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**HOLOMORPHIC FUNCTIONAL CALCULUS IN  
JORDAN-BANACH ALGEBRAS**

**J. MARTINEZ MORENO**

The purpose of this article is to establish the Holomorphic Functional Calculus for a single element in a noncommutative Jordan-Banach algebra. For Jordan-Banach algebras this was discussed in [4] and the noncommutative case in [7].

**1. PRELIMINARY**

In what follows we consider algebras over a field of characteristic different from two.

Recall that a subset  $B$  of an algebra  $A$  is a commutative subset of  $A$  if  $[a,b] = 0$  for all  $a,b$  in  $B$ .

An algebra  $A$  is called a noncommutative Jordan algebra if, for all  $a$  in  $A$ ,  $\{L_a, R_a, L_{a^2}, R_{a^2}\}$  is a commutative subset of  $L(A)$ , where  $L(A)$  denotes the algebra of linear operators on  $A$ , and  $L_a$  and  $R_a$  denote the left and right multiplication operators by an element  $a$  of  $A$ . An algebra  $A$  is said to be a Jordan algebra if  $L_a = R_a$  and  $[R_a, R_{a^2}] = 0$  for all  $a$  in  $A$ .

If  $A$  is a noncommutative Jordan algebra and if we replace the given product in  $A$  by the Jordan product  $a \cdot b = \frac{1}{2}(ab + ba)$ , then we obtain a Jordan algebra which will be denoted by  $A^+$ .

The concept of invertible element in a Jordan algebra is based on the following fact.

**Proposition 1.1** ([3], chap. I, pag. 51) Let  $A$  be an associative algebra with unit  $I$ . Then  $ab = ba = I$  if and only if  $a \cdot b = I$  and  $a^2 \cdot b = a$  for all  $a, b$  in  $A$ .

This proposition says that the concept of invertible element in an associative algebra with unit as well as the inverse of such an element can be expressed by mean of the Jordan product of  $A$ .

These considerations lead to the following definition.

If  $A$  is a Jordan algebra with unit  $I$ , then an element  $a$  in  $A$  is called invertible with  $b$  as inverse if the identities

$$ab = I \quad \text{and} \quad a^2b = a$$

hold in  $A$ .

If  $A$  is a noncommutative Jordan algebra with unit  $I$  and  $a, b$  are in  $A$  satisfying

$$(1) \quad ab = ba = I \quad \text{and} \quad a^2b = ba^2 = a$$

then  $a$  is invertible in the Jordan algebra  $A^+$  with  $b$  as inverse.

Conversely, [6], if  $a$  is invertible in  $A^+$  with  $b$  as inverse, then  $a, b$  satisfy the equalities (1).

Consistently we have :

**Definition 1.2.** Let  $A$  be a noncommutative Jordan algebra with unit  $I$ . An element  $a$  in  $A$  is called invertible with  $b$  as inverse if  $a, b$  satisfy the equalities (1).

From the above comments together with the properties of inverses in Jordan algebras [3, Chap. 1, pag. 52] it follows that if  $a$  is an invertible element of a noncommutative Jordan algebra, then the inverse of  $a$  is unique and will be denoted  $a^{-1}$ .

Lastly we shall recall that a subalgebra  $B$  of a noncommutative Jordan algebra  $A$  with unit is called a full subalgebra of  $A$ , if  $B$  contains the unit of  $A$  and the inverses of those of their elements which are invertible in  $A$ .

2. SPECTRAL THEORY AND FUNCTIONAL CALCULUS

**Definition 2.1** A subalgebra  $B$  of a Jordan algebra  $A$  is called a strongly associative subalgebra if  $[R_a, R_b] = 0$  for all  $a, b$  in  $B$ .

The condition is clearly equivalent to

$$(ax)b - a(xb) = 0 \text{ for } a, b \text{ in } B \text{ and } x \text{ in } A.$$

In particular every strongly associative subalgebra of  $A$  is associative.

If  $B$  is any subalgebra of  $A$ , we write  $R_A(B)$  for the set  $\{R_b: b \in B\}$  and  $R_A(B)'$  for the subalgebra of  $L(A)$  generated by  $R_A(B)$  and the identity mapping of  $L(A)$ . Since an associative algebra is commutative if and only if it has a set of generators which commute, it is clear that  $B$  is a strongly associative subalgebra of a Jordan algebra  $A$  if  $R_A(B)'$  is a commutative algebra of linear transformations.

The following lemma exhibits a set of generators for  $R_A(B)'$ .

**Lemma 2.2.** ([3], Chap. 1, pag. 42) Let  $A$  be a Jordan algebra with unit  $I$ ,  $B$  a subalgebra of  $A$  containing  $I$ ,  $G$  a set of generators of  $B$  containing  $I$ . Then the set of operators

$$\{U_{a,b} \equiv R_a R_b + R_b R_a - R_{ab}: a, b \text{ in } G\}$$

is a set of generators for  $R_A(B)'$ .

**Theorem 2.3.** (N. Jacobson, private communication). Let  $A$  be a Jordan algebra with unit  $I$ ,  $B$  a strongly associative subalgebra containing  $I$ ,  $C$  a subset of  $B$  such that if  $c$  is in  $C$  then the inverse  $c^{-1}$  exists in  $A$  and  $D$  the subalgebra of  $A$  generated by  $B \cup \{c^{-1}: c \in C\}$ . Then  $D$  is a strongly associative subalgebra of  $A$ .

**Proof.** By the above lemma the set

$$\{U_{a,b}: a, b \in B \cup \{c^{-1}: c \in C\}\}$$

is a system of generators for  $R_A(D)'$ .

Therefore it is enough to prove that any two of the operators  $U_{a,b}$ ,  $U_{c,d}$  with  $a,b,c,d$  in  $B \cup \{c^{-1}: c \in C\}$  commute.

If we denote  $U_x$  for  $U_{x,x}$ , it follows from the fundamental identity ([3], Chap. 1, pag. 52)

$$U_x U_y U_x = U_{U_x(y)}$$

that

$$(2) \quad U_x U_{y,z} U_x = U_{U_x(y)}, U_x(z)$$

So, if  $a$  is in  $B$  and  $c$  is in  $C$ , we have

$$U_c U_{c^{-1},a} U_c = U_{U_c(c^{-1}),U_c(a)} = U_{c,U_c(a)}$$

and since  $U_{c^{-1}} = U_c^{-1}$  ([3], Chap. I, pag. 52) we have

$$U_{c^{-1},a} = U_c^{-1} U_{c,U_c(a)} U_c^{-1}$$

Finally if  $a,b$  are in  $C$ , by the identity (2) and since  $R_A(B)'$  is commutative, we have

$$U_b U_a U_{a^{-1},b^{-1}} U_a U_b = U_b U_{U_a(a^{-1}),U_a(b^{-1})} U_b = U_b U_{a,U_a(b^{-1})} U_b =$$

$$U_{U_b(a),U_b(U_a(b^{-1}))} = U_{U_b(a),U_a(U_b(b^{-1}))} = U_{U_b(a),U_a(b)}$$

Therefore

$$U_{a^{-1},b^{-1}} = U_a^{-1} U_b^{-1} U_{U_b(a),U_a(b)} U_b^{-1} U_a^{-1}$$

Hence  $U_{c^{-1},a}$  as well as  $U_{a^{-1},b^{-1}}$  are products of an element in  $R_A(B)'$  by inverses of elements of this algebra.

The rest of the proof follows from the fact that, if  $A$  is an associative algebra with unit and if  $a$  is an invertible element of  $A$  commuting with another element  $b$ , then  $a^{-1}$  commutes with  $b$ .

The set of strongly associative subalgebras of a Jordan algebra is

inductively ordered by inclusion.

Therefore each strongly associative subalgebra is contained in a maximal strongly associative subalgebra. This argument and the above theorem lead to the following corollary.

**Corollary 2.4.** Let  $A$  be a Jordan algebra with unit. Then each maximal strongly associative subalgebra of  $A$  is a full subalgebra of  $A$ .

If  $A$  is a Jordan algebra and  $a$  is in  $A$  then ([3], Chap. I, pag. 35)  $[R_{a^n}, R_{a^m}] = 0$  for every positive integer  $n$  and  $m$ . Hence any subalgebra of  $A$  generated by a single element is a strongly associative subalgebra. Also if  $A$  has a unit then it is clear that the subalgebra generated by the unit element and any single element of  $A$  is a strongly associative subalgebra. We now have :

**Corollary 2.5.** Let  $A$  be a Jordan algebra. For each  $a$  in  $A$  there is a maximal strongly associative (obviously commutative) subalgebra  $B$  of  $A$  which includes the element  $a$ . Moreover, when  $A$  has a unit element,  $B$  is a full subalgebra of  $A$ .

We now proceed to study the general case of a noncommutative Jordan algebra.

**Proposition 2.6.** Let  $A$  be a noncommutative Jordan algebra. Then each commutative subset of  $A$  is included in a maximal commutative subset of  $A$ . Each maximal commutative subset is a subalgebra of  $A$  which is full if  $A$  has a unit element.

**Proof.** The set of commutative subsets of  $A$  is inductively ordered by inclusion. Therefore each commutative subset is contained in a maximal commutative subset. Let  $B$  a maximal commutative subset of  $A$ . Clearly  $B$  is a subspace of  $A$ . Given  $a, b$  in  $B$ , we have  $ab = a.b$  and since the mapping  $x \rightarrow [x, .]$  is a derivation of  $A^+$  ([8], Chap. V, pag. 146) we have  $[x, ab] = 0$  for all  $x$  in  $B$ . Then, by maximality of  $B$ , it follows that  $ab$  is in  $B$ , so  $B$  is a subalgebra. If  $A$  has a unit element  $I$ , clearly  $I$  is in  $B$ . Given  $a$  in  $B$  such that  $a$  is invertible in  $A$  with  $a^{-1}$  as inverse we know that  $a$  is invertible in  $A^+$  with  $a^{-1}$  as inverse. Now using again the fact that the mapping  $x \rightarrow [x, .]$  is a derivation of  $A^+$  by ([3], Chap. I, pag. 54) we have

$$[x, a^{-1}] = -U_{a^{-1}}[x, a] = 0 \text{ for all } x \text{ in } B.$$

Hence by maximality of  $B$ ,  $a^{-1}$  is in  $B$ . Therefore  $B$  is a full commutative subalgebra.

A real or complex algebra  $A$  with a norm  $\|\cdot\|$  such that  $\|ab\| \leq \|a\| \|b\|$  is called a normed algebra. If  $A$  is a normed algebra, clearly each maximal commutative subset of  $A$  is closed. Moreover if  $A$  is also a Jordan algebra each maximal strongly associative subalgebra is closed.

**Theorem 2.7.** Let  $A$  be a noncommutative Jordan algebra. For each  $a$  in  $A$  there is an associative and commutative subalgebra  $B$ , full if  $A$  has a unit element and closed if  $A$  is normed, such that  $a$  is in  $B$ .

**Proof.** Let  $C$  be a maximal commutative subset of  $A$  such that  $a$  is in  $C$ . By the above proposition  $C$  is a (commutative) subalgebra of  $A$ , full if  $A$  has a unit element. Furthermore if  $A$  is normed then  $C$  is closed. Now by corollary 2.5 there is a maximal strongly associative (and commutative) subalgebra  $B$  of the Jordan algebra  $C$  containing  $a$ . In addition if  $A$  has a unit element then  $B$  is full in  $C$  and since  $C$  is full in  $A$  we have that  $B$  is a full subalgebra of  $A$ . Finally, if  $A$  is normed then  $B$  is closed in  $A$  since  $C$  is closed in  $A$  and  $B$  is closed in  $C$ .

Let  $A$  be a noncommutative Jordan algebra with unit  $I$ ; the set of invertible elements of  $A$  is denoted by  $\text{Inv}(A)$ . If  $A$  is a complex algebra, the spectrum of an element  $a$  of  $A$  is the set  $\text{Sp}(A, a)$  of complex numbers defined as follows

$$\text{Sp}(A, a) = \{z \in \mathbb{C} : zI - a \notin \text{Inv}(A)\}$$

Since  $A^+$  is a Jordan algebra with unit and since an element  $a$  is invertible in  $A$  if and only if  $a$  is invertible in  $A^+$  and the inverse  $a^{-1}$  of  $a$  is the same for  $A$  and  $A^+$ , we have

$$\text{Inv}(A) = \text{Inv}(A^+) \text{ and } \text{Sp}(A, a) = \text{Sp}(A^+, a) \text{ for all } a \text{ in } A.$$

We now prove the following basic theorem.

**Theorem 2.8.** Let  $A$  be a complete normed noncommutative Jordan algebra with unit  $I$ . Then

- (i)  $\text{Inv}(A)$  is an open subset of  $A$
- (ii) The mapping  $a \rightarrow a^{-1}$  from  $\text{Inv}(A)$  into  $A$  is differentiable and the differential at each  $a$  in  $\text{Inv}(A)$  is  $-U_{a^{-1}}$ , where  $U_{a^{-1}}$  is the linear operator defined on the Jordan algebra  $A^+$ .
- (iii) If  $A$  is a complex algebra, then for each  $a$  in  $A$  we have :

$\text{Sp}(A, a)$  is a non-void compact subset of  $\mathbb{C}$   
 $\lim_{n \rightarrow \infty} \|a^n\|^{1/n} = \max \{|z| : z \in \text{Sp}(A, a)\}$   
 The mapping  $z \rightarrow (a-zI)^{-1}$  is a holomorphic mapping from  $\mathbb{C}-\text{Sp}(A, a)$  into  $A$  and  $\frac{d}{dz} (a-zI)^{-1} = (a-zI)^{-2}$ .

**Proof.** We may assume that  $A$  is a Jordan algebra.

(i) By ([3], Th. 13, Chap. I, pag. 52) an element  $a$  of  $A$  is in  $\text{Inv}(A)$  if and only if  $U_a$  is invertible in the Banach algebra  $\text{BL}(A)$  of all continuous linear operators on  $A$ . Therefore  $\text{Inv}(A)$  is open since  $\text{Inv}(A) = f^{-1}(\text{Inv}(\text{BL}(A)))$ , where  $f$  is the continuous mapping  $a \rightarrow U_a$  of  $A$  into  $\text{BL}(A)$  and  $\text{Inv}(\text{BL}(A))$  is an open subset of  $\text{BL}(A)$ .

(ii) For  $a$  in  $\text{Inv}(A)$  we have  $a^{-1} = U_a^{-1}(a)$  ([3], Th. 13, pag. 52). Thus the mapping  $g : a \rightarrow a^{-1}$  from  $\text{Inv}(A)$  into  $A$  is continuous (recall the continuity of the inverse in associative Banach algebras).

Now we prove that the differential of  $g$  at each  $a$  in  $\text{Inv}(A)$  is  $-U_{a^{-1}}$ .

If  $B$  is an associative algebra, then for any  $x$  and  $y$  invertible elements in  $B$  we have in  $B^+$

$$(3) \quad x^{-1} - y^{-1} + U_{y^{-1}}(x-y) = U_{y^{-1}}(U_{x^{-1}}(x-y))$$

Therefore, by the Shirshov-Cohn theorem with inverses [5], the identity (3) is verified for all invertible elements  $x, y$  in any Jordan algebra.

We now have

$$\|g(x) - g(a) + U_{a^{-1}}(x-a)\| = \|U_{a^{-1}}(U_{x^{-1}}(x-a))\| \leq 3 \|U_{a^{-1}}\| \|x-a\|^2 \|x^{-1}\|$$

Hence

$$\lim_{x \rightarrow a} \frac{\|g(x) - g(a) + U_{a^{-1}}(x-a)\|}{\|x-a\|} = 0$$

and  $-U_{a^{-1}}$  is the differential of  $g$  at  $a$ .

(iii) For  $a$  in  $A$ , by theorem 2.6 there is a full associative Banach subalgebra  $B$  of  $A$  such that  $a$  is in  $B$ . Since  $B$  is a full subalgebra of  $A$ ,  $Sp(A,a) = Sp(B,a)$ . Therefore (iii) follows from the general theory of Banach algebras [1].

Remark. The statement (iii) in our preceding theorem was first proved for Jordan algebras by Viola [9].

Once the "full closed associative" localization theorem (Theorem 2.7) has been proved and the properties of the spectrum have been recalled (Theorem 2.8), we conclude this paper by applying these results to obtain the Holomorphic Functional Calculus for a single element of a noncommutative complex Jordan-Banach algebra.

Recall that given subsets  $K, D$  of  $\mathbb{C}$  with  $K$  compact,  $D$  open and  $K \subset D$ , an envelope for  $(K, D)$  is a bounded open subset  $W$  of  $\mathbb{C}$  such that  $K \subset W \subseteq D$  and the boundary  $\partial W$  of  $W$  consists of a finite number of closed smooth curves positively oriented surrounding only once each point in  $W$ .

Theorem 2.9. Let  $A$  be a complete normed complex noncommutative Jordan algebra with unit  $I$  and  $a$  an element in  $A$ . Let  $D$  be an open neighbourhood of  $Sp(A,a)$  and let  $H(D)$  be the space of all holomorphic mappings from  $D$  into  $\mathbb{C}$ . For each  $f$  in  $H(D)$  we consider the element of  $A$  defined by

$$f(a) = \frac{1}{2\pi i} \int_{\partial W} f(z) (zI-a)^{-1} dz$$

where  $W$  is an envelope for  $(Sp(A,a), D)$ . Then

- (i)  $f(a)$  is independent of the choice of the envelope  $W$  for  $(Sp(A,a), D)$
- (ii) The mapping  $f \rightarrow f(a)$  is a homomorphism from  $H(D)$  into  $A$  that maps the unit of  $H(D)$  into the unit of  $A$  and the mapping  $z \rightarrow z$  into the element  $a$ .

(iii) If we give  $H(D)$  the topology of uniform convergence on compact subsets, the mapping  $f \rightarrow f(a)$  of  $H(D)$  into  $A$  is continuous.

(iv) (Spectral Mapping Theorem) For any  $f$  in  $H(D)$

$$\text{Sp}(A, f(A)) = \{f(z) : z \in \text{Sp}(A, a)\}.$$

Proof. By theorem 2.7 there is a full commutative Banach (associative) subalgebra  $B$  of  $A$  such that  $a$  is in  $B$ . Clearly  $f(a)$  is in  $B$  and  $\text{Sp}(A, x) = \text{Sp}(B, x)$  for all  $x$  in  $B$  since  $B$  is a full subalgebra of  $A$ . Therefore the proof can be reduced to the associative case ([2], Chap. V).

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