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Jordan algebras and mutation algebras. homotopy and von Neumann regularity

JORDAN ALGEBRAS AND MUTATION ALGEBRAS.
HOMOTOPY AND VON NEUMANN REGULARITY

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The aim of the present paper is to present Jordan algebras in the context of Santili's Mechanics. We also study the homotope algebra of a non-associative algebra (not necessarily associative algebra) with special attention to the homotope algebra of a Boolean algebra. Later the relation between homotopy and Von Neumann regularity is considered, mainly for Jordan algebras. Finally the idempotent (Jordan) algebras are studied.

DEFINITIONS AND NOTATIONS

If A is any non associative algebra there are two associated algebras A^- and A^+ having the same underlying vector space as A but with products $[x,y] = xy-yx$ and $x.y = \frac{1}{2}(xy+yx)$ respectively where we denote by juxtaposition the initial product on A. The algebra A is said Lie-admissible if A^- is a Lie-algebra, that is $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ holds for every x, y, z in A, and Jordan-admissible if A^+ is a Jordan algebra, that is $x.(y.x^2) = (x.y).x^2$ for every x, y. The algebra A is called flexible if the identity $(x,y,x) = 0$ holds for all x, y in A where $(a,b,c) = (ab)c - a(bc)$ is the associator of a, b, c. It is known ([21], p. 141) that if A is flexible and Jordan-admissible then A is a non-commutative (not necessarily commutative) Jordan algebra and conversely. Also, A is said to be power-associative if the subalgebra generated by any arbitrary element x in A is associative.

In any algebra A, the distributive and scalar laws imply that the mappings $R_x : y \mapsto yx$, $L_x : y \mapsto xy$ are linear transformation on the vector space of A. Obviously

    1) A is commutative iff $R_x = L_x$ for every x.

    2) A is associative iff $L_x R_y = R_y L_x$ for every x, y in A.

    3) A is flexible iff $L_x R_x = R_x L_x$ for every x.

    4) A is Lie iff $R_{xy} = R_x R_y - R_y R_x$ and $R_x = -L_x$.

5) A is noncommutative Jordan iff L_a , R_a , L_{a^2} and R_{a^2} commute.

6) A is Lie-admissible iff $R_{[x,y]} - L_{[x,y]} = [R_x - L_x , R_y - L_y]$.

If e is an idempotent of a flexible power-associative algebra A, then $A = A_1 \dotplus$

$A_{1/2} \dotplus A_0$ (Peirce decomposition) ([3] p. 562) with $A_i = \{x \in A | ex = xe = ix\}$, i = 0,1

and $A_{1/2} = \{x \in A \mid ex + xe = x\}$. Then A_1 and A_0 are zero or orthogonal subalgebras of A.

Moreover $A_i A_{1/2} + A_{1/2} A_i \subseteq A_{1/2} + A_{1-i}$, i = 0,1.

Finally by $\leq$ we denote a subalgebra ; by $\dotplus$ the direct sum of vector spaces and by $\oplus$ the direct sum of subalgebras.

1. JORDAN ALGEBRAS IN HADRONIC MECHANICS

In recent years an increasing number of mathematicians and experimentatl physicists tried to achieve a generalization of Atomic Mechanics, specifically conceived for the structure of strongly interacting particles (hadrons). This new mechanics is called Hadronic Mechanics and these studies include works mainly by theoretical physicists (Santilli, Mignani, Trostel, Okubo, Fronteau,...), experimental physicists (Ranchs, Solbodrian, Louzett,...) and mathematiciens (Myung, Osborn, Benkart, Tomber, Oehmke, ...). The Hadronic Journal, Nonantum, Massachusetts (USA), under the editor-ship of R.M. Santilli has played an important role in the development of Hadronic Mechanics and recently the mathematical journal : "Publications in Algebras, Groups and Geometries" under the editor-ship of H.C. Myung.

In this physical context the Santilli's generalization of Heisenberg's equation of motion $dx/dt = (i/\hbar)[H,x]$ for time-development of any observable x (see G. Loupias, this Colloque), where $\hbar$ is Planck's constant divied by 2π and $i = \sqrt{-1}$ is the imaginary unit, is $dx/dt = i\hbar$ (xpH-Hqx), where p and q are arbitrary fixed nonsingular operaotrs. This equation leads to a new product $x*y = xpy - yqx$ defined on the same linear structure of the associative algebra A and where by juxtaposition we denote the associative product in A. This new algebra is called the (p,q)-mutation of A and is denoted by A(p,q). The algebraic structure and physical applications of A(p,q) have been investigated in some details by Santilli, Myung, Ktorides, osborn, Tomber, Kalnay...

It seems to be an interesting problem to investigate the existence of a unit element in the mutation algebra. So Kalnay in ([9], p. 15) says : "Let A be an associative algebra over the field of the complex numbers $\mathbb{C}$ with unit element (in particular A could be a C*-algebra). The fundamental realization A(p,q) of the Lie-admissible algebras is due to Santilli. We shall choose a subalgebra $A^* \subseteq A(p,q)$, so A^* is also Lie-admissible. The motivation for working with A^* will be the quantum algebra : Nambu quantum algebras

needs a unit element and sometimes it is easier to find it in a proper subspace of A(p,q) than in A(p,q) itself, since in the subalgebra A* the multiplication table is smaller than in A(p,q), ..." For this reason in [5], we take up the problem of the existence of a unit element in any mutation algebra. Our aim is to get necesary and sufficient conditions on A, p and q so as to guarantee the existence of a unit element in A(p,q). The main result is that A(p,q) contains a unit element if and only if A has a unit element, p-q is invertible in A and $pxq = qxp$ for every element x in A.

If A(p,q) contains a unit element it is proved [5] that A(p,q) is flexible and Jordan-admissible (that is, $A(p,q)^+$ is a Jordan algebra), so A(p,q) is a non-commutative Jordan algebra. Also is proved the following

Theorem [5]. *If* A *contains a unit element* 1 *and* p-q *is invertible in* A *then the following properties are equivalent* :

1) A(p,q) *has a unit.*

2) A(p,q) *is isomorphic to* A(s+1,s) *whith* s *in the center of* A *(we consider* $s = (p-q)^{-1}q$ *and* $\alpha : A(s+1,s) \to A(p,q)$ *given by* $\alpha(x) = x(p-q)^{-1}$.

3) A(p,q) *is generalized quasi-associative.*

4) A(p,q) *is flexible.*

5) A(p,q) *is power associative.*

6) A(p,q) *satisfies the third power identity* $((x,x,x)^* = 0)$.

An interesting consequence of the above theorem is the following. It known that the expoential function $\exp x = \sum_{i=0}^{} x^i/i!$, $x^0 = 1$ is definable in a nonassociative but power-associative algebra A, and that this function plays an important role in the structure theory of nonassociative algebras. H.C. Myung [17] studies the exponential function in the (p-q)-mutation algebra of an associative algebra A with unit over a field F of characteristic 0. He assumes that A(p,q) is a power-associative algebra with a unit element 1. Obviously by our theorem above the interesting Myung's results are always true for any mutation algebra with identity.

Now, we have the following two natural questions : 1) When is the (p,q)-mutation A(p,q) a Lie algebra ? and 2) What happens when the mutation process is reiterated ?

With respect to the first question, since A(p,q) is a Lie-admissible algebra, it will be a Lie-algebra if and only if it is an anticommutative algebra, that is $x*y = xpy - yqx = -y*x = -ypx + xqx$. If r = p-q, it is clear that A(p,q) is an anticommutive algebra if and only if the homotope algebra A(r) (the algebra with the same underlying vector space A and the new multiplication defined by $x.y = xry$) is anticommutative. So the question is : when does an associative algebra A possesses an element $r \neq 0$ such that A(r) is

anticommutative ? In this sense we obtain in [6] that the mutation algebras $A(p,q)$ which are Lie algebras are the algebras $A(p,q)$ or "are very near" to them.

Finally, with respect to the second question. I studied in [8] the mutation algebra of a nonassociative algebra. Obviously, to obtain the flexibility and Lie-admissibility of the new algebra, it would be necessary to impose conditions over the elements p and q. Define the commutative center $K(A) = \{x \in A \mid [x,y] = 0 \ \forall \ y \in A\}$, the left nucleus $N_\ell(A) = \{x \in A \mid (x,y,z) = 0, \ \forall \ x,z \in A\}$, the right nucleus $N_r(A) = \{z \in A \mid (x,y,z) = 0, \ \forall \ x,y \in A\}$, the middle nucleus $N_m(A) = \{y \in A \mid (x,y,z) = 0, \ \forall \ x,z \in A\}$, the nucleus $N(A) = N_\ell(A) \cap N_m(A)$ and the center $Z(A) = K(A) \cap N(A)$. Then

Theorem [8]. 1) *Assume* $p,q \in N_m(A) \cap K(A)$. *Then* :

i) A *flexible implies* $A(p,q)$ *flexible*.

ii) A *flexible and Lie-admissible imply* $A(p,q)$ *flexible and Lie-admissible*.

2) *Assume* $p,q \in N(A)$. *Then*, A *flexible implies* $A(p,q)$ *Lie-admissible*.

3) *If* $p,q \in Z(A)$, A *Lie-admissible implies* $A(p,q)$ *Lie-admissible*.

However $A(p,q)$ is not necessarily flexible when A is flexible and p,q in A. Nevertheless we can obtain the following two theorems :

Theorem [8]. *If* $p = qc$ *with* $c \in K(A)$ *and* A *is flexible, then* $A(p,q)$ *is flexible too*.

Theorem [8]. *Let* A *be a flexible power associtive algebra with unit* 1 *over a field* F *of characteristic* $\neq 2,3$. *Let* p *and* q *be fixed elements of* $N(A)$ *such that one of* p *and* q *is invertible in* $N(A)$. *Then the following properties for* $A(p,q)$ *are equivalent* :

i) $A(p,q)$ *is third power-associative*.

ii) $A(p,q)$ *is flexible*.

iii) $A(p,q)$ *is power associative*.

iv) $A(p,q) \cong A(\alpha,-\beta)$ *with* $\alpha,\beta \in Z(A)$.

v) $p = \alpha q$ *or* $q = \alpha p$ *with* $\alpha \in Z(A)$.

vi) $A(p,q) \cong A(1,\beta)$ *or* $A(p,q) \cong A(\beta,1)$, $\beta \in Z(A)$.

Note that $A(p,q)^-$, that is to say $A(p,q)$ with the product $[x,y] = x*y - y*x$, is equal to the algebra $A(p,+q)^-$ where $A(p+q)$ is the (p+q)-homotope of A, and also $A(p,q)^+ = A(p-q)^+$ where $x.y = \frac{1}{2}(x*y + y*x)$. So, we are going to study the homotope algebra of a nonassociative (not necessary associative) algebra A.

2. HOMOTOPY

The concept of isotopy was suggested to Albert ([2]) by the work of N. Steenrod who, in his study of homotopy groups in topology, was led to study isotopy of division algebras, concluding that algebras related in this fashion would yield the same homotopy properties and should therefore be put into the same class. Albert defines the isotope algebra of an associative algebra A as an algebra A^0 over the same field k which satisfies $R_x^0 = PR_{Qx}S$ where R_x and R_x^0 denote the right multiplication by x in A and A^0 respectively and P,Q, S are nonsingular linear transformations of the underlying vector space.

An important example of isotopy is obtained when A is an associative algebra and, in the same vector space, a new multiplication is defined by x.y = xay where a is a fixed invertible element of A. When the assumption "a invertible" is dropped, A^0 is called *homotope algebra* of A and is also associative. H.C. Myung ([18]), in response to recent studies of physical systems via isotopic lifting of Hilbert spaces, introduces a generalized concept of hermitian and unitary operators in a Hilbert space in terms of an isotope of the associtive algebra of linear operators and of a positive operator. Finally ([19]), he studies isotopes of the tensor product of Hilbert spaces and of associative algebras of linear operators. This is applied to the real envelopping algebras of spin half integer algebras.

When the algebra A considered is a nonassocitive algebra we study in [8] the homotope algebra and we consider the conditions on the algebra A which assure that the homotope algebra is a flexible algebra. The definitions and notations used and the main results obtained are the following.

Let r be an element of a nonassociative algebra. We call (left) homotope of A, A(r), the algebra with the same underlying vector space as A ans the new multiplication xoy = (xr)y.

Theorem [8]. 1) *Let* r *be an element of* $N_m(A) \cap K(A)$. *Then* :

i) A *flexible implies* A(r) *flexible* ;

ii) A *flexible Lie-admissible implies* A(r) *flexible Lie-admissible.*

2) *Let* r *be an element of* N(A). *Then* :

i) A *flexible implies* A(r) *flexible* ;

ii) A *flexible Lie-admissible implies* A(r) *flexible Lie-admissible.*

3) *Let* r *be an element of* Z(A). *Then,* A *Lie-admissible implies* A(r) *Lie-admissible.*

Let now A be an associative algebra, $p \in A$, and $A(p) = A^0$ the homotope algebra. If q is an idempotent in A^0 then qoq = qpq = q. Hence pq and qp are

idempotents in A. Now we consider the Peirce's decomposition of A for pq and qp and the Peirce's decomposition of A^0 with respect to q. We have :

$$A = A_{00}(pq) \dotplus A_{01}(qp) \dotplus A_{10}(pq) \dotplus A_{11}(qp) = A_{00} \dotplus A_{01} \dotplus A_{10} \dotplus A_{11}$$

$$A = A'_{00}(pq) \dotplus A'_{01}(pq) \dotplus A'_{10}(pq) \dotplus A'_{11}(p,q) = A'_{00} \dotplus A'_{01} \dotplus A'_{10} \dotplus A'_{11}$$

$$A^0 = \bar{A}_{00}(q) \dotplus \bar{A}_{01}(q) \dotplus \bar{A}_{10}(q) \dotplus \bar{A}_{11}(q) = \bar{A}_{00} \dotplus \bar{A}_{01} \dotplus \bar{A}_{10} \dotplus \bar{A}_{11} .$$

Thus $A_{ij} \leq A$, $A'_{ij} \leq A$ and $\bar{A}_{ij} \leq A^0$, $i,j \in \{0,1\}$. Besides

$$\bar{A}_{00} = \{A'_{00} \dotplus A'_{10}\} \cap (A_{00} \dotplus A_{01}) \leq A$$

Similarly,

$$\bar{A}_{01} = (A_{00} \dotplus A_{01}) \cap (A'_{01} \dotplus A'_{11}) \leq A$$

$$\bar{A}_{10} = (A_{10} \dotplus A_{11}) \cap (A'_{00} \dotplus A'_{10}) \leq A$$

$$\bar{A}_{11} = (A_{10} \dotplus A_{11}) \cap (A'_{01} \dotplus A'_{11}) \leq A$$

Also A_{ij}, A'_{ij} are subalgebras of A for every $i, j \in \{0,1\}$ because we have

$$xqp = jx, \quad qpx = ix \quad (yqp = jy, qpy = iy) \quad x,y \in A_{ij}$$

$$(xoy)qp = xo(yqp) = xojy, \quad qp(xoy) = (qpx)oy = ixoy$$

Finally we have the following relations :

$$A_{00} \leq \bar{A}_{00} \dotplus \bar{A}_{01} , \quad A_{01} \leq \bar{A}_{00} \dotplus \bar{A}_{01} , A_{10} \leq \bar{A}_{10} \dotplus \bar{A}_{11} , \quad A_{11} \leq \bar{A}_{10} \dotplus \bar{A}_{11}$$

$$A'_{00} \leq \bar{A}_{00} \dotplus \bar{A}_{10} , \quad A'_{01} \leq \bar{A}_{01} \dotplus \bar{A}_{11} , \quad A'_{10} \leq \bar{A}_{00} \dotplus \bar{A}_{10} , A'_{11} \leq \bar{A}_{01} \dotplus \bar{A}_{11} .$$

Let now A be an associtive and commutative algebra with identity and let I(A) be the set of idempotent elements in A. If, $a,b \in I(A)$, ab is an idempotent element too but $a+b$ is not an idempotent element. However, if we define the new operation $aob = a+b-2ab$ we have $(aob)^2 = aob$ and $(I(A),o,.)$ is a Boolean ring. Also, if A is semiprime ring, it is known ([10], p. 110) that the set of annihilator ideals of A is a Boolean ring. Finally, if A is a p-ring (or generalized Boolean ring), that is to say a ring of fixed prime characteristic p in which $a^p = a$ for all a in A, the set B(A) of idempotents of A is a Boolean ring with the above operations. In Batbedat : p-anneaux (secrétarit de Math. de la Faculté des Sciences de Montpellier, 1968-1969, n° 34) is established, with the help of B(A), an isomorphism between the category of p-rings and the one of Boolean rings. Then, it is interesting to study the homotope algebra of a Boolean algebra. So let A be a Boolean algebra, $a \neq 1$ an element of A, and $A^0 = A(a)$ the homotope algebra. As $a^2 = a$, we have $A = A_0 \dotplus A_1$ with $A_0 = \{x \in A \mid xa = 0\}$, $A_1 = \{x \in A \mid xa = x\}$.Note that (A_0,o) is a subalgebra of A^0 since $xoy = 0$ for every $x,y \in A_0$. Similarly (A_1,o) is a subalgebra of A^0 since $xoy = xay = xy$ for every $x,y \in A_1$.

It is easy to prove that $N \leq A$, $B \leq A$ and $NB = 0$. Hence $A = N \oplus B$. Similarly $N^0 \leq A^0$, $B^0 \leq A^0$ and $A^0 = N^0 \oplus B^0$. Moreover $N^0 o A^0 = 0$ and B^0 is a boolean

algebra (N^0 is the nilradical of A and $x \circ y = xy$ for every x, y in B). Since $N^0 = 0$ if and only if a = 1, and $B^0 = 0$ if and only if a = 0, we have that a proper homotope algebra of a Boolean algebra is neither a nilpotent algebra nor a Boolean algebra.

Conversely, if C is an algebra over the field $\mathbb{Z}/2\mathbb{Z}$ such that C = N ⊕ B, N,B ≤ C, NC = 0 and B is a Boolean algebra with unit element, it is natural to ask : does there exists an element a in C and a new product * such that (C,*) = $\overline{C}$ is a Boolean algebra whose homotope $\overline{C}(a)$ is C ?

Let us consider $\{e_1, e_2, ...\}$ a basis of N and define the product by $e_i * e_i = e_i$, $e_1 * e_i = e_i$ and $e_i * e_j = 0$ in the orther cases. So N is a Boolean algebra with e_1 as unit element. On the elements of B we consider the initial product and finally x*y = 0 for every x∈ N, y∈ B. Let a be the unit element of B. Then if x = n+b, y = n'+b', x*a*y = b*a*b' = bb' = xy. Hence (C,*)(a) = C as desired.

Finally we show that a Boolean algebra is never a proper homotpe algebra of an associative algebra. Assume that A is an associative algebra and a∈ A is such that A(a) is a Boolean algebra. If A(a) has a unit element, a must be an invertible element, so that A ≅ A(a) and A is a Boolean algebra. Thus a = 1 and A(a) ≡ A. In the other case, for every x in A we have xox = xax = x ; in particular $a^3 = a$. Hence B = Aa is a subalgebra of A in which every element is idempotent and so $a = a^2 a \in B$ is an idempotent element. Since A(a) is a commutative algebra, we have :

xoa = aox implies xaa = aax, that is xa = ax for every x in A ,

xoy = yox implies xay = yax = axy = ayx for every x, y in A.

Thus x = (x-xa)+xa implies (x-xa)a = 0, so (x-xa) = (x-xa)o(x-xa) = (x-xa)a(x-xa) = 0. Then x = xa and a is the unit element. Hence A(a) ≡ A as desired.

3. VON NEUMANN REGULARITY AND HOMOTOPY

Von Neumann regular rings were originally introduced by Von Neumann in order to clarify certain aspects of operator algebras. In his book "Von Neumann regular rings" Goodearl says (p. IX) : "As would be expected with any good concepts, regular rings have also been extensively studied for their own sake, and most ring theorists are at least aware of the connections between regular rings and the rings happen to be interested in".

An associative ring with identity element 1∈ R is Von Neumann regular provided that for every x∈ R there exists y∈ R such that x = xyx. By Artin's theorem the same definition is possible for alternative rings ([16], p. 338). Finally, for a Jordan ring J, one says that J is a Von Neumann regular ring if for every x∈ R there exists y∈ R such that $U_x y = x$ where the operator U is defined by $U_x = (L_x + R_x)R_x - R_x^2$ with L_x (resp. R_x)

is the left multiplication (resp. right multiplication) by the element x. Now, we are going to study the relation between regularity and homotopy for associative, alternative ([7]) and Jordan algebras respectively.

i) The associative case

Let a be an element of the associative algebra A and let A(a) be the homotope algebra. It is clear that A(a) has identity if and only if A has identity and a is an invertible element of A. It is easy to verify the following

Theorem. *The homotope algegra* A(a) *is a Von Neumann regular algebra if and only if* A *is a regular von Neumann algebra and* a *is an invertible element of* A.

Definition. An algebra B verifies the <u>condition of regulariy (r)</u> if for every x∈ B there exists y∈ B such that x = xyx.

Theorem [7]. *Let* A *be an associative algebra and* a *an element of* A. *Then* A(a) *verifies the condition of regularity* (r) *iff* A *has identity element 1, verifies* (r) *(so* A *is von Neumann regular) and* a *is an invertible element of* A.

ii) The alternative case

Given elements u, v of an alterntive algebra, K. McCrimmon ([11]) obtains a new algebra by taking the same linear structure but a new multiplication $x*y = (xu)(vy)$. The resulting algebra, denoted $A^{(u,v)}$, is called the u,v-homotope of A. If $v = 1, A^{(u,1)}$ is called the left u-homotope. Similarly, if $u = 1, A^{(1,v)}$ is called the right v-homotope. It is clear that $L_x^{(u,v)} = L_{xu}L_v$,$R_x^{(u,v)} = R_{vx}R_u$, $U_x^{(u,v)} = L_x^{(u,v)} R_x^{(u,v)} = R_x^{(u,v)}L_x^{(u,v)} =$

U_xU_{uv} . This kind of homotope $A^{(u,v)}$ was introduced for general linear algebras by Albert ([2]), and was further investigated in the alternative case by Schafer [20] and for loops by Bruck [4]. Finally, in the theory of the Jacobson-Smiley radical this notion plays an important role. So, McCrimmon [12] proves that an element z of A is properly quasi-invertible iff z is quasi-invertible in all homotopes A(x) of A. This condition is more useful in practice than proper quasi-invertibility ifself, and is used to obtain short proofs of results such as rad(eAe) =eAe ∩ rad A for any idempotent e and rad 𝔞 = 𝔞 ∩ rad A for any ideal 𝔞 .

Theorem [11]. *The* u,v-*homotope* $A^{(u,v)}$ *of an alternative algebra* A *is again alternative.*

Theorem [11]. *A* homotope $A^{(u,v)}$ has a unit $1^{(u,v)}$ *if and only if* A *has a unit 1 and* uv *is invertible, in which case* $1^{(u,v)} = (uv)^{-1}$.

Proposition [7]. *Let* A *be a von Neumann regular alternative algebra and* z = uv *invertible. Then* $A^{(u,v)}$ *is a von Neumann regular algebra too.*

Observation. If $A^{(u,v)}$ satisfies the condition of regularity (r) A verifies also the same condition. Finally the same questions as in the associative case seem natural in the alternative case, that is : if $A^{(u,v)}$ satisfies the condition of regularity (r), has A a unit element 1_A ? is A regular von Neumann ? and is uv an invertible element ? Unfortunately we have'nt the same result. However, we can prove the following

Theorem [7]. *Let* A *be an alterntive algebra without nonzero nilpotent elements, and* $A^{(u,v)}$ *the* u,v-*homotope that verifies the condition of regularity* (r). *Then* A *is an abelian regular algebra (any idempotent* e *is* A *is central) and* $A^{(u,v)}$ *is a von Neumann regular algebra.*

iii) The Jordan case

Given a noncommutative Jordan algebra A, the (left) a-homotope A(a) determined by an element $a \in A$ has the same linear structure as A but has a new multiplication xoy = (xa)y - (x,y,a). The multiplication operators in A(a) are given by : $L_x^{(a)} = L_{xa} - [R_a, L_x]$, $R_y^{(a)} = R_{ya} - [R_a, R_y]$ and K. McCrimmon established the following

formulas for the multiplication operators in A(a) :

$$U_a L_x^{(a)} = \{L_x^{(a)} - [V_x, L_a]\}U_a = \{L_{xa} - [L_x, L_a]\}U_a \quad (V_x = L_x + R_x)$$

$$U_a R_x^{(a)} = \{R_x^{(a)} - [V_x, R_a]\}U_a = \{R_{xa} - [L_x, R_a]\}U_a$$

$$L_x^{(a)}U_x = U_x\{L_{xa} - [L_x, L_a]\}$$

$$R_x^{(a)}U_x = U_x\{R_{xa} - [R_x, R_a]\}$$

The main result of [13] is :

Theorem. *If* A *is a noncommutative Jordan algebra then the homotope* A(a) *determined by an element* $a \in A$ *is again a noncommutative Jordan algebra and* $\{A^{(a)}\}^+ = A^+(a)$.

It is clear that an element 1 is the unit element of A if and only if it is the unit element of A^+. So, $A(a)$ has an identity element if and only if $A(a)^+$ has an identity element. Thus, to study when the homotope algebras have an identity element we shall only need to study the commutative case.

Let A be a commutative Jordan algebra with identity. Then $A(a)$ has an identity if and only if a is an invertible element of A (that is, there exists b such that $ab = 1$, $a^2 b = a$). In this case a^{-1} is the identity of $A(a)$.

Now, we only suppose that $A(a)$ has identity e, that is : $eox = xoe = x$ for every x. So $U_e^{(a)} = I = U_e U_a$. Then U_a is injective and U_e is surjective. Consequently there exists z with $U_e z = a$. So, $I = U_e U_a = U_e U_{U_e z} = U_e U_e U_z U_e$ (by the Fundamental Formula). Therefore U_e is bijective with inverse U_a. Hence $e = U_e a$, $a = U_a e = 2aea - a^2 e$. But $a = aoe = a^2 e - (a,e,a) = a^2 e$ and $a = U_a e = 2aea - a^2 e$ implies $a^2 e = a = aea$, that is $(a,a,e) = 0$.

On the other hand, if $x = e$ in the above identities, we obtain the following identities : $L_e^{(a)} = I = L_{ez} - [L_a, L_e]$ and $U_e(L_{ea} - ([L_e, L_a]) = L_e^{(a)} U_e$. Then $L_{ea} - [L_e, L_a] = I = L_{ea}[L_a, L_e]$. So $[L_a, L_e] = [L_e, L_a] = 0$ and $L_{ea} = L^{(a)} = I$. Then ea is the identity in A and a is invertible with inverse e. So we have the following

Theorem. *Let* A *be a Jordan algebra, and* $a \in A$. *Then, the homotope* $A(a)$ *has an identity if and only if* A *has an identity and* a *is an invertible element.*

Using this result it is easy to prove the following

Theorem. *Let* A *be a Jordan algebra and* $a \in A$. *Then* $A(a)$ *is a regular algebra if and only if* A *is a regular algebra and* a *is invertible in* A.

Finally we shall study the conditions for a von Neumann regular Jordan ring to be idempotent. Then we shall study the behaviour of these rings.

4. CONDITIONS FOR A REGULAR JORDAN ALGEBRA TO BE IDEMPOTENT. IDEMPOTENTS (JORDAN) RINGS AND AN ORDER RELATION.

In response to a problem posed in the American Mathematical Monthly, T. Wong characterized the Boolean rings with identity 1 as commutative von Neumann regular associative rings with 1 in which 1 is the only unit. H.C. Myung extended this result to the setting of alternative rings [16] and he showed that the same characterization does not

hold for Jordan rings. He considers an algebra R with identity over $\mathbb{Z}/2\mathbb{Z}$ with basis $\{1, x_1, x_2, ...\}$ given by

$$x_i^2 = x_i \, , \quad i = 1, 2, ...$$

$$x_3 x_4 = x_4 x_3 = x_1 x_j = x_j x_1 = x_4 \, , \quad j = 2, 3, ...$$

and all other products are 0. One easily checks that R is an idempotent ring, and so a Jordan ring, but is not associative since $(x_1 x_2) x_3 = x_4$ and yet $x_1 (x_2 x_3) = 0$. It is also easy to prove that 1 is the only unit in R.

However, by modifying the definition of von Neumann regularity and replacing idempotent by Boolean, we can obtain a result similar to Wong and Myung's ones.

Definition. The nonassociative ring J is said to satisfy the condition of regulariy (R) if for each x in J there exists an $y \in J$ such that $x = xyx$ and the subring generated by x and y is associative.

Theorem. *A ring* J *with identity* 1 *is an idempotent ring (every element is idempotent) if and only if it is a Jordan ring with* 1, *verifying* (R) *and with* 1 *as the only unit.*

Proof. One implication is clear. Obviously every idempotent ring is a (commutative) Jordan ring, satisfies (R) (for any element a we consider, as element b, the same a) and 1 is the only unit.

Conversely, let a be an element of J and b the element of J such that $a = aba$ and a,b generate an associative ring. Let $u = 1 - ba + bab$. Then, $aua = a^2 - a(ba)a + ababa$ = $a^2 - a^2 + ababa = a$. Besides u is an unit element. In fact, let $u' = 1 - ba + aba$. Then $u'u = uu' = 1 - ba + bab - ba + baba - baba - babba + aba - baaba + bababa = (1 - ba + bab - ba + ba - bab + a - a + ba) = 1$, and $u^2 u' = u(uu') = u$, so u' is the inverse of u. By the hypotesis 1 is the only unit, hence $u = 1$ and so $a^2 = a$.

Order relation

The usual order relation in Boolean rings is extended to reduced rings A (no nilpotent element) when expressed as : $a \leq b$ if and only if $ab = a^2$ ([1]). H.C. Myung and Jimenez [15] extend the results of Abian to any alterntive rings. They prove that if A is an alternative ring without nonzero nilpotent elements then the relation $\leq$ is a partial order on A and any idempotent e in A is central. Later, Myung in [16] proves that an alterntive ring A equipped with the relation $\leq$ is a Boolean ring if and only if $\leq$ is a partial order on A, such that for every element $x \in A$, $\{x, x^2\}$ has an upper bound with respect of $\leq$. Unfortunately the same result is not true for Jordan rings. The above example is an idempotent ring (so a Jordan ring) which is not Boolean and where the

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relation $\leq$ is not a partial order. Obviously an idempotent ring has no nonzero nilpotent elements. However he proves the following.

Theorem. *A Jordan ring* J *equipped with the relation $\leq$ is an idempotent ring if and only if* J *has no nonzero nilpotent elements and, for every element* $x \in$ J, $\{x, x^2\}$ *has an upper bound with respect to $\leq$.*

The following question is now a natural question : is it possible to characterize an idempotent ring in which $\leq$ is a partial order ? This question (at least for me !) is very difficult to be answered. We show the different properties of idempotent rings with respect to Boolean rings and we construct, for any n, an idempotent Jordan algebra with dimension n in which $\leq$ is not a partial order.

1) An idempotent ring in which $\leq$ is a partial order has no (necessary) identity and zero divisors. For example, A = $\{0,a,b,a+b\}$ with the multiplication : $a^2 = a$, $b^2 = b$, ab = ba = a+b, a(a+b) = b, (a+b)a = b, b(a+b) = (a+b)b = a, (a+b)^2 = a+b$. Obviously, if it has an identity element then it has zero divisors, except the trivial associative case A = $\{0,1\}$.

2) There are two idempotent rings (non associative rings) in which $\leq$ is a partial order but which are not isomorphic rings. For instance, $R_1 = \{1,e_1,e_2,e_1+e_2,1+e_1,1+e_2,1+e_1+e_2\}$ with the following multiplication table :

	e_1	e_2	e_1+e_2	1	$1+e_2$	$1+e_2$	$1+e_1+e_2$
e_1	e_1	e_1+e_2	e_2	e_1	0	e_2	e_1+e_2
e_2	e_1+e_2	e_2	e_1	e_2	e_1	0	e_1+e_2
e_1+e_2	e_2	e_1	e_1+e_2	e_1+e_2	e_1	e2	0
1	e_1	e_2	e_1+e_2	1	$1+e_1$	$1+e_2$	$1+e_1+e_2$
$1+e_1$	0	e_1	e_1	$1+e_1$	$1+e_1$	1	1
$1+e_2$	e_2	0	e_2	$1+e_2$	1	$1+e_2$	1
$1+e_1+e_2$	e_1+e_2	e_1+e_2	0	$1+e_1+e_2$	1	1	$1+e_1+e_2$

and R_2, the $\mathbb{Z}/2\mathbb{Z}$-algebra with base $\{a,b,c\}$ and product : ab = a, ac = a+c, bc = b+c.

3) An idempotent ring in which $\leq$ is a partial order is not a direct product of idempotent rings without zero divisors. So it is easy to prove that R_1 verifies this affirmation.

4) Let $\{e_1,...,e_n\}$ a finite set with an order R. We suppose that $\{e_1,...,e_n\}$ is a base of the $\mathbb{Z}/2\mathbb{Z}$-algebra A with the product $e_ie_j = e_i$ if e_iRe_j ; $e_ie_j = e_j$ if e_jRe_i and

$e_i e_j = 0$ in the oder cases. It is clear that $\leq$ coincides with R for the q's elements. Besides we have the following

Theorem. A *is a non associative (idempotent) algebra if and only if there exists* e_i, e_j, e_h *so that* $e_i R e_j$, $e_i R e_h$ *and* e_j *is not related by* R *to* e_h. *Consequently,* $\leq$ *is a partial order iff* A *is an associative algebra.*

Proof. $(e_i e_j)e_h = e_i e_h = e_i$; $e_i(e_j e_h) = e_i 0 = 0$. So A is a non associative algebra. There exist $\{e_1, e_2, e_3\}$ such that $(e_1 e_2)e_3 \neq e_1(e_2 e_3)$. Then, we consider the following cases :

i) If $e_1 \leq e_2 \leq e_3$ we have $(e_1 e_2)e_3 = e_1 e_3 = e_1$, $e_1(e_2 e_3) = e_1 e_2 = e_1$.
    Contradiction.

ii) If $e_1 \leq e_3$, $e_2 \leq e_3$, e_1 no-R e_2 then $(e_1 e_2)e_3 = 0$ and $e_1(e_2 e_3) = e_1 e_2 = 0$.
    Contradiction.

iii) If $e_1 \leq e_2$, e_1 no-R e_3, e_3 no-R e_2 then $(e_1 e_2)e_3 = e_1 e_3 = 0$ and
    $(e_1(e_2 e_3)) = e_1 0 = 0$.
    Contradiction.

iv) If e_1 no-R e_3, e_2 no-R e_3, e_3 no-R e_1 then $(e_1 e_2)e_3 = e_1(e_2 e_3) = 0$.
    Contradiction.

So there exist three elements in that condition.

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