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NONCOMMUTATIVE JORDAN ALGEBRAS CONTAINING MINIMAL INNER IDEALS

Antonio FERNÁNDEZ LÓPEZ

The purpose of this exposition is to present a study on noncommutative Jordan algebras containing minimal inner ideals, with special emphasis on the cases when these algebras are endowed with a norm of algebra. The exposition is divided into five sections :

- I. Primitive nondegenerate noncommutative Jordan algebras with non-zero socle.
- II. Structure theorems for simple and for primitive noncommutative Jordan normed algebras with nonzero socle.
- III. Finiteness conditions in noncommutative Jordan Banach algebras.
- IV. Some properties of the socle of a noncommutative Jordan algebra.
- V. Modular annihilator noncommutative Jordan algebras.

I. PRIMITIVE NONDEGENERATE NONCOMMUTATIVE JORDAN ALGEBRAS WITH NON ZERO SOCLE.

This first section is devoted to show how a theory analogous to that of the primitive associative rings with non zero socle can be developed for noncommutative Jordan algebras. I start with an expository survey of the theory for associative, alternative and Jordan algebras and end by proving how these various theories can be unified in the frame of a noncommutative Jordan algebra. This last result is essentially the content of a joint work with Rodriguez Palacios [18]. All the algebras we consider here are over a field of characteristic $\neq 2$.

We recall that an associative algebra A is said to be primitive when it contains a maximal modular left ideal M such that M does not contain any nonzero (two-sided) ideal of A . Every primitive associative algebra A is semiprime ($I^2 = 0$ implies $I = 0$, I ideal of A). The socle of an associative algebra A is defined to be the sum of all its

minimal right ideals. The socle $S(A)$ of A is an ideal of A and when A is semiprime this definition is left-right symmetric.

The theory of primitive associative rings with nonzero socle was developed by N. Jacobson [22] as a generalization of the duality theory previously used by Dieudonné in studying simple rings containing minimal right ideals [11]. On the other hand, primitive associative rings are a natural generalization of the artinian simple rings.

Following [24, p. 69] let (V, W) be a pair of dual vector spaces over a division associative algebra Δ . An element $a \in \text{Hom}_\Delta(V, V)$ is said to be continuous if there exists $a^* \in \text{Hom}_\Delta(W, W)$, necessarily unique, such that $(av, w) = (v, a^*w)$ for all $v \in V$, $w \in W$. $L_W(V)$ denotes the algebra of all continuous linear transformations of A and $F_W(V)$ the ideal of $L_W(V)$ of all elements with finite rank.

Theorem 1. [24, Structure theorem, p. 75]. *An associative φ -algebra A is primitive with nonzero socle if and only if there exists a pair of dual vector spaces (V, W) over a division associative φ -algebra Δ such that A is isomorphic to a subalgebra of $L_W(V)$ containing $F_W(V)$. Moreover $S(A) = F_W(V)$ is a simple associative φ -algebra containing minimal right ideals.*

If A has additionally an involution $*$ then V is self-dual with respect to a hermitian or symplectic inner product (\mid) and the involution $*$ is the adjoint with respect to (\mid) . We remark that, in the symplectic case, Δ is a field and its involution is the identity (see [24, Theorem 2, p. 83] and [23, Theorem 9 below, p. 270]).

Let now A be a primitive alternative algebra. Kleinfeld proved [29, Theorem 2] that A is either associative or a Cayley-Dickson algebra over its centre. On the other hand, Slater developed a notion of socle for semi-prime alternative algebras analogous to the associative one [38]. These facts allow to extend Theorem 1 to the alternative case.

Theorem 2. *An alternative algebra is primitive with nonzero socle if and only if it is either isomorphic to a Cayley-Dickson algebra or to a subalgebra of $L_W(V)$ containing $F_W(V)$.*

Osborn and Racine [34] have defined a notion of socle for Jordan algebras that reconstructs the associative one. Namely, for a non-degenerate Jordan algebra J ($U_a = 0$ implies $a = 0$, $a \in J$) the socle $S(J)$ of J is defined to be the sum of all minimal inner ideals of J . When A is a semiprime associative algebra then the Jordan algebra A^+ given by the product $x \cdot y = 1/2(xy + yx)$, $x, y \in A$ is non-degenerate and $S(A^+)$ coincides with the socle of A . For an associative algebra A with an involution $*$,

$H(A,*)$ will denote the Jordan subalgebra of A^+ of all symmetric elements. The following theorem of Osborn and Racine determines the structure of the nondegenerate prime Jordan algebra J ($U_B C = 0$ implies $B = 0$ or $C = 0$, B and C ideals of J) with nonzero socle.

Theorem 3 [34, Theorem 18]. *If J is a nondegenerate prime Jordan ϕ -algebra with nonzero socle, then either J is simple unital and satisfies DCC on principal inner ideals or J is isomorphic to a Jordan subalgebra of $H(A,*)$ containing $H(S(A),*)$ or to a subalgebra of A^+ containing $S(A)$, where A is a primitive associative ϕ -algebra with nonzero socle $S(A)$, and in the first case $*$ is an involution. Conversely, if J is a simple unital Jordan algebra satisfying DCC on principal inner ideals, a Jordan subalgebra of $H(L_V(V),*)$ containing $H(F_V(V),*)$ or of $L_W(V)^+$ containing $F_W(V)$, then J is a nondegenerate prime Jordan algebra with nonzero socle.*

Osborn and Racine have also proved [34, Theorem 1] that under the hypothesis of nonzero socle an associative algebra is prime if and only if it is primitive. We have been able to prove the same for Jordan algebras [18, Theorem 12] by using the notion of primitivity given by Hogben and McCrimmon [21].

A nonassociative algebra A satisfying

- (i) $(x,y,x) = 0 \quad x,y \in A$ (Flexible law)
- (ii) $(x^2,y,x) = 0 \quad x,y \in A$ (Jordan identity)

where $(x,y,z) = (xy)z - x(yz)$ is the associator of x, y, z , is called a noncommutative Jordan algebra. Let A be a noncommutative Jordan ϕ -algebra. For any $\lambda \in \phi$ we can define a new algebraic structure on A , the λ -mutation $A^{(\lambda)}$, by $x \cdot_\lambda y = \lambda(xy) + (1-\lambda)xy$. An algebra of the form $A = B^{(\lambda)}$ for B associative is called split quasi-associative. An algebra A is quasi-associative when it has a scalar extension A_Σ which is split quasi-associative. Since the λ -mutation of a noncommutative Jordan algebra remains so, we have that quasi-associative algebras are noncommutative Jordan algebras.

Let A be a nondegenerate noncommutative Jordan algebra (A^+ is nondegenerate). Then the socle of A^+ is actually an ideal of A that we call the socle of A (see [18]). This definition of socle reconstructs the one of Slater for a semiprime alternative algebra [18, Corollary 8].

The notion of primitivity for Jordan algebras can also be extended to the noncommutative Jordan case. Indeed, a noncommutative Jordan algebra A is called primitive when it contains a maximal-modular inner ideal I of A^+ (see [21] for definition) such that I does not contain any nonzero ideal of A . As we have already pointed out, a

nondegenerate noncommutative Jordan algebra A with nonzero socle is prime if and only if it is primitive. Also, it follows from our results [18, Proposition 11] that a semiprime associative algebra A is primitive with nonzero socle if and only if so is A considered as a noncommutative Jordan algebra. A unital nonassociative ϕ -algebra A is called quadratic if for every element $a \in A$ there exist $\lambda, \mu \in \Phi$ such that $a^2 + \lambda a + \mu 1 = 0$. It is clear that every flexible quadratic algebra is a noncommutative Jordan algebra. Now we state our main result in this section, which generalizes theorem 1, 2 and 3.

Theorem 4. [18, Theorem 13]. *A noncommutative Jordan algebra is nondegenerate primitive with nonzero socle if and only if it is one of the following :*

- (i) *A noncommutative Jordan division algebra.*
- (ii) *A simple flexible quadratic algebra over its centre.*
- (iii) *A nondegenerate prime Jordan algebra with nonzero socle.*
- (iv) *A subalgebra of $L_W(V)^{(\lambda)}$ containing $F_W(V)$ or of $H(L_V(V), *)^{(\lambda)}$ containing $H(F_V(V), *)$ where in the first case (V, W) is a pair of dual vector spaces over a division algebra Δ and $\lambda \neq \frac{1}{2}$ is a central element in Δ , and where in the second case V is self-dual with respect to a hermitian inner product (\cdot, \cdot) , Δ has an involution $\alpha \rightarrow \bar{\alpha}$ and $\lambda \neq \frac{1}{2}$ is a central element in Δ with $\lambda + \bar{\lambda} = 1$.*

Remark. We note that in the latter case of Theorem 4 a symplectic inner product (\cdot, \cdot) cannot occur since the involution $\alpha \rightarrow \bar{\alpha}$ would be the identity in this case and hence $\lambda = \frac{1}{2}$, a contradiction. On the other hand, an algebra as in (iv) need not be quasiassociative, as we will see in the next section.

II. STRUCTURE THEOREMS FOR SIMPLE AND FOR PRIMITIVE NONCOMMUTATIVE JORDAN NORMED ALGEBRAS WITH NONZERO SOCLE.

In this second section the results of the preceding one are particularized to the case of a noncommutative Jordan normed algebra. Since the presence of a norm on the algebra makes the work less hard and, at the same time, reduces the number of cases that can occur, a new and independent approach of the general algebraic case will be given here.

A nonassociative real or complex algebra A is said to be normed if the underlying vector space of A is endowed with a norm $\| \cdot \|$ with respect to which the product of the algebra is continuous. A is called a Banach algebra when $\| \cdot \|$ is complete.

We recall [25, p. 206] that for any nonassociative algebra A the centroid Γ of A is defined to be the set of all the linear mappings $T : A \rightarrow A$ such that $T(ab) = aT(b) = T(a)b$ for all $a, b \in A$. If A is simple ($A^2 \neq 0$ and A does not contain any nonzero ideal) then Γ is a field, which extends the base field, and A , regarded as a Γ -algebra, is central simple, that is, A_Σ is simple for any extension field Σ of Γ .

Lemme 1. *Let A be a simple nonassociative normed algebra which contains a nonzero idempotent e . Then the centroid Γ of A is the complex field when A is a complex algebra, and either $\Gamma = \mathbb{R}$ or $\Gamma = \mathbb{C}$ when A is real. Moreover, when A is real and $\Gamma = \mathbb{C}$ then A can be regarded as a normed complex algebra with respect to a new norm $\| \cdot \|'$ which satisfies $\|a\|' \geq \|a\|$ ($a \in A$).*

Proof. We have only to prove that Γ can be endowed with a norm of algebra and then Mazur-Gelfand theorem and a result of Rickart [36, 1.3.3] are applied. Now, for each $T \in \Gamma$ we define $\|T\| = \|T(e)\|$. It is not difficult to see that $\| \cdot \|$ is a norm of algebra on Γ .

Since the complex case is simpler than the associative one we start by determining the simple noncommutative Jordan normed complex algebras which contain a completely primitive idempotent e ($U_e A$ is a division algebra). On the other hand, we note that the simple noncommutative Jordan algebras containing a completely primitive idempotent are precisely the noncommutative simple ones with nonzero socle.

We recall that the simple flexible quadratic normed algebras are, up to topological isomorphisms, the noncommutative Jordan normed algebras $A = K \oplus V$ ($K = \mathbb{R}$ or \mathbb{C}) determined by a continuous nondegenerate symmetric bilinear form $\{ | \}$ on a normed vector space V , and a continuous anticommutative product \wedge on V such that $\{x \wedge t | x\} = 0$ and $\|\alpha + x\| = |\alpha| + \|x\|$, for all $\alpha \in K$, $x, y \in V$. Moreover, $A = K \oplus V$ is a Banach algebra if and only if V is a Banach space (see the beginning of the proof of [35, Theorem 3.1]).

Theorem 5. *A is a simple noncommutative Jordan normed complex algebra containing a completely primitive idempotent if and only if it is topologically isomorphic to one of the following :*

- (i) *The complex field \mathbb{C} .*
- (ii) *A simple flexible quadratic normed complex algebra.*

- (iii) *A simple (commutative) Jordan normed complex algebra containing a completely primitive idempotent.*
- (iv) *A split quasi-associative normed complex algebra $A = B^{(\lambda)}$ where B is a simple associative normed complex algebra containing a minimal right ideal and $\lambda \in \mathbb{C}$, $\lambda \neq \frac{1}{2}$.*

Proof. Let e be a completely primitive idempotent in A . Then either e is a unit for A and hence A is a division algebra or A^+ is a simple Jordan algebra [32, Theorem 1]. If the former, $A = \mathbb{C}e$ by the noncommutative Jordan version of Mazur-Gelfand theorem [27]. If the latter, either A^+ has a capacity or contains a subalgebra of capacity n for each positive interger n [34, Corollary 7]. If A^+ has capacity two, then A is quadratic. Indeed, let $1 = e(1)+e(2)$ with $e(1)$ and $e(2)$ orthogonal completely primitive idempotents. Since $U_{e(i)}A^+$ is a division normed complex algebra it follows as above that $U_{e(i)}A^+ = \mathbb{C}e(i)$, $i = 1,2$. Thus A^+ is a simple reduced Jordan algebra with capacity two, so by [25, p. 203] $A^+ = \mathbb{C} \oplus V$, the Jordan complex algebra determined by a continuous nondegenerate symmetric bilinear form $(\cdot | \cdot)$ on a normed complex vector space V . Hence by [39, Theorem 2] A is a quadratic complex algebra. Finally we must consider the case when A contains two orthogonal completely primitive idempotents whose sum is not 1. By [32, Theorem 5], either A is commutative or quasi-associative over its centroid Γ . Since $\Gamma = \mathbb{C}$ by Lemma 1 and since \mathbb{C} has no proper quadratic extension, we have that A is split quasi-associative in the latter case, which completes the proof.

Theorem 5 can be used to determine the structure of the primitive nondegenerate noncommutative Jordan normed complex algebras with nonzero socle. The tool we will need is the next algebraic lemma whose proof can be found in [13]. Although a different approach could be given by using Theorem 4, this last one is less clear.

Lemma 2. *Let A be a primitive nondegenerate noncommutative Jordan K -algebra with nonzero socle $S(A)$. Then*

- (i) *A is commutative if $S(A)$ is commutative.*
- (ii) *$A = S(A)$ if $S(A)$ has a unit.*
- (iii) *A is split quasi-associative if $S(A)$ is split quasi-associative over K .*
- (iv) *A is quasi-associative if $S(A)$ is quasi-associative over K .*

Theorem 6. *A is a primitive nondegenerate noncommutative Jordan normed complex algebra with nonzero socle if and only if it is isomorphic to one of the following :*

- (i) *The complex field \mathbb{C} .*
- (ii) *A simple flexible quadratic normed complex algebra.*
- (iii) *A primitive (commutative) Jordan normed complex algebra with nonzero socle.*
- (iv) *A split quasi-associative normed complex algebra $A = B^{(\lambda)}$ where B is a primitive associative normed complex algebra with nonzero socle and $\lambda \in \mathbb{C}$, $\lambda \neq \frac{1}{2}$.*

Proof. Since A is primitive, then A is prime and hence $S(A)$ is a simple noncommutative Jordan normed complex algebra containing a completely primitive idempotent. Now Theorem 5 together with Lemma 2 are applied.

Remark. Primitive associative normed complex algebras with nonzero socle are well-known (see [9, p. 158, Theorem 6]). On the other hand, some informations about primitive (commutative) Jordan normed complex algebras with nonzero socle will be given in Section IV.

Let A be a noncommutative Jordan Banach algebra. It follows from [12, Theorem 4.1 and Lemma 6.5] that $\text{Rad}(A^+)$ is a closed ideal of A^+ , but $\text{Rad}(A) = \text{Rad}(A^+)$ [19, Lemma 16]. Hence every topologically-simple semisimple noncommutative Jordan Banach algebra is primitive and nondegenerate, so we can apply Theorem 6 to obtain :

Corollary 1. *Let A be a topologically-simple semisimple noncommutative Jordan Banach complex algebra with nonzero socle. Then A is, up to topological isomorphisms, either \mathbb{C} , a simple flexible quadratic Banach complex algebra, a topologically-simple semisimple Jordan Banach complex algebra with nonzero socle, or a split quasi-associative algebra $A = B^{(\lambda)}$ where B is a topologically-simple semisimple associative Banach complex algebra with nonzero socle and $\lambda \in \mathbb{C}$, $\lambda \neq \frac{1}{2}$.*

Now we are going to deal with real algebras. The following lemma is crucial to prove the "real" version of Theorem 5.

Lemma 3. *Let A be a noncommutative Jordan normed real algebra containing a completely primitive idempotent. Then the complexification $A_{\mathbb{C}}$ also contains a completely primitive idempotent.*

Proof. Let e be a completely primitive idempotent in A . Since $U_e A$ is a division algebra we have that $U_e A$ is quadratic. Hence by [37, p. 50] $U_e(A_{\mathbb{C}}) = (U_e A)_{\mathbb{C}}$ is a

simple flexible quadratic complex algebra. Then, either e is a completely primitive idempotent in $A_{\mathbb{C}}$ or $e = e_1 + e_2$, sum of two orthogonal completely primitive idempotents, which completes the proof.

Theorem 7. *Let A be a simple noncommutative Jordan normed real algebra containing a completely primitive idempotent. Then A is one of the following :*

(i) *A is either a simple flexible quadratic normed real algebra or the underlying real algebra of a simple flexible quadratic normed complex algebra.*

(ii) *A is a simple (commutative) Jordan normed real algebra containing a completely primitive idempotent.*

(iii) *A is a simple associative normed real algebra containing a minimal right ideal and $\lambda \in \mathbb{R}$, $\lambda \neq \frac{1}{2}$, a non-split quasi-associative real algebra $A = H(B, *)^{(\lambda)}$ where B is a simple associative normed complex algebra containing a minimal right ideal, $*$ a continuous conjugate linear involution on B and $\lambda \in \mathbb{C}$, $\lambda \neq \frac{1}{2}$ with $\lambda + \bar{\lambda} = 1$, or $(A, \|\cdot\|)$ is the underlying real algebra of a split quasi-associative normed complex algebra $B^{(\lambda)}$ where $(B, \|\cdot\|)$ is a simple associative normed complex algebra containing a minimal right ideal, $\lambda \in \mathbb{C} - \mathbb{R}$ and $\|a\| \leq \|a\|$ ($a \in A$).*

Proof. Let Γ be the centroid of A . By Lemma 1 either $\Gamma = \mathbb{R}$ or $\Gamma = \mathbb{C}$. If $\Gamma = \mathbb{R}$ then A is central simple and hence the normed complexification $A_{\mathbb{C}}$ is a simple noncommutative Jordan normed complex algebra which contains a completely primitive idempotent (Lemma 3). Thus, by Theorem 5, $A_{\mathbb{C}}$ is either the complex field, a simple flexible quadratic normed complex algebra, a (commutative) Jordan algebra or a split quasi-associative complex algebra. When $A_{\mathbb{C}}$ is quadratic, then A is a quadratic real algebra [37, p. 50]. If $A_{\mathbb{C}}$ is commutative, then A is clearly commutative. So we may assume that $A_{\mathbb{C}}$ is a split quasi-associative complex algebra $A = B^{(\lambda)}$, $\lambda \neq \frac{1}{2}$. By [32,

Theorem 5, p. 583] $\lambda - \lambda^2 \in \mathbb{R}$, so either $\lambda \in \mathbb{R}$ or $\lambda + \bar{\lambda} = 1$. In the first case A is a split quasi-associative real algebra. In the second case $A = H(B, *)^{(\lambda)}$ where B is a simple associative normed complex algebra containing a minimal right ideal, $*$ a conjugate linear involution and $\lambda \in \mathbb{C}$, $\lambda + \bar{\lambda} = 1$. Finally, if $\Gamma = \mathbb{C}$ then A is the underlying real algebra of a simple noncommutative Jordan normed complex algebra (Lemma 1) which contains a completely primitive idempotent, and we apply Theorem 5 again.

Unfortunately we cannot use Lemma 2 to determine the structure of the primitive nondegenerate noncommutative Jordan normed real algebras with nonzero socle from the structure theorem for simple ones, in a way similar the one used in the complex case. The reason is that, in the real case, the centroid of the socle can be the complex field and, at the same time, the algebra need not be complex. The following example casts light on this situation.

Let X be an infinite dimensional normed complex space and let $FB(X)$ denotes the normed complex algebra of all continuous linear operators with finite rank on (λ) . Then $A = FB(X)(\lambda) + \mathbb{R}I$, where I is the identity on X , is a primitive nondegenerate noncommutative Jordan normed real algebra with socle $S(A) = FB(X)(\lambda)$. The centroid of $S(A)$ is \mathbb{C} , but A is clearly not an algebra over \mathbb{C} . Moreover, A is not a quasi-associative real algebra (see [30, p. 1456]) although it is a \mathbb{R} -subalgebra of the split quasi-associative normed complex algebra $FB(X)(\lambda) + \mathbb{C}I$, which has the same socle as A .

Theorem 8. *Let A be a primitive nondegenerate noncommutative Jordan normed real algebra with nonzero socle. Then A is one of the following :*

- (i) *A simple flexible quadratic normed real algebra or the underlying real algebra of a simple flexible quadratic normed complex algebra.*
- (ii) *A primitive (commutative) Jordan normed real algebra with nonzero socle.*
- (iii) *A is either a split quasi-associative real algebra $A = B^{(\lambda)}$ where B is a primitive associative normed real algebra with nonzero socle and $\lambda \in \mathbb{R}, \lambda \neq \frac{1}{2}$, a non-split quasi-associative real algebra $A = H(B, *)^{(\lambda)}$ where B is a primitive associative normed complex algebra with nonzero socle, $*$ a continuous conjugate linear involution on B and $\lambda \in \mathbb{C}, \lambda \neq \frac{1}{2}$ with $\lambda + \bar{\lambda} = 1$, or A is a "normed" real subalgebra of a split quasi-associative complex algebra $B^{(\lambda)}$ where B is a primitive associative complex algebra with nonzero socle such that $S(A) = S(B)^{(\lambda)}$ and $\lambda \in \mathbb{C} - \mathbb{R}$.*

Proof. Let M be the socle of A . M is a simple noncommutative Jordan normed real algebra containing a completely primitive idempotent. We may assume that M is quasi-associative over its centroid $\Gamma(M)$, since otherwise we conclude the proof by applying Theorem 7 and Lemma 2. Then by Lemma 1, either $\Gamma(M) = \mathbb{R}$ or $\Gamma(M) = \mathbb{C}$. If $\Gamma(M) = \mathbb{R}$ then we can apply Theorem 7 together with Lemma 2 to get that A is either a split quasi-associative real algebra or A is a non-split quasi-associative real algebra. If $\Gamma(M) = \mathbb{C}$ then M is a split quasi-associative complex algebra $M = D^{(\lambda)}$ where D is a simple associative normed complex algebra containing a minimal right ideal and $\lambda \in \mathbb{C} - \mathbb{R}$. Now we use the structure of D (see [9, p. 158, Theorem 6]) to obtain, by the same methods

as in [4, Proof of Theorem 13] that there exists a primitive associative complex algebra B such that A is a \mathbb{R} -subalgebra of $B^{(\lambda)}$ and $S(A) = S(B)^{(\lambda)}$.

Remark. We must remark that in the last case of Theorem 8 we do not know whether the associative complex algebra B is normable. We do know that $S(B)$ is normed, which follows from Theorem 7, so that the problem reduces to the following associative one :

Is normable a primitive associative complex algebra whose socle is normable ?

III. FINITNESS CONDITIONS IN NONCOMMUTATIVE JORDAN BANACH ALGEBRAS

It is well-known [42] that certain algebraic conditions on an associative Banach complex algebra force it to be finite dimensional. For instance, semiprime Banach algebras coinciding with their socle, von Neumann regular Banach algebras and semisimple Banach algebras in which each element has a finite spectrum are finite dimensional. However these results do not hold in every Jordan Banach algebra. Indeed, the Jordan Banach algebra $J = \mathbb{C} \oplus V$ defined by a continuous nondegenerate bilinear form on an infinite dimensional Banach complex space V satisfies all the conditions above, but has infinite dimension. Nevertheless this is essentially the only pathological case that can occur.

We recall that a unital noncommutative Jordan algebra A is said to have a capacity when $1 = e_1 + \dots + e_n$, sum of orthogonal completely primitive idempotents. Every nondegenerate noncommutative Jordan algebra with a capacity coincides with its socle and the true finiteness condition for a noncommutative Jordan algebra is to have a capacity [32]. In this section we will see that certain algebraic conditions on noncommutative Jordan Banach algebras imply that they have a capacity ; then we will determine such algebras. Since every nondegenerate noncommutative Jordan algebra having a capacity splits into a direct sum $A = M_1 \oplus \dots \oplus M_r$ where each M_i is a simple noncommutative Jordan algebra with a capacity, and where the sum is topological when A is normable, we have only to consider the simple ones.

To fix notation, given a composition algebra D with involution $j : D \rightarrow D$, let $M_n(D)$ be the algebra of n th order matrices over D . Then the mapping $S : X \rightarrow \bar{X}^t$, where \bar{X}^t is the matrix obtained from X by applying the involution to each entry in X and then transposing, is an involution on $M_n(D)$. If D is associative then $H(M_n(D), S)$

is a special Jordan algebra. But if D is a Cayley-Dickson algebra then $H(M_n(D), S)$ will be Jordan only for $n \geq 3$ and exceptional for $n = 3$.

Theorem 9. *Let A be a simple noncommutative Jordan normed complex algebra with a capacity. Then A is one of the following :*

- (i) *The complex field \mathbb{C} .*
- (ii) *A simple flexible quadratic normed complex algebra.*
- (iii) *A Jordan matrix algebra $H(M_n(D), S)$ where $n \geq 3$ and D is a composition algebra over \mathbb{C} of dimension 1, 2 or 4 if $n \geq 4$ and of dimension 1, 2, 4 or 6 if $n=3$.*
- (iv) *A split quasi-associative algebra $M_n(\mathbb{C})^{(\lambda)}$, where $n \geq 3$ and $\lambda \in \mathbb{C}$, $\lambda \neq \frac{1}{2}$.*

Proof. Use Theorem 5 (Section II), the structure theorem of reduced simple Jordan algebras over \mathbb{C} [25, p. 203-204] and the Wedderburn theorem for finite-dimensional simple associative complex algebra.

As usual, the real case is something more complicated than the complex one.

Theorem 10. *Let A be a simple noncommutative Jordan normed real algebra having a capacity. Then A is one of the following :*

- (i) *A is either a simple flexible quadratic normed real algebra or the underlying real algebra of a simple flexible quadratic normed complex algebra.*
- (ii) *A is either a finite dimensional central simple Jordan real algebra of capacity ≥ 2 or A is the underlying real algebra of a matrix complex algebra $H(M_n(D), S)$ where D is a composition algebra over \mathbb{C} and $n \geq 3$.*
- (iii) *A is either a split quasi-associative real algebra $M_n(\mathbb{K})^{(\lambda)}$ where \mathbb{K} is \mathbb{R} , \mathbb{C} or \mathbb{H} (Hamilton's quaternion algebra over \mathbb{R}), $n \geq 3$ and $\lambda \in \mathbb{R}$, $\lambda \neq \frac{1}{2}$, a non-split quasi-associative real algebra $H(M_n(\mathbb{C}), S)^{(\lambda)}$ where $n \geq 3$, $\lambda \in \mathbb{C}$, $\lambda \neq \frac{1}{2}$ with $\lambda + \bar{\lambda} = 1$ and $S(X) = \bar{X}^t$ for every $X \in M_n(\mathbb{C})$, or A is the underlying real algebra of a split quasi-associative complex algebra $M_n(\mathbb{C})^{(\lambda)}$ where $n \geq 3$ and $\lambda \in \mathbb{C} - \mathbb{R}$.*

Proof. Let Z be the center of A . Since Z is a commutative associative normed division real algebra, we have by Mazur-Gelfand theorem that either $Z = \mathbb{C}$ or $Z = \mathbb{R}$. If the former, then A is the underlying real algebra of a simple noncommutative Jordan normed complex algebra having a capacity, and Theorem 9 is applied. If the latter, then A is a central simple real algebra, so $A_{\mathbb{C}}$ is a simple noncommutative Jordan normed complex algebra having a capacity (Lemma 3 of Section II). Then, by Theorem 9 again,

either $A_{\mathbb{C}}$ is a simple flexible quadratic normed complex algebra, and hence A is a quadratic real algebra, $A_{\mathbb{C}}$ is a simple finite-dimensional Jordan complex algebra with capacity $n \geq 3$, and hence A is a finite-dimensional central simple Jordan real algebra of capacity $n \geq 2$, or $A_{\mathbb{C}}$ is a split quasiassociative finite-dimensional simple complex algebra of capacity $n \geq 3$, and hence A is either a split quasiassociative real algebra of the form $A = M_n(\mathbb{K})^{(\lambda)}$ where $K = \mathbb{R}, \mathbb{C}$ ou \mathbb{H} , $\lambda \in \mathbb{R}$, $\lambda \neq \frac{1}{2}$ and $n \geq 3$, or a non-split quasi-associative real algebra of the form $A = H(M_n(\mathbb{C}), S)^{(\lambda)}$ where $\lambda \in \mathbb{C}$, $\lambda \neq \frac{1}{2}$, $\lambda + \bar{\lambda} = 1$, $n \geq 3$ and $S(X) = \bar{X}^t$, by the same arguments as in the proof of Theorem 7 of Section II.

Remark. A classification of all the finite-dimensional central simple Jordan real algebras can be found in [25, p. 211-212].

As a consequence of theorems 9 and 10 we obtain the following

Corollary 2. *Every simple noncommutative Jordan normed algebra with a capacity is either finite-dimensional or is an infinite-dimensional simple flexible quadratic normed algebra.*

Since every nondegenerate noncommutative Jordan algebra having a capacity is a direct sum of a finite number of simple ones and since every quadratic alternative algebra is a composition algebra, we also obtain :

Corollary 3. *Semiprime alternative normed algebras with a capacity are finite-dimensional.*

We end this section by showing that certain algebraic conditions on noncommutative Jordan Banach complex algebras are equivalent.

We recall that a noncommutative Jordan K -algebra A is called algebraic if every element $a \in A$ satisfies a nontrivial polynomial relation $p(a) = 0$. Clearly A is algebraic if and only if A^+ is algebraic.

Theorem 11. *Every semisimple algebraic noncommutative Jordan Banach algebra A has a capacity.*

Proof. Since the Jacobson radical of A [32] coincides with the Jacobson radical of the Jordan algebra A^+ [19, Lemma 16], we may assume that A is commutative. Now by [33, Theorem 1.15] we only need to prove that A is idempotent-finite (no infinite sequence of orthogonal idempotents). Suppose otherwise that A contains an infinite sequence $\{e_n\}$ of nonzero orthogonal idempotents and let $\{\lambda_n\}$ be an infinite set of complex numbers such that $\sum \lambda_n e_n$ converges. Write $u = \sum \lambda_n e_n$. Then there exists a nonzero polynomial $p(x) \in K[x]$ ($K = \mathbb{R}$ or \mathbb{C}) such that $p(u) = 0$. It is not difficult to see that $p(\lambda_n) = 0$ for every positive integer n , but this leads to a contradiction. Therefore A is idempotent-finite, as required.

In the next section we will see that the socle of a nondegenerate noncommutative Jordan normed algebra A is an algebraic ideal of A . This result together with Theorem 11 yields the following :

Corollary 4. *Every nondegenerate noncommutative Jordan Banach algebra which coincides with its socle has a capacity.*

Remark. A different proof of this corollary can be found in [14].

A noncommutative Jordan algebra A is called von Neumann regular if for every $a \in A$ there exists $b \in A$ such that $U_a b = a$. For an element a in a unital noncommutative Jordan \mathbb{K} -algebra A the spectrum of a is the set

$$\text{Sp}(a, A) = \{\lambda \in \mathbb{K} ; \lambda 1 - a \text{ is not invertible in } A\}.$$

When A has no unit, the spectrum of a is defined to be the set $\text{Sp}(A^1, a)$ where A^1 denotes the unital hull of A .

The following theorem has recently been proved by Benslimane and Kaidi.

Theorem 12. [8]. *Every noncommutative Jordan Banach complex algebra A which is von Neumann regular or semisimple with finite spectrum ($\text{Sp}(a, A)$ is finite for every $a \in A$) has a capacity.*

We end this section by collecting all the foregoing results in the following

Corollary 5. *For a noncommutative Jordan Banach complex algebra A the following conditions are equivalent :*

- (i) A is semisimple with finite spectrum.
- (ii) A is von Neumann regular.

- (iii) A is nondegenerate and coincides with its socle.
- (iv) A is nondegenerate and has a capacity.
- (v) A is nondegenerate and satisfies DCC on principal inner ideals.
- (vi) A is topologically isomorphic to a direct sum $A = M_1 \oplus \dots \oplus M_n$ of closed ideals, where each M_i is one of the algebras listed in Theorem 9.
- (vii) A is semisimple and algebraic.

Remark. Conditions (ii)-(vii) are also equivalent in every noncommutative Jordan Banach real algebra. On the other hand, any of these conditions implies (i). Thus it would be just necessary to show that (i) implies (iv) to close the cycle in every noncommutative Jordan Banach real algebra.

IV. SOME PROPERTIES OF THE SOCLE OF A NONCOMMUTATIVE JORDAN ALGEBRA.

In this section we study the socle of a nondegenerate noncommutative Jordan algebra with special emphasis on the normed case. Some of the results we present here have not been explicitly stated in the associative case ; the others are nontrivial noncommutative Jordan extensions of associative ones.

Since the socle of a nondegenerate noncommutative Jordan algebra A is a direct sum of simple ideals, each of which containing a completely primitive idempotent [18, Theorem 7], it is possible to assume that A is simple in a lot of cases. Also, since the socle of A coincides with the socle of the Jordan algebra A^+ we can always suppose that A is commutative. Finally, Osborn-Racine theorem [34, Theorem 9] allows to reduce the study of a lot of questions related to the socle to the cases of a simple Jordan algebra satisfying DCC on principal inner ideals and of a simple associative algebra containing a minimal right ideal.

The following proposition can be proved by making use of the foregoing ideas together with the corresponding results for associative and for simple Jordan algebras satisfying DCC on principal inner ideals. Nevertheless a full proof can be found in [15].

Proposition 1. *Let A be a nondegenerate noncommutative Jordan algebra. Then every element in the socle of A is von Neumann regular and has finite spectrum.*

We recall [41] that in every Jordan algebra J the sum of all von Neumann regular ideals of J is a von Neumann regular ideal called the maximal von Neumann regular ideal of J . For a noncommutative Jordan algebra A it is not difficult to see that the maximal von Neumann regular ideal of A^+ is in fact an ideal of A , which is also called the maximal von Neumann regular ideal of A . By Proposition 1 the socle of every nondegenerate noncommutative Jordan algebra A is contained in the maximal von Neumann regular ideal of A . However this inclusion can be strict, even in an associative normed algebra.

Indeed, let X be an infinite dimensional normed complex vector space and let $FB(X)$ denote the algebra of all continuous linear operators with finite rank on X . Then the normed associative complex algebra $A = FB(X) + CI$, where I is the identity operator on X , is von Neumann regular, but $S(A) = FB(X) \neq A$.

However, the socle and the maximal von Neumann regular ideal both coincide in every nondegenerate noncommutative Jordan Banach algebra, as is stated in the following theorem whose proof will appear in [15].

Theorem 13. *In every nondegenerate noncommutative Jordan Banach algebra A the socle coincides with the maximal von Neumann regular ideal of A .*

It is well-known (see [40] for instance) that, for every semisimple associative Banach complex algebra A , $x \in S(A)$ if and only if xAx has finite dimension.

The following proposition, whose proof can be found in [15], extends the above results in one direction.

Theorem 14. *Let A be a nondegenerate noncommutative Jordan algebra. If x is an element in A such that $U_x A$ has finite dimension then $x \in S(A)$.*

We remark that the converse of Theorem 14 does not hold even for a Jordan Banach algebra. Indeed, every infinite dimensional simple quadratic Jordan Banach algebra J coincides with its socle, but $U_1 J = J$ has infinite dimension. However, for a certain class of nondegenerate noncommutative Jordan normed complex algebras the converse of Theorem 14 is true. To prove such a result we first have to know the structure of the simple Jordan normed complex algebras containing a completely primitive idempotent, hence completing the classification of the noncommutative Jordan ones given in Theorem 5 of Section II.

Proposition 2. *Let J be a simple Jordan normed complex algebra containing a completely primitive idempotent. Then J is one of the following :*

- (i) *A finite dimensional simple Jordan complex algebra.*
- (ii) *An infinite dimensional simple quadratic Jordan normed complex algebra.*
- (iii) *J is either the Jordan algebra $F_W(V)^+$, where (V,W) is a pair of dual vector spaces over the complex field, or a Jordan algebra of symmetric elements $H(F_V(V),*)$, where V is a self-dual complex vector space with respect to a hermitian or symplectic inner product (\mid) over \mathbb{C} .*

Proof. By [34, Theorem 9] either J has a capacity or $J = H(A,*)$ for a $*$ -simple associative complex algebra A coinciding with its socle. If the former, we have by Theorem 9 of Section III that J is finite-dimensional or a (possibly infinite dimensional) quadratic Jordan normed complex algebra. Thus we may suppose that J is of the form $H(A,*)$. Now, either A is simple or $A = B \oplus B^{\text{op}}$ where B is a simple associative algebra with socle, B^{op} is the opposite algebra of B and $*$ is the exchange involution. In this case $J = B^+$ and if e is a completely primitive idempotent in J then $U_e J = (eBe)^+$ is a division Jordan normed complex algebra, so $U_e J = \mathbb{C}e$ and hence $eBe = \mathbb{C}e$ clearly. Write $V = Be$, $W = eB$ and $(xly)e = yx$ ($x \in V, y \in W$): then $(V,W,(\mid))$ is a pair of dual vector spaces over the complex field and the regular left representation $a \rightarrow L_a$ ($L_a x = ax, x \in V$) is an isomorphism from B onto $F_W(V)$ (see [24, p. 77]). Suppose now that A is simple and $J = H(A,*)$. As it was already pointed out (Theorem 1 below), $(A,*) = (F_V(V),*)$, where V is a self-adjoint vector space over a division associative complex algebra Δ , with respect to a hermitian or symplectic inner product $(\mid) : V \times V \rightarrow \Delta$ (in the symplectic case $\Delta = \Phi$ is a field). If (\mid) is hermitian then $A = F_V(V)$ contains a hermitian division idempotent e ($e = e^*$ and eAe is a division associative algebra) as it can be proved by using the arguments of Proof of Lemma 4 of [26]. Then $H(eAe,*) = U_e J = \mathbb{C}e$, as above, and hence it is not difficult to see that eAe is a quadratic division complex algebra, so $\Delta e = eAe = \mathbb{C}e$. Thus V is a complex vector space, as required. Finally, if (\mid) is symplectic we may use again the arguments of Proof of Lemma 4 of [26] to prove that $J = H(F_V(V),*)$ contains a completely primitive idempotent e such that $U_e J = \Phi e$. Hence $\Phi e = \mathbb{C}e$, so V is a complex vector space. This completes the proof.

Remark. It must be noted that Proposition 2 is not a direct consequence of Osborn-Racine result [34, Theorem 9] and the structure theorem of simple associative normed complex algebras containing a minimal right ideal since in the case when $J = H(A,*)$, for

A a *-simple associative complex algebra coinciding with its socle, A need not be normed.

For brevity, we call a noncommutative Jordan normed algebra A quasi-finite if it does not contain any infinite dimensional quadratic simple ideal. Every compact noncommutative Jordan Banach algebra A (U_x is a compact operator for all $x \in A$) and every alternative normed algebra are quasi-finite.

Theorem 15. *Let A be a quasi-finite nondegenerate noncommutative Jordan normed algebra. Then $x \in S(A)$ if and only if $U_x A$ has finite dimension.*

Proof. By Theorem 14 we only need to prove that $U_x A$ is finite dimensional for every $x \in S(A)$. Without loss of generality we may assume that $A = J$ is commutative. Now if $x \in S(J)$ then $x = x(1) + \dots + x(n)$ where each $x(i) \in M_i$ and where M_i is a simple Jordan normed complex algebra containing a completely primitive idempotent e_i . Moreover, $M_i M_j = 0$ for $i \neq j$. Hence $U_x = U_{x(1)} + \dots + U_{x(n)}$. By regularity of M_i (Proposition 1) it follows that $U_{x(i)} J = U_{x(i)} M_i$. Thus we only need to prove that $U_x : M \rightarrow M$ has finite rank. Since this is clear when M is finite dimensional, we may assume by Proposition 2 that M is a Jordan subalgebra of $F(X)$, for X a complex vector space. Now the following well-known result [7, Theorem 0.6.1] is applied.

Lemma 4. *Let $L(X)$ be the algebra all linear operators on a vector space X over a field. If $F \in L(X)$ has finite rank then the mapping $U_F : L(X) \rightarrow L(X)$ defined by $U_F(T) = FTF$, $T \in L(X)$, has finite rank.*

Smyth has proved [40, Theorem 3.2] that for a semisimple associative Banach complex algebra A the socle of A coincides with the largest algebraic ideal of A (see [24, p. 246]). However this result does not hold without completeness. Indeed, let X be an infinite dimensional Banach complex space and T a continuous linear operator with infinite rank on X and such that $T^2 = 0$. Then the primitive associative normed complex algebra $A = FB(X) + \mathbb{C}T$ is algebraic, but $T \notin S(A) = FB(X)$.

Proposition 3. *Let A be a nondegenerate noncommutative Jordan normed algebra. Then $S(A)$ is an algebraic ideal of A.*

Proof. Since $S(A) = S(A^+)$ and since $x^n = x \cdot^n$ for every $x \in A$ and any positive integer n , we may assume that $A = J$ is commutative. Also, since $S(J) = \sum M_i$ where

$M_i \cdot M_j = 0$ for $i \neq j$, we may suppose that $J = M$ is a simple Jordan normed complex algebra containing a completely primitive idempotent.

(1) Complex case. By Proposition 2 together with Theorem 15, either M is quadratic, and thus algebraic, or $U_x M$ has finite dimension for every $x \in M$. In the latter case $\{x^n : n \geq 3\}$ is linearly dependent in $U_x M$ and hence x is clearly algebraic.

(2) Real case. Let Γ be the centroid of M . By Lemma 1 of Section II, either $\Gamma = \mathbb{C}$ and M can be regarded as a normed complex algebra, or $\Gamma = \mathbb{R}$. In the former case M is algebraic over \mathbb{C} by (1). This implies that M is also algebraic over \mathbb{R} . In the latter case M is central simple. Then the normed complexification $M_{\mathbb{C}}$ is a simple Jordan normed complex algebra containing a completely primitive idempotent (Lemma 3 of Section II). Thus by (1), $M_{\mathbb{C}}$ is algebraic over \mathbb{C} , and hence it is not difficult to see that M is algebraic over \mathbb{R} , as required.

Open question. In view of Smith's result one could ask whether every algebraic ideal of a semisimple noncommutative Jordan Banach algebra is contained in the socle. A partial answer to this question is given in the following result.

Proposition 4. *Let A be a semisimple noncommutative Jordan Banach algebra and let I be an algebraic ideal of A . Then every element $x \in I$ is a sum $x = y + z$ where $y \in S(I)$ and z is nilpotent. In particular, $I/S(I)$ is nil.*

Proof. Since $\text{Rad}(A) = \text{Rad}(A^+)$ [19, Lemma 16] we may suppose that A is commutative. We first prove that every idempotent $u \in I$ lies in the socle. Indeed as $U_u A = U_u I$ is an algebraic semisimple Jordan Banach algebra, then by Theorem 11 of Section III $U_u A$ has a capacity and hence $u \in S(I)$. Let now $x \in I$ be such that x is not nilpotent; then $U_x A$ is an algebraic Jordan algebra which is not nil, so $U_x A$ contains a nonzero idempotent. We claim that $U_x A$ is idempotent-finite. Otherwise let $\{e_n\}$ be an infinite sequence of nonzero orthogonal idempotents in $U_x A$. Then $e_n = U_x(a_n)$ for some $\{a_n\} \subset A$. Choose an infinite set of scalar $\{\lambda_n\}$ such that $|\lambda_n| < 2^{-n} \|a_n\|$ for all n . Then $b = \sum \lambda_n a_n \in A$ and $U_x b = \sum \lambda_n e_n$, which leads to contradiction since $U_x b$ is algebraic. This implies that $U_x A$ is idempotent finite. Hence $U_x A$ contains a principal idempotent e ($U_x A$ does not contain any nonzero idempotent u such that $u \cdot e = 0$).

Let $x = x_1 + x_{1/2} + x_0$ be the Peirce decomposition of x relative to e . Since $e \in S(I)$ then $x_1, x_{1/2} \in S(I)$. Also $x_1 \in U_e A \subset U_x A$ and

$$(U_{e+x} - U_e - U_x)e = 2x_1 - x_{1/2} \in U_x A$$

implies $x_{1/2} \in U_x A$. Then $x_0 \in (Kx + U_x A) \cap U_{1-e} A$, where $K = \mathbb{R}$ ou \mathbb{C} . Now let

$$B = (Kx + U_x A^1) \cap U_{1-e} A$$

where $A^1 = A$ when A has a unit and $A^1 = K1 \oplus A$ otherwise. Then the subalgebra B must be nil. Since otherwise it would contain a nonzero idempotent u with $u = U_u u \in U_x A \cap U_{1-e} A$, as it is not difficult to see, which is contrary to e to be principal in $U_x A$. Therefore $x = x_1 + x_{1/2} + x_0$ where $x_1 + x_{1/2} \in S(I)$ and x_0 is nilpotent. This proves in particular that $I/S(I)$ is nil, as required.

Since a JB-algebra (see [2] for definition) does not contain any nonzero nilpotent, we get from Proposition 4 the following corollary :

Corollary 6. *In every JB-algebra J , an ideal I is algebraic if and only if it coincides with its socle. In particular, the socle of J is the largest algebraic ideal of J .*

V. MODULAR ANNIHILATOR NONCOMMUTATIVE JORDAN ALGEBRAS

We recall [4] that an associative algebra A is called modular annihilator if it is semiprime and satisfies any of the following equivalent conditions :

- (i) $\text{Ran}(M) \neq 0$ for each maximal modular left ideal M of A ;
- (ii) $\text{Lan}(N) \neq 0$ for each maximal modular right ideal N of A ;
- (iii) $A/S(A)$ is radical ;

where $\text{Ran}(M) = \{ x \in A : Mx = 0 \}$, $\text{Lan}(N) = \{ x \in A : xN = 0 \}$ and $S(A)$ is the socle.

It is well-known [6] that some important Banach algebras are modular annihilator. For example, semiprime compact Banach algebras [1], proper H^* -algebras [3] and dual B^* -algebras [28].

In view of condition (iii) above, a modular annihilator associative algebra is one that is close to its socle, but this condition is expressible in terms of Jordan algebra. Indeed, let A^+ be the Jordan algebra associated with an associative algebra A . A is semiprime if and only if A^+ is nondegenerate and since $S(A) = S(A^+)$ [12, Proposition 2.6] and $\text{Rad}(B) = \text{Rad}(B^+)$ for every associative algebra B [31, Theorem 1], we have

that A is modular annihilator if and only if A^+ is nondegenerate and $A^+/S(A^+)$ is a radical Jordan algebra. This leads to the following definition.

Definition. A Jordan algebra J is called modular annihilator if it is nondegenerate and $J/S(J)$ is radical.

Modular annihilator Jordan algebras were studied by the author in [12]. Let now A be a nondegenerate noncommutative Jordan algebra. A is said to be a modular annihilator noncommutative Jordan algebra if $A/S(A)$ is radical. Since $S(A) = S(A^+)$ and since a noncommutative Jordan algebra B is radical if and only if so is B^+ [32, Theorem 11], we have that A is modular annihilator if and only if so is A^+ .

The aim of this last section is to show that modular annihilator noncommutative Jordan algebras have interesting properties and collect some examples of them.

We recall that an inner ideal I of a noncommutative Jordan algebra A is said to be a maximal-modular inner ideal when it is one of A^+ . For every maximal-modular inner ideal I of A , the core $K(I)$ of I is defined to be the largest ideal of A contained in I . If $K^+(I)$ denotes the core of I in A^+ it is clear that $K(I)$ is contained in $K^+(I)$. Hogben and McCrimmon [21] have characterized the Jacobson radical of a Jordan algebra J as the intersection of the cores of the maximal-modular inner ideals of J . This result together with the fact that the Jacobson radical of a noncommutative Jordan algebra A is the largest ideal of A contained in $\text{Rad}(A^+)$ [32, Theorem 11] yield the following theorem.

Theorem 16. *The Jacobson radical of a noncommutative Jordan algebra is the intersection of the cores of its maximal-modular inner ideals.*

Given an ideal B of a noncommutative Jordan algebra A , the annihilator $\text{Ann}(B)$ of B is defined to be the largest ideal C of A such that $BC = CB = 0$. $\text{Ann}^+(B)$ stands for the annihilator of B in the Jordan algebra A^+ .

Theorem 17. *Let A be a nondegenerate noncommutative Jordan algebra. Then the following conditions are equivalent :*

- (i) A is modular annihilator.
- (ii) No cores of maximal-modular inner ideals of A contain $S(A)$.
- (iii) An ideal K of A is the core of some maximal-modular inner ideal of A if and only if $K = \text{Ann}(M)$ where M is the simple ideal generated by a completely primitive idempotent e in A .

- (iv) $\text{Ann}(K) \neq 0$ for every core K of a maximal-modular inner ideal of A and $\text{Rad}(A) = \text{Ann}(S(A))$.

Proof. It follows as in the commutative case (see [12, Theorem 2.4]). Since $\text{Ann}(B) = \text{Ann}^+(B)$ for every ideal B of a nondegenerate non-commutative Jordan algebra A [18, Corollary 10], it follows from (iii) and (iv) of Theorem 17.

Corollary 7. *Let A be a modular annihilator noncommutative Jordan algebra. Then the cores of the maximal-modular inner ideals of A are precisely those of A^+ . Therefore, $\text{Rad}(A) = \text{Rad}(A^+)$.*

Remark. Nodal noncommutative Jordan algebras [37] provide examples of non-commutative Jordan algebras A in which the equality $\text{Rad}(A) = \text{Rad}(A^+)$ does not hold.

Since the socle of a nondegenerate noncommutative Jordan algebra is a von Neumann regular ideal [Proposition 1 of Section IV] and since the radical of a noncommutative Jordan algebra does not contain any nonzero von Neumann regular element [32, Prop. 2], we get from the definition of modular annihilator noncommutative Jordan algebras :

Proposition 5. *Let A be a modular annihilator noncommutative Jordan algebra. Then $S(A)$ coincides with the set of regular elements of A .*

Next we state that modular annihilator alternative algebras can be characterized as the associative ones. A proof of this results can be found in [17].

Theorem 18. *Let A be a semiprime alternative algebra. Then the following conditions are equivalent*

- (i) A is modular annihilator.
- (ii) $\text{Ran}(M) \neq 0$ for every maximal-modular left ideal M of A .
- (iii) $\text{Lan}(N) \neq 0$ for every maximal-modular right ideal N of A .
- (iv) The maximal-modular left ideals M of A are of the form $M = \text{Lan}(R)$ for some minimal right ideal R of A .
- (v) The maximal-modular right ideals N of A are of the form $N = \text{Ran}(L)$ for some minimal left ideal L of A .

We end with some examples of noncommutative Jordan Banach algebras which are modular annihilator. We recall that a noncommutative Jordan Banach algebra A is said to be compact if U_x is a compact operator for every $x \in A$.

Theorem 19. *Every nondegenerate compact noncommutative Jordan Banach complex algebra is modular annihilator.*

Proof. It follows as in [12, Theorem 6.4].

We recall [10] that a nonassociative complex algebra A is said to be a J^* -algebra if A is endowed with a conjugate linear involution $*$ and the underlying vector space of A is a Hilbert space with respect to an inner product $(\cdot | \cdot)$ which satisfies

$$(xyz) = (y|x^*z) = (x|zy^*)$$

for all $x, y, z \in A$.

Theorem 20. *Every noncommutative Jordan H^* -algebra A such that $\text{Ann}^+(A) = 0$ is modular annihilator.*

Proof. It follows as in the commutative case (see [12, Theorem 6.7])

Proposition 4 of Section IV can be rephrased by saying :

Proposition 6. *Every algebraic ideal of a semisimple noncommutative Jordan Banach algebra is modular annihilator.*

A well-known result of Barnes [5] asserts that a semisimple associative Banach complex algebra A is modular annihilator if and only if the spectrum of each element $a \in A$ has no nonzero accumulation points. In the next theorem, whose proof will appear in [16], the result of Barnes is extended in one direction for noncommutative Jordan algebras.

Theorem 21. *Let A be a semisimple noncommutative Jordan Banach complex algebra such that the spectrum of every element $x \in A$ has no nonzero accumulation points. Then A is modular annihilator.*

Open question. Does every semisimple modular annihilator noncommutative Jordan Banach complex algebra have the above spectral property ?

REFERENCES

1. J.C. ALEXANDER, Compact Banach algebras. Proc. London Math. Soc. (3) 18 (1968), 1-18.
2. E.M. ALFEN, F.W. SHULTZ and E. STORMER, A Gelgand Neumark theorem for Jordan algebras. Advances in Math. 28 (1978), 11-56.
3. W. AMBROSE, Structures theorems for a special class of Banach algebra. Trans. Amer. Math. Soc. 57 (1945), 364-386.
4. B.A. BARNES, Modular annihilator algebras. Canad. J. Math. 18 (1966), 566-578.
5. B.A. BARNES, On the existence of minimal ideals in Banach algebras. Trans. Amer. Math. Soc. 133 (1968), 511-517.
6. B.A. BARNES, Examples of modular annihilator algebras. Rocky Mountain J. Math. 1 (1971), 657-665.
7. B.A. BARNES, G.J. MURPHY, M.R.F. SMITH and T.T. WEST, Riesz and Fredholm theory in Banach algebras. Research Notes in Mathematics. Pitman (1982).
8. M. BENSLIMANE and A.M. KAIDI, Structure des algèbres de Jordan Banach non commutatives complexes régulières ou semisimples à spectre fini. Submitted to J. Algebra.
9. F.F. BONSALL and J. DUNCAN, Complete Normed Algebras. Springer Verlag (1973).
10. J.A. CUENCA MIRA and A. RODRIGUEZ PALACIOS, Structure theory for noncommutative Jordan H^* -algebras. J. Algebra (to appear).
11. J. DIEUDONNE, Sur le socle d'un anneau et les anneaux simples infinis. Bull. Soc. Math. de France, 71 (1943), 1-30.
12. A. FERNANDEZ LOPEZ, Modular annihilator Jordan algebras. Commun. in Algebra 13 (12) (1985), 2597-2613.
13. A. FERNANDEZ LOPEZ, Ideals in nondegenerate noncommutative Jordan algebras. Commun. in Algebra, (to appear).
14. A. FERNANDEZ LOPEZ, Noncommutative Jordan algebras coinciding with its socle. Preprint.
15. A. FERNANDEZ LOPEZ and A. RODRIGUEZ PALACIOS, On the socle of a non-commutative Jordan algebra. Submitted to Manuscripta Mathematica.
16. A. FERNANDEZ LOPEZ, Noncommutative Jordan Riesz algebras. Preprint.
17. A. FERNANDEZ LOPEZ, Theoremas de Wedderburn-Zorn en algebras alternativas normadas completas. Tesis Doctoral. Universidad de Granada. Granada 1983.
18. A. FERNANDEZ LOPEZ and RODRIGUEZ PALACIOS, Primitive non commutative Jordan algebras with nonzero socle. Proc. Amer. Math. Soc. 96 (2) (1986), 199-206.

19. A. FERNANDEZ LOPEZ and RODRIGUEZ PALACIOS, A Wedderburn theorem for nonassociative complete normed algebras. Proc. London Math. Soc. (to appear).
20. A. FERNANDEZ LOPEZ and RODRIGUEZ PALACIOS, Coincidence of the socle and the maximal regular ideal in associative and Jordan Banach algebras. Preprint.
21. L. HOGBEN and K. McCRIMMON, Maximal modular inner ideals and the Jacobson radical of a Jordan algebra. J. Algebra 68 (1981), 155-169.
22. N. JACOBSON, On the theory of primitive rings. Ann. of Math. 48 (1947) 8-21.
23. N. JACOBSON, Lecture in Abstract Algebra II, Linear Algebra, New York, 1953.
24. N. JACOBSON, Structure of rings. Amer. Math. Soc. Colloq. Publ. 37, Providence R.I. 1964.
25. N. JACOBSON, Structure and representations of Jordan algebras. Amer. Math. Soc. Colloq. Publ. 39, Providence R.I. 1968.
26. N. JACOBSON and C.E. RICKART, Homomorphisms of Jordan rings of self-adjoint elements. Trans. Amer. Math. Soc. 72 (1952), 310-322.
27. A.M. KAIDI, Bases para una teoria de las algebras no-associativas normadas. Tesis Doctoral. Universidad de Granada. Granada 1977.
28. K. KAPLANSKY, Normed algebras. Duke Math. J. 16 (1949), 399-418.
29. E. KLEINFELD, Primitive alternative rings and semisimplicity. Amer. J. Math. 77 (4) (1955), 725-730.
30. K. McCRIMMON, A note on quasi-associative algebras. Proc. Amer. Math. Soc. 17 (1966), 1455-1459.
31. K. McCRIMMON, On Herstein's theorem relating Jordan and associative algebras. J. Algebra 13 (1969), 382-392.
32. K. McCRIMMON, Noncommutative Jordan rings, Trans. Amer. Math. Soc. 158 (1971), 1-33.
33. K. McCRIMMON, Zelmanov's prime theorem for quadratic Jordan algebras. J. Algebra 76 (1982), 297-326.