonassociative case.

Even for unital noncommutative Jordan-Banach algebras, in which a spectral theory is developed, the relationship between the spectrum and the numerical range of an element is reduced to the associative case by the "full closed associative" localization theorem (see [48]).

In view of the above comments we think that, if one wish working in numerical ranges and if one is weary of associative numerical ranges, then the best direction is to study numerical ranges in Banach spaces X (without product) in which a norm-one element u is selected in order to play the role of the multiplicative unit in the case of unital complete normed algebras. Such a couple (X,u) is called a *numerical range space*. The numerical range $V(x)$ of an element x in X is defined by $V(x) = \{f(x) : \|f\| = f(u)=1\}$. The concepts of numerical radius of x, $v(x)$, and of numerical index of X, $n(X)$, may be deduced by analogy with the case of unital complete normed algebras. We explain, with some examples, how the theory of numerical range spaces may give light even in associative and nonassociative numerical ranges. The first example is motivated by the following result of Smith.

Theorem 3.1 [72]. Let A be a unital complex Banach algebra and let K be a closed convex subset of $\mathbb{C}$ with a non-empty interior. Then, for each F in A" with $V(A'',F) \subset K$, there exists a net $\{a_\lambda\}$ in A which converges to F in the w^*-topology and such that $V(A,a_\lambda) \subset K$ for every λ.

Theorem 3.1 can be extended, by using techniques different to the ones in [72], to arbitrary real or complex numerical range spaces and so, in particular, to unital nonassociative complete normed real or complex algebras. Thus we have :

Theorem 3.2 [50]. Let X be a numerical range space and let K be a closed convex subset of the base field with non-empty interior. Then, for each F in W" with $V(F) \subset K$, there exists a net $\{x_\lambda\}$ in X which converges to F in the w^*-topology and such that $V(x_\lambda) \subset K$ for all λ.

As a consequence we obtain the following result which was not observed previously even for unital Banach algebras.

Corollary 3.1 [50]. For every numerical range space X we have that $n(X")=n(X)$.

New Proof. Clearly $n(X") \leqslant n(X)$ and so it is enough to prove the converse inequality. Let F be in X" and let ϵ be an arbitrary positive number. Then $V(F)$ is contained in $\{z \in \mathbb{K} : |z| \leqslant v(F) + \epsilon\}$ which is a closed convex subset of the base field $\mathbb{K}$ with non-empty interior. By the theorem there exists $\{x_\lambda\}$ in X such that $\{x_\lambda\} \xrightarrow{w^*} F$ and $v(x^\lambda) \leqslant v(F) + \epsilon$. Since $n(X)\|x_\lambda\| \leqslant v(x_\lambda)$ for all λ, we have also $n(X)\ \|F\| \leqslant v(F) + \epsilon$ in view of the w^*-lower-semicontinuity of the norm on X". Thus, since ϵ is arbitrary, $n(X)\|F\| \leqslant v(F)$, and so $n(X") \geqslant n(X)$ as required.

Some results on numerical ranges in unital complete normed algebras cannot be extended to the context of numerical range spaces because they involve in the statement the product of the algebra. But, even if the product is not involved, there are results on algebra numerical ranges which are not true for numerical range spaces. A simple example is the Bohnenblust--Karlin theorem, stating that $n(A) \geqslant 1/e$ for every unital complete normed complex algebra A, which fails in every complex numerical range space (X,u) with X a Hilbert space of dimension greater or equal to two. This shows that the geometry of the numerical range spaces of unital complete normed algebras is very peculiar. In this respect we must cite a relevant geometric property of unital complete normed algebras which is easy to prove and which was first noticed in [50]. To state this result we recall some concepts. For x in the unit sphere S(X) of a given Banach space X we write $D(x) = \{f \in X' : \|f\| = f(x) = 1\}$. The mapping $x \to D(x)$ from S(X) into the set of non-empty subsets of X' is called the *duality mapping* of X. Following [32], the duality mapping of a Banach space X will be said *(n,n)* -

(resp. (n,w)-) upper-semicontinuous at a point x in S(X) if for every n- (resp. w-) neighbourhood of zero U in X′ there is an $\epsilon > 0$ such that if $y \in S(X)$ and $\|x-y\| < \epsilon$, then $D(y) \subset D(x) + U$, where n (resp. w) denotes the norm (resp. weak) topology on X′. Now we have :

Proposition 3.1 [50]. Let A be a unital nonassociative complete normed algebra and let I denote the unit of A. Then the duality mapping of the Banach space of A is (n,n)- (so also (n,w)-) upper-semicontinuous at I.

Proposition 3.1 and the Bohnenblust-Karlin theorem (for the complex case) are two independent important geometric properties of unital complete normed algebras from which many others such properties can be codified. As an example : *for any numerical range space* (X,u) *the fact* "$n(X) > 0$" *and* "X′ *is the linear span of* D(u)"*are equivalent* [50], and so the Bohnenblust--Karlin theorem and the Moore-Sinclair theorem ([17; Theorem 31.1]) can be deduced each one from the other. To show another example, consider the following result :

Theorem 3.3 ([16; Theorem 12.2]). Let A be a unital Banach algebra. Then for every F in A″ we have $V(F) = \{F(f) : f \in D(I)\}^-$ (where $^-$ denotes closure).

This theorem (and even its nonassociative extension [62]) is now a direct consequence of Proposition 3.1 and the following

Theorem 3.4 [32]. Let (X,u) be a numerical range space. Then the following statements are equivalent :
i) The duality mapping of X is (n,w)-upper-semicontinuous at u.
ii) The equality $V(F) = \{F(f) : f \in D(u)\}^-$ is true for every F in X″.

We conclude this quick incursion in numerical ranges in Banach spaces by stating a result on this topic from which one can obtain, for unital complete normed algebras, an important generalization of Theorem 3.3.

Theorem 3.5. [8]. Let (X,u) be a numerical range space such that X is the dual space of a suitable Banach space X_*. Assume that the duality mapping of X is (n,n)-upper-semicontinuous at u. Then for every f in X we have $V(f) = \{f(p) : p \in X_*, \|p\| = u(p) = 1\}^-$.

Now, from Proposition 3.1 and Theorem 3.5, we obtain the following extension of Theorem 3.3.

Corollary 3.2. Let A be a unital nonassociative complete normed algebra, the Banach space of which is the dual space of a suitable Banach space A_*. Then for every f in A we have $V(f) = \{f(p) : p \in A_*, \|p\| = I(p) = 1\}^-$.

Normed algebras which satisfy the assumptions of this corollary are, among others, the biduals of unital normed algebras with Arens product (for which our corollary is just Theorem 3.3), the (associative) W^*-algebras [65], the noncommutative JW^*-algebras [58], and the extremal algebra of a compact convex subset of $\mathbb{C}$ [17; Section 24].

Remark 3.1. The content of this section may be considered as a modest project of reconciliation between Banach spacits and Banach algebrists, which sometimes are not well behaved. As an anecdote, in the paper [32], where Theorem 3.4 is proved (in an equivalent form and with different terminology), no reference is made on the application of this theorem to unital normed algebras. Fortunately, such a project of reconciliation has now many protectors once the Jordan-Banach algebrists have been nicely surprised by the recent Dìneen's proof [26] (see also [15]) that the bidual of a JB^*-triple is another JB^*-triple , by using the principle of local reflexitivity.

In spite of what was said before about the interest of Banach space numerical ranges versus algebra numerical ranges, we wish cite some aspects of algebra numerical ranges which can be explored. The reader should take a decision on wether or not this line seems interesting. The starting idea is to weaken no much the usual axiom $\|ab\| \leqslant \|a\| \, \|b\|$ for normed algebras in such a way that the classical theory of numerical ranges in unital complete normed algebras is not perturbed. Since "weakening axioms" is not a suggestive phrase in mathematics, we give a motivation which appear naturally in the context of (commutative) Jordan algebras.

Let A be a real or complex Jordan algebra with unit I which is also a Banach space satisfying $\|I\| = 1$. Taking into account the equivalent axioms for Jordan algebras given by MacCrimmon [51] in terms of the quadratic operator U_a, it seems to be more natural to assume $\|U_a(b)\| \leqslant \|a\|^2 \, \|b\|$ instead of the usual axiom $\|a.b\| \leqslant \|a\| \, \|b\|$ $(a,b \in A)$. Actually these two conditions are independent geometric axioms both ensuring the continuity of the product of A. If we assume the first axiom to be true for A, then the non classical numerical range space (A,I) is related with the numerical range space of the unital Banach algebra BL(A) as follows.

Proposition 3.2. **Let A be a real or complex Jordan algebra with unit I, which is also a Banach space satisfying $\|I\| = 1$ and $\|U_a(b)\| \leq \|a\|^2\|b\|$ for a,b in A. Then for any a in A we have $V(A,a) = V(BL(A), L_a)$.**

Proposition 3.2 follows, even if A is only a commutative (not necessarily Jordan) algebra, from the clear inclusion $V(a) \subset V(L_a)$ and the next lemma on general numerical range spaces.

Lemma 3.1. **Let (X,u) and (Y,v) be numerical range spaces and let F be a symmetric bilinear mapping from $X \times X$ into Y satisfying $\|F(x,x)\| \leq \|x\|^2$ for all x in X and $F(u,u) = v$. Then $V(Y,F(u,x)) \subset V(X,x)$ for all x in X.**

Proof. Let x be in X and let r be an arbitrary positive number. Then $\|v + 2rF(u,x) + r^2F(x,x)\| \leq \|u+rx\|^2$. Computing right derivatives at r=0, we have ([55; Lemma 1.6]) $\mathrm{MaxReV}(F(u,x)) \leq \mathrm{MaxReV}(x)$ and therefore $V(F(u,x)) \subset V(x)$, as required.

The algebra A in Proposition 3.2 shares with the unital nonassociative complete normed algebras the property that $V(a) = V(L_a) = V(R_a)$ for any a in A ([62; Corollary 2(a)]), which was the cornestone to carry results of associative numerical ranges over the nonassociative case. Thus we have find the desired weaker axiom for numerical ranges in algebras. Explicitly, we call a *numerical range algebra* a nonassociative real or complex algebra A, with unit I, which is also a Banach space such that $\|I\| = 1$, the product is continuous for the topology of the norm and $V(A) = V(L_a) = V(R_a)$ for all a in A. It can be proved that Proposition 3.1 (and so, by Theorem 3.5, also Corollary 3.2 and, by Theorem 3.4, Theorem 3.3) remains true when A is only assumed to be a numerical range algebra. However the proof of Theorem 3.3 given in [16], which in wiew of Proposition 3.1 and Theorem 3.4 is obsolete, can now be reconsidered in order to prove that the bidual of any numerical range algebra, with the Arens product, is another numerical range algebra.

On the other hand we must say that Theorem 1.2, 1.3 and 1.4 are true if we replace "V-algebra" by "complex numerical range algebra A with

$A = H(A)+iH(A)$". In verifying these facts the reader will rediscover in our new context a wide part of the theory of numerical ranges in unital complete normed algebras.

Unfortunately we do not know for numerical range algebras a result analogous to Theorem 2.2. So we conclude this section with the following

Problem 3.1. Describe all real numerical range algebras whose Banach space is smooth at the unit.

If A is such an algebra and $a \in S(A)$ with $V(a) \neq \{0\}$, then we know that A is smooth at a and that the duality mapping of A is (n,n)-upper-semicontinuous at a. Also it must be noticed that if A is a smooth complete normed algebra then for all r in $\mathbb{R}$ the algebra $A^{(r)}$ (the same Banach space that of A with product $rab + (1-r)ba$) is a numerical range algebra smooth at the unit.

4. UNIQUENESS OF NORM TOPOLOGY

In the preceding sections we have paied attention to some questions, on normed algebras, which depend on the particular norm given on the algebra to generate its topology and, in most of the cases, we have assumed also that the algebra has a norm-one multiplicative unit (that is : we have studied some "geometric" sides of the theory of normed algebras). Now we wish study normed algebras only from an algebraic and topological point of view.

In this direction, one of the more remarkable known results for (associative) Banach algebras is the following one (see [18; Theorem 25.9]).

Theorem 4.1 (Johnson). Let A be a semisimple Banach algebra. Then all the complete algebra norms on A are equivalent.

If a complete normed algebra satisfies the thesis of this theorem, then we say that the algebra *has a unique complete algebra norm topology*. We recall that an associative algebra A is said to be *semisimple* if it has zero Jacobson radical. Recall also that the *Jacobson radical* is the greatest (two-sided) ideal each element of which is quasiinvertible, and that an element a in A is called *quasiinvertible* when there is b in A such that $a \circ b = b \circ a = 0$ (where $a \circ b = a + b - ab$). The existence of such a greatest quasi-invertible ideal is a theorem in the associative case and we do not know wether or not this fact remains true in the nonassociative case. Even if this is true or if we define the Jacobson radical of a nonassociative algebra as the intersection of all the primitive ideals (which for the associative case is an equivalent definition and for the nonassociative case has no problem : maximal modular left ideals are defined as in the associative case [18; Definition 9.1] and primitive ideals are defined as the greatest two-sided ideals contained in some maximal modular left ideal), a nonassociative literal extension of Theorem 4.1 would not be satisfactory for any nonassociative Banach algebrist, because such a literal extension would not be applicable to the anticommutative algebras since, for these algebras, both the greatest quasiinvertible ideal and the intersection of the primitive ideals agree with the whole of the algebra (notice that for Banach-Lie algebras the problem of uniqueness of norm topology is posed [34]). It follows from these comments that, if we look for a satisfactory nonassociative extension of Theorem 4.1, then first we should find a suitable concept of nonassociative semisimplicity.

One may think that this problem does not appear when one is working on noncommutative Jordan algebras because, as it is well known, McCrimmon [52, 54] gave a fine extension of Jacobson radical for noncommutative Jordan algebras, by defining an element a to be quasiinvertible if these exists b such that $a \circ b = b \circ a = 0$ and $(a \circ a) \circ b = b \circ (a \circ a) = 0$ (the last two equalities follow as a consequence of the two first ones in the associative, even alternative, case), and by proving the existence of a greatest quasiinvertible ideal in any noncommutative Jordan algebra. This ideal is called the *Jacobson-McCrimmon radical* of the given noncommutative Jordan algebra. Actually we have the following noncommutative Jordan extension of Johnson's theorem :

Theorem 4.2 (Aupetit [10]). **Every complete normed noncommutative Jordan algebra with zero Jacobson-McCrimmon radical has a unique complete algebra norm topology.**

In [10] this theorem is proved only for (commutative) Jordan algebras. Our more general statement can be deduced either by the same arguments or by reducing it to the commutative case (see [64; Remark 2.4 (iii)]).

Although Aupetit's theorem is satisfactory in the (commutative) Jordan case, we do not believe so in the noncommutative case, because every anticommutative algebra is a noncommutative Jordan algebra whose Jacobson-McCrimmon radical is the whole of the algebra and, consequently, the theorem is not applicable to Banach-Lie algebras.

In order to obtain the desired concept of nonassociative semisimplicity we will "project" on the algebra the Jacobson radical of "multiplication algebra". Thus we follow the idea of Albert in [1]. But some changes on the Albert's construction are needded in order to avoid finite dimensionality (and to obtain the desired nonassociative extension of Johnson's theorem !).

Definition 4.1.

i) A subalgebra B of an associative algebra A is called a *full algebra* of A if B contains the quasiinverses of its elements that are quasiinvertible in A.

ii) Since the intersection of full subalgebras of an associative algebra A is another full subalgebra of A it follows that for any non-empty subset S of A there is a smallest full subalgebra of A which contains S. This subalgebra will be called *the full subalgebra of* A *generated by* S.

iii) Now let A be a nonassociative algebra and let L(A) denote the associative algebra of all the linear mappings from A into A. The full subalgebra of L(A) generated by the set of all right and left multiplication operators by elements of A will be called *the full multiplication algebra of* A and will be denoted by FM(A).

iv) Consider the set W(A) of those elements a in A for which L_a and R_a belong to the Jacobson radical of FM(A). W(A) is a subspace of A so it contains a largest subspace invariant under the full multiplication algebra of A. This last subspace, which is clearly a two-sided ideal of A, will be called *the weak radical of* A and denoted by w-Rad(A).

Now we can state the nonassociative extension of Johnson's theorem.

Theorem 4.3. [64]. Every nonassociative complete normed algebra with zero weak radical has a unique complete algebra norm topology.

To understand the power of this theorem we must fix the idea that the weak radical is very small and, therefore, the assumption that the weak radical is zero appears frequently. Actually the weak radical of any nonassociative algebra A is contained in the intersection of primitive ideals (called the *radical of* A) and, when A is a noncommutative Jordan algebra, the weak radical is even contained in the Jacobson-McCrimmon radical (which, in general, is smaller than the radical) [64; Proposition 2.3]. Thus our Theorem 4.3 contains the Johnson's associative theorem and the Aupetit's extension for noncommutative Jordan algebras. But, even for noncommutative Jordan algebras, Theorem 4.3 is stronger than Aupetit's theorem, because our theorem can be applied to some anticommutative algebras, as shows the following

Corollary 4.1 [64; Remark 2.8 (iii)]. The Banach-Lie algebra of all derivations of a C^*-algebra has zero weak radical, so a unique complete algebra norm topology.

Remark 4.1. The proofs of the three theorems in this section are historically very related. To prove his theorem, Johnson uses intensively the Jacobson representation theory for associative rings. In [10] Aupetit (by using the subharmonicity of the spectral radius [76]) gets a new proof of Johnson's theorem which does not make use of the representation theory and, with minor changes on this proof, obtains Theorem 4.2. Actually the arguments in [10] can be even applied to power-associative complete normed algebras, although the result obtained for them is difficult to algebrize because an extension of the concept of Jacobson-McCrimmon radical for general power-associative algebras is unknown. The proof in [64] of Theorem 4.3 turns over again on the associative arguments and consists of an ingenuous adaptation of the Aupetit's new proof of the Johnson's theorem. Actually it can be observed that most of this proof remains true for some associative noncomplete normed algebras. When this fact is applied to the full multiplication algebra of any nonassociative complete normed algebra, our Theorem 4.3 follows easily.

Once we have seen that the particularization of Theorem 4.3 to noncommutative Jordan algebras is stronger than Theorem 4.2, the power of our Theorem 4.3 in the associative and/or (commutative) Jordan case may be discussed. In the common case of associative and commutative algebras Theorem 4.3 is just the well known Gelfand's precedent of the Johnson's theorem. In fact we have :

Proposition 4.1. For every associative and commutative algebra A we have that w-Rad(A) = Rad(A).

Proof. Only the inclusion Rad(A) $\subset$ w-Rad(A) must be proved. First observe that, under our assumption of associativity and commutativity, the set of all left (= right) multiplication operators by elements of A is a subalgebra of L(A). But this subalgebra is actually a full subalgebra. Indeed : if, for a in A, L_a is quasiinvertible in L(A) with quasiinverse T, then a is quasiinvertible in A with quasiinverse b := T(a)-a and so T = L_b because the mapping x $\rightarrow$ L_x is an algebra homomorphism and the quasiinverse is unique. Now it is clear that FM(A) = $\{L_a : a \in A\}$ and therefore Rad(A) is a FM(A)-invariant subspace of A. To conclude we must only show that, for every a in Rad(A), $L_a(=R_a)$ lies in Rad(FM(A)). But this follows from the fact that x $\rightarrow$ L_x is now an algebra homomorphism from A *onto* FM(A).

For a finite dimensional associative or Jordan algebra A the conditions w-Rad(A) = 0 and M-Rad(A) = 0 are equivalent (where M-Rad(A) denotes the Jacobson-McCrimmon radical of A). Although this result is not related with the problem of uniqueness of norm topology, it is interesting to understand the philosophy of the new concept of the weak radical. We prove here this result in the unifying context of generalized standard algebras. The class of generalized standard algebras is defined by a suitable finite set of identities (see [66]) and it is the "smallest" subclass of the class of noncommutative Jordan algebras containing all the alternative (in particular, associative) and (commutative) Jordan algebras, in the sense that every prime generalized standard algebra is either alternative or Jordan [45; Theorem 1].

Proposition 4.2. For every finite dimensional generalized standard algebra A the statements w-Rad(A) = 0 and M-Rad(A) = 0 are equivalent.

To prove this proposition we need some previous results. We recall that an ideal P of an algebra A is said to be a *semiprime ideal* of A if every ideal Q of A satisfying $Q^2 \subset P$ is actually contained in P.

Lemma 4.1. Let A be a finite dimensional generalized standard algebra and

let P be a semiprime ideal of A. Then M-Rad(A) $\subset$ P.

Proof. It is well known, and easy to see, that the Jacobson-McCrimmon radical of any finite dimensional noncommutative Jordan algebra is a nil-algebra (in fact it is the largest nilideal). But every finite dimensional generalized standard nilalgebra is nilpotent [66; Theorem 4] so solvable. Write $M := $ M-rad(A). Since M is a solvable ideal of A, by [62; Theorem 10] it is Penico solvable. That is : there is a non negative integer k such that $M^{\langle k\rangle} = 0$ (where $M^{\langle n\rangle}$ is defined inductively by $M^{\langle 0\rangle} = M, M^{\langle n+1\rangle} = A(M^{\langle n\rangle})^2 + (M^{\langle n\rangle})^2$). Since $M^{\langle k\rangle} \subset P$, let i be the smallest non negative integer such that $M^{\langle i\rangle} \subset P$. Assume $i \geqslant 1$. Then $(M^{\langle i-1\rangle})^2 \subset M^{\langle i\rangle} \subset P$ and, since $M^{\langle i-1\rangle}$ is an ideal of A [66; Theorem 9] and P is a semiprime ideal of A, we have $M^{\langle i-1\rangle} \subset P$, a contradiction. Thus i=0 and $M = M^{\langle 0\rangle} \subset P$, as required.

Remark 4.2. Since the Jacobson-McCrimmon radical of any noncommutative Jordan algebra is always a semiprime ideal, the above lemma shows that, for finite dimensional generalized standard algebras, it is actually the smallest semiprime ideal. As a consequence : a finite dimensional generalized standard algebra is "semisimple" (has zero Jacobson-McCrimmon radical) if and only if it is "semiprime" (0 is a semiprime ideal).

Lemma 4.2. Let A be a nonassociative semiprime algebra with D.C.C. on two sided ideals. Then w-Rad(A) = 0.

Proof. Clearly A has the D.C.C. on FM(A) -invariant subspaces so, if w-Rad(A) $\neq 0$, there is a minimal FM(A)-invariant subspace P with $P \subset$ w-Rad(A). If PA=0, then $P^2=0$. Otherwise P is an irreductible FM(A)-module. But for $x \in$ w-Rad(A) we have $L_x \in$ Rad(FM(A)) and, since the Jacobson radical of an associative algebra can be characterized as the intersection of its irreducible representations, it follows that $L_x(P) = 0$. Thus w-Rad(A)P = 0 and we have all over $P^2=0$. In any case A is not semiprime.

Corollary 4.2. [42]. Every nonassociative simple algebra has zero weak radical.

Proof. A simple algebra is semiprime with D.C.C. on two-sided ideals.

Corollary 4.3. A finite dimensional nonassociative algebra is semiprime if

and only if it has zero weak radical.

Proof. The "only if" part follows from Lemma 4.2. If A is a finite dimensional algebra with w-Rad(A) = 0 and P is an ideal of A such that $P^2=0$, then P is FM(A)-invariant, since under the assumption of finite dimension every subalgebra of an associative algebra is a full subalgebra and therefore FM(A) agrees with the usual multiplication algebra M(A) of A. If $x \in P$ and $F \in M(A)$, then clearly we have $L_x^2 = L_xFL_x = 0$, from which we deduce that the ideal of M(A) generated by L_x is a nilideal so a quasiinvertible ideal and so $L_x \in \mathrm{Rad}(M(A))$. Analogously $R_x \in \mathrm{Rad}(M(A))$. Thus $P \subset$ w-Rad(A) = 0, P = 0, and A is semiprime, as required.

Proof of Proposition 4.2. Remark 4.2 and the preceding corollary.

Remark 4.3. For a finite dimensional associative (even alternative) algebra A it can be proved, making use of the nilpotence of the Jacobson-McCrimmon radical, of the fact that the product of two ideals is another ideal and of arguments close to the ones in the proof of Corollary 4.3, that w-Rad(A) = M-Rad(A). This improves Proposition 4.2 in this particular case. I think that such an improvement should be true for every finite dimensional generalized standard algebra, but I do not have a proof at this time.

The equality w-Rad(A) = M-Rad(A) and even the equivalence w-Rad(A) = 0 $\Leftrightarrow$ M-Rad(A) = 0 are not true in general for A an associative or Jordan algebra, as shows the following

Proposition 4.3. There exists an associative (resp. : Jordan) algebra A such that 0 = w-Rad(A) and $0 \neq A =$ M-Rad(A).

Proof. It was given by Sasiada an example of an associative simple algebra A with A = M-Rad(A) (see [35; page 125-131]). Write $B := A^+$. Then B is a Jordan algebra which is also a simple algebra [22; Lemma 2.4]. Both A and B have zero weak radical (Corollary 4.2) while B (as well as A) agrees with its Jacobson McCrimmon radical.

In [64] we posed the problem of the equality of weak and Jacobson-McCrimmon radical for associative or Jordan algebras. Proposition 4.3 shows that the answer to this problem is negative. I am indebted to N. Jacobson

who made known to me the Sasiada's example which allows us to solve this problem.

Now, for the associative (resp. : Jordan) Banach algebrists, the comparison of our Theorem 4.3 with the one of Johnson (resp. : Aupetit) leads to the following problem (once the inclusion of the weak radical in the Jacobson-McCrimmon radical has been recalled).

Problem 4.1. Does the weak radical of any associative (resp. : Jordan) Banach algebra agrees with the Jacobson-McCrimmon radical ? If not, is there an associative (resp. : Jordan) Banach algebra with zero weak radical and non zero Jacobson-McCrimmon radical ?

5. DECOMPOSITION FOR NORMED ALGEBRAS

A well known theorem of Wedderburn states that a finite dimensional semisimple associative algebra is the direct sum of ideals which are simple algebras. It can be said in terms of lattices that, for such an algebra, the greatest element of the lattice L of its ideals is the least upper bound of the set of the atoms of L. If seems natural to look for conditions under which the lattice of the closed ideals of a nonassociative normed algebra has the above property, that is : the algebra is the closure of the sum of its minimal closed ideals.

It is easy to see that, if a nonassociative normed algebra A with zero annihilator ($aA = Aa = 0 \Longrightarrow a = 0$) is the closure of the sum of its minimal closed ideals, then A is necessarily semiprime and for every proper closed ideal P of A there is a non zero ideal Q of A satisfying $PQ = QP = 0$ (In short : A is a *generalized annihilator* normed algebra). Thus the assumption of generalized annihilation is a natural requirement in order to obtain the desired decomposition for the algebra. However, even in the associative case, this requirement is not enough in general to obtain minimal closed ideals, even less to obtain the decomposition of the algebra.

Thus very significant is the following result which is the strongest we know on decomposability in the associative case (although there are some preceding particular results).

Theorem 5.1. (Civin-Yood [22]). Let A be a semisimple generalized annihilator Banach algebra. Then A is the closure of the direct sum of its minimal closed ideals and these are topologically simple semisimple Banach algebras.

We recall that a normed algebra is said to be *topological simple* if it has non zero product and no non zero proper closed ideals. Also it must be noticed that the assumption of semisimplicity in the theorem of Civin and Yood is almost essential. For, if a non semisimple Banach algebra with zero annihilator is the closure of the sum of its minimal closed ideals, then one can easily obtain a topologically simple radical Banach algebra, thus answering one of the most famous unsolved problems in the theory of Banach algebras.

As in the case of the nonassociative extension of Johnson's theorem on uniqueness of norm topology, a nonassociative extension of Theorem 5.1 compels to have available a suitable concept of nonassociative semisimplicity. But we know that the assumption w-Rad(A) = 0, on a nonassociative algebra A, is a good concept of nonassociative semisimplicity. Therefore the following theorem is easy to guess (but not to prove).

Theorem 5.2. [30]. Let A be a nonassociative generalized annihilator complete normed algebra with zero weak radical. Then A is the closure of the direct sum of its minimal closed ideals and these are topologically simple complete normed algebras.

Remark 5.1. The proof of this theorem in [30] consists of two quite different steps. In the first one we make some modifications of the associative arguments of Yood in [82] which allow us to prove that a nonassociative generalized annihilator normed algebra is the closure of the sum of its minimal closed ideals if and only if the intersection of its maximal closed ideals is zero. In the second step (the more significant one) we define the concept of weak primitive ideal of a nonassociative algebra A as follows. Let p be a primitive ideal of FM(A) and let P denote the largest FM(A)-invariant subspace of A contained in $\{a \in A : L_a, R_a \in p\}$. Such a subspace P, which is clearly an ideal of A, is said to be a weak primitive ideal of A. We prove that every weak primitive ideal of a nonassociative generalized annihilator complete normed algebra is a maximal closed ideal. Thus Theorem 5.2 follows from the first step and the fact that the weak

radical of any nonassociative algebra is the intersection of its weak primitive ideals.

In order to have in our Theorem 5.2 a complete nonassociative parallel to the Civin-Yood theorem, an affirmative answer to the following problem is missing.

Problem 5.1. Let A be as in Theorem 5.2. and let M be a minimal closed ideal of A. Does M have zero weak radical ?

In most of interesting corollaries to our problem the answer to this problem is affirmative. This is the case in the following result for noncommutative Jordan algebras, in which we recover again the associative Civin-Yood theorem.

Corollary 5.1. [30]. Let A be a noncommutative Jordan generalized annihilator complete normed algebra with zero Jacobson-McCrimmon radical. Then A is the closure of the direct sum of its minimal closed ideals and these are noncommutative Jordan topologically simple complete normed algebras with zero Jacobson-McCrimmon radical.

New Proof. In view of the inclusion w-Rad(A) $\subset$ M-Rad(A) and Theorem 5.2, it is enough to prove that the minimal closed ideals of A have zero Jacobson-McCrimmon radical. Let M be such an ideal. Since M-Rad(M) is a closed ideal of M, it is also an ideal of A [30; Proposition 2(ii)]. Now since M-Rad(M) is a quasiinvertible ideal of A, we have M-Rad(M) $\subset$ M-Rad(A) = 0.

It is clear that in Corollary 5.1 the minimal closed ideals of the algebra A have zero weak radical (recall the inclusion of the weak radical in the Jacobson-McCrimmon radical). There is another very interesting particular case of our theorem in which Problem 5.1 has also an affirmative answer. Consider a nonassociative complete normed algebra A with zero annihilator and such that for every x in A there is a summable familly $\{x_i\}$ with $x_i \in M_i$ and $x = \Sigma x_i$ (where $\{M_i\}$ denotes the set of the minimal closed ideals of A). Clearly this latter condition is the best improvement of the thesis of our Theorem 5.2 which one can expect. But this is not always true, as it is well known even in the case of an annihilator commutative (associative) Banach algebra [39]. It is easy to see that an algebra A as considered above is a *generalized complemented* normed algebra, that is : A is a normed algebra each closed ideal of which is a direct summand. In the converse direction we have :

Corollary 5.2 [30]. Let A be a nonassociative generalized complemented normed algebra with zero weak radical and let $\{M_i\}$ denote the set of the minimal closed ideals of A. Then for each x in A there is a unique summable familly $\{x_i\}$ with x_i in M_i and $x = \Sigma x_i$. Moreover each M_i is a topologically simple complete normed algebra with zero weak radical.

The proof of this corollary uses Theorem 5.2, some results by Bachelis [12] on unconditional decompositions of Banach spaces, and the following proposition which was stated without proof in [30].

Proposition 5.1. For every direct summand P of a nonassociative algebra A the equality w-Rad(P) = w-Rad(A) $\cap$ P is true.

Proof. Write $A = P \oplus Q$ for a suitable ideal Q of A and let p and q denote the full subalgebras of L(A) generated by the sets $\{L_x, R_x : x \in P\}$ and $\{L_y, R_y : y \in Q\}$, respectively. Since $\{F \in L(A) : F(A) \subset P\}$ and $\{G \in L(A) : G(P) = 0\}$ are one-sided ideals (so full subalgebras) of L(A) containing the sets of generators for p and q, respectively, we have $p \subset \{F \in L(A) : F(A) \subset P\}$ and $q \subset \{G \in L(A) : G(P) = 0\}$, from which we deduce $qp = 0$. A similar argument gives $pq = 0$. Now, by [30; Lemma 6] $p+q$ is a full subalgebra of L(A). Since clearly $p+q \subset$ FM(A) and $p+q$ contains all the multiplication operators by elements of A, we conclude that FM(A) = $p+q$ which together with $pq = qp = 0$ shows that p is an ideal of FM(A). For F in p write $\varphi(F)$ for the linear mapping $x \to F(x)$ from P into P. It is clear that φ is an homomorphism from p into L(P) which is actually valued in FM(P). Indeed, $\varphi^{-1}(FM(P))$ is a full subalgebra of p (so of L(A), by the transitivity of fullness) containing the set of generators for p, so $p \subset \varphi^{-1}(FM(P))$ and $\varphi(p) \subset FM(P)$. For G in FM(P) write $\psi(G)$ for the linear mapping $x+y \to G(x)$ $(x \in P,\ y \in Q)$ from A into A. Thus ψ is an homomorphism from FM(P) into L(A) which is actually valued in p since $\psi^{-1}(p)$ is a full subalgebra of FM(P) (so of L(P)) containing the set $\left\{L_x^P, R_x^P : x \in P\right\}$ (where L_x^P and R_x^P denote the operators of multiplication by x on P) and FM(P) is generated as a full subalgebra in L(P) by this set. Moreover φ and ψ, regarded as mapping from p into FM(P) and from FM(P) into p, respectively, are each one the inverse of the other, as it is routine to see. Now for x in w-Rad(A) $\cap$ P and G in FM(P) we have

$G(x) \in P$, $\psi(P) \in \mathcal{p} \subset FM(A)$ and so $G(x) = \psi(G) \in$ w-Rad(A) $\cap$ P because x-Rad(A) is, by definition, FM(A) -invariant. Therefore w-Rad(A) $\cap$ P is FM(P)-invariant. On the other hand, whenever $x \in$ w-Rad(A) $\cap$ P, we have that L_x and R_x belong to Rad(FM(A)) $\cap\ \mathcal{p}$ = Rad($\mathcal{p}$). Thus $L_x^P = \varphi(L_x) \in \varphi(\mathrm{Rad}(\mathcal{p}))$ = Rad(Fm(P)) and similarly $R_x^P \in$ Rad(Fm(P)). Therefore the inclusion w-Rad(A) $\cap$ P $\subset$ w-Rad(P) has been proved. A similar argument allows to prove the reverse inclusion.

Remark 5.2. Let A be a complete normed algebra and let P be a closed ideal of A. Assume that there is another closed ideal Q of A such that $A = \overline{P \oplus Q}$ (this assumption is automatically satisfied if A is in addition generalized annihilator [30; Proposition 2(i)]). Then most of the proof of the above proposition remains true. Specifically the algebras $\mathcal{p}$ and FM(P) are isomorphic via the mapping φ defined as above. Actually, to prove the existence of an inverse mapping for φ, the definition of ψ should be touched up as follows. One can prove that for each G in FM(P) there is a non negative number k_G such that $\|G(x)\| \leqslant k_G\|x+y\|$ for all x in P and y in Q. Then the mapping $x+y \to G(x)$ $(x \in P,\ y \in Q)$ is a continuous densely defined operator on A which has a unique continuous extension $\psi(G)$ to A. Unfortunately (in order to give an affirmative answer to Problem 5.1), now we only can prove that $\mathcal{p}+\mathcal{q}$ is dense in FM(A), which does not imply that $\mathcal{p}$ is an ideal of FM(A) (the last fact is needed in the proof of Proposition 5.1. to make use of the hereditarity of the Jacobson radical).

In what follows we make some improvements of our Theorem 5.2. when the algebra A is assumed to have a unit, the most important one beeing that the assumption that w-Rad(A) = 0 is then superfluous. This is a consequence of folkloric results on nonassociative complete normed algebras and the following purely algebraic

Theorem 5.3. (Fernandez, private communication). **Let A be a nonassociative semiprime algebra and assume that for each proper ideal P of A there is a nonzero ideal Q of A such that PQ = QP = 0. Then A is the direct sum of its minimal ideals and these are simple algebras.**

Proof. For each ideal P of A let <u>An(P)</u> denote the largest ideal Q of A such that PQ = QP = 0. Now our assumptions are that A is semiprime and that An(P) $\neq$ 0 for every proper ideal P of A. From these facts we deduce easily that A = P $\oplus$ An(P) for every ideal P of A. Regarded A as a module over its multiplication algebra $M(A)$, the submodules are the ideals. Thus every submodule of A has a complement and clearly $M(A)(a) = 0 \Longrightarrow a=0$. Therefore, by [38; Theorem IV.1.2], A is a complete reducible module, that is : A is the sum of its irreducible submodules (= minimal ideals). It follows from the semiprimness of A that this sum is direct. Also it is clear that the minimal ideals of A are simple algebras.

As a consequence we have :

Corollary 5.3. Let A be a nonassociative generalized annihilator complete normed algebra with unit. Then A is the direct sum of its minimal ideals (the set of which is finite) and these are complete normed simple algebras with unit.

Proof. Let P be a proper ideal of A. Then $\bar{P}$ is also a proper (closed) ideal of A (see Lemma 5.2 below) and therefore $An(P) = An(\bar{P}) \neq 0$. Now the proof is concluded by using Theorem 5.3 and the following

Lemma 5.1. Suppose that a nonassociative complete normed algebra with zero annihilator is the direct sum of a set of ideals. Then this set of ideals is finite and each of its elements is actually a closed ideal.

Proof. Let A be our complete normed algebra which is the direct sum of a set $\{M_i\}$ of ideals. That each M_i is closed follows from the equality $M_j = An\left(\bigoplus_{i\neq j} M_i\right)$. If $\{M_i\}$ is infinite we can construct a summable familly $\{x_i\}$ with $x_i \in M_i$ and such that the set $\{i : x_i \neq 0\}$ is infinite. Write $x := \Sigma x_i$. Then since $A = \oplus M_i$, we have also $x = \Sigma y_i$ where $y_i \in M_i$ and the set $\{i : y_i \neq 0\}$ is finite. Now $0 = \Sigma(x_i - y_i)$, which implies $(x_i-y_i)M_j = M_j(x_i-y_i) = 0$ for all i,j. Thus $(x_i-y_i)A = A(x_i-y_i) = 0$ and so $x_i = y_i$. Therefore the infinite set $\{i : x_i \neq 0\}$ agrees with the finite set $\{i : y_i \neq 0\}$, a contradiction.

Lemma 5.2. The closure of a proper ideal of a nonassociative complete normed algebra with unit is another proper ideal.

Proof. Let A be our complete normed algebra with unit, let I denote the unit of A and let M be an ideal of A such that $\bar{M}=A$. Then there exists m in

M such that $\|I-m\| < 1$. Thus $\|I_A-L_M\| < 1$ which implies, as it is well known, that L_m is an invertible element in the Banach algebra VL(A). Therefore $I = L_m(x) = mx \in M$, for some x in A, and so M = A also.

Remark 5.3. Let A be as in Corollary 5.3. Then from Corollary 4.2 and Proposition 5.1 we obtain that A has zero weak radical. Also, if in addition A is a noncommutative Jordan algebra, one can see easily that A has zero Jacobson-McCrimmon radical. These facts show that, under the assumption of existence of a unit, the assumption w-Rad(A) = 0 (resp. : M-Rad(A) = 0) in Theorem 5.2 (resp. : Corollary 5.1) is superfluous. Corollary 5.3 also shows that, for an algebra A under its assumptions, the minimal ideals of A are just the minimal closed ideals of A (actually one may deduce that every ideal of A is closed). This is no longer true for non unital generalized annihilator complete normed algebras even in the semisimple associative case (consider the Banach algebra of compact operators on an infinite dimensional Hilbert space).

In some interesting particular cases Corollary 5.1 can be improved. A first example is the following

Corollary 5.4 [31]. **Let A be an alternative generalized annihilator semisimple complete normed algebra. Then A is the closure of the direct sum of its minimal closed ideals and these are either topologically simple semisimple (associative) Banach algebras or isomorphic to :**
(complex case) the algebra of complex octonions
(real case) either the underlying real algebra of the algebra of complex octonions, the split algebra of real octonions, or the division algebra of real octonions.

Notice that for alternative algebras the term "semisimple" is used in a sense analogous to the associative case because of the equality of the Jacobson-McCrimmon radical (which was defined previously by Smiley [71] for alternative algebras) with the radical (intersection of primitive ideals) [87]. The preceding corollary follows easily from Corollary 5.1 by using the results in [87,44].

Another improvement of Corollary 5.1 can be obtained in the particular case of JB- (resp. : noncommutative JB^*-) algebras. Using that one to one homomorphisms (resp. : *-homomorphisms) of JB- and noncommutative JB^*-

algebras have zero Jacobson-McCrimmon radical (see [61], for example) and that closed ideals of noncommutative JB^*-algebras are $*$-invariant [58], we obtain :

Corollary 5.5. Let A be a generalized annihilator JB- (resp. : noncommutative JB^*-) algebra. Then A is the c_0-sum of its minimal closed ideals and these are topologically simple JB- (resp. : noncommutative JB^*-) algebras.

By putting together Corollary 5.4 and 5.5 and taking into account the essential uniqueness of the C^*-algebra structure on the algebra of complex octonions [58,20], we have :

Corollary 5.6. Let A be a generalized annihilator alternative C^*-algebra. Then A is the c_0- sum of its minimal closed ideals and these are either topologically simple associative C^*-algebras or the C^*-algebra of complex octonions.

Theorem 5.1 reduces in some sense the study of semisimple generalized annihilator Banach algebras to the semisimple topologically simple case and no more information can be expected from this theorem because every topologically simple normed algebra is, trivially, generalized annihilator and no much is known on the structure of general topologically simple semisimple Banach algebras. So if, following the Wedderburn work, one wish giving a description of some topologically simple Banach algebras, some stronger or additional assumptions are needed. The more suggestive one in our context is the one of "annihilator Banach algebras" by Bonsall and Goldie (see [18; Section 32]). The annihilator condition implies the generalized annihilator one and is inherited by the minimal closed ideals. Therefore, under the additional assumption of semisimplicity, Theorem 5.1 applies with the advantage that now the minimal closed ideals are topologically simple semisimple annihilator Banach algebras and for these algebras a satisfactory representation theory is known because they are prime algebras with non zero "socle" (see [18,38]).

Although the theory of the socle has been succesfully extended to noncommutative Jordan algebras [57,29,28], we do not know a coherent nonassociative (even Jordan) extension of the annihilator condition, except for alternative normed algebras and JB-algebras. Alternative annihilator complete normed algebras are defined without changes on the associative case and have the same advantages above cited for annihilator Banach algebras. Therefore, for the semisimple case, its theory is finished with Corollary 5.4 which reduces the study to the topologically simple semisimple annihilator Banach case and to the algebras of octonions (which are, trivially, annihilator because they have no proper non zero one-sided ideals). Annihilator (or dual) JB-algebras were defined and studied by Bunce [21]. Dual JB-algebras are generalized annihilator and their closed ideals are dual JB-algebras. Therefore, by Corollary 5.5, the theory of dual JB-algebras is reduced to the topologically simple case. By the way, the classification theorem in [21] for topologically simple dual JB-algebras can be improved as follows.

Theorem 5.4. The topologically simple dual JB-algebras are the spin factors, the exceptional JB-algebra M_3^8 and the JB-algebras $K(H)_{sa}$ of all selfadjoint compact linear operators on a real, complex or quaternionic Hilbert space H.

Proof. That all the algebras listed in the statement are topologically simple dual JB-algebras follows from the results in [21]. Conversely, let A be a topologically simple dual JB-algebra. As noticed in the proof of [21; Proposition 4.4] A" is a factor which, by [21; Corollary 2.2], is actually a type I factor. If A" is either a spin factor of M_3^8, then so is A because of the reflexivity of A" in this case. Otherwise, by [5; Theorem 3.1], A" is the JB-algebra $BL(H)_{sa}$ of all selfadjoint linear operators on a suitable real, complex or quaternionic Hilbert space H. Since the minimal projections of A" are the orthogonal projections onto the one-dimensional subspaces of H, it is enough to apply [21; Corollary 2.2] and the well known theory of selfadjoint compact operators on Hilbert spaces to obtain $A = K(H)_{sa}$.

A compact Jordan-Banach algebra is a Jordan-Banach algebra such that for all a in A the operator U_a is compact. Since compact JB-algebras are dual [21; Proposition 4.1] and the only non compact JB-algebras listed in the theorem are the infinite dimensional spin factors, we have :

Corollary 5.7. **The topologically simple compact JB-algebras are the finite dimensional spin factors, the JB-algebra M_3^8 and the JB-algebras $K(H)_{sa}$ for a real, complex or quaternionic Hilbert space H.**

We conclude this section by giving the noncommutative JB^*-algebra analogous to Bunce theory for JB-algebras. The Bunce's definition of annihilator condition for JB-algebras cannot be extended without changes to noncommutative JB^*-algebras. However it is shown in [21] that a JB-algebra is dual if and only if it is an ideal of its bidual. Thus the noncommutative JB^*-algebras which are ideals of their biduals are the desired analogous of dual JB-algebras.

Corollary 5.8. Let A be a noncommutative JB^*-algebra which is an ideal of its bidual. Then A is the c_0-sum of its minimal closed ideals and these are topologically simple noncommutative JB^*-algebras which are ideals of its bidual.

Proof. By Corollary 5.5 it is enough to prove that A is generalized annihilator and that the (minimal) closed ideals of A are ideals of its bidual (the last fact is clear from the Arens regularity of A [58; Corollary 1.8]). Let P be a proper closed ideal of A. Then P^{00} is a proper w^*-closed ideal of A" and so, by [58; Theorem 3.9], the annihilator of P^{00} in A", $An(P^{00})$, is non zero. Since A" has zero annihilator, and A is Arens regular and an ideal of A", it follows easily that A is an essential ideal of A". Therefore $An(P) \supset An(P^{00}) \cap A \neq 0$ and A is generalized annihilator, as required.

Remark 5.3. Another more geometric proof of Corollalry 5.8 can be given, without using the results of this section, from [58; Theorem 1.7, Corollary 1.11 and Theorem 4.3] and a result by Harmand on c_0-decomposition of Banach spaces which are M-ideals of their biduals (see [46; Corollary 9]). For the description of topologically simple noncommutative JB^*-algebras which are ideals of its bidual see a result of the author in [46].

6. H^*-ALGEBRAS

Associative complex H^* -algebras are the first infinite dimensional Banach algebras for which a complete structure theory was given (see [18; Section 34]). For them we have :

Theorem 6.1. (Ambrose). Every associative complex H^*-algebra with zero annihilator is the Hilbert sum (closure of the orghogonal sum) of its minimal closed ideals and these are isomorphic (up to a positive multiple of the inner product) to the H^*-algebra of all the Hilbert-Schmidt operators on a suitable complex Hilbert space.

We recall that a nonassociative (real or complex) *semi-H^*-algebra* is a complete normed algebra A with a vector space involution (linear in the real case and conjugate linear in the complex one) and an inner product $(\mid)$ such that $(a|a) = \|a\|^2$ and $(ab|c) = (a|cb^*) = (b|a^*c)$ for all a,b,c in A. Semi H^*- algebras whose involution is actually an algebra involution are called nonassociative *H^*-algebras*. For the associative, (commutative) Jordan and Lie case (always with zero annihilator) every semi-H^*-algebra is an H^*-algebra. But this need not be true in general, as one can see even in the two-dimensional commutative complex case. Thus the concept of semi-H^*-algebra is an essentially nonassociative weakening of the one of H^*-algebra.

Except for the description of all the nonassociative topologically simple semi-H^*-algebras (which seems to be a problem almost unsolvable), we have proved in [25] the nonassociative extension of Ambrose's theorem in the complex case. The arguments in [25], of geometric nature, can be also applied to the real case. Besides this, it can be easily seem that semi-H^*-algebras with zero annihilator are generalized annihilator complete normed algebras with zero weak radical (see [30; Lemma 14] for the complex case), hence another proof of the nonassociative extension of Theorem 6.1 can be given by applying Theorem 5.2 and making use of some specific results on semi-H^*-algebras with minor difficulties. Thus we have :

Theorem 6.2 [25]. Every nonassociative semi-H^*-algebra **with zero annihilator is the Hilbert sum** of its minimal closed ideals **and these are topologically simple** semi-H^*-algebras.

As a first application, alternative complex H^*-algebras are perfectly known from Theorem 6.1 and 6.2 and the following theorem in [60]. We give here a much more simple new proof of this result, which can also be obtained from the results in [25].

Theorem 6.3 [60].Every topologically simple alternative **complex H^*-algebra**

is either associative or isomorphic (up to a positive multiple of the inner product) to the H^*-algebra of complex octonions.

Proof. We recall that the algebra of complex octonions has a canonical structure of H^*-algebra [60] which is essentially unique in view of Corollary 6.3 below. Let A be our topologically simple alternative complex H^*-algebra. Then An(A) = 0. But, with minor changes on the associative case (assume $a = a^*$ in the proof of [18; Lemma 34.9] and take into account that M-Rad(A) is $*$-invariant), one obtains M-Rad(A) = An(A) = 0. By [87], Rad(A) = 0 also. But an alternative topologically simple complete normed complex algebra with zero radical is either associative or isomorphic to the algebra of complex octonions (Corollary 5.4).

Theorem 6.2 contains some preceding particular results by Schue [67] for Lie algebras and Viola [77] for Jordan algebras. Also Schue and Viola [67,78] have worked on the classification of Lie and Jordan topologically simple complex H^*- algebras. Cuenca and Rodriguez [25] have finished the description of Jordan topologically simple complex H^*- algebras and they have also given a classification theorem for noncommutative Jordan topologically simple complex H^*-algebras. We do not state here this classification since it gets off the aim of this paper (see [25,23] for details). We shall only note that this classification includes two types of noncommutative Jordan topologically simple complex H^*-algebras which depend, for its complete knowledge, on the description of the H-algebras introduced in Section 2 to describe the smooth complete normed real algebras. H-algebras are just the real H^*- algebras whose (algebra) involution is the mapping $x \to -x$. Thus the theory of H-algebras can be reduced to the topologically simple case, as shows the following result which can be easily deduced from Theorem 6.2.

Corollary 6.1. Every H-algebra is the Hilbert sum of closed ideals, one of which is a H-algebra with zero product (so a suitable real Hilbert space equipped with the zero product) and the others are topologically simple H-algebras.

Remark 6.1. The preceding corollary gives some additional information about all those structures whose understanding depends on the knowledge of the H-algebras. These are to our knowledge :

i) The smooth complete normed algebras (Theorem 2.2).
ii) The quadratic noncommutative JB^*-factors (which are connected with the smooth complete normed algebras [59]).
iii) The anticommutative complex H^*-algebras with isometric involution and, in particular, the anticommutative topologically simple complex H^*-algebras, which are one of the types in the classification given in [25] for the noncommutative Jordan topologically simple complex H^*-algebras (since such an anticommutative H^*- algebra A is completely determined by the H-algebra $\{a \in A : a^* = -a\}$).
iv) The flexible quadratic complex H^*-algebras with dimension not two, which depend on the anticommutative complex H^*-algebras with isometric involution (see [25]) and which are another type in the classification of [25] for the noncommutative Jordan topologically simple complex H^*-algebras.

In what follows we state a result for nonassociative complex H^*-algebras which does not have associative precedent. As a motivation, consider a complex algebra A which has been structured as a topologically simple H^*-algebra (in most of the cases A is one of the finite dimensional simple algebras interesting for the algebrist). It is desirable every H^*-algebra structure on A to be (up to a positive multiple of the inner product) totally isomorphic to the given one. This will be true as a consequence of the following

Theorem 6.4 [24]. **Every (algebraic) isomorphism F between nonassociative complex H^*-algebras with zero annihilator can be written in a unique way as $F = G\exp(D)$ with G a $*$-isomorphism and D a continuous derivation of the first algebra satisfying $D(a^*) = -D(a)^*$ for all a in the algebra.**

Corollary 6.2. **Isomorphic complex H^*-algebras with zero annihilator are $*$-isomorphic.**

Also it is proved in [24] that the $*$-isomorphisms between topologically simple complex H^*-algebras are positive multiples of isometries. So we have :

Corollary 6.3. **Up to a positive multiple of the inner product, all the H^*-algebra structures on a topologically simple complex H^*-algebra are totally isomorphic.**

As we have said above, we do not know any associative precedent for our Theorem 6.4. However, it is inspired by the analogous result by Okayasu for associative C^*-algebras (see [65; Corollary 4.1.21]) which is also true for noncommutative JB^*-algebras [58].

The proof of Theorem 6.4. uses the uniqueness of norm topology for semi-H^*-algebras with zero annihilator (Theorem 4.3 together with the fact, above mentioned, that semi-H^*-algebras with zero annihilator have zero weak radical). Also it uses Theorem 6.2 and some results, on general nonassociative complete normed complex algebras, to which we wish devote the rest of this section. One of these results is, we think, the first remarkable result on general nonassociative normed algebras which appears in the literature. It was proved, but not stated, in Dixmier's book [27] and its proof uses holomorphic functional calculus in Banach algebras and the Gelfand theory for commutative Banach algebras.

If D is a continuous derivation of a complete normed algebra, then it is easy to see that exp(D) is an automorphism of the algebra. In the reverse direction we have :

Theorem 6.5 (Dixmier [27; pag. 313-314]). Let A be a nonassociative complete normed complex algebra and let F be a continuous automorphism of A satisfying $sp(F) \subset \{z \in \mathbb{C}-\{0\} : |\arg(z)| < 2\pi/3\}$. Then there is a unique continuous derivations D of A such that $F = \exp(D)$ and $sp(D) \subset \{z \in \mathbb{C} : |\mathrm{Imag}(z)| < 2\pi/3\}$.

With the same arguments used by Dixmier in the proof of his theorem we have obtained the following result which is also needed in the proof of Theorem 6.4.

Theorem 6.6 [9]. Let F and D be respectively a continuous automorphism and a continuous derivation of a nonassociative complete normed complex algebra with non zero product. Then there are z,w in sp(F) such that zw belongs to sp(F), and u, v in sp(D) such that u+v belongs to sp(D).

Under the stronger assumption of zero annihilator the proof of the above theorem can be refined to obtain :

Corollary 6.4 [9]. Let D be a continuous derivation of a nonassociative complete normed complex algebra with zero annihilator. Then zero lies in the convex hull of the spectrum of D.

7. AUTOMATIC CONTINUITY

In this last section we will consider three classical sides of the problem of the automatic continuity for linear mappings between complete normed algebras, namely : continuity of homomorphisms onto, of homomorphisms with dense range and of derivations. In the first case we have a satisfactory general nonassociative result, in the second one we prove a power-associative result which can be considered satisfactory at least for (commutative) Jordan algebras, and in the last one (although to our knowledge there is not any associative remarkable result even for Jordan algebras) we prove some non trivial facts about derivations of general nonassociative complete normed algebras.

The original proofs of the theorems of Johnson and Aupetit on uniqueness of norm topology (Theorems 4.1 and 4.2) yield directly to the more general result that every homomorphism from a Banach (resp. : noncommutative Jordan-Banach) algebra *onto* another one with zero Jacobson-McCrimmon radical is continuous. The proof of Theorem 4.3 does not yield an analogous fact and, actually, we do not know wether or not every homomorphism from a nonassociative complete normed algebra onto another one with zero weak radical is continuous. However, a small elargement of the weak radical allows us to obtain a nonassociative result on this question which contains the Johnson and Aupetit results above cited.

Definition 7.1. Let A be a nonassociative algebra and let C be any subalgebra of L(A) such that $M(A) \subset C \subset FM(A)$, where M(A) denotes the usual multiplication algebra of A and FM(A) is the full multiplication algebra of A (Definition 4.1 (iii)). Then the largest C-invariant subspace of A consisting of elements a such that L_a and R_a lie in the Jacobson radical of C is called the C-radical of A and is denoted by C-Rad(A). The *ultra-weak radical* of A (uw-Rad(A)) is defined as the sum of all the C-radicals of A when C runs through the set of all subalgebras of L(A) satisfying $M(A) \subset C \subset FM(A)$. Since the weak radical is a C-radical (take $C = FM(A)$) it follows that w-Rad(A) $\subset$ uw-Rad(A).

Now we can state :

Theorem 7.1 [64]. Every homomorphism from a nonassociative complete normed algebra onto another one with zero ultra-weak radical is continuous.

The assumption of zero ultra-weak radical is stronger than the one of zero weak radical. Thus the uniqueness of norm topology of the range algebra in the above theorem (given by Theorem 4.3) is one of the ingredients in its proof. The ultra-weak radical is no much more larger than the weak radical and in fact we do not know any example in which both radicals do not agree (perhaps the Lie algebra of all derivations of a C^*-algebra has non zero ultra-weak radical ? Compare with Corollary 4.1). Actually for a nonassociative algebra A we have (see [64]) :
An(A) $\subset$ w-Rad(A) $\subset$ uw-Rad(A) $\subset$ M-Rad(A) $\subset$ Rad(A) $\subset$ s-Rad(A), where s-Rad(A) denotes the *strong radical* of A (intersection of modular maximal two-sided ideals) and the Jacobson-McCrimmon radical of A, M-Rad(A), is only considered when A is a noncommutative Jordan algebra. The equality w-Rad(A) = uw-Rad(A) is clearly true for finite dimensional A. Another case of coincidence of the weak and ultra-weak radical appears when A is an H^*-algebra. Then by analogous arguments to the ones in the proof of [30; Lemma 14], one obtain An(A) = w-Rad(A) = uw-Rad(A). For associative and commutative A we have (Proposition 4.1) w-Rad(A) = uw-Rad(A) = M-Rad(A) = Rad(A) = s-Rad(A). For general A the assumption uw-Rad(A) = 0 has some advantages on the assumption w-Rad(A) = 0. Actually the proof of Corollary 4.3 shows that every ideal P of A with $P^2 = 0$ is contained in the M(A)-radical so in the ultra-weak radical. Thus uw-Rad(A) = 0 implies that A is semiprime, while it remains unknow wether or not the implication "w-Rad(A) = 0 $\Longrightarrow$ A semiprime" is true. Now some results in Section 4 can be improved. As an example, the arguments in the proof of Lemma 4.2 yield to the following

Proposition 7.1. Let A be a nonassociative algebra with D.C.C. on two sided ideals. Then A is semiprime if and only if A has zero ultra-weak radical.

As a consequence, simple algebras have zero ultra-weak radical, which follows also from

Proposition 7.2. Let A be a nonassociative prime algebra with D.C.C. on two sided ideals. Then A is semiprime if and only if A has zero ultra-weak radical.

As a consequence, simple algebras have zero ultra-weak radical, which follows also from

Proposition 7.2. Let A be a nonassociative prime algebra with a minimal ideal. Then A has zero ultra-weak radical.

Proof. Let M be the given minimal ideal of A, let C be any subalgebra of L(A) with $M(A) \subset C \subset FM(A)$, and let P denote the C-invariant subspace of A generated by M. We will see that P is a minimal C-invariant subspace of A. Indeed, if Q is a C-invariant subspace of A with $Q \subset P$, then $Q \cap M = M$ or 0, because Q is an ideal and M is a minimal ideal. In the first case we have clearly $Q = P$, while in the second one $Q = 0$ since A is prime. Now P is an irreducible C-module and therefore, as in the proof of Lemma 4.2, C-Rad(A)P = 0 and, since A is prime, we deduce C-Rac(A) = 0. Thus uw-Rad(A) = 0, as required.

Abundance of minimal ideals in complete normed algebras will also imply that the ultra-weak radical is zero.

Lemma 7.1. Let A be a nonassociative generalized annihilator complete normed algebra. Then uw-Rad(A) is contained in An (ΣM_i), **where** $\{M_i\}$ **denotes the set of the minimal ideals of A.**

Proof. Let M be a minimal ideal of A, let C be any subalgebra of L(A) such that $M(A) \subset C \subset FM(A)$, and let P denote the C-invariant subspace of A generated by M. If Q is a C-invariant subspace of A with $Q \subset P$, then either $Q = P$ or $M \subset An(Q)$. If the last occurs then, since An(Q) is a closed ideal of A, it is FM(A)- (so C-) invariant [30; Proposition 5] so $P \subset An(Q)$ which, together with $Q \subset P$, implies $Q = 0$. Now P is a minimal C-invariant subspace of A and, as in the proof of the preceding proposition, we obtain $C\text{-Rad}(A) \subset An(P) \subset An(M)$. Thus $uw\text{-Rad}(A) \subset \cap An(M_i) = An(\Sigma M_i)$, as required.

Proposition 7.3. Let A be a nonassociative complete normed algebra with zero annihilator and such that the sum of its minimal ideals is dense in A. Then A has zero ultra-weak radical.

Proof. By the preceding lemma it is enough to prove that A is generalized annihilator. It is clear that A is semiprime. Let P be a proper closed ideal of A. Then $PM = MP = 0$ for some minimal ideal M of A (if not $P \cap M_i \neq 0$ for all i and so $\Sigma M_i \subset P \Longrightarrow P = A$, a contradiction). Hence A is generalized annihilator.

Either from Proposition 7.2 or from Proposition 7.3, we obtain :

Corollary 7.1. Every nonassociative topologically simple complete normed algebra with a minimal ideal has zero ultra-weak radical.

Remark.7.1. Every simple algebra is always a minimal ideal whenever it is imbedded as an ideal in any algebra. Thus, if an associative radical simple algebra B can be imbedded as an ideal into a prime Banach algebra A, we have w-Rad(A) = uw-Rad(A) = 0 (Proposition 7.2), while
Rad(A) $\supset$ Rad(A) $\cap$ B = Rad(B) = B $\neq 0$ (compare with Problem 4.1.).

Concerning the problem of automatic continuity of homomorphisms with dense range the most remarkable result, in the associative case, is the following one (see [70; Theorem 6.18]).

Theorem 7.2 (Rickart). Every densely valued homomorphism from a Banach algebra into another one with zero strong radical is continuous.

This theorem seems to be unsatisfactory even for the (associative) Banach algebrists since it has been conjectured (but not proved, I think) that the assumption in the theorem of strong semisimplicity on the range algebra can be weakened to the usual semisimplicity (see [11] for a complete information on this question). For the nonassociative case the situation is more unfortunate. For we do not know a proof of the following natural

Conjecture 7.1. Every densely valued homomorphism from a nonassociative

complete normed algebra into another one with zero strong radical is continuous.

Even if this conjecture is true, the result is not a good nonassociative extension of Rickart theorem because of its vacuity in the case of anticommutative algebras. In despite of this comments, we give here a proof of the above conjecture in the particular case of power-associative algebras.

If a is an element of a power-associative normed algebra A, the number $\lim \left\{\|a^n\|^{1/n}\right\}$ exists and will be called as usual the spectral radius of a and denoted by r(a). It was noticed in [14] that, for complete normed A, the spectral radius is an algebraic invariant, since r(a) is the maximum of the moduli of the numbers in the spectrum of a relative to any maximal associative subalgebra of A containing a. Let A and B be power-associative complete normed algebras and let F be an homomorphism from A into B. For a in A let C be a maximal associative subalgebra of A with $a \in C$ and let D be a maximal associative subalgebra of B containing the (associative) subalgebra F(C) of B. Now, by regarding F as an homomorphism from C into D, we obtain clearly $sp(D,F(a)) \subset sp(C,a)$ and so, by the preceding characterization of the spectral radius, we have

Lemma 7.2. $r(F(a)) \leqslant r(a)$, **whenever a is an element of a power-associative complete normed algebra A and F is an homomorphism from A into another one.**

Now we can prove

Theorem 7.3. Every densely valued homomorphism from a power-associative complete normed algebra into another one with zero strong radical is continuous.

Proof. Let A and B be our power-associative complete normed algebras (with s-Rad(N) = 0) and let F be our densely valued homomorphism. Since the maximal modular (two-sided) ideals of B are closed (refine the proof of Lemma 5.2), a standard application of the closed graph theorem allows us to reduce to the case that B is simple with unit. Then, since F is densely valued, the separating subspace S(F) for F is an ideal of B and, to state the continuity of F (equivalently S(F) = 0, again by the closed graph theorem), it is enough to see that $I \notin S(F)$ where I denotes the unit of B. But

this last fact is clear from
$1 - r(I) \leqslant r(I-F(a)) + r(F(a)) \leqslant \|I-F(A)\| + \|a\|$, for all $a \in A$, where we have used Lemma 7.2 and the well known result that the spectral radius is subaditive on associative and commutative subalgebras.

Remark 7.2. The reader can easily see that the proof of the theorem remains true if the complete algebras A and B are assumed only to be such that A^+ and B^+ are power-associative (B always with zero strong radical). This is a much more general situation which appears, in particular, in the case of quadratic algebras.

Let A and B be full subalgebras of suitable (associative) Banach algebras and let F be a densely valued homomorphism from A into B. Then $S(F) \subset$ s-Rad(B) [83]. Using this result a satisfactory nonassociative extension of Rickart theorem can be given if there is an affirmative answer to the following

Problem 7.1. Let X and Y be Banach spaces, T a densely valued linear mapping from X into Y, and F (resp.: G) a continuous linear operator on X (resp. : Y) such that TF = GT. Is the inequality $r(G) \leqslant r(F)$ true ? (Compare with [64; Lemma 3.1]).

Now we recall the more relevant facts about the continuity of derivations of (associative) Banach algebras. Concerning positive results we have :

Theorem 7.4. (Johnson-Sinclair [40]). Every derivation of a semisimple Banach algebra is continuous.

However it remains open the following

Problem 7.2. Is every Jordan derivation of a semisimple Banach algebra continuous ?

We recall that a *Jordan derivation* of an associative algebra A is a linear mapping D from A into A satisfying $D(a^2) = aD(a) + D(a)a$ for all a in A. From Theorem 7.4 and the fact that every continuous Jordan derivation of a semisimple Banach algebra is actually a derivation [69], it follows that the preceding problem can be equivalently reformulated as follows :

Problem 7.2 bis. Is every Jordan derivation of a semisimple Banach algebra a derivation ?

The problem of the continuity of derivations of nonassociative complete normed algebras has at this time a lamentable situation. A side from the observation in [85] that derivations of JB^*-algebras (so of noncommutative JB^*- algebras) are continuous (by adapting the classical obsolete proof of the continuity of derivations of associative C^*-algebras) and the result in [34] that derivations of "classical Banach-Lie algebras of operators on Hilbert spaces" are continuous, no much more is known. Actually Problem 7.2 is a problem of continuity of derivations of nonassociative complete normed algebras. For a Jordan derivation of a semisimple Banach algebra A is a derivation of the Jordan-Banach algebra A^+ which has zero Jacobson-McCrimmon radical [53]. Thus Problem 7.2 has an affirmative answer if the following conjecture is true.

Conjecture 7.2. Every derivation of a Jordan-Banach algebra with zero Jacobson-McCrimmon radical is continuous.

Another natural conjecture is that every derivation of a nonassociative H^*-algebra with zero annihilator is continuous. Both conjectures are particular cases of the following (possibly daring) general

Conjecture 7.3. Every derivation of a nonassociative complete normed algebra with zero weak radical is continuous.

Although we know quite well that we are very far from a proof of the preceding conjecture, we give here some new results about the continuity of derivations of general nonassociative complete normed algebras. The arguments are inspired by those of Bade and Curtis [13] to study the continuity of module valued derivations of Banach algebras.

Lemma 7.3. Let B be an associative algebra, D a derivation of B, and C a full subalgebra of B. Then $C \cap D^{-1}(C)$ is a full subalgebra of B.

Proof. Clearly $C \cap D^{-1}(C)$ is a subalgebra of B. Assume that an element c in

$C \cap D^{-1}(C)$ is quasiinvertible in B with quasiinverse c^o. Then, denoting by B_1 the unital hull of B and by D_1 the unique extension as derivation of D to B_1, we have : $D(c^o) = -D_1(I-c^o) = -D_1((I-c)^{-1}) = (I-c)^{-1} D_1(I-c)(I-c)^{-1} = -(I-c^o) D(c)(I-c^o) = -D(c) + c^o D(c) + D(c)c^o - c^o D(c)c^o \in C$. Therefore $c^o \in C \cap D^{-1}(C)$.

Proposition 7.4. Let A be a nonassociative algebra, D a derivation of A and F an element in FM(A). Then DF-FD lies in FM(A).

Proof. The mapping $d : F \to DF - FD$ is a derivation of the associative algebra L(A). By the preceding lemma $FM(A) \cap d^{-1}(FM(A))$ is a full subalgebra of L(A) . But this subalgebra contains all the multiplication operators by elements of A because of the equalities $d(L_a) = L_{D(a)}$ and $d(R_a) = R_{D(a)}$ for all a in A. Therefore $FM(A) \cap d^{-1}(FM(A)) \supset FM(A)$.

Notation. For the rest of this section A will denote a nonassociative complete normed algebra and D a derivation of A. If for a in A we write $|a| := \|a\| + \|D(a)\|$, then $|.|$ is a (non complete) algebra norm on A. We use without comment the fact that $FM(A) \subset BL(A)$ [64; Remark 1.8]. We write $H(D) := \{F \in FM(A) : DF \in BL(A)\}$.

Proposition 7.5. H(D) is a (two-sided) ideal of FM(A) and, for F in FM(A), F belongs to H(D) if and only if there is a non negative real number k_F such that $|F(a)| \leq k_F\|a\|$ for all a in A.

Proof. Let F be in FM(A). Since $DF-FD \in BL(A)$ (Proposition 7.4), we have that $DF \in BL(A)$ if and only if $FD \in BL(A)$. This proves that H(D) is a two-sided ideal of FM(A) . On the other hand if $F \in H(D)$, then $|F(a)| = \|F(a)\| + \|DF(A)\| \leq (\|F\| + \|DF\|) \|a\|$, so $|F(a)| \leq k_F\|a\|$ with $k_F := \|F\| + \|DF\|$. Conversely, if $|F(a)| \leq k_F\|a\|$, then clearly $\|DF(a)\| \leq k_F\|a\|$ and $F \in H(D)$.

Lemma 7.4. Let F be in FM(A). Then F is continuous for the topology of the norm $|.|$ on A.

Proof. For a in A we have :
$|F(a)| = \|F(a)\| + \|DF(a)\| \leq \|F(a)\| + \|(DF-FD)(a)\| + \|FD(a)\| \leq$
$\|F\| \|a\| + \|DF-FD\| \|a\| + \|F\| \|D(a)\| \leq (\|F\| + \|DF-FD\| |a|$, where we have

used Proposition 7.4.

Theorem 7.5. Let $\{F_i\}$ and $\{G_i\}$ be families of elements of FM(A) such that $F_iG_j = 0$ for $i \neq j$. Then F_iG_i belongs to H(D), except for a finite number of i's.

Proof. Assume, on the contrary, that there exists an infinite sequence $\{i_n\}$ of distinct i's such that $F_{i_n}G_{i_n} \notin H(D)$ for all n in $\mathbb{N}$. Then, by Proposition 7.5, for each n in $\mathbb{N}$ we can chose an a_n in the unit sphere of A satisfying $|F_{i_n}G_{i_n}(a_n)| \geq n2^n|F_{i_n}| \, \|G_{i_n}\|$, where $|F_{i_n}|$ denotes the norm of F_{i_n} as a continuous linear operator on the normed space (A, $|.|$) (Lemma 7.4). If we write $a := \sum_{n=1}^{\infty} \left(1/2^n \, \|G_{i_n}\|\right) G_{i_n}(a_n)$, then $F_{i_n}(a) = \left(1/2^n\|G_{i_n}\|\right) F_{i_n}G_{i_n}(a_n)$. Therefore $|F_{i_n}|\,|a| \geq |F_{i_n}(a)| = \left(1/2^n \, \|G_{i_n}\|\right) |F_{i_n}G_{i_n}(a_n)| \geq n \, |F_{i_n}|$. Thus $|a| \geq n$ for all n in $\mathbb{N}$, a contradiction.

In what follows we will make use of the preceding theorem in two different contexts. The first one appears when A is supposed to be generalized complemented. In this particular case we reduce the proof of Conjecture 7.3 to the topologically simple case (notice that topologically simple algebras are, trivially, generalized complemented). This reduction follows clearly from the next lemma and theorem (the proof of the lemma being easy, it is omitted).

Lemma 7.5. Every direct summand of a nonassociative semiprime algebra is invariant under any derivation of the algebra.

Theorem 7.6. Assume that A is generalized complemented with zero weak radical. Then A can be written as the direct sum of a finite set of closed ideals satisfying that D is continuous on one of them and the other are topologically simple complete normed algebras with zero weak radical.

Before proving the theorem we state without proof the following elementary

Lemma 7.6. Let F be in G(D) and let a be in the separating subspace S(D) for D. Then F(a) = 0.

Proof of Theorem 7.6. It is easy to see that A is semiprime and that S(D) is a closed ideal of A. Therefore $A = S(D) \oplus An(S(D))$. If D_1 denotes the restriction mapping of D to An(S(D)) , by Lemma 7.5 we have that $S(D_1) \subset S(D) \cap An(S(D)) = 0$ and so D is continuous on An(S(D)). On the other hand S(D), as any closed ideal of A, inherit from A the property of being a generalized complemented complete normed algebra with zero weak radical (recall Proposition 5.1). Thus, by Corollary 5.2, the proof of our theorem is concluded by showing that the set $\{M_i\}$ of the minimal closed ideals of S(D) is finite. For each i chose x_i and y_i in M_i such that $L_{x_i}L_{y_i}$ does not vanish on S(D) (this is possible because M_i is an algebra with zero annihilator). Clearly $L_{x_i}L_{y_i} = 0$ for $i \neq j$ and $L_{x_i}L_{y_i} \notin H(D)$ by Lemma 7.6. Now the finiteness of the set of i's follows from Theorem 7.5.

For another application of Theorem 7.5 we recall previously some concepts. Let B be an associative algebra and let e be an idempotent in B. Then e is said to be a completely primitive idempotent of B if eBe is a division algebra (notice that if M is an ideal of B and $e \in M$, then e is a completely primitive idempotent of M because of the equality eBe = eMe). The *socle* of B is defined as the sum of those ideals of B which are simple algebras containing a completely primitive idempotent, and is denoted by Soc(B).

Lemma 7.7. Let B be an infinite dimensional simple associative normed algebra containing a completely primitive idempotent. Then there exists in B an infinite sequence of mutually orthogonal non zero idempotents.

The proof of the preceding lemma depends on the representation theory for associative simple algebras with non zero socle [38; IV.15] and on the finite dimensionality of division normed algebras given by the Gelfand-Mazur theorem. With this lemma we can prove :

Proposition 7.6. $Soc(FM(A)) \subset H(D)$.

Proof. It is enough to show that $M \subset H(D)$ for all the ideals M of FM(A) which are simple algebras containing a completely primitive idempotent. If M is infinite dimensional, then by Lemma 7.7 and Theorem 7.5 we obtain $M \cap H(D) \neq 0$ and so $M \subset H(D)$ as required, because H(D) is an ideal (Proposition 7.5) and M is a minimal ideal of FM(A). If M is finite dimensional, then it has a unit F and it is enough to prove that $F \in H(D)$. But, since F is clearly a central idempotent in FM(A), we have $FL_a = L_aF$ $(\forall\ a\in A) \Leftrightarrow F(ab) = aF(b)$ $(\forall a,b \in A) \Leftrightarrow R_{F(b)} = FR_b$ $(\forall\ b\in A)$. Thus for x in F(A) we have that R_x, and similarly L_x, belongs to M. Moreover $F(An(A)) = 0$ (see the proof of [64; Proposition 2.3]), from which is follows that $F(A) \cap An(A) = 0$. Now the mapping $x \to (L_x,R_x)$ from F(A) into $M \times M$ is injective so F(A) is finite dimensional and so the restriction mapping of D to F(A) is continuous. Thus DF is continuous, that is $F \in H(D)$.

From Proposition 7.6 and Lemma 7.6 we obtain :

Theorem 7.7. Assume that the socle of FM(A) separates the points of A. Then D is continuous.

Remark 7.3. The reader can see easily that the assumption in the preceding theorem is stronger than the one of w-Rad(A) = 0 in Conjecture 7.3. Actually it is much more stronger (consider an associative and commutative semisimple Banach algebra B with zero socle, then w-Rad(B) = 0 while Soc(FM(B)) = Soc(B) = 0 since B and FM(B) are algebrically isomorphic, see the proof of Proposition 4.1). If A is topologically simple, the assumption in Theorem 7.7. is equivalent to $Soc(FM(A)) \neq 0$.

From Proposition 7.4 it follows that S(D) is invariant under FM(A) and that the mapping $F \to DF - FD$ is a derivation of FM(A). From these facts a proof of Conjecture 7.3 can be given if there is an affirmative answer to the following

Problem 7.3. Is the separating subspace of any derivation of a full subalgebra B of a Banach algebra contained in the Jacobson radical of B ?

Note added in Proof.

Since this paper was written, some additional information about the results reviewed in it has arrived to my knowledge. Thus, in relation with the nonassociative characterizations of some classes of normed algebras, refered in Section 1, it should be emphasized the recent result by D.P.

Blecher, Z. Ruan and A.M. Sinclair [88] providing a set of intrinsic axioms (which do not involve the associativity) for the not necessarily self-adjoint norm-closed (associative) algebras of bounded linear operators on some complex Hilbert space.

Concerning Corollary 5.5, we cite that topologically simple JB- or JB*-algebras are now well-described as a consequence of a recent general classification theorem for prime JB- or JB*-algebras [95].

The most easy proof of Theorem 6.3 we know at this time consists of an application of a theorem by M. Slater (see [100; Theorem 9 in p. 194]) together with the fact that the centre of a topologically simple H*-algebra is either $\mathbb{C}1$, in the case of existence of a unit 1, or zero otherwise (a consequence of [24; Theorem 1.2]).

In relation with the theory of general nonassociative H*-algebras summarized in Section 6, we refer the systemmatic study of linear (algebra) involutions on H*- algebras with zero annihilator made in [90]. The main fact in this direction is that every linear involution on such an algebra is algebraically equivalent to a *-involution [90; Theorem 1.3].

Problem 7.2 has an affirmative answer, since in fact every Jordan derivation of a semiprime associative algebra is a derivation (see [93] and [89]).

Recently the general problem of finding reasonable additional requirements on a prime normed algebra which implie the algebra to be centrally closed (see [94] for definition) has received the attention of several authors. Concerning general nonassociative algebras, we cite first that topologically simple nonassociative H*-algebras are centrally closed [91; Proposition 4] (see also [92] and [98]), a result that has been used by A.R. Villena [99] to prove the automatic continuity of derivations of nonassociative H*-algebras with zero annihilator (one of the most relevant particular cases of Conjecture 7.3). Nonassociative ultraprime normed algebras are centrally closed [92] (an extension of a result by M. Mathieu in [97] for the associative case), and complete normed nonassociative algebras which are "primitive" in a suitable sense are also centrally closed [98]. This last result implies in particular that Jordan Banach algebras which are primitive in the sense of L. Hogben and K. McCrimmon [96] are centrally closed.

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